Measurement Sliding Mode-$\mathcal{H}_\infty$ Control with Application to Decentralized Systems

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Abstract— **The robustness properties of both, sliding mode control and $\mathcal{H}_\infty$ design methodologies are exploited in the context of decentralized control, for which interconnections among subsystems are treated as perturbations. A dynamic surface is proposed for the sliding mode design. The sliding mode control successfully rejects the matched interconnections and reduces the order of the $\mathcal{H}_\infty$ problem (crucial in large-scale systems). $\mathcal{H}_\infty$ control is used to attenuate the unmatched interconnections and relaxes the assumption of full state information. It is demonstrated that the combination of these two methods simplifies the design process, as it allows to design each subsystem separately. A composite Lyapunov function, constructed as a sum of the individual functions of each subsystem is used in order to ensure stability of the overall system. Sufficient conditions for stability are given in terms of the existence of proper solutions to Riccati equations.**

Index Terms— **Sliding mode control, Variable structure systems, Large-scale systems, Distributed control.**

I. Introduction

A. Motivation

Large-scale systems may require the use of decentralized control design when one, or several of the following difficulties occur:

1) The system is widely distributed in space, so information transfer is too costly (e.g. power systems),
2) implementation of a centralized feedback law is hard or impossible due to the system's decentralized structure (e.g. aerial and terrestrial traffic control),
3) the complexity of analysis and design resulting from the system's order can be reduced by splitting the system into several subsystems (e.g. large flexible structures),
4) the design criteria is robustness in the presence of structural perturbations where subsystems are disconnected and again connected during operation.

In general terms, the problem of decentralized control is that of finding a set of controllers satisfying an information constraint: the information available at each control station is only a subset of the measurement variables. The controllers are to be designed for stabilizing the set of interconnected subsystems that comprise the overall system. Clearly, such a decentralized feedback scheme would address difficulties 1 and 2.

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As in the centralized case, several strategies have been proposed in order to solve the problem. For instance, eigenvalue assignment [1], [2] and [3]; or optimal control, which attracted considerable attention [4], [5], [6] and [7]. The main disadvantage of these methods is that, at some stage of the design procedure, a solution to a set of simultaneous equations of at least the same order of the system needs to be found. Thus, difficulty 3 is left unsolved.

An alternative strategy is to consider each system independently and treat interconnections as perturbations. This approach seeks to eliminate, or at least attenuate the perturbations using available robust techniques. A scheme like this one has the advantage of resolving difficulties 3 and 4.

Sliding mode control (SMC) is a powerful and robust technique that fits well into this framework. The sliding mode controller drives the system's state into a "custom built", sliding (switching) surface and constraints the state thereafter. A system motion in a sliding surface, named *sliding mode*, is robust with respect to uncertainties and disturbances matched by a control. Besides robustness, SMC has other features, such as the order reduction of the dynamic equations when the system is in the sliding mode. Examples of the variable structure theory applied to decentralized control can be found in [8], [9] and [10].

Nevertheless, SMC alone has a few disadvantages. One is the sensitivity to unmatched disturbances, which is usually overcome by assuming that disturbances are only of the matched type. Another disadvantage is the requirement of full state feedback, where a *straight forward estimation in place of the states results in detriment of the robust properties of the controller*. The problem of measurement feedback using SMC has been addressed in [11], for example. As an example of unmatched disturbance attenuation see [12]. Although these issues have been successfully resolved, a direct combination of the previous solutions is not satisfactory.

B. Contribution

To overcome the loss in robustness, we consider a dynamic surface, where an $\mathcal{H}_\infty$ *reduced order* observer is used to estimate the state. We show that under reasonable assumptions, the robustness properties of the SMC controller are maintained; and that when combining SMC and $\mathcal{H}_\infty$ techniques, it is possible to achieve several goals at the same time:

- Measurement feedback.
- Matched disturbance annihilation.
- Unmatched disturbance attenuation.

Moreover, because of the SMC, the Riccati equations typical of $\mathcal{H}_\infty$ problems, turn out to be of reduced order.

C. Paper Structure

In this paper, sliding mode control combined with $\mathcal{H}_\infty$ methods is applied to the decentralized control problem. In section II, we give a formal statement of the problem. A short review of the $\mathcal{H}_\infty$ theory is given in section III. In section IV the $\mathcal{H}_\infty$ and SMC techniques are combined, achieving matched disturbance rejection and unmatched disturbance attenuation. In section V the methodologies developed are applied to the decentralized control problem, where a Lyapunov function for the overall, interconnected system is used for stability analysis. Sufficient conditions for stability are then given in terms of resolvability of Riccati equations.

II. Problem Statement

Consider a linear time invariant decentralized system with ν control stations

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(y_i, t) + \sum_{j \neq i}^{\nu} A_{ij} x_j(t) \tag{1a}$$

$$y_i(t) = C_i x_i(t), \quad i = 1, 2, \ldots, \nu \tag{1b}$$

where $x_i(t) \in \Re^{n_i}$ is the state vector, $y_i(t) \in \Re^{p_i}$ are the available measurements and $u_i(x_i, t) \in \Re^{m_i}$ is the control action of the ith station at time $t \in \Re$. Note that u_i satisfies the information constraint, it depends on y_i only. A_i, B_i and C_i are matrices of appropriate dimensions. $\sum_{j \neq i}^{\nu} A_{ij} x_j$ represents the influence of the other stations, where the A_{ij}'s are, again, matrices of appropriate dimensions. In what follows, whenever the subscript i appears, it is assumed that the properties stated hold for all $i = 1, 2, \ldots, \nu$.

The objective is to design each of the control laws u_i so that system (1) is semi-globally asymptotically stable.

Assumption 1: $\operatorname{rank}(C_i B_i) = m_i$.

We will consider system (1) as a set of perturbed systems

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(y_i, t) + \eta_i(x^{\bar{i}}) \\ y_i(t) &= C_i x_i(t). \end{aligned}$$

x, the whole state, is defined as

$$x \triangleq \begin{bmatrix} x_1^T & x_2^T & \ldots & x_\nu^T \end{bmatrix}^T,$$

$x^{\bar{i}}$ is understood in the same way as x, but with x_i removed

$$x^{\bar{i}} \triangleq \begin{bmatrix} x_1^T & x_2^T & \ldots & x_\nu^T & \backslash x_i \end{bmatrix}^T,$$

and the nominal systems are

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(y_i, t) \tag{2a}$$

$$y_i(t) = C_i x_i. \tag{2b}$$

The perturbations η_i, resulting from the interconnections, are thus defined as

$$\begin{aligned} \eta_i(x^{\bar{i}}) &= \sum_{j \neq i}^{\nu} A_{ij} x_j(t) \qquad (3) \\ &= A^{\bar{i}} x^{\bar{i}}, \quad A^{\bar{i}} \triangleq \begin{bmatrix} A_{i1} & A_{i2} & \ldots & A_{i\nu} & \backslash A_{ii} \end{bmatrix} \end{aligned}$$

Assumption 2: The initial state $x(t_0)$ is bounded by a known constant q, $\|x(t_0)\| \leq q$.

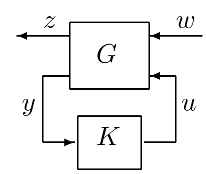
With the preceding assumptions in mind, we can restate our problem as follows: *design ν control laws for the nominal systems (2), so that perturbations (3) do not affect the stability of system (1).*

III. $\mathcal{H}_\infty$ Control, Short Review

In this section we make a short review of the standard $\mathcal{H}_\infty$ state-space control problem as presented in [13], where the methodology is paralleled to the one proposed in $\mathcal{H}_2$ theory, originally termed Linear Quadratic Gaussian (LQG).

A. Overview

According to [14], the most general block diagram of a control system is



where G is called the generalized plant and K is the controller. The output z is a penalty variable which may contain an error signal; y is composed of the available measurement variables; u is the control input and w contains all external inputs, including disturbances, sensor noise and commands. The resulting closed-loop transfer function from w to z is denoted by T_{zw}.

The objective is to minimize $\|T_{zw}\|_\infty$, where the $\mathcal{H}_\infty$ norm, defined in the frequency domain of a stable transfer matrix $G(s)$ is

$$\|G(s)\|_\infty \triangleq \sup_\omega \sigma_{max}[G(jw)].$$

With this objective in mind, the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control algorithms are put in the same setting; as it is shown in [13], the LQG problem, when translated to the frequency domain (i.e., $\mathcal{H}_2$) can be formulated in terms of minimizing $\|T_{zw}\|_2$.

An interesting result arises when the $\mathcal{H}_\infty$ control problem is translated to the time domain. In the general case, the resulting state-space controller has an estimator-state feedback structure, similar to LQG controllers.

Despite the similarities outlined, there is a fundamental difference: when the infinite norm is used, we can only find explicit solutions for $\|T_{zw}\|_\infty < \gamma$, where γ is grater than it's optimum value. For the optimum to be found, a search is required. This stands in contrast to $\mathcal{H}_2$, where the optimum γ can be calculated straight forward.

Whilst the comparison between $\mathcal{H}_2$ and $\mathcal{H}_\infty$ is not strictly necessary in our analysis, it certainly provides insight into the $\mathcal{H}_\infty$ control problem.

B. Classical $\mathcal{H}_\infty$ Control Problem

The $\mathcal{H}_\infty$ control problem can be stated as follows: find a controller K such that $\|T_{zw}\|_\infty < \gamma$. An additional feature is required, *internal stability*. By internal stability we mean that the states of G and K go to zero from all initial values when

$w = 0$. A controller K which is internally stabilizing is said to be *admissible*.

The solution to this problem requires a few assumptions. The generalized plant considered has the form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right],$$

where

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \triangleq C(sI - A)^{-1}B + D.$$

Or in the time domain, G may be written as

$$\dot{x} = Ax + B_1 w + B_2 u \tag{4a}$$

$$z = C_1 x + D_{12} u \tag{4b}$$

$$y = C_2 x + D_{21} w \tag{4c}$$

Assumption 3: (A, B_1) is stabilizable and (C_1, A) is detectable.

Assumption 4:

$$D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$$

Assumption 5:

$$\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Assumption 3 warrants that the resulting controller is admissible. Assumption 4 means that z has no cross weighting between the state and control, and that the control weight matrix is the identity. This assumption is only used to simplify the formulas derived and can be relaxed by a suitable coordinate transformation. Assumption 5 is dual to 4 and serves the same purpose.

C. Solution

The solution is given in the form of a theorem.

Theorem 1: Given assumptions 3-5, there exists an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ iff the following three conditions hold.

1) There exists a real, positive semi-definite matrix X_∞ satisfying the Riccati equation

$$X_\infty A + A^T X_\infty - X_\infty (B_2 B_2^T - \gamma^{-2} B_1 B_1^T) X_\infty + C_1 C_1^T = 0. \tag{5}$$

2) There exists a real, positive semi-definite matrix Y_∞ satisfying the Riccati equation

$$Y_\infty A + A^T Y_\infty - Y_\infty (C_2^T C_2 - \gamma^{-2} C_1 C_1^T) Y_\infty + B_1 B_1^T = 0. \tag{6}$$

3) The spectral radius of $X_\infty Y_\infty$ is less than γ^2.

Moreover, when this conditions hold, one such controller is

$$K_{sub}(s) \triangleq \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right] \tag{7}$$

where

$$\hat{A}_\infty \triangleq A + \gamma^{-2} B_1 B_1^T X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$
$$F_\infty \triangleq -B_2^T X_\infty, \quad L_\infty \triangleq -Y_\infty C_2^T$$
$$Z_\infty \triangleq (I - \gamma^{-2} Y_\infty X_\infty)^{-1} \quad \square$$

Theorem 1 is a restatement of theorem 3 in [13] and it's proof can be found there.

The (suboptimal) controller K_{sub} in equation (7) can be realized with an estimator-state feedback structure as

$$\dot{\hat{x}} = A\hat{x} + B_1 \hat{w}_{\text{worst}} + B_2 u + Z_\infty L_\infty (C_2 \hat{x} - y)$$
$$u = F_\infty \hat{x}, \quad \hat{w}_{\text{worst}} = \gamma^{-2} B_1^T X_\infty \hat{x}.$$

In [15] it is shown that the $\mathcal{H}_\infty$ norm in the frequency domain and the (truncated) $\mathcal{L}_2$ induced norm of a linear system in the time domain are equivalent. This equivalence allows to understand the $\mathcal{H}_\infty$ problem in terms of *disturbance attenuation* and to assess stability in terms of Lyapunov functions. The following theorem is basically a restatement of the results found in [15].

Theorem 2: Suppose that,

1) Assumptions 3, 4 and 5 hold.
2) The conditions of theorem 1 are satisfied.

Then, the controller K_{sub} in (7) is internally stabilizing and

$$\int_0^T z^T(s) z(s) ds \leq \gamma^2 \int_0^T w^T(s) w(s) ds$$

holds for any $T > 0$.

Moreover, the candidate Lyapunov function

$$V(x, \hat{x}) = x^T X_\infty x + \gamma^2 \tilde{x} (Z_\infty Y_\infty)^{-1} \tilde{x} \tag{8}$$

satisfies

$$\dot{V} \leq -\|C_1 x\|^2 - \|u\|^2 + \gamma^2 \|w\|^2 \quad \square \tag{9}$$

IV. Measurement Sliding Mode Control

As in [11], we approach the measurement sliding mode problem by a suitable change of coordinates. In this section however, we take into account the effect of the *unmatched* perturbations.

In what follows, we assume that the measurements have no noise, that is, the exogenous input w only contains disturbances and uncertainties.

The effect of the exogenous input w, via B_1, in system (4) can be represented as a vector consisting of two components, one that belongs to the space spanned by B_2, and another that belongs to the space orthogonal to B_2

$$B_1 w = B_2 B_2^+ B_1 w + B_2^\perp B_2^{\perp +} B_1 w$$
$$B_1 w = B_2 w_m + B_2^\perp w_u,$$

where the pseudoinverse B^+ is defined as

$$B^+ \triangleq (B^T B)^{-1} B^T$$

and

$$w_m = B_2^+ B_1 w, \quad w_u = B_2^{\perp +} B_1 w.$$

The first term, $B_2 w_m$, enters through the same channel as the control and it is said that satisfies the *standard matching* *condition*. Sliding mode control (SMC) is an effective technique with the ability to withstand disturbances of the matched type. The main idea is to use a discontinuous control action in order to force the system's state into a desired surface, regardless of the matched uncertainties. Besides it's robustness property, sliding mode control has another advantage: the dynamic equations are of lower order than the original system. This is due to the fact that the sliding surface, being embedded in the state space, is of lower dimension.

SMC design is usually done in two steps. The first one is to design a desired sliding surface, and the second one is to design the control action which drives the system into the specified surface.

A. The Dynamic Sliding Surface

When full information is available, the sliding surface can be set as

$$\sigma(x) = Sx = 0,$$

where $S \in \Re^{m\times n}$ provides stable dynamics. When the only information available is through the measurement variables y, one might be tempted to design an estimator (e.g., using the $\mathcal{H}_\infty$ techniques described in the previous section) to reconstruct the state and use the estimation in place of x

$$\sigma = S\hat{x}.$$

Following a similar analysis to the one developed in this paper, it is easy to show that an approach like this would be vulnerable to uncertainties, even of the matched type. The reasons should become clear latter, but roughly speaking, the problem is that the estimator filters out uncertainties.

Suppose that the system under consideration has the form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}w_u\\ u+\delta\end{bmatrix} \tag{10a}$$

$$y = \begin{bmatrix}C_{2_1} & C_{2_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}, \tag{10b}$$

and a transformation $\bar{y} = T_y y$ exists, such that

$$\begin{bmatrix}\bar{y}_1\\ \bar{y}_2\end{bmatrix} = \begin{bmatrix}C_{2_{11}} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

where $x_1 \in \Re^{n-m}$, $x_2 \in \Re^{m}$, $\bar{y}_1 \in \Re^{p-m}$ and $\bar{y}_2 \in \Re^{m}$. However, if the system doesn't have this structure, as long as assumption 1 holds, we can always find a suitable transformation. Such transformation is described in the appendix.

This particular form allows the use of x_2 as a virtual control u_v for the reduced order system

$$\dot{x}_1 = A_{11}x_1 + A_{12}u_v + B_{11}w$$
$$B_{11} = B_2^{\perp^+}B_1, \quad u_v = x_2.$$

x_2 can be obtained by a simple transformation on the measurement variables, so it is only necessary to estimate x_1.

Suppose that conditions of theorem 1 are satisfied with

$$\begin{array}{lll} B_1 \to B_{11} & C_1 \to I_{n-m\times n-m} & A \to A_{11}\\ B_2 \to A_{12} & C_2 \to C_{2_{11}} & \end{array}$$

Notice that the reduced dimension of the problem broadens the class of system's satisfying such conditions.

We can estimate x_1 as

$$\dot{\hat{x}}_1 = A_{11}\hat{x}_1 + A_{12}u_v + \hat{w}_{u\,\mathrm{worst}} + Z_\infty L_\infty(C_{2_{11}}\hat{x}_1 - \bar{y}_1)$$

and use the virtual control law

$$u_v = \bar{y}_2 = F_\infty \hat{x}_1,$$

so that the (dynamic) sliding surface becomes

$$\begin{aligned}\sigma &= -F_\infty\hat{x}_1 + \bar{y}_2\\ &= \underbrace{\begin{bmatrix}-F_\infty & I\end{bmatrix}}_{S}\begin{bmatrix}\hat{x}_1\\ \bar{y}_2\end{bmatrix} = 0.\end{aligned}$$

B. Enforcing the Sliding Modes

The discontinuous control proposed is

$$u(\hat{x}_1, y) = -M(\hat{x}_1, \bar{y}_2)\frac{\sigma}{\|\sigma\|},$$

where $M(\hat{x}_1, \bar{y}_2)$ is a positive scalar function satisfying

$$\begin{aligned}M > \|F_\infty A_\infty - A_{21}\|\,\|\hat{x}_1\|\\ + \|F_\infty Z_\infty L_\infty - A_{22}\|\,\|\bar{y}_2\| + \bar{\delta}\end{aligned} \tag{11}$$

with

$$\bar{\delta} > \|\delta - A_{21}\tilde{x}\|$$

By taking $V_\sigma(\sigma) = \|\sigma\|^2/2$ as a Lyapunov function we verify the stability of σ:

$$\dot{V} = \sigma^T\dot{\sigma}, \tag{12}$$

the derivative of σ along time is

$$\begin{aligned}\dot{\sigma} &= S\begin{bmatrix}A_\infty\hat{x}_1 - Z_\infty L_\infty \bar{y}_2\\ A_{21}x_1 + A_{22}\bar{y}_2 + u + \delta\end{bmatrix}\\ &= (-F_\infty A_\infty + A_{21})\hat{x}_1 + (F_\infty Z_\infty L_\infty - A_{22})\bar{y}_2\\ &\quad + \delta - A_{21}x_1 - M\frac{\sigma}{\|\sigma\|}\end{aligned}$$

Combining (12) and (13), one has

$$\begin{aligned}\dot{V}_{\sigma_i} &= -M\|\sigma\| + \sigma^T\Big[(-F_\infty A_\infty + A_{21})\hat{x}_1\\ &\qquad + (F_\infty Z_\infty L_\infty - A_{22})\bar{y}_2 + \delta - A_{21}\tilde{x}_1\Big]\\ &\le -\|\sigma\|\Big[M - \|F_\infty A_\infty - A_{21}\|\,\|\hat{x}_1\|\\ &\qquad + \|(F_\infty Z_\infty L_\infty - A_{22})\|\,\|\bar{y}_2\| + \bar{\delta}\Big]\\ &\le 0.\end{aligned}$$

C. The Sliding Dynamics

To determine the dynamics of the sliding mode, we first obtain the equivalent control u_{eq}, see [16],

$$\begin{aligned}\dot{\sigma} &= -F_\infty(A_{11}\hat{x}_1 + A_{12}x_2 + \hat{w}_{u\,\mathrm{worst}}\\ &\quad + Z_\infty L_\infty(C_{2_{11}}\hat{x}_1 - \bar{y}_1))\\ &\quad + A_{21}x_1 + A_{22}x_2 + u_{eq} + \delta = 0\\ u_{eq} &= -(A_{21}x_1 + A_{22}x_2 + \delta)\\ &\quad + F_\infty(A_{11}\hat{x}_1 + A_{12}x_2 + \hat{w}_{u\,\mathrm{worst}}\\ &\quad + Z_\infty L_\infty C_{2_{11}}(\hat{x}_1 - x_1)),\end{aligned}$$

and then substitute u_{eq} in the system's equations

$$\begin{aligned}\dot{x}_2 &= F_\infty(A_{11}\hat{x}_1 + A_{12}x_2 + \hat{w}_{u\,\text{worst}} \\ &\quad + Z_\infty L_\infty C_{2_{11}}(\hat{x}_1 - x_1)) \\ &= F_\infty(A_{11}x_1 + A_{12}x_2 + w_u + \dot{\tilde{x}}_1) \\ \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + w_u.\end{aligned}$$

Where it is verified that the matched disturbance is successfully rejected.

A similar analysis, using a full order observer would show that the system is susceptible even to matched perturbations.

V. Application to Decentralized Control

We can write system (1) using $\mathcal{H}_\infty$ notation as

$$\begin{aligned}\dot{x}_i &= A_i x_i + \underbrace{B_{i1}}_{A^{\bar{i}}}\underbrace{w_i}_{x^{\bar{i}}} + B_{i2}u_i \\ y_i &= C_{i2}x_i.\end{aligned}$$

After the transformation described in the appendix

$$\begin{aligned}\begin{bmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\end{bmatrix} &= \begin{bmatrix}A_{i11} & A_{i12}\\ A_{i21} & A_{i22}\end{bmatrix}\begin{bmatrix}x_{i1}\\ x_{i2}\end{bmatrix} + \begin{bmatrix}w_{iu}\\ u_i+\delta\end{bmatrix} \\ \begin{bmatrix}\bar{y}_{i1}\\ \bar{y}_{i2}\end{bmatrix} &= \begin{bmatrix}C_{i2_{11}}x_{i1}\\ x_{i2}\end{bmatrix}.\end{aligned}$$

Select a candidate Lyapunov function for the interconnected system

$$V = \sum_{i=1}^{\nu} V_i,$$

with each V_i defined as in (8). For each Lyapunov function

$$\begin{aligned}\dot{V}_i &\le -\|x_{i1}\|^2 - \|x_{i2}\|^2 + \gamma_i^2\|x^{\bar{i}}\|^2 \\ &= -\|x_i\|^2 + \gamma_i^2\|x^{\bar{i}}\|^2,\end{aligned}$$

so the derivative of V along time is

$$\begin{aligned}\dot{V} &\le -\sum_{i=1}^{\nu}\left(\|x_i\|^2 - \gamma_i^2\|x^{\bar{i}}\|^2\right) \\ &= -\|x\|^2 + \sum_{i=1}^{\nu}\left(\gamma_i^2\sum_{j\neq i}^{\nu}\|x_j\|^2\right) \\ &= -\|x\|^2 + \sum_{i=1}^{\nu}\left(\gamma_i^2\sum_{j=1}^{\nu}\|x_j\|^2\right) - \sum_{i=1}^{\nu}\left(\gamma_i^2\|x_i\|^2\right) \\ &\le -\|x\|^2\left(1 - \sum_{i=1}^{\nu}\gamma_i^2\right) \\ &= -\|x\|^2\left(1 - \|\gamma\|^2\right), \quad \gamma \triangleq \begin{bmatrix}\gamma_1 & \dots & \gamma_\nu\end{bmatrix}^T.\end{aligned}$$

So, if $\|\gamma\| < 1$, where each γ_i satisfies the conditions of theorem 1, we have an asymptotically stable system.

We only need to determine the bounds $\bar{\delta}_i$. From the appendix,

$$\delta_i = w_{im} + C''_{21}w_{iu}$$

If the initial estimation error is known, it can be used to estimate $\bar{\delta}$. If not, we may take a conservative approach (setting $\hat{x}(0) = 0$)

$$\bar{\delta}_i = \|B_{i2}^{+}B_{i1} + C''_{i_{21}}B_{i2}^{\perp^+}B_{i1} - A_{i_{21}}\|q$$

Thus, for any initial condition $x(t_0)$ we can find a set $\bar{\delta}_i$. So, if conditions of theorem 1 are satisfied, we obtain a set of controllers achieving semi-global stability for system (1).

A. Step by Step Algorithm Description

To summarize the analysis just developed, we provide the step by step algorithm needed for the design process:

1) Propose γ such that $\|\gamma\|^2 < 1$.
2) Verify conditions of theorem 1. If they hold, determine the sliding surfaces σ_i as in (7).
3) Calculate gain matrices $M_i(\hat{x}_{i1}, \bar{y}_{i2})$ as in (11). Set $u_i(\bar{y}_i) = -M_i(\hat{x}_{i1}, \bar{y}_{i2})\frac{\sigma}{\|\sigma\|}$.

B. Numerical Example

Consider two identical interconnected systems

$$\begin{aligned}\begin{bmatrix}\dot{x}_{i1}\\ \dot{x}_{i2}\\ \dot{x}_{i3}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ -2 & -3 & 4\\ -11 & -6 & -1\end{bmatrix}\begin{bmatrix}x_{i1}\\ x_{i2}\\ x_{i3}\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}u_i + \\ + \begin{bmatrix}0.2 & -0.5 & 0.1\\ -1 & 0.1 & 0.2\\ 12 & 5.5 & 3\end{bmatrix}\begin{bmatrix}x_1^{\bar{i}}\\ x_2^{\bar{i}}\\ x_3^{\bar{i}}\end{bmatrix}\end{aligned} \quad (15)$$

with outputs

$$\begin{bmatrix}y_{i1}\\ y_{i2}\end{bmatrix} = \begin{bmatrix}5 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad i = 1, 2.$$

Notice that the system is already in the form of equation (10), so we proceed directly with the algorithm above. A possible set of γ's is

$$\gamma_i = 0.67, \quad \|\gamma\|^2 = 2\gamma_i^2 = 0.9$$

The reduced order problem is given by

$$\begin{gathered}A = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} \\ B_1 = \begin{bmatrix}0.2 & -0.5 & 0.1\\ -1 & 0.1 & 0.2\end{bmatrix}, \quad B_2 = \begin{bmatrix}0\\4\end{bmatrix}, \\ C_1 = I, \quad C_2 = \begin{bmatrix}5 & 0\end{bmatrix}.\end{gathered}$$

The conditions of theorem 1 are satisfied with

$$X_{i\infty} = \begin{bmatrix}2.864 & 0.480\\ 0.478 & 0.222\end{bmatrix}, \quad Y_{i\infty} = \begin{bmatrix}0.155 & -0.060\\ -0.060 & 0.149\end{bmatrix}.$$

Since the control is scalar, we can set $u_i = -M\,\text{sign}(\sigma_i)$. A conservative approach for the gain (given $\hat{x}_{i1}(0) = \hat{x}_{i2}(0) = 0$, $x_{i1}(0) = x_{i2}(0) = x_{i3}(0) = 1$) is

$$M = 10$$

A simulation of the interconnected system is shown in Figure 1, where the stability of the system is verified. In Figure 2, there is a plot of the control action (smoothed by a linear function and a saturator). It can be seen that the control effort is reasonable.

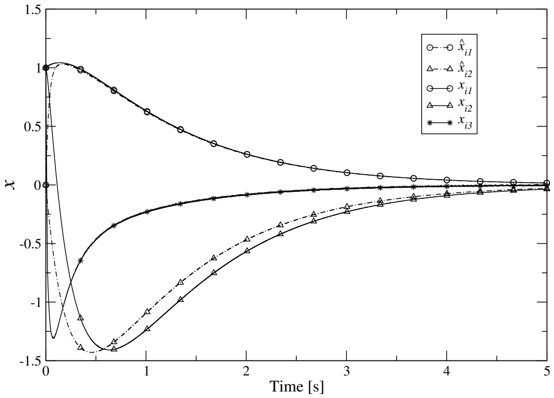


Fig. 1. Estimated state (dashed line) and actual state (solid line). $\hat{x}_1(0) = \hat{x}_2(0) = 0$, $x_1(0) = x_2(0) = x_3(0) = 1$

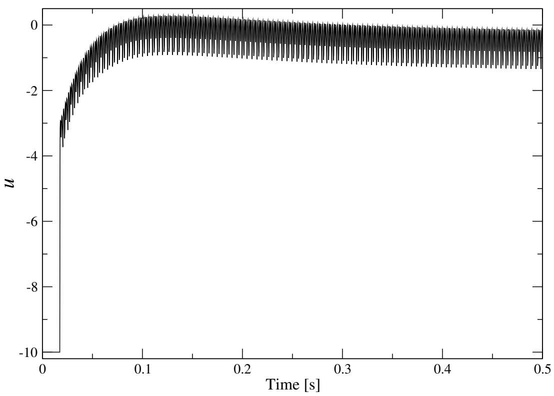


Fig. 2. Control Action. The function $\mathrm{sign}(\sigma_i)$ was smoothed by the linear function $1000\sigma_i$ followed by a saturator with minimal and maximal values of -10 and 10 respectively.

VI. Conclusions

Two robust techniques were combined in order to amend the interactions held between a set of subsystems comprising an interconnected system. A control law, obtained by combining $\mathcal{H}_\infty$ and sliding mode theory, proved useful in the stabilization of the decentralized system by measurement feedback. Sufficient conditions for stability have $\mathcal{H}_\infty$ flavor, as they are given in terms of resolvability of Riccati equations. These equations however, are of reduced dimension. The analysis developed proved semi-global exponential stability.

Appendix

A. System's Coordinate Transformation

As in [16], we use a transformation $x' = T_1 x$ to put system (4) in regular form. A possible T_1 is

$$T_1 = \begin{bmatrix} B_2^{\perp^+} \\ B_2^+ \end{bmatrix},$$

After such a transformation, the system can be written as

$$\begin{bmatrix} \dot{x}'_1 \\ \dot{x}'_2 \end{bmatrix} = \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} + \begin{bmatrix} w_u \\ u + w_m \end{bmatrix}$$
$$y = \begin{bmatrix} C'_1 & C'_2 \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$$

Where $x'_1 \in \Re^{n-m}$, $x'_2 \in \Re^m$ and all matrices are of appropriate dimensions. Let's make now a transformation T_y on the *output*

$$T_y = \begin{bmatrix} {C'}_2^{\perp} \\ {C'}_2^{+} \end{bmatrix}$$
$$\bar{y} = T_y y = \begin{bmatrix} C''_{11} & 0 \\ C''_{21} & I \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$$
$$C''_{11} = {C'}_2^{\perp} C'_1, \quad C''_{21} = {C'}_2^{+} C'_1.$$

Notice that assumption 1 implies that $\mathrm{rank}(C'_2) = m$, which warranties the existence of ${C'}_2^{+}$. If we further make another transformation on the state

$$T_2 = \begin{bmatrix} I_{(n-m)\times(n-m)} & 0_{(n-m)\times m} \\ C''_{21} & I_{m\times m} \end{bmatrix}$$

we get

$$\begin{bmatrix} \dot{x}''_1 \\ \dot{x}''_2 \end{bmatrix} = \begin{bmatrix} A''_{11} & A''_{12} \\ A''_{21} & A''_{22} \end{bmatrix} \begin{bmatrix} x''_1 \\ x''_2 \end{bmatrix} + \begin{bmatrix} w_u \\ u + \delta \end{bmatrix}$$
$$\begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \end{bmatrix} = \begin{bmatrix} C''_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x''_1 \\ x''_2 \end{bmatrix}$$

with $\delta = w_m + C''_{21} w_u$.

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