Decomposition and Robustness Properties of Integral Sliding Mode Controllers

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***Abstract*— The concept of integral sliding mode (ISM) is revised. ISM ensures insensitivity of the robustified system's trajectory with respect to the matched uncertainty, starting from the beginning of the process. Modification of the ISM approach allows to design *the desired dynamics for the matching variables* and guarantees that *the unmatched part of the uncertainties is minimized and not amplified*.**

I. INTRODUCTION

A. Motivation

Sliding Mode Control is a powerful nonlinear control technique that has been intensively developed during the last 35 years [1]. A system's motion in a sliding surface, named *sliding mode,* turns out to be robust with respect to disturbances and uncertainties matched by a control but sensitive to unmatched ones. The sliding mode design approach consist of two steps [1]. First, the switching surface is designed such that the system's motion in the sliding mode satisfies design specifications. Second, a control function that makes the switching function attractive to the system's state is designed.

A disadvantage of such a control design is that the trajectory of the designed solution is not robust even with respect to the matched disturbances on a time interval preceding the sliding motion.

In [2],[1],[3] and [4] a new sliding mode design concept, namely integral sliding mode (ISM), *without any reaching phase* was proposed. As a result, the robustness of the trajectory for a nominal system can be guaranteed throughout an entire response of the system starting from the initial time instant. The main disadvantage of ISM is that ISM *does not have the decomposition property* typical for sliding mode controllers because the robustified trajectory requires the design of the control law in the complete state space. Moreover, for systems with both matched and unmatched uncertainties it is necessary to be sure that during ISM design the unmatched part of the uncertainties is not amplified.

B. Main Contribution

In this paper the concept of the ISM concept is modified. It is shown that in exchange of the decomposition property of the sliding mode control it is possible to design the ISM ensuring desired dynamics for the linear part of the matched variables.

Moreover it is shown that the proposed ISM design:

- *does not amplify the unmatched uncertainties in the sense that its Euclidean norm is not bigger than the Euclidean norm of the original unmatched perturbation*;
- *ensures that the Euclidean norm of the resulting unmatched perturbation is minimal*.

II. CONTROL DESIGN CHALLENGE

Let us consider a controlled linear uncertain system

$$\dot{x}\left(t\right) = Ax\left(t\right) + Bu\left(t\right) + \xi\left(x,t\right), \quad x(0) = x^0, \qquad (1)$$

where $x\left(t\right) \in \Re^n$ is the state vector at time $t \in [0, t_1]$, $u\left(t\right) \in \Re^m$ is a control action, $B^+ := \left[B^\intercal B\right]^{-1} B^\intercal$ is the pseudoinverse matrix for B, $\|A\| \leq a^+$, $\|B^+\| \leq b^+$, and $\xi(x,t)$ is an external disturbance (or uncertainty) consisting of matched and unmatched parts. Suppose that $\xi(0,t)$ is bounded and $\xi(x,t)$ can not grow faster than a linear function.

The control design problem can be formulated as follows: *design the control law $u = u\left(t\right)$ in the form*

$$u\left(t\right) = u_0\left(t\right) + u_1\left(t\right), \qquad (2)$$

where $u_1\left(t\right)$ is the "integral sliding-mode" control part, providing:

- the complete compensation of the unmeasured matched part of the uncertainty *starting from the initial time instant* ($t_{comp} = 0$);
- dependency of the dynamics of the matched part of the system solely on the control component $u_0\left(t\right)$ (control design for the function $u_0\left(t\right)$ will be discussed bellow).

Substitution of the control law (2) into system (1) yields

$$\dot{x}\left(t\right) = Ax\left(t\right) + Bu_0 + Bu_1 + \xi\left(x,t\right), \quad x(0) = x^0. \qquad (3)$$

A. Projection matrix design

Let us define the auxiliary sliding function

$$s(t) = \sigma(t) + Gx(t), \qquad (4)$$

where $\sigma(t)$ is some auxiliary variable and G is a projection matrix, which will be defined bellow. Then

$$\dot{s}(t) = \dot{\sigma}(t) + G[Ax\left(t\right) + Bu_0 + Bu_1 + \xi(x,t)].$$

Suppose that $\det GB \neq 0$ and that we can enforce the sliding mode in the surface $s = 0$ via the ISM control $u_1\left(s\right)$. In order to find the ISM dynamics one has

$$\begin{aligned} u_{1eq} \quad &= \quad -[GB]^{-1}G(Ax\left(t\right) + Bu_0 + \xi(x,t)) \\ &\quad - [GB]^{-1}\dot{\sigma}(t) \end{aligned}$$

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Then the ISM dynamics equations have the form

$$\begin{aligned}\dot{x}(t) &= \left[I - B(GB)^{-1}G\right][Ax(t) + Bu_0 + \xi(t)] - \\ &- B(GB)^{-1}\dot{\sigma}(t). \end{aligned} \tag{5}$$

Let us design a projection matrix G such that

- does not amplify the unmatched uncertainties $\xi_{eq}(x,t) = \left[I - B(GB)^{-1}G\right]\xi(x,t)$ in the sense that its Euclidean norm is not bigger than the Euclidean norm of the original unmatched perturbation;
- ensures that the Euclidean norm $\xi_{eq}(x,t)$ of the resulting unmatched perturbation is minimal.

Proposition 1: B^+ is a matrix which minimizes the norm of $\xi_{eq}(x,t)$, i.e.

$$B^+ = \arg\min_{G\in\Re^{m\times n}} \left\|\left[I - B(GB)^{-1}G\right]\xi(x,t)\right\|_2 \tag{6}$$

Let us remark that

$$\begin{aligned}\left\|\left[I - B(GB)^{-1}G\right]\xi(x,t)\right\|_2 &= \\ \left\|\xi(t) - B(GB)^{-1}G\xi(x,t)\right\|_2 &= \|\xi(x,t) - B\varphi\|_2\end{aligned}$$

where $\varphi = (GB)^{-1}G\xi(x,t)$. Thus, problem (6) can be rewritten in the form:

$$\varphi_0 = \arg\min_{\varphi\in\Re^m} \|\xi(x,t) - B\varphi\|_2,$$

which has $\varphi_0 = B^+\xi(x,t)$ as solution (see [5]). Taking $G = B^+$ we will have:

$$(GB)^{-1}G\xi(x,t) = B^+\xi(x,t) = \varphi_0$$

which implies (6).

Proposition 2: The unmatched perturbation $\xi_{eq}(x,t) = [I - BB^+]\xi(x,t)$ is not amplified, i.e.

$$\left\|[I - BB^+]\right\|_2 = 1.$$

Let $\mu(D)$ be the largest eigenvalue of D and $\nu(D)$ the smallest eigenvalue of D. Let us denote the Euclidean norm of a real matrix as

$$\begin{aligned}\|D\|_2 &= \left(\text{largest eigenvalue of } D^TD\right)^{1/2} \\ &= \left(\mu(D^TD)\right)^{1/2}\end{aligned}$$

we have that

$$\left\|[I - BB^+]\right\|_2 = \left(\mu\left([I - BB^+]^T[I - BB^+]\right)\right)^{1/2}$$

since $[I - BB^+]^T[I - BB^+] = [I - BB^+]$, and from the properties of eigenvalues $(I + D)x = (1 + \lambda)x$ (λ is an eigenvalue of D). Then

$$\begin{aligned}\left\|[I - BB^+]\right\|_2 &= \left(\mu(I - BB^+)\right)^{1/2} & (7)\\ &= \left(1 - \nu(BB^+)\right)^{1/2}. & (8)\end{aligned}$$

Now, let λ be any eigenvalue of BB^+, since the matrix $\left(B^TB\right)^{-1}$ can be represented as $\left(B^TB\right)^{-1/2}\left(B^TB\right)^{-1/2}$, the following is obtained

$$\begin{aligned}\lambda x &= BB^+x = B\left(B^TB\right)^{-1}B^Tx \\ \lambda x^Tx &= x^TB\left(B^TB\right)^{-1/2}\left(B^TB\right)^{-1/2}B^Tx \\ &= \left\|\left(B^TB\right)^{-1/2}B^Tx\right\|^2 \geq 0,\end{aligned}$$

which means that $\lambda \geq 0$. The matrix BB^+ is singular, that is why at least one eigenvalue is equal to zero, hence $\nu(BB^+) = 0$. Then from (8) it follows that

$$\left\|[I - BB^+]\right\|_2 = 1.$$

So it is reasonable to select $G = B^+$.

B. Transformation of the state

Let us split system (3) in two parts, one corresponding to the matched and another to the unmatched coordinates of uncertainties. Let us do so by defining the following nonsingular transformation:

$$T := \begin{bmatrix} B^\perp \\ B^+ \end{bmatrix},$$

where $B^\perp = \mathrm{In}(I - BB^+)$; $\mathrm{In}(X)$ means the linearly independent rows of X, $\|A\| \leq a^+$, $\|B^+\| \leq b^+$.

Applying transformation T to system (3)

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := Tx(t) = \begin{bmatrix} B^\perp x(t) \\ B^+x(t) \end{bmatrix}$$

one has

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} F_{11}z_1 + F_{12}z_2 + B^\perp\xi\left(T^{-1}z,t\right) \\ F_{21}z_1 + F_{22}z_2 + u_0 + u_1 + B^+\xi\left(T^{-1}z,t\right) \end{bmatrix} \tag{9}$$

C. Design of the auxiliary function

Let's try to define an integral sliding mode surface such that the linear part of the equation for z_2 in system (9) will have the form $F_{21}^dz_1 + F_{22}^dz_2$.

Define the auxiliary "sliding" function $s(z,t) \in \Re^m$ as

$$s(z,t) = \sigma(z,t) + z_2, \tag{10}$$

where $\sigma(z,t)$ is an auxiliary variable which will be defined bellow. Then, it follows that

$$\begin{aligned}\dot{s}(z,t) &= \dot{\sigma}(z,t) + F_{21}z_1 + F_{22}z_2 + \\ &+ u_0 + u_1 + B^+\xi\left(T^{-1}z,t\right). \end{aligned} \tag{11}$$

The next step is to select the auxiliary variable σ as the solution to the following Cauchy problem

$$\begin{aligned}\dot{\sigma}(z,t) &= -u_0 - F_{21}^dz_1 - F_{22}^dz_2, \\ \sigma(z(0),0) &= -z_2(0).\end{aligned} \tag{12}$$

Then the equation for the function $s(z,t)$ becomes

$$\begin{aligned}\dot{s}(z,t) &= -(F_{21}^d - F_{21})z_1 - (F_{22}^d - F_{22})z_2 + \\ &+ u_1(z,t) + B^+\xi\left(T^{-1}z,t\right) \qquad (13)\\ s(z(0),0) &= 0.\end{aligned}$$

Suppose that $\left\|\xi\left(T^{-1}z,t\right)\right\| \leq q||z|| + p$, $q, p > 0$. In order to realize a *sliding mode dynamics*, let us design the relay control in the form

$$u_1(z,t) = -M(z)\frac{s(t)}{\|s(t)\|},$$
$$M(z) = \bar{q}||z(t)|| + \bar{p}, \quad \bar{q} \geq q + b^+a^+, \quad \bar{p} \geq p + b^+\xi^+ \tag{14}$$

That implies

$$\dot{s}\left(z,t\right) = -M(z)\frac{s(t)}{\|s(t)\|} + (F_{21}^d - F_{21})z_1 + (F_{22}^d - F_{22})z_2.$$

D. ISM stability

For the Lyapunov function $V = \dfrac{1}{2}\left\|s\right\|^2$ we have

$$\begin{aligned}
\frac{d}{dt}V &= (s,\dot{s}) \\
&= \left(s, B^+\xi\left(T^{-1}z,t\right) - M(z)\frac{s(t)}{\|s(t)\|}\right) + \\
&\quad + \left(s, (F_{21}^d - F_{21})z_1 + (F_{22}^d - F_{22})z_2\right) \\
&\leq -\left\|s\right\|\left[M(z) - \left\|B^+\xi\left(T^{-1}z,t\right)\right\|\right] - \\
&\quad - \left\|s\right\|\left[-b^+\left(\left\|F\right\| - \left\|z\left(t\right)\right\|\right)\right] \\
&\leq -\left\|s\right\|\left[\left(\bar{q} - q - b^+a^+\right)||z(t)||\right] - \\
&\quad - \left\|s\right\|\left[\bar{p} - p\right] \leq 0,
\end{aligned}$$

where

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21}^d - F_{21} & F_{22}^d - F_{22} \end{bmatrix}.$$

So we realize that

$$\begin{aligned}
V\left(s\left(z\left(t\right),t\right)\right) &\leq V\left(s\left(z\left(0\right),0\right)\right) = \frac{1}{2}\left\|s\left(z\left(0\right),0\right)\right\|^2 \\
&= 0.
\end{aligned}$$

That implies that for all $t \in [0, t_1]$ the following identities

$$s\left(t\right) = 0, \quad \dot{s}\left(t\right) = 0 \tag{15}$$

hold; which in turn means that *the integral sliding mode control (14) completely compensates the effect of the matched uncertainty $B^+\xi\left(T^{-1}z,t\right)$ from the beginning of the process*.

E. Nominal system design

Taking into account (14), we will find the equivalent control for the ISM dynamics as follows:

$$u_{1eq} = (F_{21}^d - F_{21})z_1 + (F_{22}^d - F_{22})z_2 - B^+\xi\left(T^{-1}z,t\right).$$

Applying u_{1eq} in (9) we obtain the ISM dynamics in the form

$$\begin{aligned}
\begin{bmatrix}\dot{z}_{10}\left(t\right) \\ \dot{z}_{20}\left(t\right)\end{bmatrix} &:= \begin{bmatrix} F_{11} & F_{12} \\ F_{21}^d & F_{22}^d \end{bmatrix}\begin{bmatrix} z_{10}\left(t\right) \\ z_{20}\left(t\right)\end{bmatrix} + \\
&\quad + \begin{bmatrix} B^{\perp}\xi\left(T^{-1}z,t\right) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u_0(t) \end{bmatrix},
\end{aligned} \tag{16}$$

which will be called the nominal system.

III. About the smooth control law design

The nominal system (16)

- contains the uncertainties in the equations for the unmatched variable $z_{10}\left(t\right)$ only;
- for the case when $F_{21}^d = F_{22}^d = 0$, $\dot{z}_{20}$ in (16) does not depend on the state vectors z_{10}, and z_{20}, but depends on the control u_0 only.

Design of the smooth control law for the nominal system (16) allows to make the decomposition of the min-max multimodel problem [6]. A similar approach simplified the $\mathcal{H}_\infty$ control design for a decentralized system [7].

IV. Conclusions

ISM ensures the insensitivity of the robustified control law with respect to the matched uncertainties, starting from the beginning of the process. On the other hand, the ISM approach does not have the decomposition properties typical of conventional sliding mode control, i.e. the sliding dynamics for the ISM has the same order as the original system. Suggested modification of the ISM approach allows to design the *desired dynamics for the linear part of the matched variables*.

Moreover, the proposed ISM design guarantees that *the unmatched part of the uncertainties is minimized and not amplified*. The proposed results are useful for control design in a system with both matched and unmatched uncertainties.

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