Passive Properties of Dynamic Neural Networks

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Abstract

In this paper the passivity approach is used to access several stability properties of dynamic neural networks. By using a simple gradient learning law, the conditions for passivity, stability, asymptotic stability and input-to-state stability are established.

Keywords: dynamic neural networks, stability, passivity

1 INTRODUCTION

In many application, learning-based control using neural networks has emerged as a viable tool for the control of nonlinear systems with unknown dynamics. This model-free approach is presented as a nice feature of neural networks, but the lack of model for the controlled plant makes hard to obtain theoretical results on the stability and performance of neuro system. For the engineers it is very important to assure the neuro controller can be stabilized in theory before they apply it to a real system.

There are not many results on stability analysis of neural networks in spite of their successful applications. The global asymptotic stability (GAS) of dynamic neural networks has been developed during the last decade. Diagonal stability [6] and negative semi-definiteness [7] of the interconnection matrix may make Hopfield-Tank neuro circuit GAS. Multilayer perceptron (MLP) and recurrent neural networks can be related to the Lur'e systems, the absolute stabilities were developed by [13] and [9]. Input-to-state stability (ISS) analysis method is an effective tool for dynamic neural networks, [12] stated that if the weights are small enough, neural networks are ISS and GAS with zero input.

Many publishes investigate the stability of identification error and tracking error of neural networks. [5] studied the stability conditions when multilayer perceptrons are used to identify and control a nonlinear system. Lyapunov-like analysis is suitable for dynamic neural network, the signal-layer case were discussed in [11] and [14], the high-order networks and multilayer networks may be found in [8] and [10].

Another stability analysis tool is passivity theory, it may deal with the poor define nonlinear systems, usually by means of sector bounds. But it offers elegant solutions for the proof of absolute stability. A promising approach to stability analysis of neuro systems may be the passivity framework, because it can lead to general conclusions on the stability using only input-output characteristics. The passivity properties of MLP were examined in [2]. By means of analyzing the interconnected of error models, they derived the relationship between passivity and closed-loop stability. To the best of our knowledge, open loop analysis based on the passivity method for dynamic neural networks has not yet been established in the literature.

In this paper, the passivity method is used to develop the stability properties of dynamic neural network. It is shown that a gradient-like learning law will make the dynamic neural network stable. With additional conditions, th neural networks are GAS and ISS. The paper is organized as follows. Section 2 discusses the preliminaries and main stability proprieties. Simulation results are presented in Section 3. The conclusions and highlights of this paper are summarized in Section 4.

2 Stability Properties of Dynamic Neural Networks

Consider a class of nonlinear systems given by

\[ \begin{align*}
    x_t &= f(x_t, u_t) \\
    y_t &= h(x_t, u_t)
\end{align*} \] (1)

where \( x_t \in \mathbb{R}^n \) is the state, \( u_t \in \mathbb{R}^m \) is the input vector, \( y_t \in \mathbb{R}^m \) is the output vector. \( f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is locally Lipschitz, \( h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \) is continuous. It is also assumed that for any \( x_0 = x_0 \in \mathbb{R}^n \), the output \( y_t = h(\Phi(t, x_0, u)) \) of system (1) is such that \( \int_0^t \| u_s \|_2 \, ds < \infty \), for all \( t \geq 0 \), i.e. the energy
stored in system (1) is bounded. Following to [1] and [3], let us now recall some passivity properties as well as some stability properties of passive systems.

**Definition 1** A system (1) is said to be passive if there exists a \( C^1 \) nonnegative function \( S(x_t) : \mathbb{R}^n \rightarrow \mathbb{R} \), called storage function, such that, for all \( u_t, \) all initial conditions \( x^0 \) and all \( t \geq 0 \) the following inequality holds:

\[
\dot{S}(x_t) \leq u_t^T y_t - \varepsilon u_t^T u_t - \delta y_t^T y_t - \rho \psi(x_t)
\]

where \( \varepsilon, \delta \) and \( \rho \) are nonnegative constants, \( \psi(x_t) \) is positive semidefinite function of \( x_t \) such that \( \psi(0) = 0 \). \( \rho \psi(x_t) \) is called state dissipation rate. Furthermore, the system is said to be

- **lossless if** \( \varepsilon = \delta = \rho = 0 \) and \( \dot{S}(x_t) = u_t^T y_t; \)
- **input strictly passive if** \( \varepsilon > 0 \)
- **output strictly passive if** \( \delta > 0 \)
- **state strictly passive if** \( \rho > 0 \)
- **strictly passive if** there exists a positive definite function \( V(x_t) : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( \dot{S}(x_t) = u_t^T y_t - V(x_t) \)

**Property 1.** If the storage function \( S(x_t) \) is differentiable and the dynamic system is passive, storage function \( S(x_t) \) satisfies

\[
\dot{S}(x_t) \leq u_t^T y_t
\]

**Definition 2** A system (1) is said to be globally input-to-state stable if there exists a \( C^1 \)-function \( \gamma(s) \) (continuous and strictly increasing \( \gamma(0) = 0 \)) and \( K \) -function \( \beta(s,t) \) \( (K \) -function and for each fixed \( s_0 \geq 0 \), \( \lim_{t \to \infty} \beta(s_0,t) = 0 \) ), such that, for each \( u \in L_{\infty} \) \( (\sup \{ ||u(t)||, t \geq 0 \} < \infty) \) and each initial state \( x^0 \in \mathbb{R}^n \), it holds that

\[
||x(t,x^0,u_t)|| \leq \beta(||x^0||,t) + \gamma(||u_t||)
\]

for each \( t \geq 0 \).

**Property 2.** If a system is input-to-state stable, the behavior of the system should remain bounded when its inputs are bounded.

Let us consider the following dynamic neural networks (see Figure 1)

\[
\dot{x}_t = Ax_t + W_{1,t} \sigma(V_{1,t}x_t) + W_{2,t} \phi(V_{2,t}x_t)u_t
\]

where \( \forall t \in [0,\infty) \), the vector \( x_t \in \mathbb{R}^n \) is the state of the neural network, \( u_t \in \mathbb{R}^m \) is the given control vector field. The matrix \( A \in \mathbb{R}^{n \times n} \) is stable. The matrices \( W_{1,t} \in \mathbb{R}^{n \times m}, W_{2,t} \in \mathbb{R}^{m \times n} \) and \( V_{1,t} \in \mathbb{R}^{m \times n}, V_{2} \in \mathbb{R}^{m \times n} \) are the weights describing output and hidden layers connections. The vector field \( \sigma(x_t) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is assumed to have the elements increasing monotonically. The function \( \phi(\cdot) \) is the transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^{m \times n} \). The typical presentation of the elements \( \sigma_i(\cdot) \) and \( \phi_{ij}(\cdot) \) are as sigmoid functions

\[
\sigma_i(x_{i,t}) = a_i/ (1 + e^{-h\cdot x_{i,t}}) - c_i.
\]

**Remark 1** The neural networks have been discussed by many authors, for example [11], [8], [10] and [14]. One may see that Hopfield model [4] is the special case of this networks with \( A = \text{diag} \{ a_i \} \), \( a_i := -1/R_i C_i, R_i > 0 \) and \( C_i > 0 \). \( R_i \) and \( C_i \) are the resistance and capacitance at the \( i \)th node of the network respectively.

**Remark 2** The sub-structure \( W_{1,t} \sigma(V_{1,t}x_t) \) and \( W_{2,t} \phi(V_{2,t}x_t)u_t \) are multilayer perceptrons. The continuous-time learning law for these kinds of neural networks were studied in [5].

**Theorem 1** If the weights \( W_{1,t} \) and \( W_{2,t} \) are updated as

\[
W_{1,t} := -P x_t \sigma^T + \frac{1}{2} x_t u_t^T
\]

\[
W_{2,t} := -P x_t (\phi u_t)^T + \frac{1}{2} x_t u_t^T
\]

where \( P \) is the solution of Lyapunov function

\[
PA + A^T P = -Q, \quad Q = Q^T > 0
\]
then the dynamic neural network (2) is strictly passive from input $u_t$ to output
\[ y_t = (W_{1,t}^T + W_{2,t}^T) x_t \] (5)

**Proof:** Select a Lyapunov function (storage function) as
\[ S_t = x_t^T P x_t + \text{tr} \left\{ W_{1,t}^T W_{1,t} \right\} + \text{tr} \left\{ W_{2,t}^T W_{2,t} \right\} \] (6)
where $P \in \mathbb{R}^{n \times n}$ is positive definite matrix. According to (2), the derivative is
\[
\dot{S}_t = x_t^T (PA + AP^T) x_t + 2x_t^T P W_{1,t} \sigma(V_{1,t} x_t) + 2x_t^T P W_{2,t} \phi(V_{2,t} x_t) u_t + 2\text{tr} \left\{ W_{1,t}^T W_{2,t} \right\}
\]
Adding and subtracting $x_t^T (W_{1,t} + W_{2,t}) u_t$ and using (4), we obtain
\[
\dot{S}_t = -x_t^T Q x_t + x_t^T (W_{1,t} + W_{2,t}) u_t + 2\text{tr} \left\{ W_{1,t}^T W_{2,t} \right\}
\]

**Remark 3** Since $\sigma(\cdot)$ and $\phi(\cdot)$ are bounded, the passivity property has no relationship with $V_{1,t}$ and $V_{2,t}$. The weights of hidden layer may be fixed. We can conclude that the stability properties of dynamic neural networks (2) are not influenced by hidden layers.

**Corollary 1** Using the updating law as (3), the equilibrium $x_t = 0$ of dynamic neural networks (2) with $u_t = 0$ is stable.

**Proof:** Since the dynamic system (2) is passive, from the Property 1 the storage function $S(x_t)$ satisfies
\[ \dot{S}(x_t) \leq u_t^T y_t = 0 \]
Since $S(x_t)$ is positive define, the equilibrium $x_t = 0$ of $x = f(x_t, 0)$ is stable.

**Corollary 2** If the feedback control $u_t$ is selected as
\[ u_t = -\mu y_t, \quad \mu > 0 \] (8)
where $y_t$ is defined as (5), using the updating law as (3), the equilibrium $x_t = 0$ of dynamic neural networks (2) is asymptotic stability.

**Proof:** Because $y_t = h(x_t)$ is independent of $u_t$, the feedback loop with $u_t = -\mu y_t$ is well posed. For $u_t = -\mu y_t$, the time derivative of $S$ satisfies
\[ \dot{S}(x_t) \leq -\mu y_t^T y_t \leq 0 \]
So the equilibrium $x_t = 0$ of $x_t = f(x_t, -y_t)$ is stable. Based on the Invariance Principle, the bounded solutions of $x_t = f(x_t, y_t)$ converge to the largest invariant set of $x_t = f(x_t, 0)$ contained in $E = \{x_t | h(x_t) = 0\}$, this set is $x_t = 0$, so the asymptotic stability is proved.

**Theorem 2** If the upper bound of the weights satisfies
\[ \lambda_{\max} (\overline{W}_1 + \overline{W}_2) \leq \lambda_{\min} (Q) \] (9)
where $\overline{W}_1 = \max ||W_{1,t}||_A^{-1}$, $\overline{W}_2 = \max ||W_{2,t}||_A^{-1}$, $\Lambda_1$ and $\Lambda_2$ are positive defined matrices, using the updating law as (3), the dynamic neural networks (2) is input-to-state stability (ISS).

**Proof:** In view of the matrix inequality
\[ X^T Y + (X^T Y)^T \leq X^T \Lambda^{-1} X + Y^T \Lambda Y \] (10)
which is valid for any $X, Y \in \mathbb{R}^{n \times k}$ and for any positive defined matrix $0 < \Lambda = \Lambda^T \in \mathbb{R}^{n \times n}$, (7) can be represented as
\[
\dot{V}_t = -x_t^T Q x_t + x_t^T (W_{1,t} + W_{2,t}) u_t \leq -\lambda_{\min} (Q) ||x_t||^2 + x_t^T W_{1,t} \Lambda_1^{-1} W_{1,t}^T x_t + x_t^T W_{2,t} \Lambda_2^{-1} W_{2,t}^T x_t + u_t^T \Lambda_1 u_t + u_t^T \Lambda_2 u_t \leq -\lambda_{\min} (Q) ||x_t||^2 + x_t^T (\overline{W}_1 + \overline{W}_2) x_t + u_t^T \Lambda_1 u_t + u_t^T \Lambda_2 u_t \leq -\alpha (Q) ||x_t||^2 + \beta (\Lambda_1 + \Lambda_2) ||u_t||^2 \]
where $\alpha (Q) := \lambda_{\min} (Q) - \lambda_{\max} (\overline{W}_1 + \overline{W}_2)$, $\beta (\Lambda_1 + \Lambda_2) := \lambda_{\max} (\Lambda_1 + \Lambda_2)$. Obviously $\alpha (Q)$ and $\beta (\Lambda_1 + \Lambda_2)$ are $\mathcal{L}_2$ functions. So $V_t$ is also an ISS-Lyapunov function which means that the dynamic neural networks (2) is input to state stability.

**Remark 4** Consider Property 2, ISS means that the behavior of the dynamic neural networks should remain bounded when its input is bounded. If we regard the bounded disturbances as other inputs, the system is also bounded with the disturbances.
Remark 5 If $P$ is selected big enough, the condition (9) is not difficult to be satisfied. From (4) we have

$$TPA + TA^TP = -TQ, \quad Q = Q^T > 0, T = T^T > 0$$

(11)

If the positive defined matrix $T$ is big enough, (9) is satisfied, i.e.

$$\lambda_{\max} (W_1 + W_2) \leq \lambda_{\min} (TQ)$$

3 Simulation

To illustrate the theory results, we give following simulations.

The dynamic neural networks is

$$\dot{x}_t = Ax_t + W_{1,t}\sigma(V_{1,t}x_t) + W_{2,t}\phi(V_{2,t}x_t)u_t$$

(12)

where $x_t \in \mathbb{R}^2$, $u_t \in \mathbb{R}^3$, $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. The initial conditions are $x_0 = [2, 5]^T$, $W_{1,0} = \begin{bmatrix} -5 & -4 & 2 \\ 1 & -2 & -2 \end{bmatrix}$, $W_{2,0} = \begin{bmatrix} 1 & -5 & 2 \\ -6 & 2 & -6 \end{bmatrix}$. $V_{1,t}$ and $V_{2,t}$ are fixed as $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$. The sigmoid functions are selected as

$$\sigma(x) = 2/(1 + e^{-2x}) - \frac{1}{2}$$

$$\phi(x) = 0.2/(1 + e^{-0.2x}) - \frac{1}{20}$$

First, let check the passivity of the neural networks (12). If we select $Q = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$, the solution of Lyapunov equation (4) is $P = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$. The updating law for the weights $W$ are used as (3). Figure ?? shows the bounded inputs responses of the dynamic neural network (12), here $u_1$ is unit step input started at $t = 2$; $u_2$ and $u_3$ are $\sin$ and $\cos$ waves with amplitude 2 and phase 0. Figure 2 shows the zero-input responses. Figure 3 shows the closed-loop responses with the additional output feedback (8) for $\mu = 1$. Clearly, the results obtained confirm that the dynamic neural networks are passivity with the learning law (3), stable with zero-input, and asymptotic stability with output feedback.

Second, let check input-to-state stability (ISS). From (11) we know that the solution of Lyapunov equation (11) is $T$ times as that of (4). Let $T$ as a positive scalar $T = 3$, so $P = \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$. The updating law for the weights $W$ are used as (3). The bounded inputs are used the same as Figure ???. Figure 4 presents the...
If we select $T = 0.1$, so $P = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$. The updating law for the weights $W$ are used as (3), Figure 5 shows the results. One can see that condition (9) is very important for input-to-state stability.

4 Conclusion

In this brief, as our contribution, in order to guarantee stability, asymptotic stability and input-to-state stability, we establish updating conditions for the weights of dynamic neural networks by passivity technique. These results can be easily extended to neuro identification and neuro control.

References


