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Sujet :

Cyclo Passivité et Commande par Interconnexion.

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Résumé

Nous abordons le problème de commande de systèmes non linéaires. Nous nous adhérons à un nouveau changement de paradigme où l'énergie est l'objet central dans le problème de commande. Elle joue le rôle principal dans l'étape de modélisation, la spécification des objectifs de performance et l'étape de conception.

On verra que la fonction d'énergie d'un système détermine son comportement stationnaire ainsi que le comportement transitoire, par le transfert d'énergie entre les soussystèmes. De ce point de vue, les systèmes (processus et dispositif de commande) ne sont plus regardés comme des processeurs de signaux, mais plutôt comme des dispositifs transformateurs d'énergie, ou des « processeurs d'énergie » qui sont interconnectés pour atteindre le comportement désiré. Les avantages d'une perspective fixée autour de l'énergie sont multiples, les modèles mathématiques sont plus simples et la synergie résulte de l'intuition physique et la rigueur mathématique.

On définira la cyclo passivité pour formaliser mathématiquement la notion d'énergie (ou plus généralement, énergie abstraite) et l'on utilisera l'idée de « façonnement de l'énergie » comme principe central de conception.

Dans la première partie, nous traitons le problème de façonnement de l'énergie par retour d'état statique. Ensuite, nous présentons l'approche de commande par interconnexion, où le dispositif de commande est construit comme un système physique dynamique fictif ayant une fonction d'énergie obtenue par application de la méthodologie présentée. Finalement, nous faisons une comparaison des deux approches.

Mots Clés

Science de l'automatique, commande fondée sur la passivité, façonnement de l'énergie, systèmes Hamiltoniens à port, commande par interconnexion, approche *behaviors*.

Keywords

Control theory, passivity-based control, energy-shaping, port-Hamiltonian systems, control by interconnection, behavioral approach.

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1 Résumé Détaillé

1.1 Introduction

Une des questions les plus importantes dans la science de l'automatique est : étant donné un système $\mathcal{S}_1$ (le processus), comment concevoir un deuxième système $\mathcal{S}_2$ (le contrôleur), de sorte qu'une fois interconnectés, ils se comportent d'une certaine façon désirée ? Comme les systèmes de régulation automatique pour les horloges d'eau étaient à l'usage depuis l'année -270, cette question a au moins 2.000 ans. Naturellement, il y en a plusieurs réponses, mais la notion de rétroaction est sans doute un élément clé dans n'importe quelle conception réussie.

Pendant les années 1930, des ingénieurs de l'information comme Black, Nyquist, Bode et d'autres ont reconnu explicitement la rétroaction comme principe de conception. En utilisant une approche fréquentielle, ils ont créé des outils mathématiques pour l'analyse et la synthèse d'amplificateurs et d'autres dispositifs électroniques pour la communication. Pendant la Seconde Guerre mondiale, Bode et une équipe d'ingénieurs ont appliqué ces outils pour résoudre le problème de conception des canons de défense antiaérienne, un problème complexe concernant la transmission de données sans fil, des ordinateurs électriques, des principes statistiques et des servomécanismes. À la suite de leur succès, l'approche fréquentielle est devenue *le* paradigme pendant la période classique de l'automatique (1935 – 1960) [36, 5].

Avec l'arrivée de la commande optimale et le besoin d'incorporer des effets non linéaires et des phénomènes variant dans le temps, la représentation d'états a remplacé l'approche fréquentielle et a marqué le début de la science de l'automatique moderne. Pourtant, la théorie de l'information imprègne la littérature de l'automatique : typiquement, on voit les systèmes physiques comme s'il s'agissait des processeurs des signaux et le schéma fonctionnel de la Fig. 1.1 est toujours une icône de la science de l'automatique.

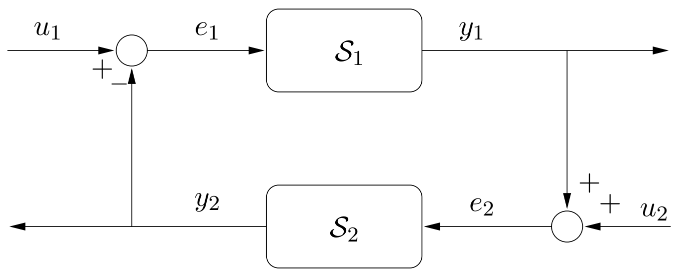

Fig. 1.1: Interconnexion à rétroaction.

Cette thèse s'adapte à un nouveau changement de paradigme où l'énergie est l'objet central dans le problème de commande. Elle joue le rôle principal dans l'étape de modélisation, la spécification des objectifs de performance et l'étape de conception. La fonction d'énergie d'un système détermine son comportement stationnaire ainsi que le comportement transitoire, par le transfert d'énergie entre les sous-systèmes. De ce point de vue, les systèmes (processus et contrôleur) sont vus plutôt comme des dispositifs transformateurs d'énergie, ou des « processeurs d'énergie » qui sont interconnectés pour atteindre le comportement désiré.

Les avantages d'une perspective fixée autour de l'énergie sont multiples [44], les modèles mathématiques sont plus simples et la synergie résulte de l'intuition physique et de la rigueur mathématique. Cette perspective énergétique offre aussi des avantages pratiques : 'les praticiens venant de domaines différents sont familiarisés avec les concepts d'énergie, qui peuvent servir comme lingua franca et faciliter la communication avec les théoriciens de l'automatique, incorporant ainsi des connaissances préalables et apportant des interprétations physiques au mode de régulation' [44].

Échange d'énergie, façonnement de l'énergie et dissipation

La notion de passivité sera formellement définie dans le chapitre 3, mais à ce stade on peut dire que la passivité est étroitement liée à la propriété fondamentale de conservation de l'énergie.

La commande basée sur la passivité [42] est une méthode de conception qui, à partir des propriétés de passivité du processus, propose un contrôleur qui transforme le processus en autre système passif ayant une fonction d'énergie différente, « façonnée ». Afin de stabiliser un point d'équilibre, on vise une fonction d'énergie positive définie pour qu'elle agisse à titre de fonction Lyapunov. Dans une deuxième étape, on rajoute de l'amortissement au système pour améliorer la réponse transitoire ou atteindre la stabilité asymptotique (voir théorème 3.12). Cette procédure est connue comme *façonnement de l'énergie* (energy shaping) et *injection d'amortissement* (damping injection). L'injection d'amortissement se réduit à choisir une fonction ϕ, de l'espace des sorties $\mathbb{Y}$ dans l'espace d'entrées $\mathbb{U}$, tel que $y^\top \phi(y) > 0$. Le vrai défi est alors de résoudre le problème de façonnement de l'énergie.

Pour illustrer les principales idées relatives au façonnement de l'énergie, supposons que $\mathcal{S}_1$ et $\mathcal{S}_2$ sont donnés dans la forme entrée-état-sortie

$$\mathcal{S}_i : \left\{ \begin{array}{rcl} \dot{x}_i & = & f_i(x_i, e_i) \\ y_i & = & h_i(x_i, e_i) \end{array} \right. \quad , \quad i = 1, 2 \; . \tag{1.1}$$

Si les systèmes sont passifs, alors il existe des fonctions $H_i(x_i)$ quantifiant l'énergie stockée quand les systèmes se trouvent aux états x_i, $i = 1, 2$. De l'énergie additionnelle peut être fournie par ou extraite des ports à puissance (e_i, y_i). Une partie de l'énergie est normalement dissipée vers l'environnement. En langage mathématique : $\dot{H}_i(x_i) \leq y_i^\top e_i$.

L'interconnexion à rétroaction

$$\begin{array}{rcl} e_1 & = & u_1 - y_2 \\ e_2 & = & u_2 + y_1 \end{array}$$

produit le nouveau système

$$\dot{x} = f(x, u) \quad (1.2a)$$
$$y = h(x, u) , \quad (1.2b)$$

où

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} , \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{et} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(voir la Fig. 1.1). Un résultat important de la théorie de la passivité est le *théorème de la passivité*, formulé ci-dessous et prouvé dans, p. ex., [28, 41, 60].

Théorème 1.1. *L'interconnexion à rétroaction de deux systèmes passifs est encore passive.*

De plus, le système composé (1.2) a une fonction d'énergie égale à la somme des fonctions d'énergie $H_i(x_i)$ des systèmes originels. Le façonnement de l'énergie peut être accompli en sélectionnant d'abord une $H_2(x_2)$ qui fournit l'énergie totale appropriée $H_1(x_1)+H_2(x_2)$, et après en construisant un $\mathcal{S}_2$ ayant $H_2(x_2)$ pour fonction de stockage. Cette idée est intuitive et conceptuellement intéressante, mais on ne peut pas l'implanter directement. Du point de vue du concept, les problèmes principaux sont :

(i) Une loi de commande statique par retour d'état $\mathring{u}_c(x_1)$ est incompatible avec (1.1). Il n'y a pas d'état x_2, donc, il n'y a pas d'énergie stockée $H_2(x_2)$. On peut encore parler de la passivité des applications passives [28], mais il n'est pas évident comment choisir les variables de port e_2 et y_2.

(ii) Supposons que le contrôleur a, effectivement, la dynamique (1.1). La fonction de stockage du régulateur dynamique dépendra seulement de x_2, et non pas de x_1. Ceci nous empêche de choisir directement une $H_2(x_2)$ qui façonne $H_1(x_1)$ tel que requis.

Ces problèmes motivent cette thèse. À côté de ces problèmes, il y a, bien entendu, plusieurs sous-problèmes, questions ouvertes et détails à résoudre.

Portée

La conception et implantation d'un système de régulation demande la solution de problèmes comme : l'estimation d'état s'il n'est pas complètement mesurable, l'estimation de paramètres du processus (si possible estimation en ligne), discrétisation du contrôleur (s'il est implanté dans un ordinateur), etc. Une fois un régulateur est proposé, il faut évaluer la robustesse vis-à-vis des perturbations externes, le bruit, l'incertitude du modèle et les effets de quantification. Considérer tous ces problèmes dans une seule œuvre serait exagérément ambitieux et peu réaliste. Cette thèse se focalise sur le développement des modèles basés sur l'énergie et sur la question de comment utiliser l'action de commande pour façonner leur énergie, en mettant l'accent sur le problème de stabilité asymptotique.

Cadre mathématique

Dans un niveau plus abstrait, il y a un autre changement de paradigme dans la science de l'automatique : *l'approche comportemental* (behaviors) proposée par Willems [68, 46].

En gros, en modélisant un phénomène donné on commence avec un *universum* $\mathcal{U}$, l'ensemble de tous les évènements (non modélisés). À partir des principes fondamentaux, on dérive un sous-ensemble $\mathcal{B} \subset \mathcal{U}$, appelé le comportement, qui contient tous les évènements réalisables. Un modèle mathématique d'un phénomène est alors donné par une paire $(\mathcal{U}, \mathcal{B})$. Nous allons travailler uniquement avec des systèmes dynamiques à temps continu, mais il est important de mentionner qu'un modèle avec ce degré de généralité couvre une large gamme de phénomènes comme langages, systèmes dynamiques continus ou discrets, systèmes à paramètres répartis, automates, etc.

Le cadre comportemental s'écarte aussi du point de vue du traitement des signaux. De plus, il efface la distinction entre entrées et sorties et met en question le principe de causalité [72]. Les modèles comportementaux nous permettent de séparer les variables qui nous intéressent (dans notre cas, les variables de port) des variables auxiliaires (dans notre cas, l'état). Les modèles comportementaux fournissent un cadre adéquat pour les m-ports définis dans le chapitre 3. De plus, l'approche comportemental partage (inspire, plutôt) notre vision de commande comme interconnexion de systèmes.

Caveat lector. Une affirmation récurrente dans la littérature de comportements est que tout ce qui est dit sur les systèmes (contrôlabilité, observabilité, équivalence de modèles, propriétés de modèles, symétries, identification de systèmes, etc.) doit être intrinsèque : il faut faire référence au comportement, l'ensemble $\mathcal{B}$ lui-même, et non pas à une de ses représentations particulières. Pour être concrets, mais contrairement à l'affirmation précédente, nous avons utilisé une représentation particulière (dans l'espace d'états) en développant les résultats que nous allons présenter.

Aperçu

Dans le chapitre 3 nous présentons les modèles mathématiques que nous allons utiliser pour représenter le processus et le contrôleur. Nous définissons la cyclo passivité, une version généralisée de la passivité qui capture aussi la propriété de conservation d'énergie, mais qui n'impose aucune restriction sur la forme de la fonction d'énergie, élargissant ainsi le rang d'application de nos méthodes. Nous traitons les modèles Hamiltoniens à port, qui sont basés sur l'énergie, de nature cyclo passive et parfaits pour traiter le problème de façonnement de l'énergie. Nous considérons les systèmes mécaniques et électriques d'une façon plus détaillée et nous montrons qu'il s'agit de cas particuliers des systèmes Hamiltoniens. Nous examinons brièvement l'interconnexion de systèmes et nous formulons le problème de commande.

Le problème (i), figurant ci-dessus, peut être contourné avec la commande à *bilan énergétique* (energy-balancing). L'idée est de chercher une commande $\mathring{u}_c$ telle que pour une H_2 propice, le système en boucle fermée satisfasse le bilan de puissance

$$\dot{H}_2(x_1) = y_2^\top e_2$$

avec les variables de port $e_2 = h_1(x_1)$ et $y_2 = -\mathring{u}_c(x_1)$, imitant ainsi un système cyclo passif $\mathcal{S}_2$ qui « partage » les variables d'état x_1 et rend une fonction d'énergie totale $H_1(x_1) + H_2(x_1)$.

Le problème (ii) peut être résolu avec l'approche *commande par interconnexion*. L'idée est de chercher des fonctions invariantes de la forme $z(x_1, x_2)$. Si ces fonctions sont bien invariantes, alors $H_1(x_1)+z(x_1, x_2)+H_2(x_2)$ est aussi une fonction d'énergie du système commandé et peut être façonnée convenablement.

De manière surprenante, les solutions aux problèmes (i) et (ii) susmentionnées sont équivalentes, dans un sens qui est précisé dans la section 5.7. Dans les chapitres 4 et 5 nous montrons que, dans leur formulation standard, les deux solutions sont excessivement restrictives, car elles subissent *l'obstruction de la dissipation*. On examine ces méthodes en détail et l'on présente plusieurs solutions pour surmonter l'obstacle de la dissipation.

Dans le chapitre 4 nous apportons une caractérisation algébrique de tous les dispositifs statiques qui accomplissent le façonnement de l'énergie, y compris d'autres régulateurs comme *assignation d'interconnexion et d'amortissement* (IDA). Nous montrons aussi que, pour une certaine d_2 et des variables de port judicieusement choisies, tous les régulateurs façonnant l'énergie satisfont un bilan de puissance de la forme

$$\dot{H}_2(x_1) = y_2^\top e_2 - d_2 \,.$$

Le chapitre 5 montre que la commande par interconnexion pose un problème concernant la convergence asymptotique de l'état vers l'équilibre désiré. Une des solutions présentées est l'incorporation d'un algorithme adaptatif dans le régulateur. D'autres solutions sont aussi présentées.

Les notions d'invariance de flux et de dissipation sont les axes de la discussion entière. Les invariances de flux et de dissipation sont des caractéristiques importantes des régulateurs et apportent un repère pour comparer les cas statique et dynamique.

Contributions

Les contributions principales sont :

- Chapitre 3
  1. L'identification d'une classe de réseaux RLC non linéaires qui peuvent être représentés par des modèles Hamiltoniens (section 3.3.3).
- Chapitre 4
  1. La caractérisation algébrique de tous les régulateurs basés sur la passivité (théorème 4.5).
  2. Une taxonomie, axée sur les fonctions de flux et de dissipation, des régulateurs basés sur la passivité reportés dans la littérature.
  3. L'équivalence entre bilan d'énergie et invariance du flux et de dissipation (proposition 4.7).
  4. Des conditions suffisantes pour la propriété de passivité relative (proposition 4.10) et son application à la régulation des réseaux RLC non linéaires (théorème 4.13).

   5. Une extension de la méthode d'assignation d'interconnexion et d'amortissement (théorème 4.24) ainsi que la solution explicite de ses équations algébriques (théorème 4.24).
   6. L'identification des fonctions de flux et de dissipation particulières qui sont invariantes à l'action d'assignation d'interconnexion et d'amortissement basique (proposition 4.36).
- Chapitre 5
   1. L'utilisation du flux invariant du chapitre 4 pour généraliser les hypothèses nécessaires à l'application de la commande par interconnexion (section 5.1.1).
   2. Des conditions suffisantes pour résoudre le problème de façonnement de l'énergie (théorème 5.9).
   3. L'utilisation d'un algorithme adaptatif et l'injection additionnelle d'amortissement pour résoudre le problème de convergence asymptotique.
   4. L'incorporation d'un sous-système pour détruire la préservation du flux et de la dissipation (section 5.4).
   5. Une comparaison rigoureuse, basée sur les solutions des équations aux dérivées partielles, des approches statiques et dynamiques (proposition 5.18).
- La reformulation de ces résultats dans un cadre comportemental est elle-même nouvelle et apporte une perspective unifiée.

1.2 Systèmes cyclo passifs

Pour aborder le problème de modélisation et de commande, nous prenons l'approche comportemental [46]. Ceci est un court exposé sur les concepts à utiliser dans la suite. Pour une discussion plus ample, nous recommandons [67, 68, 46, 70].

Definition 1.2. Un *système* Σ est défini comme un triplet

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B}) \ ,$$

avec $\mathbb{T}$ un sous-ensemble de $\mathbb{R}$, appelé l'*axe du temps*, $\mathbb{W}$ est l'*espace de signaux* et $\mathcal{B}$, un sous-ensemble de $\mathbb{W}^{\mathbb{T}}$, appelé le *comportement* ($\mathbb{W}^{\mathbb{T}}$ est notation mathématique standard pour représenter la collection de toutes les applications de $\mathbb{T}$ dans $\mathbb{W}$).

Pour spécifier $\mathcal{B}$, nous nous servirons parfois de variables auxiliaires, ou latentes. Les variables latentes sont impliquées dans le modèle, mais elles ne sont pas les variables visées par le modèle.

Definition 1.3. Un *système à variables latentes* est défini comme

$$\hat{\Sigma} = \left(\mathbb{T}, \mathbb{W}, \mathbb{L}, \hat{\mathcal{B}}\right) \ ,$$

où $\mathbb{T}$ et $\mathbb{W}$ sont comme dans la définition 1.2, $\mathbb{L}$ est l'*ensemble de variables latentes* et $\hat{\mathcal{B}} \subset (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$ est le comportement complet.

Pour souligner la différence entre un système et un système à variables latentes, nous appellerons $\mathbb{W}$ l'*ensemble de variables manifestes*.

Definition 1.4. Soit $\mathcal{B}$ un comportement avec un espace de signaux $\mathbb{W}$. Divisons l'espace de signaux comme $\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2$. On dit que w_1 est *libre* si, pour n'importe quel $w_1 \in \mathcal{L}_\infty^{\text{loc}}(\mathbb{T}, \mathbb{W}_1)$, il existe un $w_2 \in \mathcal{L}_\infty^{\text{loc}}(\mathbb{T}, \mathbb{W}_2)$ tel que $(w_1, w_2) \in \mathcal{B}$.

Notation. Comme nous traiterons exclusivement le cas du temps continu, nous fixons $\mathbb{T} = \mathbb{R}_+$ une fois et on élimine $\mathbb{T}$ de la discussion.

Avant de définir les m-ports que nous utiliserons toute la longue de la suite, fixons les variables d'intérêt comme $\mathbb{W} = \mathbb{U} \times \mathbb{Y}$, où $\mathbb{U} = \mathbb{R}^m$ est l'*ensemble de valeurs des efforts* et $\mathbb{Y} = \mathbb{R}^m$ est l'*ensemble de valeurs des flux*.

Definition 1.5. Un *m-port* Π est un système

$$\Pi = (\mathbb{W}, \mathcal{B}) \ , \quad \mathbb{W} = \mathbb{U} \times \mathbb{Y} \ .$$

Nous dirons que Π est *actionné par les efforts* si les variables $u \in \mathcal{L}_\infty^{\text{loc}}(\mathbb{R}_+, \mathbb{U})$ sont libres. De façon duale, nous dirons qu'il est *actionné par les flux* si les variables $y \in \mathcal{L}_\infty^{\text{loc}}(\mathbb{R}_+, \mathbb{Y})$ sont libres.

Comme dans le cas de systèmes à variables latentes, nous utiliserons des variables auxiliaires, ou variables d'état.

Definition 1.6. Un *m-port à variables d'état* $\hat{\Pi}$ est un système

$$\hat{\Pi} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}) \ , \quad \mathbb{W} = \mathbb{U} \times \mathbb{Y}$$

où $\mathbb{U}$ et $\mathbb{Y}$ sont définis comme ci-dessus et $\mathbb{X} = \mathbb{R}^n$ est l'espace d'état.

Notre objet d'étude principal sera le m-port $\hat{\Pi}$ à variables d'état spécifié par

$$\dot{x} = f(x) + g(x)u \tag{1.3a}$$
$$y = h(x) + j(x)u \ , \tag{1.3b}$$

et

$$\hat{\mathcal{B}} = \left\{ (u, y, x) : \mathbb{R}_+ \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \,\middle|\, \text{Eq. (1.3) est satisfaite} \right\} \ . \tag{1.4}$$

Notation. L'équation (1.3) contient une équation différentielle ordinaire où t est la variable indépendante. Strictement, u, y et x sont fonctions du temps, c'est-à-dire, $(u, y, x) : \mathbb{R}_+ \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$. Étant donné que les arguments de $f : \mathbb{X} \to \mathbb{R}^n$ et $g : \mathbb{X} \to \mathbb{R}^{n \times m}$ sont membres de $\mathbb{X}$ (non pas de fonctions du temps), on devrait plutôt écrire $\dot{x} = f \circ x + (g \circ x) \cdot u$ à la place de (1.3a) (le symbole '$\circ$' dénote la composition). Pour éviter une notation si lourde, nous insisterons sur (1.3a) et nous écrirons simplement, $u \in \mathbb{U}$, $y \in \mathbb{Y}$ et $x \in \mathbb{X}$. Par conséquent, nous replacerons (1.4) par

$$\hat{\mathcal{B}} = \left\{ (u, y, x) \in \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \,\middle|\, \text{Eq. (1.3) est satisfaite} \right\} \ . \tag{1.5}$$

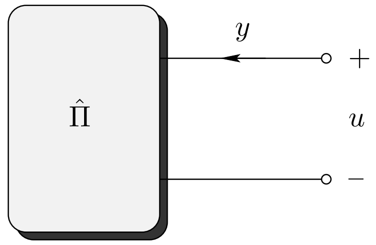

FIG. 1.2: m-port. La puissance positive rentre toujours dans le port. Seulement les variables manifestes sont montrées.

Cet abus de notation est habituel dans la littérature.

Pour simplifier les diagrammes, nous omettons l'état et nous attachons plusieurs bornes dans la même paire de vecteurs. Par convention, nous assignons la direction des flux et le signe des efforts d'une façon telle que la puissance positive rentre toujours dans les ports (voir la figure 1.2).

La cyclo passivité fait allusion à la propriété physique de conservation d'énergie. Comme un système physique ne peut pas créer de l'énergie, il ne peut pas stocker plus d'énergie que celle fournie par l'environnement. En reformulant cette propriété dans un langage mathématique, on peut faire la généralisation aux systèmes qui ne sont pas forcement physiques.

Definition 1.7. Un m-port $\hat{\Pi}$ à variables d'état est *cyclo passif* s'il existe une fonction $H : \mathbb{X} \to \mathbb{R}$ de classe $\mathcal{C}^1$, appelée *fonction de stockage*, tel que, pour tout $x_0 \in \mathbb{X}$, tout $t \geq 0$ et tous les triplets $(u, y, x) \in \hat{\mathcal{B}}$,

$$H(x) - H(x_0) \leq \int_0^t y^\top u \mathrm{d}\tau \ , \quad x(0) = x_0 \ . \tag{1.6}$$

Si, de plus, H admet un minimum, on dit que $\hat{\Pi}$ est *passif.* $\hat{\Pi}$ est *sans pertes* si (1.6) est satisfaite avec égalité.

Le théorème de Hill-Moylan fournit, dans l'esprit du lemme de Kalman-Yakubovich-Popov, une caractérisation *algébrique* des systèmes cyclo passifs.

Théorème 1.8. *[22] Considérons un m-port* $\hat{\Pi} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}})$ *qui satisfait (1.5).* $\hat{\Pi}_\mathrm{p}$ *est cyclo passif avec une fonction de stockage* H *si, et seulement si, pour une certaine* $q \in \mathbb{N}$*, il existe des fonctions* $l : \mathbb{R}^n \to \mathbb{R}^q$ *et* $w : \mathbb{R}^n \to \mathbb{R}^{q\times m}$ *telles que*

$$\nabla H^\top(x) f(x) = -|l(x)|^2 \tag{1.7a}$$
$$h(x) = g^\top(x) \nabla H(x) + 2w^\top(x) l(x) \tag{1.7b}$$
$$w^\top(x) w(x) = \frac{1}{2}(j^\top(x) + j(x)) \ . \tag{1.7c}$$

Notation. Tous les vecteurs sont des vecteurs colonne, même le gradient d'une fonction scalaire, que nous dénotons avec l'opérateur $\nabla_x \triangleq \frac{\partial}{\partial x}$. Quand le contexte est clair, nous omettons le sous-index de l'opérateur.

Nous ne démontrerons pas le théorème, mais nous pouvons aisément vérifier la partie 'si'. Appelons

$$d(x,u) \triangleq |l(x) + w(x)u|^2 \tag{1.8}$$

la *fonction de dissipation*. La dérivée de H par rapport au temps est

$$\dot{H}(x) = \nabla H^\top(x)\,[f(x) + g(x)u] = -|l(x)|^2 + \Big[h(x) - 2w^\top(x)l(x)\Big]^\top u\ .$$

On complet le carre en sommant le terme nul $u^\top j(x)u - u^\top w^\top(x)w(x)u$. Alors,

$$\begin{aligned}\dot{H}(x) &= h^\top(x)u + u^\top j(x)u - \Big(l^\top(x)l(x) + 2l^\top(x)w(x)u + u^\top w^\top(x)w(x)u\Big) \\ &= y^\top u - d(x,u)\ . \end{aligned} \tag{1.9}$$

Finalement, la non-negativité de $d(x,u)$ établit (1.6).

Pour le cas spécial

$$\begin{aligned}\dot{x} &= f(x) + g(x)u & (1.10a)\\ y &= h(x) & (1.10b)\end{aligned}$$

(c'est-à-dire, pour $j(x) = 0$), nous avons le corollaire suivant.

Corollaire 1.9. *Considérons un m-port $\hat{\Pi} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}})$ qui satisfait*

$$\hat{\mathcal{B}} = \big\{(u,y,x) \in \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \,\big|\, \textit{Eq. (1.10)}\big\}\ . \tag{1.11}$$

$\hat{\Pi}$ est cyclo passif avec fonction de stockage H si, et seulement si, il existe une fonction de dissipation $d : \mathbb{X} \to \mathbb{R}_+$ telle que

$$\begin{aligned}\nabla H^\top(x)f(x) &= -d(x) & (1.12a)\\ h(x) &= g^\top(x)\nabla H(x)\ . & (1.12b)\end{aligned}$$

Nous nous focalisons sur les systèmes de la forme

$$\begin{aligned}\dot{x} &= [J(x) - R(x)]\nabla H(x) + g(x)u & (1.13a)\\ y &= g^\top(x)\nabla H(x)\ , & (1.13b)\end{aligned}$$

où $x \in \mathbb{X}$, $u \in \mathbb{U}$ et $y \in \mathbb{Y}$ sont définis comme ci-haut. La matrice $J(x) = -J^\top(x)$ est appelée *matrice d'interconnexion* et la matrice $R(x) = R^\top(x) \geq 0$ reflète les effets de la dissipation. Ces modèles sont appelés modèles *Hamiltoniens à port* [60] et peuvent être incorporés facilement dans des m-ports

$$\hat{\mathcal{B}} = \big\{(u,y,x) \in \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \,\big|\, \text{Eq. (1.13)}\big\}\ . \tag{1.14}$$

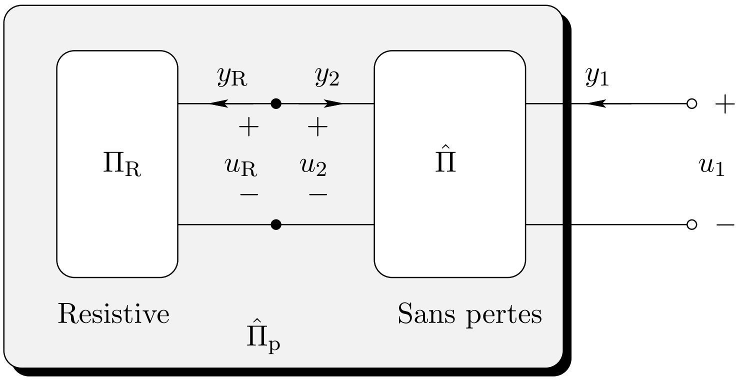

FIG. 1.3: Système Hamiltonien à port avec un port dissipatif.

Notons que la dérivée de l'énergie est

$$\dot{H}(x) = y^\top u - \nabla H^\top(x) R(x) \nabla H(x)$$

et que le terme $\nabla H(x)^\top R(x) \nabla H(x)$, étant non-négatif, établit l'inégalité de la dissipation. Ce modèle nous permet, par exemple, de décrire des systèmes mécaniques classiques.

Dans (1.13b), le flux y a la forme requise pour satisfaire le corollaire 1.9. En utilisant la version complète du théorème de Hill-Moylan, c'est-à-dire, en utilisant (1.7b), on peut étendre la définition de modèle Hamiltonien à port aux systèmes pour lesquels les efforts apparaissent explicitement dans l'équation du flux. Dans [62], les modèles Hamiltoniens à port sont étendus encore plus loin en substituant $R(x)\nabla H(x)$ avec un port externe résistif, ce qui permet d'incorporer d'autres effets dissipatifs dans le modèle. On part d'un système de la forme

$$\dot{x} = J(x)\nabla H(x) + g_1(x)u_1 + g_2(x)u_2 \tag{1.15a}$$
$$y_1 = h_1(x) + j_{11}(x)u_1 + j_{12}(x)u_2 \tag{1.15b}$$
$$y_2 = h_2(x) + j_{21}(x)u_1 + j_{22}(x)u_2 \ , \tag{1.15c}$$

où on appelle la paire $(u_2, y_2) \in \mathbb{R}^{m_{\mathrm{R}}} \times \mathbb{R}^{m_{\mathrm{R}}}$ le *port résistif*. Si on établit

$$g(x) = \begin{pmatrix} g_1(x) & g_2(x) \end{pmatrix} \ , \quad h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} \quad \text{et} \quad j(x) = \begin{pmatrix} j_{11}(x) & j_{12}(x) \\ j_{11}(x) & j_{11}(x) \end{pmatrix} \ ,$$

alors le système d'équations (1.15) peut être réécrit comme

$$\begin{aligned} \dot{x} &= J(x)\nabla H(x) + g(x)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= h(x) + j(x)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \ . \end{aligned}$$

Si $g(x)$, $h(x)$ et $j(x)$ satisfont les conditions du théorème 1.8, alors l'égalité

$$\dot{H}(x) = y_1^\top u_1 + y_2^\top u_2 \tag{1.16}$$

est valable et le système est sans pertes. Considérons maintenant un m_{R}-port résistif $\Pi_{\mathrm{R}} = (\mathbb{U}_{\mathrm{R}} \times \mathbb{Y}_{\mathrm{R}}, \mathcal{B}_{\mathrm{R}})$ qui satisfait la condition de dissipation

$$(u_{\mathrm{R}}, y_{\mathrm{R}}) \in \mathcal{B}_{\mathrm{R}} \quad \Longrightarrow \quad y_{\mathrm{R}}^\top u_{\mathrm{R}} \geq 0 \; .$$

En interconnectant le $(m + m_{\mathrm{R}})$-port sans pertes avec Π_{R} comme montré dans la figure 1.3, c'est-à-dire, en faisant $u_2 = u_{\mathrm{R}}$ et $y_2 = -y_{\mathrm{R}}$, on obtient le système Hamiltonien à port avec dissipation

$$\dot{x} = J(x)\nabla H(x) + g_1(x)u_1 + g_2(x)u_{\mathrm{R}} \tag{1.17a}$$

$$y_1 = h_1(x) + j_{11}(x)u_1 + j_{12}(x)u_{\mathrm{R}} \tag{1.17b}$$

$$y_{\mathrm{R}} = -h_2(x) - j_{21}(x)u_1 - j_{22}(x)u_{\mathrm{R}} \; . \tag{1.17c}$$

De (1.16), $y_2^\top u_2 = -y_{\mathrm{R}}^\top u_{\mathrm{R}}$ et $y_{\mathrm{R}}^\top u_{\mathrm{R}} \geq 0$, nous obtenons la condition de cyclo passivité $\dot{H}(x) \leq y_1^\top u_1$.

Les modèles Lagrangiens et Hamiltoniens s'étendent au-delà du domaine de la mécanique classique, ils sont utilisés aussi dans la littérature de la mécanique relativiste et quantique [19]. D'autres domaines importants sont la théorie de circuits électriques (voir le chapitre 3), les systèmes électromécaniques [37] et thermodynamiques [16]. Des efforts ont également été dirigés vers la recherche des conditions sous lesquelles un système générique peut être mis sous forme Hamiltonienne [58, 35, 63].

1.3 Passivation

Definition 1.10. Soient

$$\hat{\Pi}_{\mathrm{a}} = (\mathbb{W}_1 \times \mathbb{W}_2, \mathbb{X}_1 \times \mathbb{X}_2, \hat{\mathcal{B}}_{\mathrm{a}}) \quad \text{et} \quad \hat{\Pi}_{\mathrm{b}} = (\mathbb{W}_2 \times \mathbb{W}_3, \mathbb{X}_2 \times \mathbb{X}_3, \hat{\mathcal{B}}_{\mathrm{b}}) \tag{1.18}$$

deux ports ayant un espace de signaux $\mathbb{W}_2$ et un espace d'état $\mathbb{X}_2$ communs. L'*interconnexion* entre $\hat{\Pi}_{\mathrm{a}}$ et $\hat{\Pi}_{\mathrm{b}}$, notée $\hat{\Pi}_{\mathrm{a}} \wedge \hat{\Pi}_{\mathrm{b}}$, est définie par

$$\hat{\Pi}_{\mathrm{t}} = \hat{\Pi}_{\mathrm{a}} \wedge \hat{\Pi}_{\mathrm{b}} \triangleq (\mathbb{W}_1 \times \mathbb{W}_3, \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3, \hat{\mathcal{B}}_{\mathrm{t}}) \; ,$$

où

$$\hat{\mathcal{B}}_{\mathrm{t}} = \Big\{ (w_1, w_3, x_1, x_2, x_3) \in \mathbb{W}_1 \times \mathbb{W}_3 \times \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \;\Big|\; \exists\, w_2 \in \mathbb{W}_2 \text{ t.q. } (w_1, w_2, x_1, x_2) \in \hat{\mathcal{B}}_{\mathrm{a}},\ (\bar{w}_2, w_3, x_2, x_3) \in \hat{\mathcal{B}}_{\mathrm{b}} \Big\} \; . \tag{1.19}$$

avec $w_2 = (u_2, y_2)$ et $\bar{w}_2 = (u_2, -y_2)$.

En langage courant, l'interconnexion de $\hat{\Pi}_{\mathrm{a}}$ avec $\hat{\Pi}_{\mathrm{b}}$ équivaut à partager les variables communes w_2 et x_2. Après l'interconnexion des ports, on ne s'intéresse plus à w_2 et on l'élimine du modèle explicite.

Théorème 1.11. *Soient $\hat{\Pi}_\mathrm{a}$ et $\hat{\Pi}_\mathrm{b}$ deux ports décrits par (1.18). Si $\hat{\Pi}_\mathrm{a}$ et $\hat{\Pi}_\mathrm{b}$ sont cyclo passifs, $\hat{\Pi}_\mathrm{a} \wedge \hat{\Pi}_\mathrm{b}$ le sera aussi.*

Du point de vue des comportements, le problème de commande est vu comme le problème de concevoir un système (un contrôleur), tel que le système qui résulte de l'interconnexion entre le contrôleur et le processus originel (c'est-à-dire, le système commandé) ait les propriétés désirées. Nous suivons cette approche, mais nous l'ajustons aux m-ports.

Definition 1.12. Considérons un m-port $\hat{\Pi}_\mathrm{p}$. Un *contrôleur* pour $\hat{\Pi}_\mathrm{p}$ est un $2m$-port

$$\hat{\Pi}_\mathrm{c} = (\mathbb{W} \times \mathbb{W}_\mathrm{t}, \mathbb{X} \times \mathbb{X}_\mathrm{c}, \hat{\mathcal{B}}_\mathrm{c}) \ , \quad \mathbb{W}_\mathrm{t} = \mathbb{U}_\mathrm{t} \times \mathbb{Y}_\mathrm{t} \ ,$$

où $\mathbb{U}_\mathrm{t} = \mathbb{R}^m$, $\mathbb{Y}_\mathrm{t} = \mathbb{R}^m$, $\mathbb{X}_\mathrm{c} = \mathbb{R}^q$ pour une certaine $q \in \mathbb{N}$, et $u_\mathrm{t} \in \mathbb{U}_\mathrm{t}$ est une variable libre.

Le processus et le contrôleur partagent tout l'état et les variables manifestes du processus. Donc, d'après la définition 1.10, la connexion qui en résulte est de la forme

$$\hat{\Pi}_\mathrm{t} = \hat{\Pi}_\mathrm{p} \wedge \hat{\Pi}_\mathrm{c} = (\mathbb{W}_\mathrm{t}, \mathbb{X} \times \mathbb{X}_\mathrm{c}, \hat{\mathcal{B}}_\mathrm{t}) \ .$$

En particulier, un contrôleur basé sur la passivité exploite la propriété de cyclo passivité d'un processus donné.

Definition 1.13. Soit $\hat{\Pi}_\mathrm{p}$ un m-port cyclo passif. On dit que le contrôleur $\hat{\Pi}_\mathrm{c}$ est *basé sur la passivité* (abrégé par $\hat{\Pi}_\mathrm{c} \in$ PBC) si la connexion qui en résulte est encore cyclo passive.

Comme la plupart du reste du document porte sur le problème spécifique de stabilisation d'un point d'équilibre, il nous sera utile de définir l'ensemble d'équilibres assignables $\mathcal{E}_x$, qui est l'ensemble de points $x \in \mathbb{X}$ pour lesquels il existe une commande constante $u \in \mathbb{U}$ qui fait de x un équilibre. Pour les m-ports décrits par (1.3), cet ensemble est

$$\mathcal{E}_x = \left\{ x \in \mathbb{X} \,\middle|\, \exists \ u \in \mathbb{U} \text{ t.q. } f(x) + g(x)u = 0 \right\} \ , \tag{1.20}$$

ce qui équivaut à

$$\mathcal{E}_x = \left\{ x \in \mathbb{X} \mid g^\perp(x) f(x) = 0 \right\} \ . \tag{1.21}$$

Notons que la représentation (1.21) ne contient pas la variable u, elle est donc plus simple à calculer que la définition (1.20). Pourtant, pour un $x_\star \in \mathcal{E}_x$ donné, il nous sera utile de calculer la commande qui apparaît dans (1.20) :

$$u_\star \triangleq -g^+(x_\star) f(x_\star) \ . \tag{1.22}$$

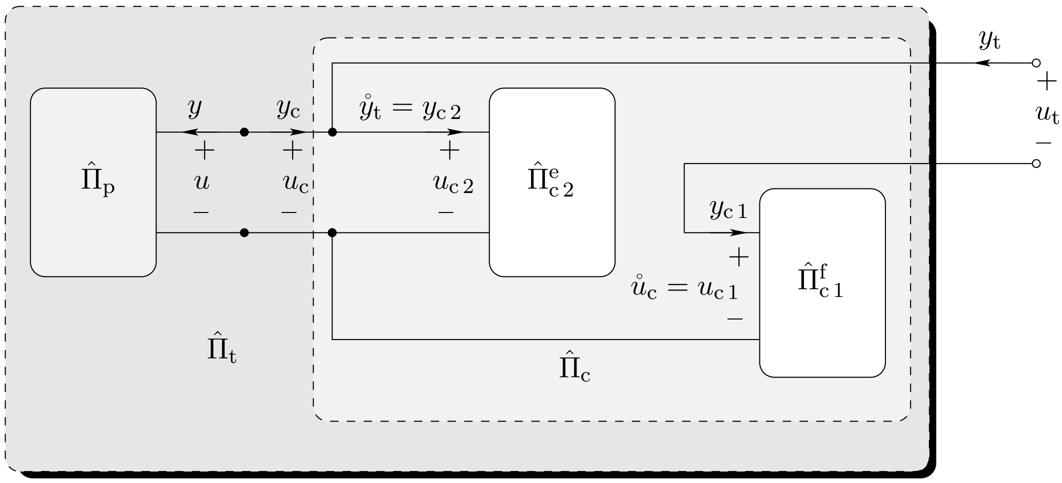

FIG. 1.4: Le contrôleur peut être implanté comme l'interconnexion de deux m-ports (l'un actionné par les flux et l'autre actionné par les efforts).

1.4 Passivation statique

Nous commençons par caractériser l'ensemble de contrôleurs basés sur la passivité. De cet ensemble nous choisissons des cas spéciaux, nous commençons avec les plus simples (mais plus restrictifs), en augmentant ensuite le degré de complexité (et de généralité). Nous traitons l'invariance du flux et de dissipation, qui sont cruciaux pour établir la propriété de bilan d'énergie et qui joueront un rôle important dans la suite.

Hypothèse 1.14. *Les processus considérés sont des m-ports cyclo passifs*

$$\hat{\Pi}_p = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_p) \tag{1.23}$$

avec $\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ *et*

$$\hat{\mathcal{B}}_p = \left\{ (u, y, x) \in \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \,\middle|\, \textit{Eq. (1.10)} \right\} \ . \tag{1.24}$$

Notation. Comme la structure de (1.24) peut être déduite de (1.23), à partir de maintenant nous remplacerons (1.24) (et des comportements généraux aussi) par la notation abrégée

$$\hat{\mathcal{B}}_p = \left\{ (u, y, x) \,\middle|\, \text{Eq. (1.10)} \right\} \ . \tag{1.25}$$

Nous considérons des contrôleurs statiques n'ayant pas d'état interne $\mathbb{X}_c$, c'est-à-dire,

$$\hat{\Pi}_c = (\mathbb{W} \times \mathbb{W}_t, \mathbb{X}, \hat{\mathcal{B}}_c) \ . \tag{1.26}$$

De plus, nous fixons notre attention sur les comportements satisfaisant l'hypothèse suivante.

Hypothèse 1.15. *Le comportement du contrôleur est de la forme*

$$\hat{\mathcal{B}}_\mathrm{c} = \left\{(u_\mathrm{c}, y_\mathrm{c}, u_\mathrm{t}, y_\mathrm{t}, x) \,\middle|\, u_\mathrm{c} = \mathring{u}_\mathrm{c}(x) + u_\mathrm{t},\ y_\mathrm{t} = \mathring{y}_\mathrm{t}(x) - y_\mathrm{c}\right\} \tag{1.27}$$

pour une certaine $\mathring{u}_\mathrm{c} : \mathbb{X} \to \mathbb{U}$ *et* $\mathring{y}_\mathrm{t} : \mathbb{X} \to \mathbb{Y}$.

Notons que le comportement (1.27) admet y_c, x et u_t comme variables libres, et que y et x sont précisément les variables qu'on *ne peut pas* assigner librement dans $\hat{\Pi}_\mathrm{p}$. Pour cette raison, l'hypothèse 1.15 simplifie la tâche de trouver $\hat{\mathcal{B}}_\mathrm{t}$ sans avoir besoin de calculer explicitement les trajectoires de $\hat{\Pi}_\mathrm{p}$. Elle exclut aussi les cas pathologiques comme $\hat{\mathcal{B}}_\mathrm{t} = \emptyset$. Comme on le verra plus tard, cette hypothèse induit peu de perte de généralité, car elle est vérifiée pour la plupart de contrôleurs basés sur la passivité qu'on trouve dans la littérature.

En raison de la même hypothèse, il est possible d'implanter $\hat{\Pi}_\mathrm{c}$ à partir d'un m-port actionné par les flux

$$\hat{\Pi}^\mathrm{f}_{\mathrm{c}\,1} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{c}\,1})\ , \quad \hat{\mathcal{B}}_{\mathrm{c}\,1} = \left\{(u_{\mathrm{c}\,1}, y_{\mathrm{c}\,1}, x) \,\middle|\, u_{\mathrm{c}\,1} = \mathring{u}_\mathrm{c}(x)\right\}$$

et d'un m-port actionné par les efforts

$$\hat{\Pi}^\mathrm{e}_{\mathrm{c}\,2} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{c}\,2})\ , \quad \hat{\mathcal{B}}_{\mathrm{c}\,2} = \left\{(u_{\mathrm{c}\,2}, y_{\mathrm{c}\,2}, x) \,\middle|\, y_{\mathrm{c}\,2} = \mathring{y}_\mathrm{t}(x)\right\}\ ,$$

soumis aux contraintes d'interconnexion (voir la figure 1.4)

$$\begin{array}{rcl} u_\mathrm{c} &=& u_{\mathrm{c}\,1} + u_\mathrm{t} \\ 0 &=& y_{\mathrm{c}\,1} + y_\mathrm{t} \end{array} \quad \text{et} \quad \begin{array}{rcl} u_\mathrm{c} &=& u_{\mathrm{c}\,2} + 0 \\ y_\mathrm{c} &=& y_{\mathrm{c}\,2} - y_\mathrm{t}\ . \end{array} \tag{1.28}$$

Remarque 1.16. On dit que l'interconnexion (1.28) préserve la puissance, car la puissance rentrant dans l'interconnexion est la même que la puissance qui en sort. Autrement dit, $y_\mathrm{c}^\top u_\mathrm{c} + y_\mathrm{t}^\top u_\mathrm{t} = y_{\mathrm{c}\,2}^\top u_{\mathrm{c}\,2} + y_{\mathrm{c}\,1}^\top u_{\mathrm{c}\,1}$.

Lemme 1.17. *Considérons le processus* $\hat{\Pi}_\mathrm{p}$ *(1.25) et un contrôleur* $\hat{\Pi}_\mathrm{c}$ *qui satisfait l'hypothèse 1.15. Le m-port composé est* $\hat{\Pi}_\mathrm{t} = \hat{\Pi}_\mathrm{p} \wedge \hat{\Pi}_\mathrm{c} = (\mathbb{W}_\mathrm{t}, \mathbb{X}, \hat{\mathcal{B}}_\mathrm{t})$ *et* $(u_\mathrm{t}, y_\mathrm{t}, x) \in \hat{\mathcal{B}}_\mathrm{t}$ *équivaut à*

$$\dot{x} = f_\mathrm{t}(x) + g(x) u_\mathrm{t} \tag{1.29a}$$

$$y_\mathrm{t} = h_\mathrm{t}(x)\ , \tag{1.29b}$$

où $f_\mathrm{t}(x) \triangleq f(x) + g(x)\mathring{u}_\mathrm{c}(x)$ *et* $h_\mathrm{t}(x) \triangleq h(x) + \mathring{y}_\mathrm{t}(x)$.

Théorème 1.18. *Soient* $\hat{\Pi}_\mathrm{c}$ *un contrôleur qui satisfait l'hypothèse 1.15 et* $\hat{\Pi}_\mathrm{p}$ *un processus qui satisfait l'hypothèse 1.14. Les affirmations suivantes sont équivalentes :*

1. $\hat{\Pi}_\mathrm{c} \in$ *PBC.*
2. *Il existe* $H_\mathrm{a} : \mathbb{X} \to \mathbb{R}$ *et* $d_\mathrm{a} : \mathbb{X} \to \mathbb{R}$, *avec* $d_\mathrm{a}(x) \geq -d(x)$, *tel que*

$$\dot{H}_\mathrm{a}(x) = y_\mathrm{c}^\top u_\mathrm{c} + y_\mathrm{t}^\top u_\mathrm{t} - d_\mathrm{a}(x) \qquad \forall\ (u_\mathrm{c}, y_\mathrm{c}, u_\mathrm{t}, y_\mathrm{t}, x) \in \hat{\mathcal{B}}_\mathrm{c}\ ,\ (u_\mathrm{c}, -y_\mathrm{c}, x) \in \hat{\mathcal{B}}_\mathrm{p}\ . \tag{1.30}$$

3. Il existe $H_\mathrm{a} : \mathbb{X} \to \mathbb{R}$ et $d_\mathrm{a} : \mathbb{X} \to \mathbb{R}$, avec $d_\mathrm{a}(x) \geq -d(x)$, tel que

$$h^\top(x)\mathring{u}_\mathrm{c}(x) = -\nabla H_\mathrm{a}^\top(x)(f(x) + g(x)\mathring{u}_\mathrm{c}(x)) - d_\mathrm{a}(x) \tag{1.31a}$$
$$\mathring{y}_\mathrm{t}(x) = g^\top(x)\nabla H_\mathrm{a}(x) . \tag{1.31b}$$

Definition 1.19. Soit $\hat{\Pi}_\mathrm{p}$ un processus qui satisfait l'hypothèse 1.14. On dit que le contrôleur $\hat{\Pi}_\mathrm{c}$ est *à bilan d'énergie* si, pour une certaine $H_\mathrm{a} : \mathbb{X} \to \mathbb{R}$,

$$\dot{H}_\mathrm{a}(x) = y_\mathrm{c}^\top \mathring{u}_\mathrm{c}(x) \qquad \forall\, (u_\mathrm{c}, y_\mathrm{c}, u_\mathrm{t}, y_\mathrm{t}, x) \in \hat{\mathcal{B}}_\mathrm{c}\ ,\ (u_\mathrm{c}, -y_\mathrm{c}, x) \in \hat{\mathcal{B}}_\mathrm{p} . \tag{1.32}$$

Proposition 1.20. *Considérons un contrôleur $\hat{\Pi}_\mathrm{c}$ qui satisfait 1.15 et un processus $\hat{\Pi}_\mathrm{p}$ qui satisfait l'hypothèse 1.14. Les affirmations suivantes sont équivalentes :*

1. $\hat{\Pi}_\mathrm{c}$ est à bilan d'énergie.

2. $\hat{\Pi}_\mathrm{c} \in PBC$ et

$$y_\mathrm{t} = y \quad \text{et} \quad d_\mathrm{t}(x) = d(x) , \tag{1.33}$$

(on dit, respectivement, que $\hat{\Pi}_\mathrm{c}$ préserve le flux et la dissipation).

3. Il existe une $H_\mathrm{a} : \mathbb{X} \to \mathbb{R}$ tel que

$$h^\top(x)\mathring{u}_\mathrm{c}(x) = -\nabla H_\mathrm{a}^\top(x) f(x) \tag{1.34a}$$
$$\mathring{y}_\mathrm{t}(x) = g^\top(x)\nabla H_\mathrm{a}(x) = 0 . \tag{1.34b}$$

Quand on utilise la commande basée sur la passivité pour stabiliser un équilibre $x_\star \in \mathcal{E}_x$, la fonction de stockage est normalement utilisée comme fonction de Lyapunov ; on impose donc

$$x_\star = \arg\min H_\mathrm{t}(x) . \tag{1.35}$$

Comme $\nabla H_{\mathrm{t}\star} \triangleq \nabla H_\mathrm{t}(x_\star) = 0$ est une condition nécessaire pour (1.35), il est clair de (4.13b) que le flux y_t doit être égal à zéro à l'équilibre, c'est-à-dire, $h_{\mathrm{t}\star} \triangleq h_\mathrm{t}(x_\star) = 0$. De même, de l'équation (4.13a) on a que la dissipation à l'équilibre doit être égale à zéro, c'est-à-dire, $d_{\mathrm{t}\star} \triangleq d_\mathrm{t}(x_\star) = 0$. Les contrôleurs à bilan d'énergie, préservant le flux et la dissipation, imposent

$$d_\star = 0 \quad \text{et} \quad h_\star = 0 \tag{1.36}$$

au processus en *boucle ouverte*. C'est que l'on appelle *obstacle de la dissipation*.

Il est clair que pour stabiliser des systèmes qui dissipent l'énergie à l'équilibre désiré, il faut modifier la dissipation. Une fonction candidate de dissipation d_t qui est compatible avec cette restriction ($d_{\mathrm{t}\star} = 0$ et ainsi, surmonte l'obstacle de la dissipation) est présentée dans la proposition suivante, où l'on récupère la célèbre méthode *d'assignation d'interconnexion et d'amortissement* (IDA).

Proposition 1.21. *Considérons un contrôleur $\hat{\Pi}_\mathrm{c}$ qui satisfait l'hypothèse 1.15 et un processus $\hat{\Pi}_\mathrm{p}$ qui satisfait l'hypothèse 1.14. Fixons H_t et la fonction de dissipation*

$$d_\mathrm{t}(x) = \nabla H_\mathrm{t}^\top(x) R_\mathrm{t}(x) \nabla H_\mathrm{t}(x) \tag{1.37}$$

avec $R_\mathrm{t}(x) \in \mathbb{R}^{n\times n}$, $R_\mathrm{t}(x) = R_\mathrm{t}^\top(x) \geq 0$. Alors, *(i)* $\hat{\Pi}_{\rm c} \in PBC$ *si, et seulement si,*

$$\mathring{y}_{\rm t}(x) = g^\top(x)\nabla H_{\rm a}(x) \tag{1.38}$$

et

$$g(x)\mathring{u}_{\rm c}(x) = -f(x) - R_{\rm t}(x)\nabla H_{\rm t}(x) + \gamma(x) \tag{1.39}$$

pour un certain champ vectoriel γ *tel que* $\nabla H_{\rm t}^\top(x)\gamma(x)$ *est identiquement égal à zéro.*

(ii) Si $x_\star \in \mathcal{E}_x$ *satisfait (1.35), alors* $\gamma_\star \triangleq \gamma(x_\star) = 0$.

(iii) Pour n'importe quelle matrice anti symétrique $J_{\rm t}(x) \in \mathbb{R}^{n\times n}$, *le champ vectoriel*

$$\gamma(x) = J_{\rm t}(x)\nabla H_{\rm t}(x) \tag{1.40}$$

satisfait les deux restrictions : $\gamma_\star = 0$ *et* $\nabla H_{\rm t}^\top(x)\gamma(x) = 0$. *De plus,* $\hat{\Pi}_{\rm t}$ *est Hamiltonien à port avec* $(u_{\rm t}, y_{\rm t}, x) \in \hat{\mathcal{B}}_{\rm t}$ *équivalent à*

$$\begin{aligned} \dot{x} &= F_{\rm t}(x)\nabla H_{\rm t}(x) + g(x)u_{\rm t}\ , \quad F_{\rm t}(x) \triangleq J_{\rm t}(x) - R_{\rm t}(x) \\ y_{\rm t} &= g^\top(x)\nabla H_{\rm t}(x)\ . \end{aligned}$$

L'affirmation (iii) correspond à la stratégie d'assignation d'interconnexion et d'amortissement. Cela revient à trouver une $\mathring{u}_{\rm c}$ et des $R_{\rm t}$, $J_{\rm t}$ et $H_{\rm t}$ convenables et qui satisfassent

$$g(x)\mathring{u}_{\rm c}(x) = -f(x) + (J_{\rm t}(x) - R_{\rm t}(x))\nabla H_{\rm t}(x)\ ,$$

ce qui équivaut à

$$g^\perp(x)\left[(J_{\rm t}(x) - R_{\rm t}(x))\nabla H_{\rm t}(x) - f(x)\right] = 0 \tag{1.42}$$

et

$$\mathring{u}_{\rm c}(x) = g^+(x)\left[(J_{\rm t}(x) - R_{\rm t}(x))\nabla H_{\rm t}(x) - f(x)\right]\ .$$

Il y a plusieurs approches pour résoudre (1.42) [39]. Une possibilité est de fixer un $H_{\rm t}$ désiré et de résoudre (1.42) comme une équation algébrique en $J_{\rm t}$ et $R_{\rm t}$ [18]. Une autre possibilité est de fixer $J_{\rm t}$ et $R_{\rm t}$ et de résoudre (1.42) comme une équation aux dérivées partielles avec $H_{\rm t}$ l'inconnue [45]. Une troisième approche est de contraindre la fonction d'énergie désirée dans une certaine classe. Par exemple, pour les systèmes mécaniques la somme d'énergies potentielle et cinétique. Fixer la structure de la fonction d'énergie produit une nouvelle équation aux dérivées partielles pour ses termes inconnues et, au même temps, impose des restrictions sur les matrices d'interconnexion et d'amortissement [1].

Pour les systèmes Hamiltoniens à port, l'équation (1.42) devient

$$g^\perp(x)F_{\rm t}(x)\nabla H_{\rm a}(x) = g^\perp(x)\left[F(x) - F_{\rm t}(x)\right]\nabla H(x)\ .$$

Comme

$$\nabla H_{\rm t}^\top(x)\gamma(x) = 0\ ,$$

le champ vecteur γ ne joue aucun rôle dans le bilan de puissance

$$\dot{H}_{\rm t}(x) = y^\top u - \nabla H_{\rm t}^\top(x)R_{\rm t}(x)\nabla H_{\rm t}\ ,$$

on dit alors que γ est *à travail nul.* D'un autre coté, γ joue un rôle important dans la consistance algébrique de (1.39), c'est-à-dire, on peut le choisir pour satisfaire

$$g^{\perp}(x)\left[f(x) + R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) - \gamma(x)\right] = 0 \; .$$

Dans la section 4.5 nous présentons une méthode pour résoudre ces équations algébriques.

Bien que dans certains cas le choix de matrices J_{t} et R_{t} peut être motivé par des considérations physiques, mise à part la condition d'existence des solutions, il n'y a pas des indications générales. Si le système originel est déjà Hamiltonien à port, un choix naturel pour F_{t} est simplement

$$F_{\mathrm{t}}(x) = F(x) \; .$$

Dan ce cas le contrôleur est dit IDA basique (BIDA) et l'équation à résoudre est, d'après (1.39),

$$\begin{aligned} g(x)\mathring{u}_{\mathrm{c}}(x) &= -F(x)\nabla H(x) - R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) + J_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) \\ &= F(x)\nabla H_{\mathrm{a}}(x) \; . \end{aligned} \qquad (1.43)$$

En ce qui concerne le flux, on a

$$\mathring{y}_{\mathrm{t}}(x) = g^{\top}(x)\nabla H_{\mathrm{a}}(x) \; .$$

Notons que si on utilise BIDA, on modifie la dissipation de

$$d(x) = -\nabla H^{\top}(x) f(x) = -\nabla H^{\top}(x)F(x)\nabla H(x) = \nabla H^{\top}(x)R(x)\nabla H(x)$$

à

$$d_{\mathrm{t}}(x) = -\nabla H_{\mathrm{t}}^{\top}(x) f_{\mathrm{t}}(x) = -\nabla H_{\mathrm{t}}^{\top}(x)F(x)\nabla H_{\mathrm{t}}(x) = \nabla H_{\mathrm{t}}^{\top}(x)R(x)\nabla H_{\mathrm{t}}(x) \; .$$

Les contrôleurs BIDA possèdent une propriété intéressante qui relie les invariances de flux et de dissipation.

Proposition 1.22. *Un contrôleur qui préserve le flux préserve nécessairement la dissipation. Par conséquent, il est à bilan d'énergie.*

Jusqu'à maintenant, nous avons utilisé le corollaire 1.9, c'est-à-dire, le théorème de Hill-Moylan pour les systèmes n'ayant pas d'action directe des efforts sur les flux. Nous montrerons que l'incorporation d'un terme direct permet de générer de nouveaux flux cyclo passifs. En particulier, d'en identifier un qui est invariant à l'action du BIDA. Le premier pas est de généraliser l'hypothèse 1.14.

Hypothèse 1.23. *Les processus considérés sont des m-ports cyclo passifs*

$$\hat{\Pi}_{\mathrm{p}}^{j} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{p}})$$

avec $\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ *et*

$$\hat{\mathcal{B}}_{\mathrm{p}} = \left\{ (u, y, x) \;\middle|\; \textit{Eq. (1.3)tient} \right\} \qquad (1.44)$$

(notons que $j(x) = 0$ *n'est plus présent).*

Il est possible d'utiliser la version complète du théorème 1.8 pour *construire* de nouveaux flux cyclo passifs. En effet, le théorème fournit un moyen de paramétrer la fonction du flux h et la fonction de dissipation d en des termes de la matrice carrée libre j (d'où la notation). La construction des flux cyclo passifs peut être accomplie comme suit. Commençons par un système cyclo passif ayant $j = 0$ (que nous dénotons par $\hat{\Pi}_\mathrm{p}^j = \hat{\Pi}_\mathrm{p}$). L'hypothèse de cyclo passivité implique que l'équation (1.7a) est valable pour un certaine l — Donc, l est fixe. Alors, pour n'importe quelle j ayant une composante symétrique non nulle, il existe toujours une w qui satisfait (1.7c). La matrice w peut être utilisé pour redéfinir, via (1.7b) et (1.8), h et d.

Bien que le théorème 1.8 soit applicable aux systèmes non linéaires génériques, nous nous intéressons pour l'instant aux systèmes Hamiltoniens.

Hypothèse 1.24. $\hat{\Pi}_\mathrm{p} = (\mathbb{U}\times\mathbb{Y}, \mathbb{X}, \hat{\mathcal{B}}_\mathrm{p})$ *est Hamiltonien à port avec* $\hat{\mathcal{B}}_\mathrm{p}$ *donné par (1.14) et il satisfait*

$$F^\top(x)(F^-)^\top(x)F(x) = F(x) \tag{1.45}$$

et

$$\mathrm{span}\ g(x) \subseteq \mathrm{span}\ F(x)\ . \tag{1.46}$$

Proposition 1.25. *Considérons un m-port* $\hat{\Pi}_\mathrm{p}$ *qui satisfait l'hypothèse 1.24 et définissons*

$$Z(x) \triangleq -(F^-)^\top(x)F(x)F^-(x)\ . \tag{1.47}$$

Le m-port $\hat{\Pi}_\mathrm{p}^j$ *décrit par les équations*

$$\dot{x} = F(x)\nabla H(x) + g(x)u \tag{1.48a}$$

$$y = g^\top(x)Z(x)\left[F(x)\nabla H(x) + g(x)u\right] \tag{1.48b}$$

est cyclo passif avec fonction de stockage H.

Proposition 1.26. *Considérons un contrôleur* $\hat{\Pi}_\mathrm{c}$ *qui satisfait l'hypothèse 1.15 et un processus* $\hat{\Pi}_\mathrm{p}^j$ *qui satisfait l'hypothèse 1.23.* $\hat{\Pi}_\mathrm{c}$ *est à bilan d'énergie si le flux et la dissipation demeurent invariants, c'est-à-dire, si*

$$\dot{H}_\mathrm{t}(x) = y_\mathrm{t}^\top u_\mathrm{t} - d_\mathrm{t}(x, u_\mathrm{t}) \tag{1.49}$$

est satisfaite avec

$$y = y_\mathrm{t} \quad et \quad d(x,u) = d_\mathrm{t}(x,u)\ . \tag{1.50}$$

Proposition 1.27. *Considérons un m-port cyclo passif décrit par (1.47) et (1.48). Supposons que l'hypothèse 1.24 est satisfaite. Le contrôleur BIDA (1.27), avec* $\mathring{u}_\mathrm{c}(x)$ *comme dans (1.43)*

$$\mathring{y}_\mathrm{t}(x) = 0\ , \tag{1.51}$$

est basé sur la passivité. Le contrôleur préserve le flux et la dissipation ; donc, il est à bilan d'énergie.

Le tableau 1.1 contient un résumé de quelques contrôleurs décrits auparavant.

Dispositif de commande	**Flux de $\hat{\mathbf{\Pi}}_t$**	**Dissipation de $\hat{\mathbf{\Pi}}_t$**	**EDP**
EB	Invariant	Invariant	$\begin{pmatrix} g^\perp F \\ g^\top \end{pmatrix} \nabla H_a = 0$
BIDA	$g^\top \nabla H_t$	$\nabla H_t^\top R \nabla H_t$	$g^\perp F \nabla H_a = 0$
BIDA avec le flux invariant	Invariant	Invariant	$g^\perp F \nabla H_a = 0$
IDA	$g^\top \nabla H_t$	$\nabla H_t^\top R_t \nabla H_t$	$g^\perp F_t \nabla H_a = g^\perp (F - F_t) \nabla H$
Fonction de stockage de $\hat{\mathbf{\Pi}}_t$ $H_t(x) = H(x) + H_a(x)$			

TAB. 1.1: Contrôleurs basés sur la passivité et leur EDP correspondante quand $\hat{\Pi}_p$ est Hamiltonien.

1.5 Passivation dynamique

Nous abordons la stabilisation de systèmes Hamiltoniens à port en utilisant la commande par interconnexion. Dans la commande par interconnexion (CbI), le façonnement de l'énergie est accompli en choisissant un contrôleur qui est aussi Hamiltonien à port. Le dispositif est interconnecté au processus en utilisant une interconnexion qui préserve la puissance, ce qui entraîne un m-port composé qui est, lui-même, Hamiltonien à port avec fonction de stockage égale à la somme des fonctions d'énergie du processus et du dispositif de commande.

La composante de base est un système Hamiltonien à port actionné par les *flux*

$$\hat{\Pi}^{\mathrm{f}}_{\mathrm{c}\,1} = (\mathbb{W}, \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}\,1}) \ , \tag{1.52}$$

où $\mathbb{X}_c = \mathbb{R}^m$. Les équations

$$\dot{\xi} = -y_{\mathrm{c}\,1} \tag{1.53a}$$
$$u_{\mathrm{c}\,1} = -\nabla H_{\mathrm{c}}(\xi) \tag{1.53b}$$

définissent son comportement comme

$$\hat{\mathcal{B}}_{\mathrm{c}\,1} = \left\{ (u_{\mathrm{c}\,1}, y_{\mathrm{c}\,1}, \xi) \,\middle|\, \text{Eq. (1.53)} \right\} \ . \tag{1.54}$$

Dans (1.53), ξ est l'état du contrôleur et $H_c : \mathbb{X}_c \to \mathbb{R}$ est une fonction de stockage à déterminer.

La composante de base $\hat{\Pi}_{c\,1}$ est interconnectée aux bornes externes du contrôleur selon la figure 5.1, ce qui produit le contrôleur dynamique

$$\hat{\Pi}_{\mathrm{c}} = (\mathbb{W} \times \mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}}) \tag{1.55}$$

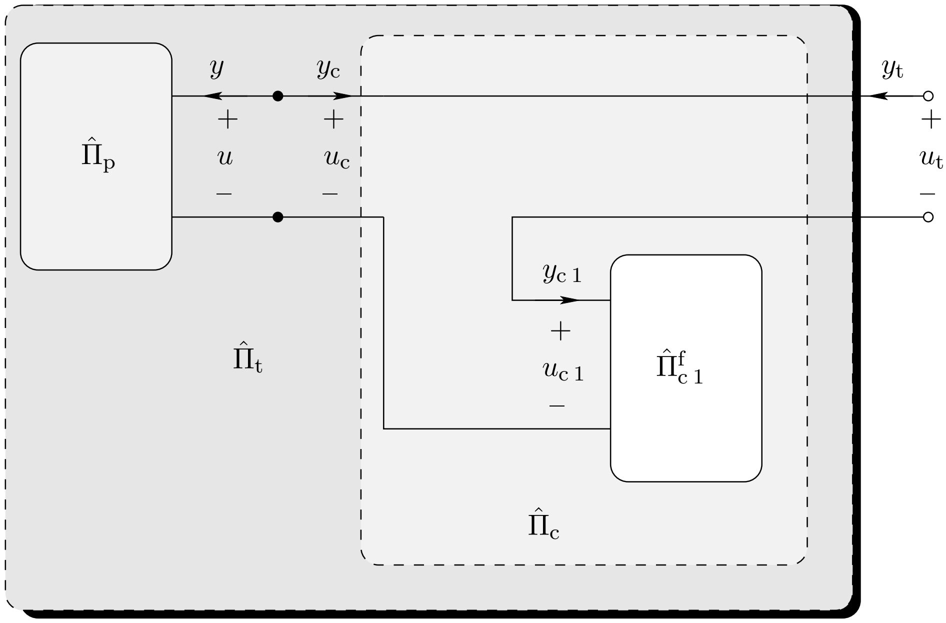

FIG. 1.5: Commande par interconnexion classique.

avec le comportement

$$\hat{\mathcal{B}}_c = \Big\{ (u_c, y_c, u_t, y_t, x, \xi) \;\Big|\; (u_c - u_t, y_c, \xi) \in \hat{\mathcal{B}}_{c\,1},\ y_c = -y_t \Big\} . \tag{1.56}$$

Lemme 1.28. *Considérons un système Hamiltonien à port* $\hat{\Pi}_p = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_p)$*, avec* $\hat{\mathcal{B}}_p$ *donné par (1.14). Le dispositif de commande (1.55) avec le comportement (1.56) est basé sur la passivité avec*

$$H_t(x, \xi) = H(x) + H_c(\xi) . \tag{1.57}$$

Hypothèse 1.29. *Il existe une application différentiable* $C : \mathbb{X} \to \mathbb{X}_c$ *telle que*

$$\begin{pmatrix} F^\top(x) \\ g^\top(x) \end{pmatrix} \nabla C(x) = \begin{pmatrix} g(x) \\ 0 \end{pmatrix} . \tag{1.58}$$

Lemme 1.30. *Sous l'hypothèse 1.29, le système Hamiltonien à port* $\hat{\Pi}_t$ *avec* $\hat{\mathcal{B}}_t$ *donné par*

$$\hat{\mathcal{B}}_t = \big\{ (u_t, y_t, x, \xi) \;\big|\; \textit{Eq. (5.10)} \big\} \tag{1.59}$$

est cyclo passif ayant

$$W(x, \xi) \triangleq H(x) + \Phi(C(x) - \xi) + H_c(\xi) \tag{1.60}$$

pour fonction de stockage, où $\Phi : \mathbb{X}_c \to \mathbb{R}$ *est une application différentiable quelconque.*

Notons que, par construction, CbI préserve le flux (voir la figure 1.5) et la dissipation (rappelons que $\hat{\Pi}^{\mathrm{f}}_{\mathrm{c}\,1}$ est sans pertes). On peut donc prévoir que la condition (1.58) sera au moins aussi restrictive que celles imposées par la commande à bilan d'énergie. On formalise cette idée dans la section 5.7, mais on vérifie tout de suite que

$$(1.58) \quad \Longrightarrow \quad \nabla C^\top(x) F^\top(x) \nabla C(x) = 0 \quad \Longrightarrow \quad \nabla C^\top(x) R(x) \nabla C(x) = 0\ .$$

Comme $R(x)$ est symétrique et non négative, on a que

$$R(x)\nabla C(x) = 0\ . \tag{1.61}$$

De (1.61) on voit que le Hamiltonien ajouté

$$H_{\mathrm{a}}(x,\xi) = W(x,\xi) - H(x) = \Phi(C(x)-\xi) + H_{\mathrm{c}}(\xi)$$

satisfait

$$R(x)\nabla_x H_{\mathrm{a}}(x,\xi) = R(x)\nabla C(x)\nabla\Phi(C(x)-\xi) = 0\ ,$$

ce qui évoque la condition nécessaire pour la préservation de la dissipation dans BIDA, c'est-à-dire,

$$\nabla H_{\mathrm{a}}^\top(x) R(x) \nabla H_{\mathrm{a}}(x) = 0\ .$$

Ceci suggère que l'on applique la stratégie CbI aux systèmes Hamiltoniens à port avec le flux construit antérieurement, qui garantit la préservation du flux et de la dissipation sans contraintes additionnelles.

Hypothèse 1.31. *Il existe une application différentiable* $C : \mathbb{X} \to \mathbb{X}_{\mathrm{c}}$ *telle que*

$$F(x)\nabla C(x) = -g(x) \tag{1.62}$$

Le lemme suivant relie les hypothèses 1.29 et 1.31.

Lemme 1.32. *L'équation (1.58) équivaut à*

$$\begin{pmatrix} F(x) \\ g^\top(x) \end{pmatrix} \nabla C(x) = -\begin{pmatrix} g(x) \\ 0 \end{pmatrix}\ . \tag{1.63}$$

Lemme 1.33. *Considérons un port Hamiltonien* $\hat{\Pi}^{j}_{\mathrm{p}} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{p}})$ *avec*

$$\hat{\mathcal{B}}_{\mathrm{p}} = \big\{(u,y,x)\ \big|\ \textit{Eq. (1.48)}\big\} \tag{1.64}$$

et un contrôleur (1.56), (1.55). Sous les hypothèses 1.31 et 1.24, le système composé $\hat{\Pi}_{\mathrm{t}}$ *est cyclo passif avec fonction de stockage*

$$W(x,\xi) = H(x) + \Phi(C(x)-\xi) + H_{\mathrm{c}}(\xi)\ ,$$

où $\Phi : \mathbb{X}_{\mathrm{c}} \to \mathbb{R}$ *est une application différentiable quelconque.*

Comme prévu, le flux invariant permet de façonner l'énergie convenablement sous des hypothèses moins sévères.

Notons que, si l'on utilise le flux standard avec l'hypothèse 1.29, ou bien l'on utilise le flux modifié avec les hypothèses 1.31 et 1.24, le comportement de $\hat{\Pi}_t$ est déterminé par les équations

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} &= F_t(x)\nabla H_t(x,\xi) + g_t(x)u_t \qquad (1.65a) \\ y_t &= g_t^\top(x)\nabla H_t(x) \; , \qquad (1.65b) \end{aligned}$$

où

$$F_t(x) \triangleq \begin{pmatrix} I \\ \nabla C^\top(x) \end{pmatrix} \begin{pmatrix} F(x) & -g(x) \end{pmatrix} \quad \text{et} \quad g_t \triangleq \begin{pmatrix} I \\ \nabla C^\top(x) \end{pmatrix} g(x) \; . \qquad (1.66)$$

Dans la suite, nous traitons seulement (1.65), en accordant que, selon les hypothèses satisfaites, nous ferons référence à la fonction de flux correspondante.

Hypothèse 1.34. *Le processus est Hamiltonien à port et donné par, soit*
(a) $\hat{\Pi}_p = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_p)$ *avec (1.14) et satisfait (1.63) avec une* $C : \mathbb{X} \to \mathbb{X}_c$ *différentiable, soit par*
(b) $\hat{\Pi}_p^j = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_p)$ *avec (1.14) et satisfait (1.62) avec une* $C : \mathbb{X} \to \mathbb{X}_c$ *différentiable,*
La matrice $F(x)$ *est non singulière.*

Maintenant, nous pouvons unifier les lemmes 1.30 and 1.33.

Théorème 1.35. *Sous l'hypothèse 1.34, le contrôleur (1.55), (1.56) produit le port Hamiltonien* $\hat{\Pi}_t = (\mathbb{W}_t, \mathbb{X} \times \mathbb{X}_c, \hat{\mathcal{B}}_t)$ *avec le comportement*

$$\hat{\mathcal{B}}_t = \left\{ (u_t, y_t, x) \;\middle|\; \textit{Eq. (1.65)} \right\} \; , \qquad (1.67)$$

Pour n'importe quel $\Phi : \mathbb{X}_c \to \mathbb{R}$ *différentiable,* $\hat{\Pi}_t$ *est cyclo passif avec fonction de stockage*

$$W(x,\xi) = H(x) + \Phi(C(x) - \xi) + H_c(\xi) \; . \qquad (1.68)$$

Dans certains cas, il est possible de choisir systématiquement H_c et Φ dans le théorème 1.35 pour stabiliser un point d'équilibre assignable. Dans un premier pas, définissons l'ensemble $\mathcal{E}$ d'équilibres à effort nul (c'est-à-dire, avec $u_t = 0$). D'après (1.65) et (1.66),

$$\mathcal{E} = \left\{ (x,\xi) \;\middle|\; F(x)\nabla H(x) - g(x)\nabla H_c(\xi) = 0 \right\} \; . \qquad (1.69)$$

D'après le théorème 1.35, W satisfait

$$\dot{W}(x,\xi) \leq y_t^\top u_t \; . \qquad (1.70)$$

Selon la théorie standard de Lyapunov, si W a un minimum strict au point $(x_\star, \xi_\star) \in \mathcal{E}$ et si l'on fixe $u_t = 0$, alors $(x_\star, \xi_\star)$ est stable. L'objectif est alors de trouver les bons Φ et H_c, en imposant des conditions sur C, tels que

$$(x_\star, \xi_\star) = \arg\min W(x,\xi) \; .$$

Naturellement, la négativité de $\dot{W}(x,\xi)$ peut être renforcée en faisant

$$u_t = -K_v y_t \; , \quad K_v = K_v^\top > 0 \; . \tag{1.71}$$

Comme on l'avait déjà dit, cette injection d'amortissement peut être adoptée pour essayer de faire l'équilibre *asymptotiquement* stable, ce qui est le cas si y_t est un flux détectable. Malheureusement, nous montrerons que la dernière condition n'est pas satisfaite par CbI et qu'il faut adopter d'autres stratégies qui seront présentées ci-dessus, mais d'abord, nous proposons une solution au problème de stabilisation d'un élément arbitraire de $\mathcal{E}_x$.

Proposition 1.36. *Considérons le m-port* $\hat{\Pi}_t$ *avec le comportement (1.67) et* $u_t = 0$. *Fixons un point* $x_\star \in \mathcal{E}_x$ *et calculons la* $u_\star$ *correspondante via (1.22). Soit*

$$H_c(\xi) = \frac{1}{2} \left\| \xi - K_c^{-1} u_\star \right\|_{K_c}^2 \; , \tag{1.72}$$

où $K_c = K_c^\top > 0$ *et choisissons*

$$\Phi(z) = -u_\star^\top z \; . \tag{1.73}$$

Alors, $(x_\star, 0)$ *est un équilibre de* $\hat{\Pi}_t$*, c'est-à-dire,* $(x_\star, 0) \in \mathcal{E}$*. De plus,* $(x_\star, 0)$ *est un point d'équilibre stable si*

$$\nabla^2 H(x_\star) - \sum_{i=1}^{m} u_{\star i} \nabla^2 C_i(x_\star) > 0 \; . \tag{1.74}$$

Notons que la propriété fondamentale qui permet de façonner la fonction de stockage le long de x est l'invariance de $C(x) - \xi$. En d'autres mots, le pas fondamental d'un façonnement de l'énergie réussie est la génération des variétés invariantes

$$\mathcal{M}_\kappa = \left\{ (x, \xi) \;\middle|\; C(x) - \xi = \kappa \right\} \; .$$

Malheureusement, ceci pose le problème suivant : supposons que le système commence d'une condition initiale (x_0, ξ_0) arbitraire. Il n'y a aucune raison pour laquelle l'équilibre désiré $(x_\star, \xi_\star)$ doit satisfaire

$$C(x_\star) - \xi_\star = C(x_0) - \xi_0 \tag{1.75}$$

Une possibilité pour remplir la condition (1.75) est d'initialiser le contrôleur avec la valeur ξ_0 qui met le système dans la variété invariante appropriée. Cette approche est simple, mais la dépendance des conditions initiales rend le système extrêmement non robuste. En général, $(x_\star, \xi_\star)$ n'appartient pas à l'orbite de la solution qui commence à (x_0, ξ_0), donc le flux y_t n'est pas détectable et l'équilibre désiré est peut-être stable, mais non pas asymptotiquement stable, même avec l'injection d'amortissement (1.71).

Il est clair qu'une autre façon de satisfaire la contrainte (1.75) est de décaler la valeur désirée de ξ (originalement établi à zéro) à la nouvelle valeur

$$\xi_\star = C(x_\star) - C(x_0) + \xi_0 \; . \tag{1.76}$$

Ceci revient à changer H_c par

$$H_c(\xi) = \frac{1}{2}\left\|\xi - \xi_\star - K_c^{-1}u_\star\right\|_{K_c}^2 , \tag{1.77}$$

pour que $\nabla H_c(\xi_\star) = -u_\star$. Géométriquement, nous décalons le locus $\mathcal{E}$ des équilibres le long de l'axe ξ, pour que l'intersection entre $\mathcal{E}$ et la variété où la trajectoire commence (c'est-à-dire, $\mathcal{M}_{\kappa_0}$ avec $\kappa_0 \triangleq C(x_0) - \xi_0$) ait lieux au $x_\star$ désiré.

En principe, ce schéma dépend encore de la connaissance de la condition initiale, mais cette adversité peut être éliminée en la reformulant comme un problème d'estimation de paramètres. Si l'on estime le paramètre κ_0 et l'on utilise l'invariance de la variété $\mathcal{M}_{\kappa_0}$, on peut concevoir un schéma qui assure la convergence du paramètre. Pour formuler l'affirmation précise, considérons l'ensemble d'équations

$$\dot{\xi} = -y_{c\,1} \tag{1.78a}$$
$$\dot{\hat{\kappa}}_0 = -\Gamma\left(\hat{\kappa}_0 - C(x) + \xi\right) \tag{1.78b}$$
$$u_{c\,1} = -\nabla_\xi \hat{H}_c(\xi, \hat{\kappa}_0) \tag{1.78c}$$

avec $\hat{H}_c(\xi, \hat{\kappa}_0) \triangleq \frac{1}{2}\left\|\xi - C(x_\star) + \hat{\kappa}_0 - K_c^{-1}u_\star\right\|_{K_c}^2$. Définissons la composante de base adaptative $\hat{\Pi}_{c\,1}^{f} = (\mathbb{W}, \mathbb{X}_c \times \mathbb{X}_c, \hat{\mathcal{B}}_{c\,1})$ avec le comportement

$$\hat{\mathcal{B}}_{c\,1} = \left\{(u_{c\,1}, y_{c\,1}, \xi, \hat{\kappa}_0) \,\middle|\, \text{Eq. (1.78)}\right\}$$

et construisons le contrôleur

$$\hat{\Pi}_c = (\mathbb{W} \times \mathbb{W}_t, \mathbb{X} \times \mathbb{X}_c \times \mathbb{X}_c, \hat{\mathcal{B}}_c) \tag{1.79}$$

avec

$$\hat{\mathcal{B}}_c = \left\{(u_c, y_c, u_t, y_t, x, \xi, \hat{\kappa}_0) \,\middle|\, (u_c - u_t, y_c, \xi, \hat{\kappa}_0) \in \hat{\mathcal{B}}_{c\,1},\ y_c = -y_t\right\} . \tag{1.80}$$

Proposition 1.37. *Considérons un $\hat{\Pi}_p$ qui satisfait l'hypothèse 1.34. Le m-port composé $\hat{\Pi}_t$, résultant du dispositif de commande (5.39), (5.40) satisfait :*

(i) La convergence exponentielle des paramètres est assurée, plus précisément,

$$\|\hat{\kappa}_0 - \kappa_0\| \leq e^{-\lambda_{min}\{\Gamma\}t}\|\hat{\kappa}_0(0) - \kappa_0\|$$

pour toute $t \geq 0$.

Fermons le port avec $u_t = -K_v y_t$. Alors,

(ii) Pour n'importe quel $x_\star \in \mathcal{E}_x$, le point $(x_\star, \xi_\star, \kappa_0)$, où $\xi_\star$ est donné par (1.76), est un équilibre stable si (1.74) est satisfaite.

(iii) Les orbites de la dynamique résiduelle sont confinées à l'ensemble $\mathcal{Z} \times \{\xi = \bar{\xi}\}$, où $\bar{\xi}$ est une constante et

$$\mathcal{Z} \triangleq \left\{x \,\middle|\, \begin{pmatrix}\nabla H^\top(x)\\ \nabla C^\top(x)\end{pmatrix}\left[F(x)\nabla H(x) - g(x)\left(K_c(C(x) - C(x_\star)) - u_\star\right)\right] = 0\right\} .$$

(iv) *Supposons qu'aucune trajectoire x ne peut pas rester identiquement dans $\mathcal{Z}$, autre que des points isolés, alors, $(x_\star, \xi_\star, \kappa_0)$ est un point d'équilibre asymptotiquement stable. Il sera global et asymptotiquement stable si c'est le seul point dans $\mathcal{Z}$ et si W est radialement non bornée.*

Une autre possibilité d'accomplir la convergence est de détruire l'invariance des Casimirs en ajoutant de l'amortissement au dispositif de commande. L'idée est de retourner à la fonction de stockage originale

$$H_\mathrm{c}(\xi) = \frac{1}{2} \left\| \xi - K_\mathrm{c}^{-1} u_\star \right\|_{K_\mathrm{c}}^2 , \tag{1.81}$$

et de munir $\hat{\Pi}^\mathrm{f}_{\mathrm{c}\,1}$ avec une paire de variables de port $(w, z) \in \mathbb{W}$ additionnelles. Ceci est accompli de la façon suivante : redéfinissons la composante de base comme

$$\hat{\Pi}^\mathrm{f}_{\mathrm{c}\,1} = (\mathbb{W} \times \mathbb{W}, \mathbb{X} \times \mathbb{X}_\mathrm{c}, \hat{\mathcal{B}}_{\mathrm{c}\,1}) .$$

Les équations

$$\begin{aligned} \dot{\xi} &= -y_{\mathrm{c}\,1} + w && (1.82a) \\ u_{\mathrm{c}\,1} &= -\nabla H_\mathrm{c}(\xi) && (1.82b) \\ z &= \nabla_\xi W(x, \xi) , && (1.82c) \end{aligned}$$

avec W comme dans (5.25), définissent le comportement de $\hat{\Pi}^\mathrm{f}_{\mathrm{c}\,1}$ comme

$$\hat{\mathcal{B}}_{\mathrm{c}\,1} = \left\{ (u_{\mathrm{c}\,1}, y_{\mathrm{c}\,1}, w, z, x, \xi) \,\middle|\, \text{Eq. (1.82)} \right\} .$$

Le contrôleur $\hat{\Pi}_\mathrm{c} = (\mathbb{W} \times \mathbb{W}_\mathrm{t} \times \mathbb{W}_\mathrm{t}, \mathbb{X} \times \mathbb{X}_\mathrm{c}, \hat{\mathcal{B}}_\mathrm{c})$ est alors construit comme

$$\hat{\mathcal{B}}_\mathrm{c} = \left\{ (u, y, u_\mathrm{c}, y_\mathrm{c}, z, w, x, \xi \,\middle|\, (u - u_\mathrm{t}, y_\mathrm{c}, w, z, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,1},\ y_\mathrm{c} = -y_\mathrm{t} \right\}$$

On vérifie aisément que $\hat{\Pi}_\mathrm{t}$ est déterminé par

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} &= F_\mathrm{t} \nabla H_\mathrm{t} + g_\mathrm{t} u_\mathrm{t} + \begin{pmatrix} 0 \\ I \end{pmatrix} w && (1.83a) \\ y_\mathrm{t} &= g_\mathrm{t}^\top \nabla H_\mathrm{t} && (1.83b) \\ z &= \begin{pmatrix} 0 & I \end{pmatrix} \nabla W . && (1.83c) \end{aligned}$$

Notons que pour n'importe quel $w \neq 0$, l'invariance des variétés $\mathcal{M}_\kappa$ a été détruite, car $\dot{C}(x) - \dot{\xi} = -w$. Pourtant, la dérivée de W par rapport au temps est

$$\dot{W}(x, \xi) = y_\mathrm{t}^\top u_\mathrm{t} + w^\top z - d_\mathrm{t}(x, \xi) , \tag{1.84}$$

donc, le nouveau système est aussi cyclo passif avec fonction de stockage W et variables à port $((u_\mathrm{t}, z), (y_\mathrm{t}, w))$.

En ce qui concerne la stabilité, nous avons l'homologue de la proposition 1.36.

Proposition 1.38. *Considérons le* $2m$*-port* $\hat{\Pi}_{\mathrm{t}} = \left\{\mathbb{W}_{\mathrm{t}} \times \mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{t}}\right\}$*, avec le comportement donné par (5.51),* H_{c} *donné par (1.81) et* Φ *par (1.73). Fermons les ports avec*

$$u_{\mathrm{t}} = -K_{\mathrm{v}} y_{\mathrm{t}} \ , \quad K_{\mathrm{v}} = K_{\mathrm{v}}^{\top} > 0 \tag{1.85}$$

et

$$w = -K_{\mathrm{w}} z \ , \quad K_{\mathrm{w}} = K_{\mathrm{w}}^{\top} > 0 \ . \tag{1.86}$$

(i) Pour n'importe quel $x_\star \in \mathcal{E}_x$*, le point* $(x_\star, 0)$ *est un équilibre stable si (1.74) est satisfaite.*

(ii) Les orbites de la dynamique résiduelle sont confinées à l'ensemble $\mathcal{Z}_{\mathrm{w}} \times \{\xi = 0\}$*, où*

$$\mathcal{Z}_{\mathrm{w}} = \left\{ x \mid \begin{pmatrix} \nabla H^{\top}(x) \\ \nabla C^{\top}(x) \end{pmatrix} [F(x)\nabla H(x) - g(x)u_\star] = 0 \right\} \ .$$

(iii) Si aucune trajectoire x *ne peut pas rester identiquement dans* $\mathcal{Z}_{\mathrm{w}}$*, autre que des points isolés, alors,* $(x_\star, 0)$ *est un point d'équilibre asymptotiquement stable. Il sera global et asymptotiquement stable si c'est le seul point dans* $\mathcal{Z}_{\mathrm{w}}$ *et si* W *est radialement non bornée.*

Nous avons montré que, dans sa formulation standard, CbI impose fortes restrictions découlant des préservations du flux et de la dissipation. Nous avons montré aussi que la restriction

$$\nabla C^{\top}(x) g(x) = 0 \tag{1.87}$$

peut être éliminée en utilisant le flux invariant (1.48b), qui préserve naturellement le flux et la dissipation sous BIDA. Maintenant nous montrons que, en construisant délibérément un contrôleur qui ne les préserve pas, il est aussi possible d'éliminer (1.87) sans faire appel au flux invariant.

Hypothèse 1.39. $\hat{\Pi}_{\mathrm{p}} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{p}})$ *est Hamltonien à port avec (1.14) et il satisfait (1.62) pour une application différentiable* $C : \mathbb{X} \to \mathbb{X}_{\mathrm{c}}$*.*

Considérons encore une fois le m-port actionné par les flux $\hat{\Pi}^{\mathrm{f}}_{\mathrm{c}\,1}$, décrit par (1.52) et (1.54). Faisons W comme dans (1.68) et construisons un m-port supplémentaire actionné par les efforts

$$\hat{\Pi}^{\mathrm{e}}_{\mathrm{c}\,2} = (\mathbb{W}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}\,2})$$

avec le comportement

$$\hat{\mathcal{B}}_{\mathrm{c}\,2} = \left\{ (u_{\mathrm{c}\,2}, y_{\mathrm{c}\,2}, x, \xi) \mid y_{\mathrm{c}\,2} = g^{\top}(x) \nabla C(x) \nabla \Phi(C(x) - \xi) \right\} \ .$$

Utilisons $\hat{\Pi}^{\mathrm{f}}_{\mathrm{c}\,1}$ et $\hat{\Pi}^{\mathrm{e}}_{\mathrm{c}\,2}$ pour construire un contrôleur dynamique $\hat{\Pi}_{\mathrm{c}} = (\mathbb{W} \times \mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}})$ avec

$$\hat{\mathcal{B}}_{\mathrm{c}} = \left\{ (u_{\mathrm{c}}, y_{\mathrm{c}}, u_{\mathrm{t}}, y_{\mathrm{t}}, x, \xi) \mid (u_{\mathrm{c}} - u_{\mathrm{t}}, y_{\mathrm{t}}, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,1}, \ (u_{\mathrm{c}}, y_{\mathrm{c}} + y_{\mathrm{t}}, x, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,2} \right\} \ . \tag{1.88}$$

Dispositif de commande	**Flux de $\hat{\Pi}_t$**	**Dissipation de $\hat{\Pi}_t$**	**EDP**
CbI_{sm}	Invariant	Invariant	$\begin{pmatrix} g^{\perp}F \\ g^{\top} \end{pmatrix} \nabla C = 0$
CbI_{sm} avec le flux invariant	Invariant	Invariant	$g^{\perp}F\nabla C = 0$
CbI_{sm} avec $\hat{\Pi}_{c\,2}$	$g^{\top}\nabla_x W$	$\nabla_x^{\top} W R \nabla_x W$	$g^{\perp}F\nabla C = 0$
Fonction de stockage de $\hat{\Pi}_t$ $W(x) = H(x) + \Phi(C(x) - \xi) + H_c(\xi)$			

TAB. 1.2: Contrôleurs basés sur la passivité et leur EDP correspondante.

Proposition 1.40. *Sous l'hypothèse 5.14,*

(i) le contrôleur (1.88) est basé sur la passivité. De plus, $\hat{\Pi}_t$ *est Hamiltonien à port avec la fonction Hamiltonienne*

$$W(x, \xi) = H(x) + \Phi(C(x) - \xi) + H_c(\xi) .$$

Faisons $H_c(\xi) = \frac{1}{2} \left\| \xi - K_c^{-1} u_\star \right\|_{K_c}^2$ *et* $\Phi(z) = -u_\star^{\top} z$*, alors,*

(ii) pour n'importe quelle $x_\star \in \mathcal{E}_x$*, le point est un équilibre stable si (1.74) est satisfaite.*

Si l'on utilise CbI avec le flux décrit dans la page 18 ou avec la composante supplémentaire, on accomplit le façonnement de l'énergie par l'intermédiaire d'une application C qui satisfait (1.62). On peut généraliser (1.62) et la remplacer par

$$F(x)\nabla C(x) = -g(x)\alpha(x)$$

pour quelque $\alpha : \mathbb{X} \to \mathbb{X}_c$ (ceci équivaut à $g^{\perp}(x)F(x)\nabla C(x) = 0$, bien entendu). Sous cette hypothèse plus faible, on accomplit le façonnement de l'énergie en remplaçant (1.53) par

$$\dot{\xi} = -\alpha(x) y_{c\,1} \tag{1.89a}$$

$$u_{c\,1} = -\alpha^{\top}(x) \nabla H_c(\xi) \tag{1.89b}$$

(ce qui satisfait encore $\dot{H}_c(x) = y_{c\,1}^{\top} u_{c\,1}$). Alors, le théorème 1.35, les propositions 1.36, 1.37, 1.38 et 1.40, et leurs preuves respectives suivent *mutatis mutandis*. Ce genre de commande est appelé commande par interconnexion à modulation d'état (CbI_{sm}).

Le tableau 1.2 contient un résumé de quelques contrôleurs décrits ci-dessus. À partir du tableau, il est facile d'établir, en des termes des équations différentielles partielles, l'implication suivante :

$$CbI_{sm} \implies CbI_{sm} \text{ avec le flux invariant} \iff CbI_{sm} \text{ avec } \hat{\Pi}_{c\,2} .$$

1.6 Comparaison entre les cas statique et dynamique

À partir des tableaux 5.1 et 4.2, il est clair que CbI et IDA sont connectés d'une certaine manière. La proposition suivante relie les passivations statique et dynamique dans le cadre du problème de stabilisation.

Proposition 1.41. *Considérons un triplet (F, g, H) et un point $x_\star \in \mathcal{E}_x$. Les affirmations suivantes sont équivalentes :*

1. *Il existe Φ, H_c que C tels que*
$$\begin{pmatrix} g^\perp(x)F(x) \\ g^\top(x) \end{pmatrix} \nabla C(x) = 0 \;, \tag{1.90}$$
$W(x, \xi) = H(x) + \Phi(C(x) - \xi) + H_\mathrm{c}(\xi)$ *est strictement convexe et minimal à* $(x_\star, \xi_\star)$, *avec* $\xi_\star = C(x_\star)$).
2. *Il existe H_a tel que*
$$\begin{pmatrix} g^\perp(x)F(x) \\ g^\top(x) \end{pmatrix} \nabla H_\mathrm{a}(x) = 0 \;, \tag{1.91}$$
$H_\mathrm{t}(x) = H(x) + H_\mathrm{a}(x)$ *est strictement convexe et minimal à* $x_\star$.

1.7 Conclusions

Les résultats présentés dans cette thèse suggèrent une discussion à deux niveaux, un essentiellement théorique et l'autre plus technique, tous les deux profondément reliés. Du point de vue de la théorie, nous nous intéressons à comprendre la commande comme l'acte d'interconnecter systèmes dans le but d'obtenir un comportement désiré. Nous affirmons que l'énergie joue un rôle fondamental dans la modélisation et que le façonnement de l'énergie devrait être incorporé comme principe de conception. *A priori*, ce paradigme repose sur la prémisse que, dans le but de commander, l'interconnexion de systèmes et l'échange d'énergie sont plus intuitifs, conceptuellement plus attirants et plus *naturels* que le traitement de signaux et la pensée entrée–sortie.

Au niveau technique, nous voulons simplement résoudre le problème de commande. Nous voulons formuler les équations correctes, établir si elles admettent des solutions et, si c'est le cas, les déterminer, pour qu'on puisse implanter le contrôleur. Finalement, nous voulons savoir si les résultats techniques confirment, *a posteriori*, nos affirmations sur le façonnement de l'énergie et l'interconnexion de systèmes.

L'idée de façonner l'énergie a été présente depuis l'apparition des premiers articles sur la commande basée sur la passivité [42] et a donné lieu aux lois de commande par retour d'état, comme la commande à bilan d'énergie et IDA. L'importance de compter sur une interprétation physique du contrôleur (ce qui dans un sens est relié à l'idée de commande comme interconnexion de systèmes) a été soulignée aussi depuis le début [41]. En particulier, les régulateurs à bilan d'énergie établissent un bilan de puissance de la forme
$$\dot{H}_\mathrm{c}(x) = y_{\mathrm{c}\,1}^\top u_{\mathrm{c}\,1}$$

(voir la Fig. 4.3), ce qui suggère que derrière la loi de commande par retour d'état il y a un système abstrait ou virtuel sans pertes. L'application de cette idée est bloquée par l'obstacle de la dissipation, ce qui motive l'utilisation de méthodes plus générales comme IDA. Bien qu'IDA soit applicable à une gamme plus large de problèmes, IDA est plus proche de l'interprétation de la commande comme une application de l'espace d'états dans l'espace des entrées. Dans une première approche, IDA n'admet pas un bilan de puissance et est incompatible avec la notion de commande comme interconnexion.

Dans une certaine mesure, on a développé le matériel présenté dans le chapitre 3 pour que, en plus des régulateurs à bilan d'énergie, d'autres régulateurs basés sur la passivité puissent être intégrés dans le cadre de la commande comme interconnexion (voir le chapitre 4). Ceci a été fait dans le but de mieux comprendre les méthodes existantes, plutôt que dans l'esprit de justifier artificieusement le nouveau paradigme — autrement il s'agirait de la mauvaise science. Comme résultat de cet effort théorique, nous comprenons mieux le rôle de l'invariance du flux et de la dissipation, ce qui établit un niveau de référence pour classifier les méthodes existantes. Au niveau technique, nous avons obtenu une caractérisation de tous les régulateurs basés sur la passivité, nous avons identifié un flux invariant à l'action de IDA basique et nous avons aménagé une version étendue de IDA, ainsi qu'une solution explicite à ses équations algébriques et une condition nécessaire concernant la dissipation et la fonction d'énergie désirées.

La commande par interconnexion, en revanche, est un cas particulier de l'idée de commande comme interconnexion[1]. Le contrôleur n'est plus interprété comme un système virtuel : il est bien un système dynamique réel satisfaisant son propre bilan de puissance. Pour développer le matériel du chapitre 5, nous avons commencé avec une méthode de conception sortie directement des affirmations théoriques et nous avons résolu les détails techniques, en utilisant quelques résultats du chapitre 4 pour élargir le domaine d'application. Contrairement au cas de retour d'état statique, nous sommes partis de la théorie et nous nous sommes dirigés vers l'application.

Ces lignes de recherche convergent à la section 5.7, où l'on énonce que : tant qu'on n'essaie pas de modifier la matrice de structure F, un régulateur statique et un régulateur à commande par interconnexion sont équivalents, en termes de la résolubilité des équations différentielles partielles. Rappelons que pour façonner H efficacement, il est par fois nécessaire de modifier F aussi (p. ex. le cas de systèmes mécaniques sous actionnés [1]). La question la plus importante est alors :

> Notre incapacité de modifier F, est-elle une limitation fondamentale de la commande comme interconnexion, ou c'est tout juste une obstruction technique, voire temporaire ?

Sur un autre plan fondamental, mais avec une perspective à plus long terme, il y a le problème de tout formuler dans un cadre purement comportemental, sans recourir à une représentation d'état particulière (voir le *caveat lector*). La propriété de passivité a été caractérisée de façon intrinsèque dans [71], mais le problème de définir la cyclo passivité sans utiliser la notion d'état est encore ouvert. Le lecteur intéressé trouvera dans [11] un

[1] Malheureusement, le seul moyen pour distinguer la méthode particulière présentée dans le chapitre 5 de la perspective général proposé par Willems est par l'utilisation des prépositions *par* et *comme*.

exposé sur le façonnement de l'énergie dans un cadre intrinsèque.

1.8 Publications issues de cette thèse

Revues :

- F. Castaños, B. Jayawardhana, R. Ortega, and E. García-Canseco, *Proportional plus integral control for set-point regulation of a class of nonlinear RLC circuits*, Circuits Syst. Signal Process., 28 (2009), pp. 609 – 623.
- F. Castaños, R. Ortega, A. J. van der Schaft, and A. Astolfi, *Asymptotic stabilization via control by interconnection of port-Hamiltonian systems*, Automatica, 45 (2009), pp. 1611 – 1618.
- F. Castaños and R. Ortega, *Energy-balancing passivity-based control is equivalent to dissipation and output invariance*, Systems and Control Letters, 58 (2009), pp. 553 – 560.
- R. Ortega, A. J. van der Schaft, F. Castaños, and A. Astolfi, *Control by interconnection and standard passivity-based control of port-Hamiltonian systems*, IEEE Trans. Automat. Contr., 53 (2008), pp. 2527 – 2542.
- B. Jayawardhana, R. Ortega, E. García-Canseco, and F. Castaños, *Passivity of nonlinear incremental systems : Application to PI stabilization of nonlinear RLC circuits*, Systems and Control Letters, 56 (2007), pp. 618 – 622.

Ce travail a été présenté dans les conférences suivantes :

- F. Castaños and R. Ortega, *Energy-balancing passivity-based control is equivalent to dissipation and output invariance*, in Proc. European Control Conference, page WeC2.4, Budapest, Hungary, Aug. 2009.
- F. Castaños, R. Ortega, A. J. van der Schaft, and A. Astolfi, *Asymptotic stabilization via control by interconnection of port-Hamiltonian systems*, in Congreso Latinoamericano de Control Automático, Mérida, Venezuela, Nov. 2008.
- F. Castaños, B. Jayawardhana, R. Ortega, and E. García-Canseco, *A class of nonlinear RLC circuits globally stabilizable by proportional plus integral controllers*, in Proc. of the IFAC World Congress, Seoul, Korea, June 2008, pp. 6202 – 6207.
- B. Jayawardhana, R. Ortega, E. García-Canseco, and F. Castaños, *Passivity of nonlinear incremental systems : Application to PI stabilization of nonlinear RLC circuits*, in Proc. Conference on Decision and Control, San Diego, Dec. 2006, p. ThIP2.17.
- R. Ortega, A. J. van der Schaft, F. Castaños, and A. Astolfi, *Control by (state-modulated) interconnection of port-Hamiltonian systems*, in Proc. IFAC Symposium on Nonlinear Control Systems, Pretoria, South Africa, Aug. 2007, pp. 47 – 54.

2 Introduction

One of the main questions in control theory is: Given a system $\mathcal{S}_1$ (the plant), how to design a second system $\mathcal{S}_2$ (the controller), so that when interconnected they behave in a certain, prespecified way? Automatic control systems for water clocks have been used since the year 270 B.C., so this question is at least 2,000 years old [5]. Naturally, there are multiple answers, but the notion of feedback is indisputably one of the key elements in any successful design.

During the 1930's, communication engineers like Black, Nyquist, Bode and others, explicitly recognized feedback as a design principle. Using a frequency domain approach, they built a set of mathematical tools for analyzing and synthesizing amplifiers and other electronic communication devices. During World War II, Bode and a team of engineers applied these tools to the problem of designing anti-aircraft guns, a complicated design scenario involving wireless data communications, electrical computers, statistics principles and servomechanisms — Bode humorously referred to this interdisciplinary linkage as a 'shotgun marriage'. As a result of their success, the frequency domain approach became *the* paradigm during the classical control period (1935 – 1960) [36, 5].

With the advent of optimal control and with the need for bringing non-linear effects and time-varying phenomena into the picture, the state-space approach took over the frequency domain and marked the beginning of the modern control era. Nevertheless, communication theory still pervades the control literature: physical systems are typically regarded as signal processors and the block diagram of Fig. 2.1 is still the icon of control theory.

This thesis adheres to a new paradigm shift in which energy is the central object in the control problem. It plays the main role in the modelling stage, the specification of performance objectives and the design stage. The energy function of a system determines both, its static and transient behaviors, via the energy transfer between subsystems.

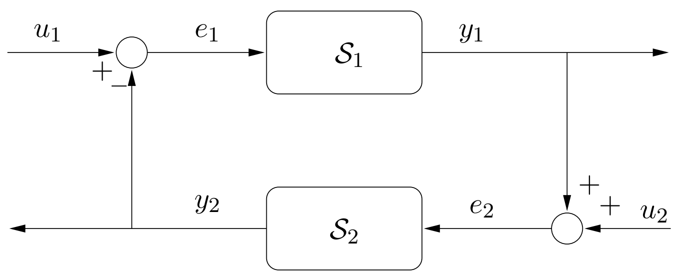

Figure 2.1: Feedback interconnection.

Systems (plants and controllers) are here viewed as energy transformation devices, or 'energy processors', which are interconnected to achieve the desired behavior.

The advantages of taking an energy-based perspective are manifold [44], the mathematical models are simpler and synergy results from physical intuition and mathematical rigor. The energy-based perspective also offers practical advantages: 'practitioners from different fields are familiar with energy concepts, which can serve as a lingua franca to facilitate communication with control theorists, incorporating prior knowledge and providing physical interpretations of the control action' [44].

2.1 Energy exchange, energy-shaping and dissipation

The notion of passivity will be properly defined in the following chapter, but at this stage we can say that passivity is closely related to the fundamental property of energy conservation.

Passivity-based control [42] refers to a controller design methodology that, based on the passivity properties of the plant, proposes a controller that transforms the plant into another passive system having a different, 'shaped', energy function. For the purpose of stabilizing an equilibrium point, one aims at a positive definite energy function that serves as a Lyapunov function. On a second stage, damping is added to the system to improve the transient or achieve asymptotic stability (cf. theorem 3.12). This procedure is also known as *energy-shaping* and *damping injection*. Damping injection simply amounts to selecting a function ϕ, defined on the output set $\mathbb{Y}$ and taking its values on the input set $\mathbb{U}$, such that $y^\top \phi(y) > 0$. So the real challenge is to solve the problem of shaping the energy.

To illustrate the basic ideas behind energy-shaping, suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are given in the input-state-output form

$$\mathcal{S}_i : \begin{cases} \dot{x}_i &= f_i(x_i, e_i) \\ y_i &= h_i(x_i, e_i) \end{cases} \quad , \quad i = 1, 2 \; . \tag{2.1}$$

If the systems are passive, then there exists functions $H_i(x_i)$ quantifying their stored energy while at states x_i, $i = 1, 2$. Additional energy can be supplied by or extracted from the power ports (e_i, y_i). Some of the energy is usually dissipated to the environment. In equations: $\dot{H}_i(x_i) \leq y_i^\top e_i$.

The feedback interconnection

$$\begin{aligned} e_1 &= u_1 - y_2 \\ e_2 &= u_2 + y_1 \end{aligned}$$

produces a new system

$$\dot{x} = f(x, u) \tag{2.2a}$$

$$y = h(x, u) \; , \tag{2.2b}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} , \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

(see Fig. 2.1). An important result in passivity theory is the *passivity theorem*, stated below and proved in, e.g., [28, 41, 60].

Theorem 2.1. *The feedback interconnection of two passive systems is again passive.*

Furthermore, the compound system (2.2) has energy function equal to the sum of the energy functions $H_i(x_i)$ of the original systems. Energy shaping can be accomplished by first selecting an $H_2(x_2)$ that furnishes the appropriate total energy $H_1(x_1) + H_2(x_2)$, and then constructing an $\mathcal{S}_2$ having storage function $H_2(x_2)$. This idea is intuitive and conceptually appealing, but it cannot be implemented in a straightforward manner. From a conceptual point of view, the main problems are:

(i) A static, state-dependent control law $\mathring{u}_c(x_1)$ is incompatible with (2.1). There is no state x_2, so there is no stored energy $H_2(x_2)$. One can still talk about passivity of static maps [28], but it is not evident what the power port variables e_2 and y_2 should be.

(ii) Suppose the controller has, in effect, the dynamics (2.1). The storage function of the dynamic controller will depend on x_2 only, not on x_1. This prevents us from directly choosing an $H_2(x_2)$ that shapes $H_1(x_1)$ as required.

These problems are the motivation for this thesis. Together with these problems, there are of course many sub-problems, open questions and details to be resolved.

2.2 Scope

Designing and implementing a control system requires the solution to problems like: estimation of the state if no complete measurements are available, estimation of the plant parameters (possibly on-line estimation), discretization of the controller if it is to be implemented in a computer, etc. Once a design is proposed, robustness with regard to external disturbances, noise, model uncertainty and quantization effects should be assessed. To consider all these problems in a single piece of work would be overly ambitious and unrealistic. Instead, this thesis focuses on the derivation of energy-based models and on the question of how to use control to shape their energy, with special emphasis on the problem of asymptotic stability.

2.3 Outline

In chapter 3 we introduce the mathematical models we will use for representing the plant and the controller. We define cyclo-passivity, a generalized version of passivity that also captures the property of energy conservation, but does not impose any restriction on the shape of the energy function, considerably widening the application range of our methods. We discuss port-Hamiltonian models, which are energy-based, cyclo-passive by nature, and ideal for treating the energy-shaping problem. We consider mechanical and electrical systems in more detail, and shown they are particular cases of port-Hamiltonian systems. We briefly discusses the subject of systems interconnection and formulate the control problem.

Problem (i), above, can be circumvented with *energy-balancing* control. The idea is to look for a $\mathring{u}_c$ such that, for some suitable H_2, the closed-loop system satisfies the power balance

$$\dot{H}_2(x_1) = y_2^\top e_2$$

with port variables $e_2 = h_1(x_1)$ and $y_2 = -\mathring{u}_c(x_1)$, 'emulating' in this way a cyclo-passive system $\mathcal{S}_2$ that 'shares' the state variables x_1 and yields a total energy function $H_1(x_1) + H_2(x_1)$.

Problem (ii) can be solved with the *control by interconnection* approach. The idea is to look for invariant functions of the from $z(x_1, x_2)$. If such functions are indeed invariant, then $H_1(x_1) + z(x_1, x_2) + H_2(x_2)$ is also an energy function of the closed-loop system and can be shaped effectively.

Quite surprisingly, the solutions to problems (i) and (ii) mentioned above, are equivalent, in a sense that will be made precise in section 5.7. In chapters 4 and 5 we show that, in their standard formulation, both solutions are overly restrictive, as they both suffer from the *dissipation obstacle*. These methods are discussed in detail and several alternatives to overcome the dissipation obstacle are given.

In chapter 4 we provide an algebraic characterization of all static controllers that perform energy-shaping, including other non-energy-balancing controllers like *interconnection and damping assignment*. We also show that, for some d_2 and properly chosen port variables, all energy-shaping controllers satisfy a power balance of the form

$$\dot{H}_2(x_1) = y_2^\top e_2 - d_2 \ .$$

Chapter 5 shows that control by interconnection poses a problem regarding the asymptotic convergence of the states towards a desired equilibrium. As one of the solutions presented, an adaptive algorithm is incorporated to the controller. Other solutions are also given.

Central to the whole discussion are the notions of flow and dissipation invariance, which are important characteristics of the controllers and provide a baseline for establishing comparisons between the static and the dynamic scenarios.

2.4 Mathematical framework

On a more abstract level, there is another paradigm shift in control theory: the *behavioral approach* proposed by Willems [68, 46].

Roughly speaking, when modelling a given phenomenon, one starts with a *universum* $\mathcal{U}$, the set of all (unmodelled) events. From first principles one derives a subset $\mathcal{B} \subset \mathcal{U}$, called the behavior, which contains all the events that are possible. A mathematical model for a phenomenon is then given by a pair $(\mathcal{U}, \mathcal{B})$. We will only work with continuous dynamical systems, but it is worth mentioning that a model with this degree of generality covers a wide range of phenomena like languages, discrete and continuous dynamical systems, distributed parameter systems, automata and so forth.

The behavioral setting also moves apart from the signal processing point of view. Moreover, it blurs the distinction between inputs and outputs and seriously questions the principle of causality [72]. Behavioral models allow us to separate the variables of interest (in our case, the power-port variables) from the auxiliary variables (in our case, the state). Behavioral models provide a nice compatible framework for the m-ports defined in the following chapter. Also, the behavioral approach shares (more precisely, inspires) our view of control as interconnection of systems [69].

Caveat lector. A recurrent statement in the literature of behaviors is that everything that is said about systems (controllability, observability, equivalence of models, properties of models, symmetries, system identification etc) must be intrinsic: it must refer to the behavior, the set $\mathcal{B}$ itself, and not to a particular representation of it. For the of sake concreteness but contrary to the previous statement, we have used particular (state-space) representations while deriving the results we are about to present.

2.5 Contributions

The main contributions are:

- Chapter 3
  1. The identification of a class of non-linear RLC networks that can be put in port-Hamiltonian form (section 3.3.3).
- Chapter 4
  1. The algebraic characterization of all passivity-based controllers (theorem 4.5).
  2. A taxonomy, based on the closed-loop flow and dissipation functions, of the existing static passivity-based controllers.
  3. The equivalence between energy balance and flow and dissipation invariance (proposition 4.7).
  4. Sufficient conditions for the property of relative passivity (proposition 4.10) and its application to non-linear RLC networks (theorem 4.13).
  5. An extension to interconnection and damping assignment (theorem 4.24) and an explicit solution of its algebraic equations (proposition 4.23).
  6. The identification of particular flow and dissipation functions that are invariant under the action of basic interconnection and damping assignment (proposition 4.36).
- Chapter 5

1. The use of the invariant flow of chapter 4 to relax the the assumptions required to apply control by interconnection (section 5.1.1).
2. Sufficient conditions to solve the energy-shaping problem (theorem 5.9).
3. The use of an adaptive algorithm and additional damping injection for solving the problem of asymptotic convergence.
4. The incorporation of another building block to destroy the flow and dissipation preservation (section 5.4)
5. A rigorous comparison, based on the solutions of the corresponding PDE's, of the static and dynamic approaches (proposition 5.18).

- The reformulation of these results in a behavioral setting is its self new and provides a unified perspective.

2.6 Publications

The journal publications that resulted from this work are:

- F. Castaños, B. Jayawardhana, R. Ortega, and E. García-Canseco, *Proportional plus integral control for set-point regulation of a class of nonlinear RLC circuits*, Circuits Syst. Signal Process., 28 (2009), pp. 609 – 623.

- F. Castaños, R. Ortega, A. J. van der Schaft, and A. Astolfi, *Asymptotic stabilization via control by interconnection of port-Hamiltonian systems*, Automatica, 45 (2009), pp. 1611 – 1618.

- F. Castaños and R. Ortega, *Energy-balancing passivity-based control is equivalent to dissipation and output invariance*, Systems and Control Letters, 58 (2009), pp. 553 – 560.

- R. Ortega, A. J. van der Schaft, F. Castaños, and A. Astolfi, *Control by interconnection and standard passivity-based control of port-Hamiltonian systems*, IEEE Trans. Automat. Contr., 53 (2008), pp. 2527 – 2542.

- B. Jayawardhana, R. Ortega, E. García-Canseco, and F. Castaños, *Passivity of nonlinear incremental systems: Application to PI stabilization of nonlinear RLC circuits*, Systems and Control Letters, 56 (2007), pp. 618 – 622.

This work has been presented in the following conferences:

- F. Castaños and R. Ortega, *Energy-balancing passivity-based control is equivalent to dissipation and output invariance*, in Proc. European Control Conference, page WeC2.4, Budapest, Hungary, Aug. 2009.

- F. Castaños, R. Ortega, A. J. van der Schaft, and A. Astolfi, *Asymptotic stabilization via control by interconnection of port-Hamiltonian systems*, in Congreso Latinoamericano de Control Automático, Mérida, Venezuela, Nov. 2008.

- F. Castaños, B. Jayawardhana, R. Ortega, and E. García-Canseco, *A class of nonlinear RLC circuits globally stabilizable by proportional plus integral controllers*, in Proc. of the IFAC World Congress, Seoul, Korea, June 2008, pp. 6202 – 6207.

- B. Jayawardhana, R. Ortega, E. García-Canseco, and F. Castaños, *Passivity of nonlinear incremental systems: Application to PI stabilization of nonlinear RLC circuits*, in Proc. Conference on Decision and Control, San Diego, Dec. 2006, p. ThIP2.17.

- R. Ortega, A. J. van der Schaft, F. Castaños, and A. Astolfi, *Control by (state-modulated) interconnection of port-Hamiltonian systems*, in Proc. IFAC Symposium on Nonlinear Control Systems, Pretoria, South Africa, Aug. 2007, pp. 47 – 54.

3 Cyclo-Passive Systems

This chapter introduces the mathematical models we will use while representing static and dynamical systems. We choose to use the *behavioral* approach to system theory, a theoretical body that — as many successful mathematical constructions — lays its foundations on set theory.

We define cyclo-passivity, formalizing in this way the notion of energy conservation, a property we will take for granted and exploit throughout this thesis. We focus on systems described by port-Hamiltonian models, which are cyclo-passive by nature. We devote some attention to the problem of modelling mechanical and electrical systems, which are particular instances of port-Hamiltonian systems.

3.1 Models

In the current control literature, one usually begins by considering dynamical systems of the form

$$\dot{x} = f(x) + g(x)u \tag{3.1a}$$

$$y = h(x) + j(x)u \,, \tag{3.1b}$$

with $x \in \mathbb{X} = \mathbb{R}^n$ the state variable, $u \in \mathbb{U} = \mathbb{R}^m$ the input, $y \in \mathbb{Y} = \mathbb{R}^p$ the output and f, g, h and j, functions of appropriate dimensions. In addition, g is assumed to be full rank, uniformly on x. From a conceptual standpoint, u is typically regarded as a *cause* that produces (via x) an *effect* on the measured output y. In this thesis we keep (3.1), but we depart from the cause–effect interpretation. We do so by constructing a framework from which u and y can be regarded as a power-conjugated pair of *port-variables*.

When dealing with complex systems, it is common in engineering practice to decompose a model into several, simpler submodels. Inherent to this decomposition process is the notion of port, or as it was accurately put by Breedveld [7]:

> The concept of a port is generated by the fact that submodels in a model have to interact with each other by definition and accordingly need some form of conceptual interface. In physical systems, such an interaction is always (assumed to be) coupled to an exchange of energy, i.e. a power.

In the present context, the pair (u, y) constitutes the interface Breedveld refers to. This of course implies that y no longer represents the measured variables. In fact, we will always assume that the whole state is measured. When speaking about a 'power-conjugated pair of port-variables' we mean, in addition, that we restrict (3.1) to the case $p = m$ (same number of inputs and outputs) and that the product $y^\top u$ denotes power. Typical

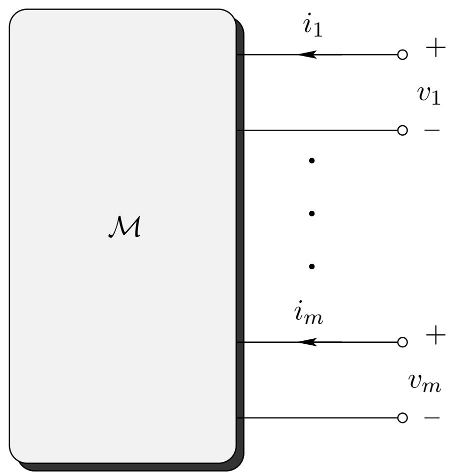

Figure 3.1: An m-port

examples of pairs (u, y) are forces and velocities for mechanical systems and voltages and currents for electrical systems.

In the field of circuit theory, the term *n-port* is used to refer to the interface *together* with the subsystem associated to it. In what follows, we will try to formalize this notion and extend it over systems which are not necessarily electrical. It is worth mentioning that generalization of electrical n-ports has already proved fruitful in the area of teleoperation, where mechanical manipulators are represented as mechanical n-ports and stability is proved using scattering representations, a notion which originally was exclusive of electrical networks [2].

Before proceeding with the actual definition, it will be useful to create a mental picture, so we start by recalling some of the particular meanings that are attributed in circuit theory.

3.1.1 Electrical m-ports

Notation. We will reserve the variable n for the dimension of the state, so we will use the term m-port instead of n-port.

Although the idea of an m-port is intuitively simple and has been used since the early stages of network theory, it is a concept that took a long time to formalize. For example, to Carlin (1956) [10], they are 'linear electrical networks having m-accessible terminal pairs'. To Chua (1973) [12], 'an electrical m-port is a black box with m-pairs of external terminals called "ports" such that the current entering a terminal of each port is equal to the current leaving the second terminal'. One of the early formal definitions, due to Youla, Castriota and Carlin (1959) [74], is the following[1]:

[1] Please refer to p. 146 for notation.

Definition 3.1. An *m-port* is an operator Φ in $\mathcal{H}(\mathbb{R}, \mathbb{R}^m)$. The correspondence $i(t) = \Phi v(t)$, $v(t) \in D(\Phi)$ is depicted schematically in Fig. 3.1.

This simple definition allowed the authors to accomplish the monumental task of constructing a completely rigorous theory of linear passive time-invariant m-ports from an axiomatic point of view. The main arguments implicated in the construction of this theory make use of frequency-domain tools such as the Fourier transform. Unfortunately, these arguments do not generalize easy to the nonlinear case. Thus, we will abandon the notion of operator, but we will retain three points:

1. The variables at play are power variables (in the present case, currents i and voltages v).
2. The definition serves equally well to static (memoryless) and dynamical m-ports.
3. The pictorial representation of Fig. 3.1.

At a latter time, Wyatt et al. (1981) [73] defined m-ports using a state-space representation similar to (3.1). This kind of representation is useful for nonlinear dynamical systems, but it excludes static controllers (which will be the main topic of chapter 4).

3.1.2 Behaviors

Instead of defining m-ports in an operator setting, we will use the behavioral approach to systems modelling and control [46]. This subsection, which is largely taken from [70], presents a brief exposition of the behavioral concepts used in subsequent parts. For a complete treatment, we recommend [67, 68, 46, 70].

In a behavioral setting, dynamical systems are modeled by specifying the set of all possible time-trajectories (or signals) that are compatible with the system.

Definition 3.2. A *system* Σ is defined as a triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B}) ,$$

with $\mathbb{T}$ a subset of $\mathbb{R}$, called the *time axis*, $\mathbb{W}$ a set called the *signal space* and, $\mathcal{B}$ a subset of $\mathbb{W}^{\mathbb{T}}$ called the *behavior* ($\mathbb{W}^{\mathbb{T}}$ is standard mathematical notation for the collection of all maps from $\mathbb{T}$ to $\mathbb{W}$).

In Willems's words [70]:

> The behavior $\mathcal{B}$ specifies which trajectories $w : \mathbb{T} \to \mathbb{W}$ are possible, according to the model. For continuous-time systems [that is, when $\mathbb{T} = \mathbb{R}$], behaviors $\mathcal{B}$ of interest consist of a strict subset of $\mathbb{W}^{\mathbb{T}}$. Typically, elements of $\mathcal{B}$ are required to be well behaved maps from $\mathbb{T}$ to $\mathbb{W}$, at least measurable or locally integrable.

On a first read, this definition may seem fairly abstract, but its meaning will become clear when we define the behaviors explicitly.

In order to specify $\mathcal{B}$, we will sometimes require the use of auxiliary, or latent variables. Latent variables are involved in the model but are not the variables the model aims at.

Definition 3.3. A *system with latent variables* is defined as

$$\hat{\Sigma} = \left(\mathbb{T}, \mathbb{W}, \mathbb{L}, \hat{\mathcal{B}}\right) \ ,$$

were $\mathbb{T}$ and $\mathbb{W}$ are as in Definition 3.2, $\mathbb{L}$ is the *set of latent variables* and $\hat{\mathcal{B}} \subset (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$ is the full behavior.

To emphasize the difference between a system and a system with latent variables, we may also call $\mathbb{W}$ the *set of manifest variables.*

Remark 3.4. Definitions 3.2 and 3.3 where originally meant for dynamical systems [70]. In general, when dealing with continuous dynamical systems, the behaviors $\mathcal{B}$ and $\hat{\mathcal{B}}$ involve the derivatives, up to a certain order, of w and/or l. We have chosen to remove the qualifier 'dynamical', since by considering zero-order behaviors one can also model static systems (cf. point 2 in Section 3.1.1).

Definition 3.5. Let $\mathcal{B}$ be a behavior with signal space $\mathbb{W}$, partition the signal space as $\mathbb{W} = \mathbb{W}_1 \times \mathbb{W}_2$. We say that w_1 is *free* if for every $w_1 \in \mathcal{L}_\infty^{\text{loc}}(\mathbb{T}, \mathbb{W}_1)$ there is a $w_2 \in \mathcal{L}_\infty^{\text{loc}}(\mathbb{T}, \mathbb{W}_2)$ such that $(w_1, w_2) \in \mathcal{B}$.

3.1.3 Generalized m-ports

Notation. Since we will always use continuous time, we set $\mathbb{T} = \mathbb{R}_+$ once and for all and eliminate $\mathbb{T}$ from the discussion.

Now we have enough material to construct an object satisfying points 1 to 3 in Section 3.1.1. Instead of limiting ourselves to voltages and currents, we will use a domain-independent framework and speak about efforts and flows. We partition the variables of interest as $\mathbb{W} = \mathbb{U} \times \mathbb{Y}$, where $\mathbb{U} = \mathbb{R}^m$ is the *set of effort values* and $\mathbb{Y} = \mathbb{R}^m$ is the *set of flow values.* This covers the first point, since the product of efforts and flows yields power [7].

Definition 3.6. An *m-port* Π is a system

$$\Pi = (\mathbb{W}, \mathcal{B}) \ , \quad \mathbb{W} = \mathbb{U} \times \mathbb{Y} \ .$$

We will say that Π is *effort-driven* if it has free variables $u \in \mathcal{L}_\infty^{\text{loc}}(\mathbb{R}_+, \mathbb{U})$. Dually, we will say that it is *flow-driven* if it has free variables $y \in \mathcal{L}_\infty^{\text{loc}}(\mathbb{R}_+, \mathbb{Y})$.

Similarly to systems with latent variables, we will use auxiliary, or state variables.

Definition 3.7. An *m-port with state variables* $\hat{\Pi}$ is a system

$$\hat{\Pi} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}) \ , \quad \mathbb{W} = \mathbb{U} \times \mathbb{Y} \tag{3.2}$$

where $\mathbb{U}$ and $\mathbb{Y}$ are as before and $\mathbb{X} = \mathbb{R}^n$ is the *state.*

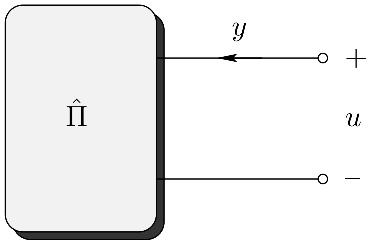

Figure 3.2: Generalized m-port. Positive power always enters the port. Only the manifest variables are shown.

As our main object of study, take an m-port $\hat{\Pi}$ with state variables given by

$$\hat{\mathcal{B}} = \left\{ (u, y, x) : \mathbb{R}_+ \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \,\middle|\, \text{Eq. (3.1) holds} \right\} . \tag{3.3}$$

For $\hat{\mathcal{B}}$ to be well defined, we will of course require the usual conditions on existence and uniqueness of solutions of the ODEs. These conditions have been put in appendix A.1 for completeness, but we will not drift along these lines any further. Since for every $u \in \mathcal{L}_\infty^{\text{loc}}(\mathbb{R}_+, \mathbb{U})$ there is a $y \in \mathcal{L}_\infty^{\text{loc}}(\mathbb{R}_+, \mathbb{Y})$ such that $(u, y) \in \mathcal{B}$, the m-port (3.3) is effort-driven.

Notation. Equation (3.1) contains an ODE with independent variable t, so strictly speaking, u, y and x are functions of time, i.e., $(u, y, x) : \mathbb{R}_+ \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X}$. Since the arguments of $f : \mathbb{X} \to \mathbb{R}^n$ and $g : \mathbb{X} \to \mathbb{R}^{n \times m}$ are members of $\mathbb{X}$, not functions of time, one should write $\dot{x} = f \circ x + (g \circ x) \cdot u$ instead of (3.1a) (the symbol '$\circ$' denotes composition). To avoid this burdensome notation, we will insist on (3.1a) and simply write, as we did at the beginning, $u \in \mathbb{U}$, $y \in \mathbb{Y}$ and $x \in \mathbb{X}$. Consequently, we will replace (3.3) by

$$\hat{\mathcal{B}} = \left\{ (u, y, x) \in \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \,\middle|\, \text{Eq. (3.1) holds} \right\} . \tag{3.4}$$

This slight abuse of notation is rather common in the literature.

To simplify our diagrams, we will omit the state and we will bind together several terminals into a single vector pair. By convention, we assign the direction of the flows and the sign of the efforts in such a way that positive power always enters the m-port (see Fig. 3.2).

3.2 Cyclo-passivity

Cyclo-passivity refers to the physical property of energy conservation. Since a physical system cannot create energy, it cannot store more energy than the one provided by the environment. By restating this property in a mathematical language, we can generalize it to systems which are not necessarily physical.

Definition 3.8. An m-port $\hat{\Pi}$ with state variables is *cyclo-passive* if there exists a $\mathcal{C}^1$ function $H : \mathbb{X} \to \mathbb{R}$, called the *storage function*, such that for all $x_0 \in \mathbb{X}$, all $t \geq 0$ and all triples $(u, y, x) \in \hat{\mathcal{B}}$,

$$H(x) - H(x_0) \leq \int_0^t y^\top u \mathrm{d}\tau \ , \quad x(0) = x_0 \ . \tag{3.5}$$

If, in addition, H admits a minimum, we will say that $\hat{\Pi}$ is *passive.* $\hat{\Pi}$ is *cyclo-lossless* if (3.5) is satisfied with equality.

Under the differentiability assumption, inequality (3.5) is of course equivalent to

$$\dot{H}(x) \leq y^\top u \ , \quad \forall\, (u, y, x) \in \hat{\mathcal{B}} \ , \tag{3.6}$$

which states that the rate of change of the storage (or energy) function cannot be greater than the power supplied through the port.

If a system is passive and starts from a minimal state of energy $x(0) = \arg\min H(x)$, then the left-hand side of (3.5) is non-negative, and according to the inequality the system exhibits a net absorption of energy ($\int y^\top u \mathrm{d}\tau$ is also non-negative). This property does not apply to general cyclo-passive systems since H may not have a minimum. Nevertheless, equation (3.5) shows that cyclo-passive systems exhibit a net absorption of energy along *closed* trajectories satisfying $x(T) = x(0)$ for some T — hence the prefix 'cyclo' [22].

Hill-Moylan's theorem gives, in the spirit of Kalman-Yakubovich-Popov's lemma, an *algebraic* characterization of cyclo-passive systems.

Theorem 3.9. *[22] Consider a port $\hat{\Pi} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}})$ satisfying (3.4). $\hat{\Pi}_\mathrm{p}$ is cyclo-passive with storage function H if, and only if, for some $q \in \mathbb{N}$, there exist functions $l : \mathbb{R}^n \to \mathbb{R}^q$ and $w : \mathbb{R}^n \to \mathbb{R}^{q \times m}$ such that*

$$\nabla H^\top(x) f(x) = -|l(x)|^2 \tag{3.7a}$$

$$h(x) = g^\top(x) \nabla H(x) + 2 w^\top(x) l(x) \tag{3.7b}$$

$$w^\top(x) w(x) = \frac{1}{2}(j^\top(x) + j(x)) \ . \tag{3.7c}$$

Notation. All vectors are taken to be column vectors, even the gradient of a scalar function, that we denote with the operator $\nabla_x \triangleq \frac{\partial}{\partial x}$. When clear from context the subindex of the operator will be omitted.

We will not prove the complete theorem, but we can easily verify the 'if' part. Denote by

$$d(x, u) \triangleq |l(x) + w(x) u|^2 \tag{3.8}$$

the *dissipation function*. The derivative of H with respect to time is

$$\dot{H}(x) = \nabla H^\top(x) \left[f(x) + g(x) u \right] = -|l(x)|^2 + \left[h(x) - 2 w^\top(x) l(x) \right]^\top u \ .$$

We complete the square by adding the null term $u^\top j(x)u - u^\top w^\top(x)w(x)u$. Then,

$$\begin{aligned}\dot{H}(x) &= h^\top(x)u + u^\top j(x)u - \Big(l^\top(x)l(x) + 2l^\top(x)w(x)u + u^\top w^\top(x)w(x)u\Big) \\ &= y^\top u - d(x,u)\ . \end{aligned} \tag{3.9}$$

Finally, the non-negativity of $d(x,u)$ establishes (3.6).

For the special case

$$\begin{aligned}\dot{x} &= f(x) + g(x)u && (3.10a)\\ y &= h(x) && (3.10b)\end{aligned}$$

(that is, for $j(x) = 0$), we have the following corollary.

Corollary 3.10. *Consider a port* $\hat{\Pi} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}})$ *satisfying*

$$\hat{\mathcal{B}} = \Big\{(u,y,x) \in \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \,\Big|\, \textit{Eq. (3.10) holds}\Big\}\ . \tag{3.11}$$

$\hat{\Pi}$ *is cyclo-passive with storage function* H *if, and only if, there exists a dissipation function* $d : \mathbb{X} \to \mathbb{R}_+$ *such that,*

$$\begin{aligned}\nabla H^\top(x) f(x) &= -d(x) && (3.12a)\\ h(x) &= g^\top(x)\nabla H(x)\ . && (3.12b)\end{aligned}$$

3.2.1 Passivity

Cyclo-passivity is a fairly general property that does not imply any form of stability whatsoever. Passivity, on the other hand, is related to several forms of stability, like input-output stability [37]. If the minimum of the energy function is strict, it can serve as a Lyapunov function and we can also expect stability in Lyapunov's sense. We close this chapter by recalling some of the basic connections between passivity and Lyapunov theory. The interested reader is referred to [65, 66, 22, 8, 37, 60].

Definition 3.11. An m-port (3.2) is *locally zero-state detectable* if there exists a neighborhood $\mathcal{X}$ of 0 such that, for all $x_0 \in \mathcal{X}$,

$$(0,0,x) \in \hat{\mathcal{B}}\ ,\ x(0) = x_0 \quad \Longrightarrow \quad \lim_{t\to\infty} x = 0\ .$$

If $\mathcal{X} = \mathbb{X}$, the m-port is *zero-state detectable.* The m-port is *locally zero-state observable* if there exists a neighborhood $\mathcal{X}$ of 0 such that, for all $x_0 \in \mathcal{X}$,

$$(0,0,x) \in \hat{\mathcal{B}}\ ,\ x(0) = x_0 \quad \Longrightarrow \quad x_0 = 0\ .$$

If $\mathcal{X} = \mathbb{X}$, the m-port is *zero-state observable.*

The following theorems apply to m-ports having behaviors of the form (3.11).

Theorem 3.12. *[8] Suppose* $\hat{\Pi}$ *is passive with a storage function* H *which is positive-definite. Suppose that* $\hat{\Pi}$ *is locally zero-state detectable. Let* $\phi : \mathbb{Y} \to \mathbb{U}$ *be any smooth function such that* $\phi(0) = 0$ *and* $y^\top \phi(y) > 0$ *for each nonzero* y*. The control law*

$$u = -\phi(y) \tag{3.13}$$

asymptotically stabilizes the equilibrium $x = 0$*. If* $\hat{\Pi}$ *is zero-state detectable and* H *is radially unbounded, the control law (3.13) globally asymptotically stabilizes the equilibrium* $x = 0$.

Proposition 3.13. *[21] Suppose* $\hat{\Pi}$ *is passive with storage function* H*. Suppose* $\hat{\Pi}$ *is zero-state observable. Then* H *is positive definite.*

Definition 3.14. A point x° is a *regular* point for a port $\hat{\Pi}$ if rank $\{L_g h(x)\}$ is constant in a neighborhood of x°.

Theorem 3.15. *[8] Suppose* $\hat{\Pi}$ *is passive with a* $\mathcal{C}^2$ *storage function* H *which is positive definite. Suppose* $x^\circ = 0$ *is a regular point for* $\hat{\Pi}$*. Then* $L_g h(0)$ *is non-singular and* $\hat{\Pi}$ *has relative degree* $\{1, \cdots, 1\}$ *at* $x^\circ = 0$*.*

Theorem 3.16. *[8] Suppose* $\hat{\Pi}$ *is passive with a* $\mathcal{C}^2$ *storage function* H *which is positive definite. Suppose that either* $x^\circ = 0$ *is a point of regularity for* $\hat{\Pi}$ *or that* H *is non-degenerate*[2]*. Then the zero-dynamics of* $\hat{\Pi}$ *locally exist at* $x^\circ = 0$ *and* $\hat{\Pi}$ *is weakly minimum phase.*

3.3 Port-Hamiltonian systems

This section, which is mainly based on [19, 60], serves as a motivation of the energy-based approach to modelling and as a short introduction to port-Hamiltoninan systems [57].

3.3.1 Mechanical systems

Let us briefly recall Newton's second law of motion for a particle moving in Euclidean space under the influence of a force $F \in \mathbb{R}^3$:

$$F = ma\ . \tag{3.14}$$

Let $r \in \mathbb{R}^3$ be the position of the particle, m its mass, $v = \dot{r}$ its velocity and $a = \ddot{r}$ its acceleration. By definition, the work done by the external force F upon taking the particle from the point r_1 to the point r_2 is

$$W_{12} = \int_{r_1}^{r_2} F \cdot \mathrm{d}r\ . \tag{3.15}$$

[2]Recall that a function is non-degenerate if its Hessian is non singular at each of its critical points.

From (3.14) and (3.15) we can see that

$$W_{12} = \int_{r_1}^{r_2} m\dot{v} \cdot \mathrm{d}r = m \int_{t_1}^{t_2} \dot{v} \cdot v\mathrm{d}t = m \int_{v_1}^{v_2} v \cdot \mathrm{d}v = \frac{1}{2} m(|v_2|^2 - |v_1|^2) \; .$$

The *kinetic energy* is defined as

$$K(v) = \frac{1}{2} m \, |v|^2 \; , \tag{3.16}$$

so that the work done is equal to the change in kinetic energy:

$$W_{12} = K(v_2) - K(v_1) \; . \tag{3.17}$$

If the work is the same for any physically possible path between points r_1 and r_2, then the force is said to be *conservative.* It is well-known that independence of the integration path is equivalent to F being the gradient of some scalar or *potential energy* function P, that is,

$$F = -\nabla P(r) \; .$$

If this is the case, then the work (3.15) can also be calculated as

$$W_{12} = -\int_{r_1}^{r_2} \nabla P(r) \cdot \mathrm{d}r = -P(r_2) + P(r_1) \; . \tag{3.18}$$

By equating (3.17) and (3.18), one arrives at the conservation of the total energy $E = P + K$, i.e.,

$$P(r_2) + K(v_2) = P(r_1) + K(v_1) \; .$$

The Euler-Lagrange equations

Let us define the function

$$L(r, v) = K(v) - P(r) \; .$$

One can easily verify that the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial v}(r, v) - \frac{\partial L}{\partial r}(r, v) = 0 \tag{3.19}$$

implies (3.14). In other words, substitution of

$$\frac{\partial L}{\partial v}(r, v) = mv \quad \text{and} \quad \frac{\partial L}{\partial r}(r, v) = -\nabla P(r)$$

into (3.19) gives $\frac{\mathrm{d}}{\mathrm{d}t}(mv) + \nabla P(r) = 0$, which is equivalent to (3.14). Hence, equation (3.19) also describes the motion of the particle. Determining the motion of a single free particle is of course simpler with Newton's law than with (3.19), but this is typically not the case when the motion is constrained or while analyzing complex bodies.

If the motion constraints can be expressed as k equations having the form

$$c_j(r_1, r_2, \ldots, r_N) = 0 \; , \quad j \in 1, 2, \ldots, k \tag{3.20}$$

where $r_1, r_2, \ldots, r_N$ are the coordinates of the N particles conforming the body, then the constraints are said to be *holonomic*[3]. A particle constrained to move along a curve is a simple example.

Since each coordinate $r_1, r_2, \ldots, r_N$ is made up of tree components, there are $3N$ variables. According to (3.20), we can choose (at least locally) $3N - k$ independent variables q_i. These are called *generalized coordinates* and are such that

$$\begin{aligned} r_1 &= \hat{r}_1(q_1, q_2, \ldots, q_{3N-k}) \\ &\vdots \\ r_N &= \hat{r}_N(q_1, q_2, \ldots, q_{3N-k}) \,. \end{aligned}$$

Consider the *Lagrangian* $L(q, \dot{q}) = K(q, \dot{q}) - P(q)$ and the *Euler Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}) - \frac{\partial L}{\partial q_i}(q, \dot{q}) = u_i \ , \quad i = 1, \ldots, 3N - k \ , \tag{3.21}$$

where u_i represents the sum of external forces. It turns out that if the constraint and internal forces (the ones between particles) are orthogonal to any possible virtual displacement, then the Euler-Lagrange equations determine the actual motion of the body. This can be proved using either a differential principle (D'Alambert's principle) or an integral one (Hamilton's principle). See [19, 37] for details. Typically, the potential energy depends on q only and the kinetic energy has the quadratic form

$$K(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} \ , \quad M(q) = M^\top(q) > 0 \ . \tag{3.22}$$

Example, a simple pendulum

Consider a simple pendulum like the one shown in Fig. 3.3. We model the pendulum as an ideal particle of mass m linked to a joint by an ideal massless bar of length l. The particle is constrained to move on a vertical plane and on a circle of radius l:

$$\begin{aligned} r_z &= 0 \\ r_x^2 + r_y^2 - l^2 &= 0 \,. \end{aligned}$$

There are three variables ($3N = 3$) and two holonomic constraints ($k = 2$). Since $3N - k = 1$, we can describe the position of the mass using a single generalized coordinate q:

$$\begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} l\sin(q) \\ l\cos(q) \\ 0 \end{pmatrix} \,.$$

The potential energy of the particle is $P(q) = mgl(1 + \cos q)$, where g is the acceleration due to earth's gravity. Since

$$v = \dot{r} = \begin{pmatrix} l\cos(q)\dot{q} \\ -l\sin(q)\dot{q} \\ 0 \end{pmatrix} \quad \text{and} \quad |v|^2 = l^2\dot{q}^2 \ ,$$

[3] The distinctiveness of (3.20) is that it does not contain derivatives of the coordinates nor inequalities.

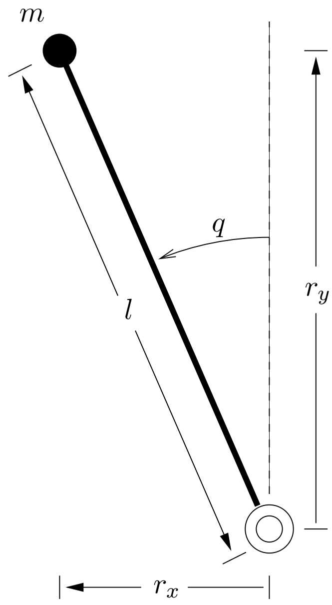

Figure 3.3: A simple pendulum with generalized coordinate q.

the kinetic energy (3.16) is $K(q, \dot{q}) = \frac{1}{2} ml^2 \dot{q}^2$ (which has the form (3.22) with $M(q) = ml^2$). The constraint force that keeps the particle on the circle acts along the bar, so it is orthogonal to any possible (i.e., tangent to the circle) virtual displacement. This means that (3.21) accurately describes the motion of the ideal pendulum. Direct substitution of

$$L(q, \dot{q}) = \frac{1}{2} ml^2 \dot{q}^2 - mgl(1 + \cos q) \tag{3.23}$$

in (3.21) yields $\frac{\mathrm{d}}{\mathrm{d}t}(ml^2 \dot{q}) - mgl \sin(q) = 0$ or, equivalently,

$$ml^2 \ddot{q} - mgl \sin(q) = 0 \; .$$

From Euler-Lagrange to Hamilton equations

One way to re-write the second-order Euler-Lagrange equations in a first-order state-space form like (3.1a) is simply to take q and $\dot{q}$ as state variables. In a Hamiltonian framework, one instead defines the *generalized momenta*

$$p = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \tag{3.24}$$

and takes q and p as state variables. The advantages of this particular choice will become clear shortly. Let us write the differential of the Lagrangian $L(q, \dot{q})$ as

$$\mathrm{d}L = \sum_{i=1}^{m} \frac{\partial L}{\partial q_i}(q, \dot{q}) \mathrm{d}q_i + \sum_{i=1}^{m} \frac{\partial L}{\partial \dot{q}_i}(q, \dot{q}) \mathrm{d}\dot{q}_i \; . \tag{3.25}$$

By substituting (3.24) into the Euler-Lagrange equations (3.21) we obtain

$$\dot{p} - u = \frac{\partial L}{\partial q}(q, \dot{q}) \ . \tag{3.26}$$

Using (3.24) and (3.26) we can write (3.25) as

$$\mathrm{d}L = \sum_{i=1}^{m} (\dot{p}_i - u_i)\mathrm{d}q_i + \sum_{i=1}^{m} p_i \mathrm{d}\dot{q}_i \ . \tag{3.27}$$

The Hamiltonian $H(q, p)$ is generated by the Legendre transformation

$$H(q, p) = \sum_{i=1}^{m} \dot{q}_i p_i - L(q, \dot{q}) \ , \tag{3.28}$$

in which $\dot{q}$ is expressed in terms of q and p by (3.24). Its differential is given by

$$\begin{aligned}
\mathrm{d}H &= \sum_{i=1}^{m} \dot{q}_i \mathrm{d}p_i + \sum_{i=1}^{m} p_i \mathrm{d}\dot{q}_i - \mathrm{d}L \\
&= \sum_{i=1}^{m} \dot{q}_i \mathrm{d}p_i + \sum_{i=1}^{m} p_i \mathrm{d}\dot{q}_i - \sum_{i=1}^{m} (\dot{p}_i - u_i)\mathrm{d}q_i - \sum_{i=1}^{m} p_i \mathrm{d}\dot{q}_i \\
&= \sum_{i=1}^{m} (u_i - \dot{p}_i)\mathrm{d}q_i + \sum_{i=1}^{m} \dot{q}_i \mathrm{d}p_i \ ,
\end{aligned} \tag{3.29}$$

where we have used (3.27) on the second line. Since dH can also be written as

$$\mathrm{d}H = \sum_{i=1}^{m} \frac{\partial H}{\partial q_i}(q, p)\mathrm{d}q_i + \sum_{i=1}^{m} \frac{\partial H}{\partial p_i}(q, p)\mathrm{d}p_i \ , \tag{3.30}$$

we obtain from (3.29) and (3.30),

$$\dot{q} = \nabla_p H(q, p) \tag{3.31a}$$
$$\dot{p} = -\nabla_q H(q, p) + u \ , \tag{3.31b}$$

the *canonical equations of Hamilton*[4].

A useful feature of Hamilton's equations is that cyclo-passivity with respect to the pair $(u, \dot{q})$ is immediately revealed. It suffices to take the derivative of H with respect to time:

$$\begin{aligned}
\dot{H}(q, p) &= \sum_{i=1}^{m} \frac{\partial H}{\partial q_i}(q, p)\dot{q}_i + \sum_{i=1}^{m} \frac{\partial H}{\partial p_i}(q, p)\dot{p}_i \\
&= \sum_{i=1}^{m} \frac{\partial H}{\partial q_i}(q, p)\frac{\partial H}{\partial p_i}(q, p) - \sum_{i=1}^{m} \frac{\partial H}{\partial p_i}(q, p)\frac{\partial H}{\partial q_i}(q, p) + \sum_{i=1}^{m} \frac{\partial H}{\partial p_i}(q, p)u_i \\
&= \sum_{i=1}^{m} \dot{q}_i u_i = \dot{q}^\top u \ .
\end{aligned}$$

[4] The actual canonical equations of Hamilton do not contain u, but to simplify the discussion we take the liberty to include it.

Furthermore, the Hamiltonian equals the total energy written in (q, p) coordinates [19, p. 339].

Example, a simple pendulum (continued)

Let us derive the Hamiltonian model for the pendulum described on p. 48. The Hamiltonian is obtained from (3.23) and (3.28) as

$$\begin{aligned} H(q,p) &= \dot{q}p - \frac{1}{2}ml^2\dot{q}^2 + mgl(1+\cos(q)) \\ &= (ml^2)^{-1}p^2 - \frac{1}{2}(ml^2)^{-1}p^2 + mgl(1+\cos(q)) \\ &= \frac{1}{2}(ml^2)^{-1}p^2 + mgl(1+\cos(q)) \ , \end{aligned}$$

which equals the total energy. If we include a torque u on the joint, then the equations of motion are, according to (3.31),

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} (ml^2)^{-1}p \\ -mgl\sin(q) \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u \ .$$

3.3.2 General port-Hamiltonian systems

Equations (3.31) can be written in compact form as

$$\begin{aligned} \dot{x} &= J\nabla H(x) + \begin{pmatrix} 0 \\ I \end{pmatrix} u \\ y &= \begin{pmatrix} 0 & I \end{pmatrix} \nabla H(x) = \dot{q} \end{aligned}$$

with

$$x = \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \ .$$

Notice that cyclo-passivity (more precisely, cyclo-losslessness) hinges upon the skew symmetry of J. From a purely mathematical point of view, we can generalize (3.32) by considering systems of the form

$$\dot{x} = [J(x) - R(x)]\nabla H(x) + g(x)u \tag{3.33a}$$

$$y = g^\top(x)\nabla H(x) \ , \tag{3.33b}$$

where $x \in \mathbb{X}$, $u \in \mathbb{U}$ and $y \in \mathbb{Y}$ as before. The matrix $J(x) = -J^\top(x)$ is called the *interconnection matrix* and the matrix $R(x) = R^\top(x) \geq 0$ reflects the effects of dissipation. These models are called *port-Hamiltonian* [60] and can be easily embedded in m-ports by setting

$$\hat{\mathcal{B}} = \left\{ (u, y, x) \in \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \,\middle|\, \text{Eq. (3.33)} \right\} \ . \tag{3.34}$$

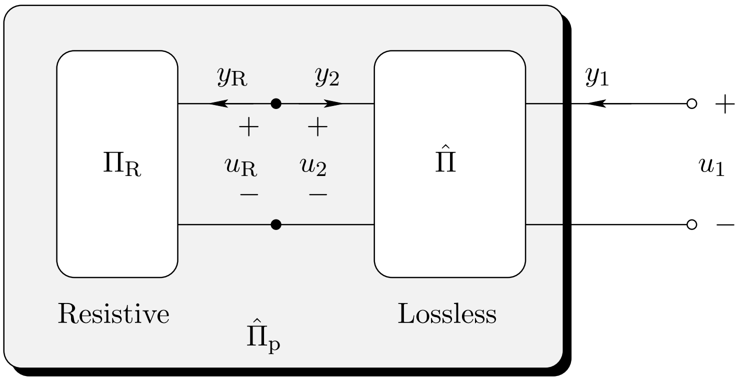

Figure 3.4: Port-Hamiltonian system with dissipation port.

The derivative of the energy is

$$\dot{H}(x) = y^\top u - \nabla H^\top(x) R(x) \nabla H(x)$$

and the term $\nabla H(x)^\top R(x)\nabla H(x)$, being non-negative, establishes the dissipation inequality. Interestingly, by letting J depend on x, one can use (3.33) to model mechanical systems with a class of nonholonomic constraints [61, 33, 59, 60]. The matrix $g(x)$, on the other hand, is useful for modelling under-actuated mechanical systems (systems with less controls u_i than generalized coordinates).

Notice that in (3.33b), the flow y has the form required to satisfy corollary 3.10. By using the full version of Hill-Moylan's theorem, that is, by using (3.7b), the definition of port-Hamiltonian models can be extended to systems having efforts u appearing explicitly in the equation of the flow y. In [62], port-Hamiltonian models are further extended by replacing $R(x)\nabla H(x)$ with an external resistive port, so that other dissipation effects can be brought into the model. The argument starts with a system of the form

$$\dot{x} = J(x)\nabla H(x) + g_1(x)u_1 + g_2(x)u_2 \tag{3.35a}$$
$$y_1 = h_1(x) + j_{11}(x)u_1 + j_{12}(x)u_2 \tag{3.35b}$$
$$y_2 = h_2(x) + j_{21}(x)u_1 + j_{22}(x)u_2 \ , \tag{3.35c}$$

where the pair $(u_2, y_2) \in \mathbb{R}^{m_R} \times \mathbb{R}^{m_R}$ is the resistive port. By setting

$$g(x) = \begin{pmatrix} g_1(x) & g_2(x) \end{pmatrix} \ , \quad h(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} \quad \text{and} \quad j(x) = \begin{pmatrix} j_{11}(x) & j_{12}(x) \\ j_{11}(x) & j_{11}(x) \end{pmatrix} \ ,$$

the system of equations (3.35) can be written as

$$\begin{aligned} \dot{x} &= J(x)\nabla H(x) + g(x)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= h(x) + j(x)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \ . \end{aligned}$$

If $g(x)$, $h(x)$ and $j(x)$ satisfy the conditions of theorem 3.9, then the cyclo-lossless equality

$$\dot{H}(x) = y_1^\top u_1 + y_2^\top u_2 \tag{3.36}$$

holds. Consider now a resistive m_R-port $\Pi_\mathrm{R} = (\mathbb{U}_\mathrm{R} \times \mathbb{Y}_\mathrm{R}, \mathcal{B}_\mathrm{R})$ satisfying the dissipation condition

$$(u_\mathrm{R}, y_\mathrm{R}) \in \mathcal{B}_\mathrm{R} \quad \Longrightarrow \quad y_\mathrm{R}^\top u_\mathrm{R} \geq 0 \ .$$

By interconnecting the cyclo-lossless $(m+m_\mathrm{R})$-port with Π_R as in Fig. 3.4, i.e., by setting $u_2 = u_\mathrm{R}$ and $y_2 = -y_\mathrm{R}$, one obtains the port-Hamiltonian system with dissipation

$$\dot{x} = J(x)\nabla H(x) + g_1(x)u_1 + g_2(x)u_\mathrm{R} \tag{3.37a}$$

$$y_1 = h_1(x) + j_{11}(x)u_1 + j_{12}(x)u_\mathrm{R} \tag{3.37b}$$

$$y_\mathrm{R} = -h_2(x) - j_{21}(x)u_1 - j_{22}(x)u_\mathrm{R} \ . \tag{3.37c}$$

From (3.36), $y_2^\top u_2 = -y_\mathrm{R}^\top u_\mathrm{R}$ and $y_\mathrm{R}^\top u_\mathrm{R} \geq 0$ we obtain the cyclo-passivity condition $\dot{H}(x) \leq y_1^\top u_1$.

Lagrangian and Hamiltonian models extend beyond the domain of classical mechanics, they are also used in relativistic and quantum mechanics [19]. Other important domains are circuit theory [25] (the subject of what follows), electro-mechanical [37] and thermodynamical [16] systems. Some efforts have also been directed towards finding conditions under which a general system can be put in Hamiltonian form [58, 35, 63].

3.3.3 Back to electrical circuits

In this section we determine sufficient conditions for writing a nonlinear circuit in the port-Hamiltonian form (3.37). A procedure for constructing a Lagrangian model of nonlinear circuits can be found in [64]. As with mechanical systems, the Legendre transformation can then be used to transform the Lagrangian model into a Hamiltonian one. A direct constructive method for obtaining Hamiltonian models for LC circuits has been proposed in [6]. Following the suggestion of [34], in this section we extend this method by adding ports to account for voltage and current sources and resistive elements (the inclusion of resistive ports was independently proposed by B. Maschke and published in [32] without a proof). This is shown in Fig. 3.5.

We consider RLC networks satisfying the following assumptions.

Assumption 3.17. *Capacitors are charge controlled and inductors are flux controlled with characteristics given by*

$$v_\mathrm{q} = \hat{v}_\mathrm{q}(q) \quad \textit{and} \quad i_\phi = \hat{i}_\phi(\phi) \ , \tag{3.38}$$

where $q, v_\mathrm{q} \in \mathbb{R}^{n_\mathrm{q}}$ *are the capacitors charges and voltages, and* $\phi, i_\phi \in \mathbb{R}^{n_\phi}$ *are the inductors fluxes and currents. Furthermore,* $\hat{v}_\mathrm{q}(q)$ *and* $\hat{i}_\phi(\phi)$ *have symmetric Jacobians—that is, they are gradients of scalar functions.*

Remark 3.18. The vector-valued model (3.38) is not restricted to 2-terminal components only; it can describe 4-terminal elements as well, such as nonlinear transformers.

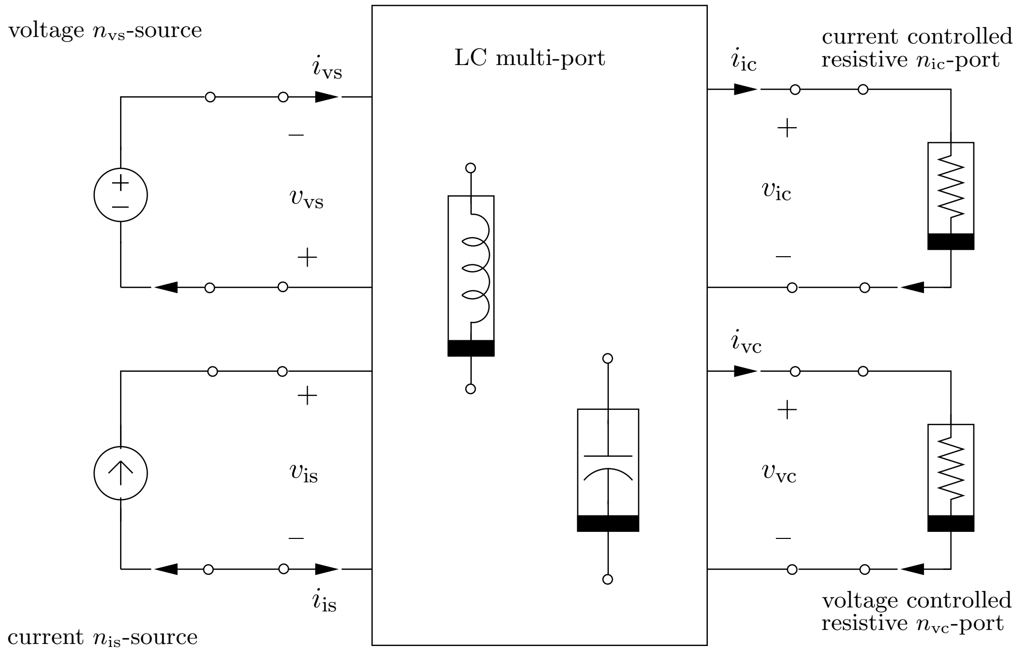

Figure 3.5: An RLC network with sources and dissipation elements attached through external ports.

The first condition of the assumption allows us to write the electric and magnetic energies as functions of q and ϕ, i.e.,

$$E_{\mathrm{q}}(q) = \int_0^q \hat{v}_{\mathrm{q}}^\top(\tau)\mathrm{d}\tau + E_{\mathrm{q}}(0) \quad \text{and} \quad E_\phi(\phi) = \int_0^\phi \hat{i}_\phi^\top(\tau)\mathrm{d}\tau + E_\phi(0) \ ,$$

while the second one guarantees that these functions do not depend on the integration path. This is of course automatically satisfied if the energy-storing elements are all single-port. To define the rest of the network components we denote by

$$v_{\mathrm{ic}} = \hat{v}_{\mathrm{ic}}(i_{\mathrm{ic}}), \quad i_{\mathrm{ic}}, v_{\mathrm{ic}} \in \mathbb{R}^{n_{\mathrm{ic}}}$$

and

$$i_{\mathrm{vc}} = \hat{i}_{\mathrm{vc}}(v_{\mathrm{vc}}), \quad v_{\mathrm{vc}}, i_{\mathrm{vc}} \in \mathbb{R}^{n_{\mathrm{vc}}}$$

the current and voltage controlled resistors. Time-varying voltage and current sources are represented by $v_{\mathrm{vs}} \in \mathbb{R}^{n_{\mathrm{vs}}}$ and $i_{\mathrm{is}} \in \mathbb{R}^{n_{\mathrm{is}}}$, respectively.

Assumption 3.19. *The graph $\mathcal{G}$ associated to the network has a tree $\mathcal{T}$ containing all capacitors, voltage sources and current controlled resistors. For future reference, denote by $\mathcal{L}$ the set of links corresponding to $\mathcal{T}$.*

This assumption excludes loops formed exclusively by capacitors and voltage sources as well as cut sets formed exclusively by inductors and current sources. This in turn means that we can choose ϕ, q, v_{vs} and i_{is} independently without violating Kirchhoff's laws.

It is possible to write Kirchhoff's voltage law in compact form as $Bv = 0$, where B is the fundamental loop matrix and v is the vector of branch voltages [14]. Moreover, if v is partitioned as $v = \mathrm{col}(v_{\mathcal{L}}, v_{\mathcal{T}})$, where $v_{\mathcal{L}}$ and $v_{\mathcal{T}}$ are the branch voltages of $\mathcal{L}$ and $\mathcal{T}$ respectively, then B takes the form $B = \begin{pmatrix} I & F \end{pmatrix}$ (with I and F of appropriate dimensions). If we further partition $v_{\mathcal{L}}$, $v_{\mathcal{T}}$ and F as

$$v_{\mathcal{L}} = \begin{pmatrix} v_{\phi} \\ v_{\mathrm{is}} \\ v_{\mathrm{vc}} \end{pmatrix} = \begin{pmatrix} \dot{\phi} \\ v_{\mathrm{is}} \\ v_{\mathrm{vc}} \end{pmatrix}, \quad v_{\mathcal{T}} = \begin{pmatrix} \hat{v}_{\mathrm{q}}(q) \\ -v_{\mathrm{vs}} \\ \hat{v}_{\mathrm{ic}}(i_{\mathrm{ic}}) \end{pmatrix}$$

(the negative sign of the voltage sources indicates that voltage drops across these branches are opposite to current flow) and

$$F = \begin{pmatrix} F_{\phi-q} & F_{\phi-\mathrm{vs}} & F_{\phi-\mathrm{ic}} \\ F_{\mathrm{is}-q} & F_{\mathrm{is}-\mathrm{vs}} & F_{\mathrm{is}-\mathrm{ic}} \\ F_{\mathrm{vc}-q} & F_{\mathrm{vc}-\mathrm{vs}} & F_{\mathrm{vc}-\mathrm{ic}} \end{pmatrix},$$

then we can write

$$\dot{\phi} = -F_{\phi-q}\hat{v}_{\mathrm{q}}(q) + F_{\phi-\mathrm{vs}}v_{\mathrm{vs}} - F_{\phi-\mathrm{ic}}\hat{v}_{\mathrm{ic}}(i_{\mathrm{ic}}) \tag{3.39a}$$

$$v_{\mathrm{is}} = -F_{\mathrm{is}-q}\hat{v}_{\mathrm{q}}(q) + F_{\mathrm{is}-\mathrm{vs}}v_{\mathrm{vs}} - F_{\mathrm{is}-\mathrm{ic}}\hat{v}_{\mathrm{ic}}(i_{\mathrm{ic}}) \tag{3.39b}$$

$$v_{\mathrm{vc}} = -F_{\mathrm{vc}-q}\hat{v}_{\mathrm{q}}(q) + F_{\mathrm{vc}-\mathrm{vs}}v_{\mathrm{vs}} - F_{\mathrm{vc}-\mathrm{ic}}\hat{v}_{\mathrm{ic}}(i_{\mathrm{ic}}) \,. \tag{3.39c}$$

Similarly, we can partition the currents as $i = \mathrm{col}(i_{\mathcal{L}}, i_{\mathcal{T}})$ with

$$i_{\mathcal{L}} = \begin{pmatrix} \hat{i}_{\phi}(\phi) \\ -i_{\mathrm{is}} \\ \hat{i}_{\mathrm{vc}}(v_{\mathrm{vc}}) \end{pmatrix}, \quad i_{\mathcal{T}} = \begin{pmatrix} i_{\mathrm{q}} \\ i_{\mathrm{vs}} \\ i_{\mathrm{ic}} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ i_{\mathrm{vs}} \\ i_{\mathrm{ic}} \end{pmatrix}$$

and write Kirchhoff's current law as $i = B^{\top} i_{\mathcal{L}}$. Simple matrix bookkeeping shows that

$$\dot{q} = F_{\phi-q}^{\top}\hat{i}_{\phi}(\phi) - F_{\mathrm{is}-q}^{\top}i_{\mathrm{is}} + F_{\mathrm{vc}-q}^{\top}\hat{i}_{\mathrm{vc}}(v_{\mathrm{vc}}) \tag{3.40a}$$

$$i_{\mathrm{vs}} = F_{\phi-\mathrm{vs}}^{\top}\hat{i}_{\phi}(\phi) - F_{\mathrm{is}-\mathrm{vs}}^{\top}i_{\mathrm{is}} + F_{\mathrm{vc}-\mathrm{vs}}^{\top}\hat{i}_{\mathrm{vc}}(v_{\mathrm{vc}}) \tag{3.40b}$$

$$i_{\mathrm{ic}} = F_{\phi-\mathrm{ic}}^{\top}\hat{i}_{\phi}(\phi) - F_{\mathrm{is}-\mathrm{ic}}^{\top}i_{\mathrm{is}} + F_{\mathrm{vc}-\mathrm{ic}}^{\top}\hat{i}_{\mathrm{vc}}(v_{\mathrm{vc}}) \,. \tag{3.40c}$$

By setting $H(q, \phi) = E_{\phi}(\phi) + E_{\mathrm{q}}(q)$ and noting that

$$\hat{v}_{\mathrm{q}}(q) = \nabla_q H(q, \phi) \quad \text{and} \quad \hat{i}_{\phi}(\phi) = \nabla_{\phi} H(q, \phi) \,,$$

we can write (3.39) and (3.40) as the port-Hamiltonian system

$$\dot{x} = J\nabla H(x) + g_1 u_1 + g_2 u_{\mathrm{R}}(y_{\mathrm{R}}) \tag{3.41a}$$

$$y_1 = g_1^{\top}\nabla H(x) + h_{11}u_1 + h_{12}u_{\mathrm{R}}(y_{\mathrm{R}}) \tag{3.41b}$$

$$y_{\mathrm{R}} = -g_2^{\top}\nabla H(x) + h_{12}^{\top}u_1 + h_{22}u_{\mathrm{R}}(y_{\mathrm{R}}) \,, \tag{3.41c}$$

with state $x = \mathrm{col}(\phi, q)$, efforts $u_1 = \mathrm{col}(v_{\mathrm{vs}}, i_{\mathrm{is}})$ and $u_{\mathrm{R}} = \mathrm{col}(\hat{v}_{\mathrm{ic}}(i_{\mathrm{ic}}), \hat{i}_{\mathrm{vc}}(v_{\mathrm{vc}}))$, flows $y_1 = \mathrm{col}(i_{\mathrm{vs}}, v_{\mathrm{is}})$ and $y_{\mathrm{R}} = \mathrm{col}(i_{\mathrm{ic}}, v_{\mathrm{vc}})$, and topological relationships

$$J = \begin{pmatrix} 0 & -F_{\phi-q} \\ F_{\phi-q}^{\top} & 0 \end{pmatrix} , \quad g_1 = \begin{pmatrix} F_{\phi-\mathrm{vs}} & 0 \\ 0 & -F_{\mathrm{is}-q}^{\top} \end{pmatrix}$$
$$g_2 = \begin{pmatrix} -F_{\phi-\mathrm{ic}} & 0 \\ 0 & F_{\mathrm{vc}-q}^{\top} \end{pmatrix} , \quad h_{11} = \begin{pmatrix} 0 & -F_{\mathrm{is-vs}}^{\top} \\ F_{\mathrm{is-vs}} & 0 \end{pmatrix}$$
$$h_{12} = \begin{pmatrix} 0 & F_{\mathrm{vc-vs}}^{\top} \\ -F_{\mathrm{is-ic}} & 0 \end{pmatrix} , \quad h_{22} = \begin{pmatrix} 0 & F_{\mathrm{vc-ic}}^{\top} \\ -F_{\mathrm{vc-ic}} & 0 \end{pmatrix} .$$

The power balance in (3.41) is now clearly revealed. The time derivative of the energy is

$$\begin{aligned} \dot{H}(x) &= \nabla H^{\top}(x)(J\nabla H(x) + g_1 u_1 + g_2 u_{\mathrm{R}}) \\ &= \nabla H^{\top}(x) J \nabla H(x) + (y_1 - h_{11}u_1 - h_{12}u_{\mathrm{R}})^{\top} u_1 - (y_{\mathrm{R}} - h_{12}^{\top}u_1 - h_{22}u_{\mathrm{R}})^{\top} u_{\mathrm{R}} . \end{aligned}$$

By noting that J, h_{11} and h_{22} are skew symmetric, we verify that

$$\dot{H}(x) = y_1^{\top} u_1 - y_{\mathrm{R}}^{\top} u_{\mathrm{R}} , \tag{3.42}$$

which shows that the rate at which the stored energy increases equals the difference between the power delivered by the sources and the power dissipated by the resistors.

Remark 3.20. Notice that for system (3.41) to be well defined, it is necessary that for every x and u_1, a unique y_{R}, solution of (3.41c), exists. We assume that such a y_{R} exists. The interested reader is referred to [54, 49], where sufficient conditions for existence and uniqueness can be found.

Remark 3.21. If the characteristics of inductors and capacitors (3.38) are bijective, it is possible to relax assumption 3.19 by finding a reduced equivalent network containing no inductor cut sets or capacitor loops. A precise notion of equivalence, as well as the explicit procedure to carry out the transformation can be found in [50], see also [9]. However, in the general nonlinear case, practical use of these procedures is impeded by the requirement of an explicit solution of (3.41). An alternative way to relax assumption 3.19 is to enforce Kirchhoff's laws using the notion of port-Hamiltonian models with constraints [27].

Example, a nonlinear RLC circuit

As an example, consider the circuit shown in Fig. 3.6. We model the diode as a nonlinear, voltage controlled resistor characterized by

$$i_{\mathrm{vc}_1} = I_{\mathrm{s}_1} \left(\exp\left(\frac{v_{\mathrm{vc}_1}}{V_{\mathrm{T}}} \right) - 1 \right) ,$$

where I_{s_1} is the saturation current and $V_{\mathrm{T}} = 25\mathrm{mV}$. The other dissipative elements are a linear conductance governed by $i_{\mathrm{vc}_2} = G v_{\mathrm{vc}_2}$ and the linear resistor $v_{\mathrm{ic}} = R i_{\mathrm{ic}}$. The

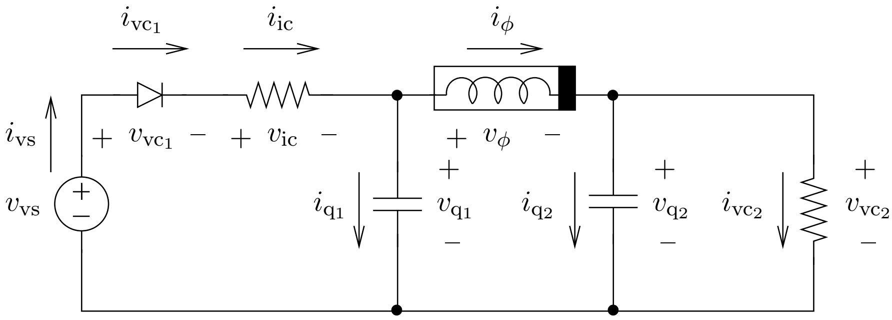

Figure 3.6: A nonlinear RLC circuit.

nonlinear inductor, described by

$$i_\phi = I_{\mathrm{s}_\phi} \tanh\left(\frac{\phi}{\delta_\phi}\right) ,$$

saturates at a current I_{s_ϕ} and has, at the origin, an incremental inductance of $\delta_\phi / I_{\mathrm{s}_\phi}$. For simplicity, we consider linear capacitors of the form $v_{\mathrm{q}_1} = q_1/C_1$ and $v_{\mathrm{q}_2} = q_2/C_2$.

The directed graph corresponding to the circuit is given in Fig. 3.7. The tree that satisfies assumption 3.19 has been highlighted with thick lines.

The energy of the circuit is given by

$$H(q_1, q_2, \phi) = I_{\mathrm{s}_\phi} \delta_\phi \ln\left(\left|\cosh\left(\frac{\phi}{\delta_\phi}\right)\right|\right) + \frac{q_1^2}{2C_1} + \frac{q_2^2}{2C_2} \tag{3.43}$$

and its gradient is

$$\nabla H(q_1, q_2, \phi) = \begin{pmatrix} I_{\mathrm{s}_\phi} \tanh(\phi/\delta_\phi) \\ q_1/C_1 \\ q_2/C_2 \end{pmatrix} .$$

Recall that the fundamental loop matrix $F = \{F_{\lambda-\tau}\}$, $\lambda \in \mathcal{L}$, $\tau \in \mathcal{T}$, is constructed according to the following rule:

$$F_{\lambda-\tau} = \begin{cases} 1 & \text{if } \tau \text{ is in the loop formed by } \lambda \text{ and} \\ & \text{their directions are equal.} \\ 0 & \text{if } \tau \text{ is not in the loop formed by } \lambda. \\ -1 & \text{if } \tau \text{ is in the loop formed by } \lambda \text{ but} \\ & \text{their directions are opposite.} \end{cases} .$$

Since there are no current sources, the matrix F reduces to

$$F = \left[\begin{array}{c|c|c} F_{\phi-q} & F_{\phi-\mathrm{vs}} & F_{\phi-\mathrm{ic}} \\ \hline F_{\mathrm{vc}-q} & F_{\mathrm{vc}-\mathrm{vs}} & F_{\mathrm{vc}-\mathrm{ic}} \end{array}\right] = \left[\begin{array}{cc|c|c} -1 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{array}\right] .$$

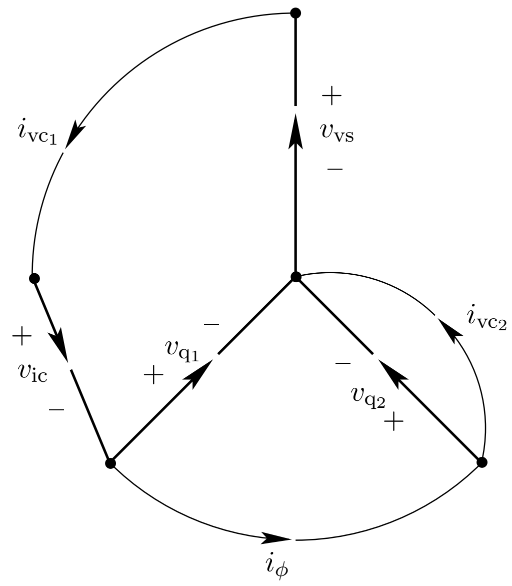

Figure 3.7: Directed graph corresponding to the circuit of Fig. 3.6. The thick straight lines represent the branches of $\mathcal{T}$. The thin arcs represent the links that belong to $\mathcal{L}$.

Finally, equation (3.41) becomes

$$\begin{pmatrix} \dot{\phi} \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \nabla H(q_1, q_2, \phi) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} u_{\mathrm{R}} \tag{3.44a}$$

$$i_{\mathrm{vs}} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} u_{\mathrm{R}} \tag{3.44b}$$

$$\begin{pmatrix} i_{\mathrm{ic}} \\ v_{\mathrm{vc}_1} \\ v_{\mathrm{vc}_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \nabla H(q_1, q_2, \phi) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} v_{\mathrm{vs}} + \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u_{\mathrm{R}} \,, \tag{3.44c}$$

where

$$u_{\mathrm{R}} = \hat{u}_{\mathrm{R}}(y_{\mathrm{R}}) = \begin{pmatrix} R i_{\mathrm{ic}} \\ I_{\mathrm{s}_1} \left(\exp\left(v_{\mathrm{vc}_1} / V_{\mathrm{T}} \right) - 1 \right) \\ G v_{\mathrm{vc}_2} \end{pmatrix} .$$

3.4 Passivation

Let us now state the central problem of this thesis. We formulate the control problem as the problem of devising a controller that, by means of an energy exchange, appropriately alters the cyclo-passive properties of a given plant.

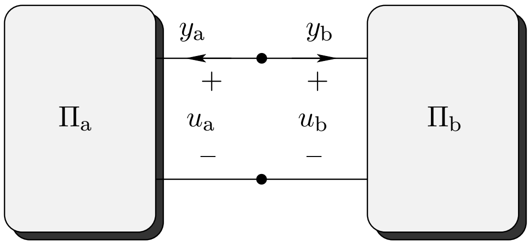

Figure 3.8: Interconnecting two manifest ports.

3.4.1 Interconnection of m-ports

Consider first two m-ports $\Pi_a = (\mathbb{W}, \mathcal{B}_a)$ and $\Pi_b = (\mathbb{W}, \mathcal{B}_b)$, interconnected as illustrated in Fig. 3.8. Let $(u_a, y_a) \in \mathcal{B}_a$ and $(u_b, y_b) \in \mathcal{B}_b$. According to the figure,

$$u_a = u_b \quad \text{and} \quad y_a + y_b = 0\ . \tag{3.45}$$

We say that (3.45) is a power-preserving interconnection because the instantaneous power entering the interconnection (which in this case is zero) equals the power leaving it[5], that is,

$$y_a^\top u_a + y_b^\top u_b = y_a^\top u_a + (-y_a)^\top u_a = 0\ . \tag{3.46}$$

The behavior of the compound system is, from (3.45),

$$\mathcal{B} = \left\{ (u, y) \in \mathbb{W} \,\middle|\, (u, y) \in \mathcal{B}_a,\ (u, -y) \in \mathcal{B}_b \right\}\ , \tag{3.47}$$

which can be interpreted as Π_a and Π_b *sharing* the port variables u and y [69]. Consider an element $w = (u, y) \in \mathbb{W}$ and define the operator $\bar{w} \triangleq (u, -y)$. The compound behavior (3.47) can be written more compactly as $\mathcal{B} = \left\{ w \in \mathbb{W} \,\middle|\, w \in \mathcal{B}_a,\ \bar{w} \in \mathcal{B}_b \right\}$.

Notice that the compound system is closed (it has no external terminals) so it is not an m-port. But then our control objective is to transform a cyclo-passive m-port $\hat{\Pi}_p$ into a passive one with desired storage function. For this purpose, we will construct the controller $\hat{\Pi}_c$ as a $2m$-port and interconnect it to the plant as shown in Fig. 3.9. This will effectively create the new m-port $\hat{\Pi}_t$. Besides interconnecting the port variables, we will share the state variables of $\hat{\Pi}_p$ with the controller and possibly let the controller have some state variables of its own. This process calls for a more elaborated notion of interconnection as the one described before. Inspired from [69], we propose the following definition.

Definition 3.22. Let

$$\hat{\Pi}_a = (\mathbb{W}_1 \times \mathbb{W}_2, \mathbb{X}_1 \times \mathbb{X}_2, \hat{\mathcal{B}}_a) \quad \text{and} \quad \hat{\Pi}_b = (\mathbb{W}_2 \times \mathbb{W}_3, \mathbb{X}_2 \times \mathbb{X}_3, \hat{\mathcal{B}}_b) \tag{3.48}$$

[5]This type of interconnections are termed *neutral* in [65].

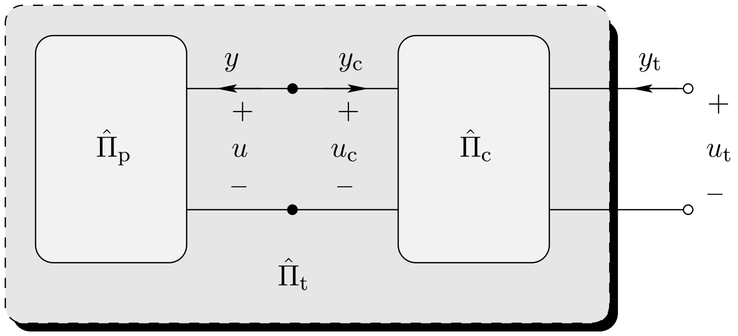

Figure 3.9: Control as interconnection of the plant $\hat{\Pi}_p$ and the controller $\hat{\Pi}_c$, a $2m$-port. (Only the manifest variables are shown).

be two ports having a common signal space $\mathbb{W}_2$ and a common state space $\mathbb{X}_2$. The *interconnection* of $\hat{\Pi}_a$ and $\hat{\Pi}_b$, denoted as $\hat{\Pi}_a \wedge \hat{\Pi}_b$, is defined by

$$\hat{\Pi}_t = \hat{\Pi}_a \wedge \hat{\Pi}_b \triangleq (\mathbb{W}_1 \times \mathbb{W}_3, \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3, \hat{\mathcal{B}}_t) ,$$

where

$$\hat{\mathcal{B}}_t = \Big\{ (w_1, w_3, x_1, x_2, x_3) \in \mathbb{W}_1 \times \mathbb{W}_3 \times \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \,\Big|\, \exists\, w_2 \in \mathbb{W}_2 \\ \text{s.t. } (w_1, w_2, x_1, x_2) \in \hat{\mathcal{B}}_a,\ (\bar{w}_2, w_3, x_2, x_3) \in \hat{\mathcal{B}}_b \Big\} . \quad (3.49)$$

In words, interconnecting $\hat{\Pi}_a$ with $\hat{\Pi}_b$ amounts to sharing the common variables w_2 and x_2. After interconnecting the ports, we loose interest in w_2 and we suppress it from the explicit model.

An important consequence of the interconnection (3.45) being power-preserving is that cyclo-passivity is inherited to the compound m-port.

Theorem 3.23. *Let $\hat{\Pi}_a$ and $\hat{\Pi}_b$ be two ports described by (3.48). If $\hat{\Pi}_a$ and $\hat{\Pi}_b$ are cyclo-passive, so is the interconnection $\hat{\Pi}_a \wedge \hat{\Pi}_b$.*

Proof: Since each individual port is cyclo-passive, the dissipation inequalities

$$H_a(x_1, x_2) \le H_a(x_{1\,0}, x_{2\,0}) + \int_0^t (y_1^\top u_1 + y_2^\top u_2) \mathrm{d}\tau , \quad \forall\, (u_1, y_1, u_2, y_2, x_1, x_2) \in \hat{\mathcal{B}}_a \quad (3.50)$$

and

$$H_b(x_2, x_3) \le H_b(x_{2\,0}, x_{3\,0}) + \int_0^t (y_2^\top u_2 + y_3^\top u_3) \mathrm{d}\tau , \quad \forall\, (u_2, y_2, u_3, y_3, x_2, x_3) \in \hat{\mathcal{B}}_a \quad (3.51)$$

hold true. Equations (3.50), (3.51) and (3.49) imply that the total storage function

$$H_t(x_1, x_2, x_3) = H_a(x_1, x_2) + H_b(x_2, x_3)$$

satisfies

$$H_{\mathrm{t}}(x_1, x_2, x_3) \leq H_{\mathrm{t}}(x_{1\,0}, x_{2\,0}, x_{3\,0}) + \int_0^t (y_1^\top u_1 + y_3^\top u_3)\mathrm{d}\tau \ , \qquad \forall\, (u_1, y_1, u_3, y_3, x_1, x_2, x_3) \in \hat{\mathcal{B}}_{\mathrm{t}} \ .$$

■

This theorem generalizes the passivity theorem (p. 33) by taking into account storage functions which are not necessarily bounded from below. It can be further generalized to an arbitrary number of ports and to general power-preserving interconnections [65, theorem 5].

3.4.2 Control as interconnection and the (cyclo) passivation problem

From a behavioral perspective, control is viewed as the problem of designing a system (a controller), such that the resulting interconnection of the controller and the original plant (i.e., the controlled system) has some desired properties. In this section we adhere to this approach but we tailor it to m-ports (again, cf. Fig. 3.9).

Definition 3.24. Consider an m-port $\hat{\Pi}_{\mathrm{p}}$. A *controller* for $\hat{\Pi}_{\mathrm{p}}$ is a $2m$-port

$$\hat{\Pi}_{\mathrm{c}} = (\mathbb{W} \times \mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}}) \ , \quad \mathbb{W}_{\mathrm{t}} = \mathbb{U}_{\mathrm{t}} \times \mathbb{Y}_{\mathrm{t}} \ ,$$

where $\mathbb{U}_{\mathrm{t}} = \mathbb{R}^m$, $\mathbb{Y}_{\mathrm{t}} = \mathbb{R}^m$, $\mathbb{X}_{\mathrm{c}} = \mathbb{R}^q$ for some $q \in \mathbb{N}$ and $u_{\mathrm{t}} \in \mathbb{U}_{\mathrm{t}}$ is a free variable.

The plant and the controller share the whole state and manifest variables of the plant. Thus, according to definition 3.22, the resulting interconnection has the form

$$\hat{\Pi}_{\mathrm{t}} = \hat{\Pi}_{\mathrm{p}} \wedge \hat{\Pi}_{\mathrm{c}} = (\mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{t}}) \ .$$

In particular, a passivity-based controller exploits the cyclo-passivity properties of a given plant.

Definition 3.25. Let $\hat{\Pi}_{\mathrm{p}}$ be a cyclo-passive m-port. The controller $\hat{\Pi}_{\mathrm{c}}$ is said to be *passivity-based* (shorthand notation: $\hat{\Pi}_{\mathrm{c}} \in \mathrm{PBC}$) if the resulting interconnection $\hat{\Pi}_{\mathrm{t}}$ is again cyclo-passive.

Depending on the particular application, we will add certain specifications to the storage function H_{t}. Typically, we will require H_{t} to be positive-definite with respect to some desired equilibrium point. Once this energy-shaping step is achieved, one can simply add extra damping by terminating $\hat{\Pi}_{\mathrm{t}}$ with a resistive m-port (cf. theorem 3.12). Stabilization of equilibria is not the only problem that can be solved using this method. One can, for example, choose an H_{t} with a 'Mexican sombrero' shape [39] to achieve stabilization of periodic orbits, like in the example on p. 90.

3.4.3 Assignable equilibrium points

Since most of the material presented in the following chapters refer to the specific problem of stabilizing an equilibrium point, we will devote a few lines to define the set of assignable equilibria $\mathcal{E}_x$, which is the set of points $x \in \mathbb{X}$ for which there exists a constant control $u \in \mathbb{U}$ that makes an equilibrium out of x. For m-ports described by (3.1) this set is

$$\mathcal{E}_x = \left\{ x \in \mathbb{X} \,\middle|\, \exists\, u \in \mathbb{U} \text{ s.t. } f(x) + g(x)u = 0 \right\} \ . \tag{3.52}$$

In other words, $\mathcal{E}_x$ is the set of points x for which the equation

$$g(x)u = -f(x) \tag{3.53}$$

has at least one solution u, that is, such that (3.53) is consistent. According to the first part of theorem A.6, equation (3.53) is consistent if, and only if[6],

$$g(x)g^{+}(x)f(x) = f(x) \ ,$$

which is equivalent to

$$\left[I - g(x)g^{+}(x) \right] f(x) = 0 \ . \tag{3.54}$$

By bringing remark A.10 into the discussion, we can rewrite (3.54) as

$$g^{\perp}_{+}(x)g^{\perp}(x)f(x) = 0 \ . \tag{3.55}$$

Since $g^{\perp}_{+}(x)$ is of full rank, we conclude that (3.55) is equivalent to $g^{\perp}(x)f(x) = 0$ and that,

$$\mathcal{E}_x = \left\{ x \in \mathbb{X} \mid g^{\perp}(x)f(x) = 0 \right\} \ . \tag{3.56}$$

Notice that the representation (3.56) does not involve the extra variable u, so it is simpler to compute that definition (3.52). However, for a given $x_\star \in \mathcal{E}_x$, it will be useful to compute the actual control that appears in (3.52). From the second part of theorem A.6, we know that if (3.53) is consistent (which it is, since we impose $x_\star \in \mathcal{E}_x$) the solutions are given by

$$u = -g^{+}(x_\star) - (I - g^{+}(x_\star)g(x_\star))\zeta \ ,$$

where $\zeta \in \mathbb{U}$ is arbitrary. Since $g^{+}(x_\star)$ is a left inverse of $g(x_\star)$ (remark A.10) the right-most part vanishes and we are left with a uniquely defined constant control

$$u_\star \triangleq -g^{+}(x_\star)f(x_\star) \ . \tag{3.57}$$

[6] The superscripts '+' and '$\perp$' denote the Moore-Penrose pseudo inverse and a full rank left annihilator, respectively. (See p. 134).

4 Static Passivation

We present several solutions to the control problem stated in the previous chapter. We consider static controllers only, leaving the use of dynamic controllers for the next chapter.

We begin by characterizing the set of passivity-based controllers. From this set we select special cases, starting with the simplest (but most restrictive) ones, increasing the degree of complexity (and generality) as the discourse develops. We discuss flow and dissipation invariance, which are crucial for establishing the property of energy balance and will also play an important role in the next chapter.

4.1 The set of passivity-based controllers

For ease of reference, let us begin by repeating (3.10):

$$\begin{aligned}\dot{x} &= f(x) + g(x)u \\ y &= h(x)\ ,\end{aligned}$$

and let us state a basic assumption on the plant.

Assumption 4.1. *The plants under consideration are cyclo-passive m-ports*

$$\hat{\Pi}_{\mathrm{p}} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{p}}) \tag{4.1}$$

with $\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ *and*

$$\hat{\mathcal{B}}_{\mathrm{p}} = \left\{(u, y, x) \in \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \,\middle|\, \textit{Eq. (3.10) holds}\right\}\ . \tag{4.2}$$

Notation. Since the structure of (4.2) can be inferred from (4.1), from now on we will replace (4.2) (and general behaviors as well) by the shorter notation

$$\hat{\mathcal{B}}_{\mathrm{p}} = \left\{(u, y, x) \,\middle|\, \text{Eq. (3.10) holds}\right\}\ . \tag{4.3}$$

As we already mentioned, we consider static controllers having no internal state $\mathbb{X}_{\mathrm{c}}$, that is,

$$\hat{\Pi}_{\mathrm{c}} = (\mathbb{W} \times \mathbb{W}_{\mathrm{t}}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{c}})\ . \tag{4.4}$$

We will further restrict our attention to behaviors satisfying the following.

Assumption 4.2. *The controller's behavior is of the form*

$$\hat{\mathcal{B}}_{\mathrm{c}} = \left\{(u_{\mathrm{c}}, y_{\mathrm{c}}, u_{\mathrm{t}}, y_{\mathrm{t}}, x) \,\middle|\, u_{\mathrm{c}} = \mathring{u}_{\mathrm{c}}(x) + u_{\mathrm{t}},\ y_{\mathrm{t}} = \mathring{y}_{\mathrm{t}}(x) - y_{\mathrm{c}}\right\} \tag{4.5}$$

for some $\mathring{u}_{\mathrm{c}} : \mathbb{X} \to \mathbb{U}$ *and* $\mathring{y}_{\mathrm{t}} : \mathbb{X} \to \mathbb{Y}$.

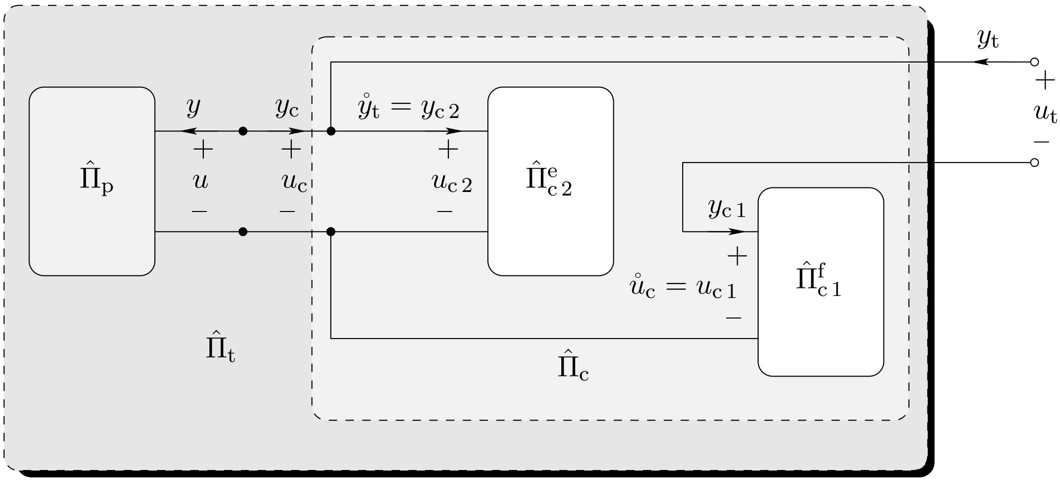

Figure 4.1: The $2m$-port controller can be realized as a series and parallel interconnection of two m-ports (one FD and the other ED).

Notice that the behavior (4.5) admits y_c, x and u_t as free variables, and that y and x are precisely the variables that are *not* freely assignable in $\hat{\Pi}_p$. For this reason, assumption 4.2 simplifies the task of obtaining $\hat{\mathcal{B}}_t$ without explicitly calculating the trajectories of $\hat{\Pi}_p$. It also excludes pathological cases like $\hat{\mathcal{B}}_t = \emptyset$. As we shall see, this assumption induces little loss of generality, since most of the PBC strategies reported in the literature can be derived from it.

Owing to the same assumption, it is possible to realize $\hat{\Pi}_c$ as a flow-driven m-port

$$\hat{\Pi}^f_{c\,1} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{c\,1}) \ , \quad \hat{\mathcal{B}}_{c\,1} = \left\{ (u_{c\,1}, y_{c\,1}, x) \mid u_{c\,1} = \mathring{u}_c(x) \right\}$$

and an effort-driven m-port

$$\hat{\Pi}^e_{c\,2} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{c\,2}) \ , \quad \hat{\mathcal{B}}_{c\,2} = \left\{ (u_{c\,2}, y_{c\,2}, x) \mid y_{c\,2} = \mathring{y}_t(x) \right\} \ ,$$

subjected to the interconnection constraints (see Fig. 4.1)

$$\begin{array}{rcl} u_c & = & u_{c\,1} + u_t \\ 0 & = & y_{c\,1} + y_t \end{array} \quad \text{and} \quad \begin{array}{rcl} u_c & = & u_{c\,2} + 0 \\ y_c & = & y_{c\,2} - y_t \ . \end{array} \tag{4.6}$$

Remark 4.3. The interconnection (4.6) is power-preserving because the power entering the interconnection is equal to the power leaving it. That is, $y_c^\top u_c + y_t^\top u_t = y_{c\,2}^\top u_{c\,2} + y_{c\,1}^\top u_{c\,1}$.

Lemma 4.4. *Consider the plant $\hat{\Pi}_p$ (4.3) and a controller $\hat{\Pi}_c$ satisfying assumption 4.2. The compound m-port is $\hat{\Pi}_t = \hat{\Pi}_p \wedge \hat{\Pi}_c = (\mathbb{W}_t, \mathbb{X}, \hat{\mathcal{B}}_t)$ and $(u_t, y_t, x) \in \hat{\mathcal{B}}_t$ is equivalent to*

$$\dot{x} = f_t(x) + g(x) u_t \tag{4.7a}$$

$$y_t = h_t(x) \ , \tag{4.7b}$$

where $f_t(x) \triangleq f(x) + g(x)\mathring{u}_c(x)$ and $h_t(x) \triangleq h(x) + \mathring{y}_t(x)$.

The proof is straightforward and given in appendix B.1. The following theorem, which constitutes the main thread of this chapter, provides an account of the energy exchange that takes place in a passivity-based controller. It also gives an *algebraic* characterization of the set PBC associated to a particular $\hat{\Pi}_{\mathrm{p}}$.

Theorem 4.5. *Consider a controller $\hat{\Pi}_{\mathrm{c}}$ satisfying assumption 4.2 and a plant $\hat{\Pi}_{\mathrm{p}}$ satisfying assumption 4.1. The following statements are equivalent:*

1. $\hat{\Pi}_{\mathrm{c}} \in PBC$.
2. *There exists $H_{\mathrm{a}} : \mathbb{X} \to \mathbb{R}$ and $d_{\mathrm{a}} : \mathbb{X} \to \mathbb{R}$, with $d_{\mathrm{a}}(x) \geq -d(x)$, such that*

$$\dot{H}_{\mathrm{a}}(x) = y_{\mathrm{c}}^\top u_{\mathrm{c}} + y_{\mathrm{t}}^\top u_{\mathrm{t}} - d_{\mathrm{a}}(x) \qquad \forall\, (u_{\mathrm{c}}, y_{\mathrm{c}}, u_{\mathrm{t}}, y_{\mathrm{t}}, x) \in \hat{\mathcal{B}}_{\mathrm{c}}\ , (u_{\mathrm{c}}, -y_{\mathrm{c}}, x) \in \hat{\mathcal{B}}_{\mathrm{p}}\ . \quad (4.8)$$

3. *There exists $H_{\mathrm{a}} : \mathbb{X} \to \mathbb{R}$ and $d_{\mathrm{a}} : \mathbb{X} \to \mathbb{R}$, with $d_{\mathrm{a}}(x) \geq -d(x)$, such that*

$$\begin{aligned} h^\top(x)\mathring{u}_{\mathrm{c}}(x) &= -\nabla H_{\mathrm{a}}^\top(x)(f(x) + g(x)\mathring{u}_{\mathrm{c}}(x)) - d_{\mathrm{a}}(x) && (4.9a)\\ \mathring{y}_{\mathrm{t}}(x) &= g^\top(x)\nabla H_{\mathrm{a}}(x)\ . && (4.9b) \end{aligned}$$

Proof: From assumption 4.1 and the power-preservation condition $y^\top u + y_{\mathrm{c}}^\top u_{\mathrm{c}} = 0$ (cf. equation (3.46)), we know that

$$\dot{H}(x) = y^\top u - d(x) = -y_{\mathrm{c}}^\top u_{\mathrm{c}} - d(x)\ . \quad (4.10)$$

First we prove the statement ($1 \Rightarrow 2$). Since $\hat{\Pi}_{\mathrm{t}}$ is cyclo-passive, there exist functions $H_{\mathrm{t}} : \mathbb{X} \to \mathbb{R}$ and $d_{\mathrm{t}} : \mathbb{X} \to \mathbb{R}$ satisfying

$$\dot{H}_{\mathrm{t}}(x) = y_{\mathrm{t}}^\top u_{\mathrm{t}} - d_{\mathrm{t}}(x)\ , \quad d_{\mathrm{t}}(x) \geq 0\ . \quad (4.11)$$

Let us define

$$H_{\mathrm{a}}(x) \triangleq H_{\mathrm{t}}(x) - H(x) \quad \text{and} \quad d_{\mathrm{a}}(x) \triangleq d_{\mathrm{t}}(x) - d(x)\ . \quad (4.12)$$

By subtracting (4.10) from (4.11), one arrives at (4.8). The condition $d_{\mathrm{a}}(x) \geq -d(x)$ follows from $d_{\mathrm{t}}(x) \geq 0$. To verify ($2 \Rightarrow 1$), add (4.8) and (4.10) to obtain (4.11), which establishes the cyclo-passivity of $\hat{\Pi}_{\mathrm{t}}$.

In order to prove ($1 \Rightarrow 3$), assume that $\hat{\Pi}_{\mathrm{t}}$ is cyclo-passive with storage function H_{t} and flow function h_{t}. From Hill-Moylan's theorem we know that

$$\begin{aligned} \nabla H_{\mathrm{t}}^\top(x) f_{\mathrm{t}}(x) &= -d_{\mathrm{t}}(x) && (4.13a)\\ h_{\mathrm{t}}(x) &= g^\top(x)\nabla H_{\mathrm{t}}(x) && (4.13b) \end{aligned}$$

for some dissipation $d_{\mathrm{t}}(x) \geq 0$. Again, define H_{a} and d_{a} as in (4.12). Equation (4.13a) becomes

$$[\nabla H_{\mathrm{a}}(x) + \nabla H(x)]^\top [f(x) + g(x)\mathring{u}_{\mathrm{c}}(x)] = -d_{\mathrm{a}}(x) - d(x)\ ,$$

which can be re-ordered as

$$\nabla H^\top(x)g(x)\mathring{u}_{\mathrm{c}}(x) = -\nabla H_{\mathrm{a}}^\top(x)[f(x)+g(x)\mathring{u}_{\mathrm{c}}(x)]-d_{\mathrm{a}}(x)-\nabla H^\top(x)f(x)-d(x)\ . \quad (4.14)$$

Substituting

$$g^\top(x)\nabla H(x) = h(x) \quad \text{and} \quad \nabla H^\top(x)f(x)+d(x)=0 \quad (4.15)$$

in (4.14), one obtains (4.9a). Substituting (4.12) in equation (4.13b) yields

$$h_{\mathrm{t}}(x) = g^\top(x)[\nabla H(x)+\nabla H_{\mathrm{a}}(x)] = h(x)+g^\top(x)\nabla H_{\mathrm{a}}(x)\ ,$$

which implies (4.9b).

Regarding the statement (3 ⇒ 1), assume that (4.9a) holds for some H_{a} and d_{a}. Define

$$H_{\mathrm{t}}(x) \triangleq H_{\mathrm{a}}(x)+H(x) \quad \text{and} \quad d_{\mathrm{t}}(x) \triangleq d_{\mathrm{a}}(x)+d(x)\ ,$$

so that (4.9a) can be rewritten as

$$h^\top(x)\mathring{u}_{\mathrm{c}}(x) = -[\nabla H_{\mathrm{t}}(x)-\nabla H(x)]^\top[f(x)+g(x)\mathring{u}_{\mathrm{c}}(x)]+d(x)-d_{\mathrm{t}}(x)$$

or, equivalently, as

$$[h(x)-g^\top(x)\nabla H(x)]^\top\mathring{u}_{\mathrm{c}}(x) = d(x)+\nabla H^\top(x)f(x)-\nabla H_{\mathrm{t}}^\top(x)f_{\mathrm{t}}(x)-d_{\mathrm{t}}(x)\ . \quad (4.16)$$

From (4.15), equation (4.16) reduces to

$$\nabla H_{\mathrm{t}}^\top(x)f_{\mathrm{t}}(x) = -d_{\mathrm{t}}(x)\ .$$

From (4.9b) and $y_{\mathrm{t}} = \mathring{y}_{\mathrm{t}}(x)+y$ we have that

$$y_{\mathrm{t}} = g^\top(x)\nabla H_{\mathrm{a}}(x)+g^\top(x)\nabla H(x) = g^\top(x)\nabla H_{\mathrm{t}}(x)\ .$$

According to Hill-Moylan's theorem, $\hat{\Pi}_{\mathrm{t}}$ is cyclo-passive with storage function H_{t}. ■

4.2 Energy balancing

To introduce the notion of energy balance, let us step back for a moment and take a look at the input-state-output approach to PBC. One considers a plant $\mathcal{S}_{\mathrm{p}}$, a controller $\mathcal{S}_{\mathrm{c}}$ connected as in Fig. 4.2, and aims at a passive closed-loop system $\mathcal{S}_{\mathrm{t}}$. Within this setting, the product $-y^\top\mathring{u}_{\mathrm{c}}(x)$ is associated with the power supplied by (or extracted from) the controller. If the difference between the total energy and the energy stored at the plant equals the energy supplied by the controller, i.e., if

$$H_{\mathrm{a}}(x(t)) = -\int_0^t (y^\top(\tau)\mathring{u}_{\mathrm{c}}(x(\tau))\mathrm{d}\tau + H_{\mathrm{a}}(x_0)\ , \quad (4.17)$$

for all t with H_{a} defined as in (4.12), then we say that the controller is energy balancing [44]. We use the differential equivalent of (4.17) to translate this notion to the port interconnection setting.

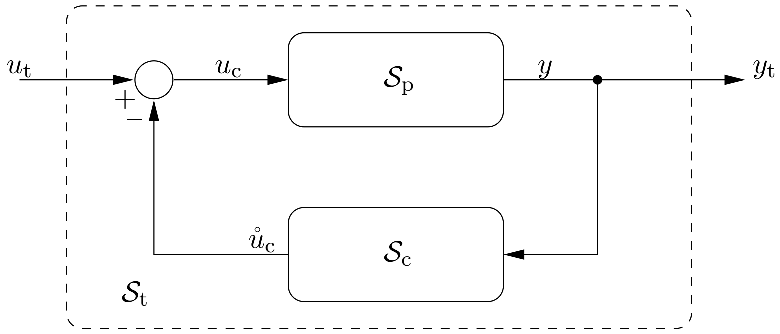

Figure 4.2: Classical, input-state-output PBC.

Definition 4.6. Consider a plant $\hat{\Pi}_p$ satisfying assumption 4.1. A controller $\hat{\Pi}_c$ is said to be *energy balancing* if, for some $H_a : \mathbb{X} \to \mathbb{R}$,

$$\dot{H}_a(x) = y_c^\top \mathring{u}_c(x) \qquad \forall\, (u_c, y_c, u_t, y_t, x) \in \hat{\mathcal{B}}_c\ ,\ (u_c, -y_c, x) \in \hat{\mathcal{B}}_p\ . \tag{4.18}$$

The following proposition characterizes all energy balancing controllers in terms of the resulting flow and dissipation functions.

Proposition 4.7. *Consider a controller $\hat{\Pi}_c$ satisfying assumption 4.2 and a plant $\hat{\Pi}_p$ satisfying assumption 4.1. The following statements are equivalent:*

1. *$\hat{\Pi}_c$ is energy balancing.*
2. *$\hat{\Pi}_c \in PBC$ and equation (4.11) holds with*

$$y_t = y \quad \text{and} \quad d_t(x) = d(x)\ , \tag{4.19}$$

(we say that $\hat{\Pi}_c$ is flow-preserving and dissipation-preserving, respectively).

3. *There exists an $H_a : \mathbb{X} \to \mathbb{R}$ such that*

$$\begin{aligned} h^\top(x)\mathring{u}_c(x) &= -\nabla H_a^\top(x) f(x) && (4.20a)\\ \mathring{y}_t(x) &= g^\top(x)\nabla H_a(x) = 0\ . && (4.20b) \end{aligned}$$

Proof: To prove $(1 \Rightarrow 2)$, develop (4.18) using $u_c = \mathring{u}_c(x) + u_t$ to obtain

$$\nabla H_a^\top(x)[f(x) + g(x)(\mathring{u}_c(x) + u_t)] = -h^\top(x)\mathring{u}_c(x)\ .$$

Since u_t is arbitrary, we conclude that $\nabla H_a^\top(x)g(x) = 0$, which is equivalent to $y_t = y$. Now, since $y = -y_c$, equation (4.18) can be rewritten as

$$\begin{aligned} \dot{H}_a(x) &= y_c^\top(u_c - u_t)\\ &= y_t^\top u_t + y_c^\top u_c\ . \end{aligned}$$

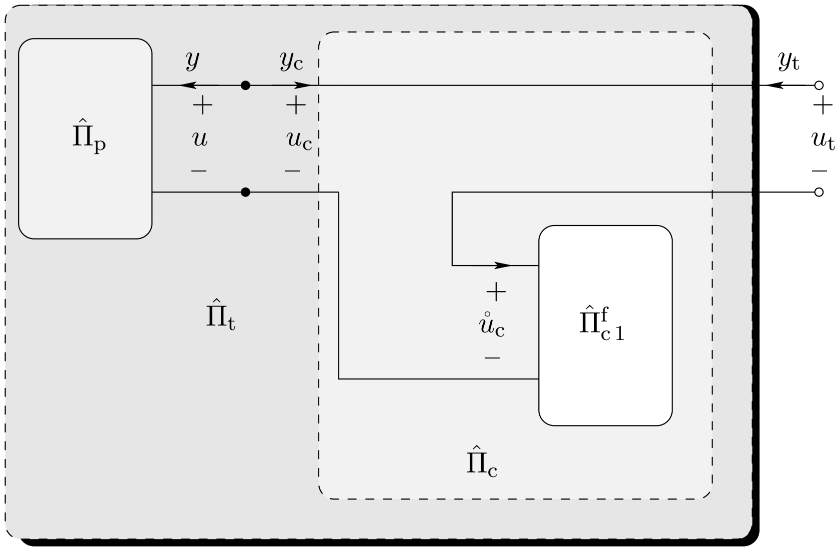

Figure 4.3: An energy balancing controller.

According to theorem 4.5, $\hat{\Pi}_{\mathrm{c}} \in \mathrm{PBC}$ with $d_{\mathrm{t}}(x) = d(x)$.

Let us show ($2 \Rightarrow 1$). Assume that $\hat{\Pi}_{\mathrm{c}} \in \mathrm{PBC}$ and that equation (4.19) holds. Substituting $d_{\mathrm{a}}(x) = d_{\mathrm{t}}(x) - d(x) = 0$ in (4.8) yields

$$\dot{H}_{\mathrm{a}}(x) = y_{\mathrm{c}}^{\top} u_{\mathrm{c}} + y_{\mathrm{t}}^{\top} u_{\mathrm{t}} \ . \tag{4.21}$$

By substituting $h_{\mathrm{t}}(x) = h(x)$ and $u_{\mathrm{c}} = \mathring{u}_{\mathrm{c}}(x) + u_{\mathrm{t}}$ in (4.21), one recovers (4.18).

The statement ($2 \Rightarrow 3$) can be verified by substituting $d_{\mathrm{a}}(x) = 0$ and $\mathring{y}_{\mathrm{t}}(x) = 0$ directly in (4.9). The converse statement is established by noting that (4.20) is a particular case of (4.9), with $d_{\mathrm{a}}(x) = 0$ and $\mathring{y}_{\mathrm{t}}(x) = 0$. ■

Remark 4.8. An energy balancing controller, being flow-preserving, can be realized with a single flow-driven m-port $\hat{\Pi}^{\mathrm{f}}_{\mathrm{c}\,1}$, as shown in Fig. 4.3.

Remark 4.9. When $\hat{\Pi}_{\mathrm{p}}$ is port-Hamiltonian, equation (4.20) translates into

$$\begin{aligned} \nabla H^{\top}(x) \left[F^{\top}(x) \nabla H_{\mathrm{a}}(x) + g(x) \mathring{u}_{\mathrm{c}}(x) \right] &= 0 \\ g^{\top}(x) \nabla H_{\mathrm{a}}(x) &= 0 \ . \end{aligned} \tag{4.22}$$

A particular instance of (4.22) is $F^{\top}(x) \nabla H_{\mathrm{a}}(x) = -g(x) \mathring{u}_{\mathrm{c}}(x)$, in which case we have

$$\begin{pmatrix} F^{\top}(x) \\ g^{\top}(x) \end{pmatrix} \nabla H_{\mathrm{a}}(x) = \begin{pmatrix} -g(x) \mathring{u}_{\mathrm{c}}(x) \\ 0 \end{pmatrix} \ . \tag{4.23}$$

It is easy to see that (4.23) is equivalent to

$$\begin{pmatrix} F(x) \\ g^{\top}(x) \end{pmatrix} \nabla H_{\mathrm{a}}(x) = \begin{pmatrix} g(x) \mathring{u}_{\mathrm{c}}(x) \\ 0 \end{pmatrix} \tag{4.24}$$

(see the proof in appendix B.2), which can be further simplified as

$$\begin{pmatrix} g^{\perp}(x)F(x) \\ g^{\top}(x) \end{pmatrix} \nabla H_{\mathrm{a}}(x) = 0$$

by setting $\mathring{u}_{\mathrm{c}}(x) = g^{+}(x)F(x)\nabla H_{\mathrm{a}}(x)$.

4.2.1 Stabilization and the dissipation obstacle

When PBC is used for stabilization of an equilibrium $x_\star \in \mathcal{E}_x$, the storage function is typically used as a Lyapunov function, so it is required that

$$x_\star = \arg\min H_{\mathrm{t}}(x) \; . \tag{4.25}$$

Since $\nabla H_{\mathrm{t}\star} \triangleq \nabla H_{\mathrm{t}}(x_\star) = 0$ is a necessary condition for (4.25), it is clear from (4.13b), that the flow y_{t} must be zero at the equilibrium, that is, $h_{\mathrm{t}\star} \triangleq h_{\mathrm{t}}(x_\star) = 0$. Likewise, from equation (4.13a), we also have that the dissipation at the equilibrium must be zero, that is $d_{\mathrm{t}\star} \triangleq d_{\mathrm{t}}(x_\star) = 0$. Energy balancing controllers, being flow and dissipation-preserving, impose

$$d_\star = 0 \quad \text{and} \quad h_\star = 0 \tag{4.26}$$

to the *open-loop* system. This is the so-called *dissipation obstacle* (refer to [44] for a discussion on this issue).

Example, set-point regulation of fully-actuated robotic manipulators

Consider a fully actuated mechanical system

$$\begin{aligned} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} &= \begin{pmatrix} 0 & I \\ -I & -R \end{pmatrix} \nabla H(q,p) + \begin{pmatrix} 0 \\ I \end{pmatrix} u \; , \quad R > 0 \\ y &= \nabla_p H(q,p) = \dot{q} \; , \end{aligned}$$

where

$$H(q,p) = \frac{1}{2} p^{\top} M^{-1}(q) p + P(q) \; , \quad M(q) > 0 \; . \tag{4.27}$$

According to (3.56), $\mathcal{E}_x$ is equal to the set of points (q,p) satisfying

$$\begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & -R \end{pmatrix} \nabla H(q,p) = \begin{pmatrix} 0 & I \end{pmatrix} \nabla H(q,p) = \nabla_p H(q,p) = 0 \; . \tag{4.28}$$

Since $\nabla_p H = M^{-1}(q)p$ and $M(q)$ is non-singular, equation (4.28) is equivalent to $p = 0$, so any point of the form $(q, 0)$ is an assignable equilibrium. This is consistent with physical intuition (for any particular equilibrium state the momentum is zero). For any $(q_\star, 0)$, we have that $\nabla_p H_\star = 0$, so the dissipation and flow functions evaluate at

$$d_\star = \nabla_p H_\star^{\top} R \nabla_p H_\star = 0 \quad \text{and} \quad h_\star = \nabla_p H_\star = 0 \; .$$

Thus, the necessary conditions for energy balance are satisfied.

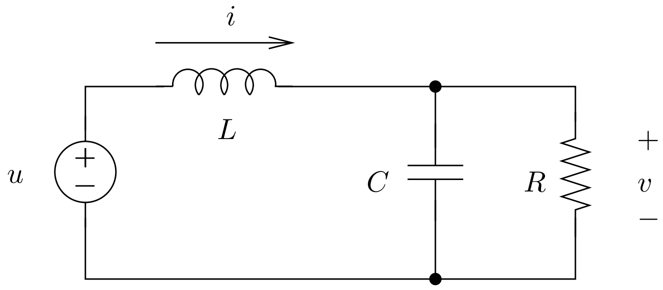

Figure 4.4: A simple, linear circuit.

Notice that the kinetic energy in (4.27) is quadratic in p and already has a unique global minimum at $p = 0$. This motivates the search for a controller that renders the system passive with Hamiltonian

$$H_{\mathrm{t}}(q,p) = \frac{1}{2}p^\top M(q)p + P_{\mathrm{t}}(q) \ , \tag{4.29}$$

that is, a controller that shapes the potential energy only. In (4.29), P_{t} is any $\mathcal{C}^1$ function having a unique global isolated minimum at $q = q_\star$, e.g.,

$$P_{\mathrm{t}}(q) = \frac{1}{2}(q - q_\star)^\top K_{\mathrm{p}}(q - q_\star) \ , \quad K_{\mathrm{p}} > 0 \ .$$

The aggregated Hamiltonian is $H_{\mathrm{a}}(q,p) = H_{\mathrm{t}}(q,p) - H(q,p) = P_{\mathrm{t}}(q) - P(q)$. Notice that H_{a} is a function of q only, so

$$g^\top \nabla H_{\mathrm{a}}(q,p) = \begin{pmatrix} 0 & I \end{pmatrix} \begin{pmatrix} \nabla(P_{\mathrm{t}}(q) - P(q)) \\ 0 \end{pmatrix} = 0$$

and equation (4.20b) is satisfied. Then, according to proposition 4.7, to devise an energy balancing controller, it suffices to find a $\mathring{u}_{\mathrm{c}}(x)$ that satisfies (4.20a), which in this case corresponds to

$$\begin{aligned} \nabla_p H^\top(q,p)\mathring{u}_{\mathrm{c}}(q,p) &= -\begin{pmatrix} \nabla\left(P_{\mathrm{t}}(q) - P(q)\right)^\top & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & R \end{pmatrix} \nabla H(q,p) \\ &= -\nabla\left(P_{\mathrm{t}}(q) - P(q)\right)^\top \nabla_p H(q,p) \ . \end{aligned}$$

Clearly, $\mathring{u}_{\mathrm{c}}(q) = -\nabla\left(P_{\mathrm{t}}(q) - P(q)\right)$ solves the control problem — this is the celebrated controller by Takegaki and Arimoto [55]. The behavior of the controller is then

$$\hat{\mathcal{B}}_{\mathrm{c}} = \left\{ (u_{\mathrm{c}}, y_{\mathrm{c}}, u_{\mathrm{t}}, y_{\mathrm{t}}, q, p) \mid u_{\mathrm{c}} = -\nabla\left(P_{\mathrm{t}}(q) - P(q)\right) + u_{\mathrm{t}}, \ y_{\mathrm{t}} = -y_{\mathrm{c}} \right\} \ .$$

Example, a simple parallel circuit

Consider a linear circuit composed of a voltage source in series with an inductor and a resistor connected in parallel with a capacitor, like the one shown in Fig. 4.4. The voltage over the capacitor and the current through the inductor are, respectively,

$$v = \frac{q}{C} \quad \text{and} \quad i = \frac{\phi}{L}\,, \quad C, L > 0\,.$$

The variable q denotes the charge of the capacitor and ϕ the inductor's flux. The circuit can be modeled as a port-Hamiltonian system

$$\begin{aligned} \begin{pmatrix} \dot{q} \\ \dot{\phi} \end{pmatrix} &= \begin{pmatrix} -\frac{1}{R} & 1 \\ -1 & 0 \end{pmatrix} \nabla H(q,\phi) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \frac{\phi}{L} = i\,, \end{aligned}$$

with $R > 0$ and

$$H(q,\phi) = \frac{1}{2C}q^2 + \frac{1}{2L}\phi^2\,.$$

Since L and C are positive, the Hamiltonian is bounded from below and the system is passive.

The set $\mathcal{E}_x$ is defined through the equation

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{R} & 1 \\ -1 & 0 \end{pmatrix} \nabla H(q,\phi) = \begin{pmatrix} -\frac{1}{R} & 1 \end{pmatrix} \begin{pmatrix} \frac{q}{C} \\ \frac{\phi}{L} \end{pmatrix} = 0\,,$$

so $\mathcal{E}_x = \left\{ \left(q, \frac{L}{RC}q\right) \;\middle|\; q \in \mathbb{R} \right\}$. The circuit's dissipation is given by the function

$$d(q,\phi) = -\begin{pmatrix} \nabla_q H \\ \nabla_\phi H \end{pmatrix}^\top \begin{pmatrix} -\frac{1}{R} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \nabla_q H \\ \nabla_\phi H \end{pmatrix} = \frac{1}{R}\left(\frac{q}{C}\right)^2\,. \tag{4.30}$$

From (4.26) and (4.30) we infer that, unfortunately, the only assignable equilibrium that can be stabilized with an energy balancing controller is the trivial equilibrium $(0,0)$.

This example attests that energy balance, although simple (cf. remark 4.8 and the example on p. 69), imposes rather stringent restrictions, even to 'well behaved' linear *passive* systems.

4.3 Relative passivity

If $\hat{\Pi}_\mathrm{p}$ is already passive, it is natural to expect a simple solution to the problem of stabilization. Indeed, let $(u_\star, y_\star)$ be the pair of port variables that correspond to the desired equilibrium. Under some additional conditions, the problem can be solved by establishing passivity with respect to the pair $(u - u_\star, y - y_\star)$, that is, *relative* to $(u_\star, y_\star)$ [15].

Proposition 4.10. *Consider a plant $\hat{\Pi}_\mathrm{p}$ satisfying assumption 4.1 and let $x_\star \in \mathcal{E}_x$ be an assignable equilibrium to be stabilized. Suppose that* *(i) The matrix g is constant.*

(ii) $H(x)$ is strictly convex.

(iii) $(\nabla H(x) - \nabla H_\star)^\top (f(x) - f(x_\star)) \leq 0$.

Set a constant $\mathring{u}_{\mathrm{c}}(x) = u_\star$ *with ($u_\star$ as in (3.57)) and a* constant $\mathring{y}_{\mathrm{t}}(x) = -h_\star$*. Then, the controller (4.5) is passivity-based with*

$$H_{\mathrm{t}}(x) = H(x) - (x - x_\star)^\top \nabla H_\star - H_\star \ , \quad H_\star \triangleq H(x_\star) \ . \tag{4.31}$$

Moreover, H_{t} is radially unbounded and has a strict and unique minimum at $x_\star$.

Proof: We will prove passivity of $\hat{\Pi}_{\mathrm{t}}$ by establishing (4.13) with

$$f_{\mathrm{t}}(x) \ = \ f(x) + g u_\star \tag{4.32a}$$
$$h_{\mathrm{t}}(x) \ = \ g^\top \nabla H(x) - g^\top \nabla H_\star \tag{4.32b}$$

(see lemma 4.4). From (4.31), let us compute

$$\nabla H_{\mathrm{t}}(x) = \nabla H(x) - \nabla H_\star \ . \tag{4.33}$$

From the definition of $\mathcal{E}_x$ (see p. 62), it follows that $f(x_\star) + g u_\star = 0$ and, according to (4.32a),

$$f_{\mathrm{t}}(x) = f(x) - f(x_\star) \ . \tag{4.34}$$

Equations (4.33) and (4.34), together with condition (iii) imply (4.13a). From (4.32b) and (4.33), one obtains (4.13b).

Now, take any $a, b \in \mathbb{X}$, $\lambda \in \mathbb{R}$ and compute

$$H_{\mathrm{t}}\left(a\lambda + (1-\lambda)b\right) = H\left(a\lambda + (1-\lambda)b\right) - \left(a\lambda + (1-\lambda)b - x_\star\right)^\top \nabla H_\star - H_\star \tag{4.35}$$

and

$$\lambda H_{\mathrm{t}}(a) + (1-\lambda) H_{\mathrm{t}}(b) = \lambda \left[H(a) - (a - x_\star)\nabla H_\star - H_\star\right] + \\ + (1-\lambda)\left[H(b) - (b - x_\star)\nabla H_\star - H_\star\right] \ .$$

After simplifying terms in the previous equation, one obtains

$$\lambda H_{\mathrm{t}}(a) + (1-\lambda) H_{\mathrm{t}}(b) = \\ \lambda H(a) + (1-\lambda) H(b) + (1-\lambda)\left[H(b) - (b - x_\star)\nabla H_\star - H_\star\right] \tag{4.36}$$

By substracting (4.36) from (4.35), one arrives at

$$H_{\mathrm{t}}\left(a\lambda + (1-\lambda)b\right) - \left[\lambda H_{\mathrm{t}}(a) + (1-\lambda) H_{\mathrm{t}}(b)\right] = \\ H\left(a\lambda + (1-\lambda)b\right) - \left[\lambda H(a) + (1-\lambda) H(b)\right] \ ,$$

which shows that (strict) convexity of $H(x)$ is equivalent to (strict) convexity of $H_{\mathrm{t}}(x)$. This, and $\nabla H_{\mathrm{t}\star} = 0$, are sufficient conditions for $H_{\mathrm{t}\star} \triangleq H_{\mathrm{t}}(x_\star) = 0$ to be a strict and unique minimum.

Recall that H_{t} is radially unbounded if

$$\lim_{|x|\to\infty} H_{\mathrm{t}}(x) \to \infty \; . \tag{4.37}$$

Denote by $S(x_\star, r) \triangleq \{x \in \mathbb{X} \mid |x - x_\star| = r\}$ the sphere centered at $x_\star$ and having radius r. To prove (4.37), we will show that, for all $r \geq 1$,

$$a \in S(x_\star, r) \quad \Longrightarrow \quad \alpha r \leq H_{\mathrm{t}}(a) \; ,$$

where $\alpha = \min_{x\in S(x_\star,1)} H_{\mathrm{t}}(x)$. Let a be any point in $S(x_\star, r)$. Let b be the point found at the intersection between $S(x_\star, 1)$ and the convex combination of $x_\star$ and a,

$$b = (1-\lambda)x_\star + \lambda a \; , \quad \lambda = \frac{1}{r} \leq 1 \; ,$$

that is,

$$b = x_\star + \frac{1}{r}(a - x_\star) \; .$$

Since $b \in S(x_\star, 1)$ and H_{t} is strictly convex, we have that

$$\alpha \leq H_{\mathrm{t}}(b) \leq \left(1 - \frac{1}{r}\right) H_{\mathrm{t}}(x_\star) + \frac{1}{r} H_{\mathrm{t}}(a) = \frac{1}{r} H_{\mathrm{t}}(a) \; ,$$

which implies $\alpha r \leq H_{\mathrm{t}}(a)$. ∎

Invoking Kalman–Yakubovich–Popov's lemma [60], it is easy to establish that all passive linear time invariant systems

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

satisfy conditions (i), (ii) and (iii). Indeed, if $H(x) = \frac{1}{2}x^\top P x$, with $P \in \mathbb{R}^{n\times n}$, $P = P^\top > 0$, is a storage function for the system, then condition (iii) amounts to

$$(Px - Px_\star)^\top (Ax - Ax_\star) = (x - x_\star)^\top PA(x - x_\star) \leq 0 \; ,$$

which is always satisfied since $PA + A^\top P \leq 0$ for any passive system. The storage function H_{t} is then

$$H_{\mathrm{t}}(x) = \frac{1}{2}x^\top P x - (x - x_\star)^\top P x_\star - \frac{1}{2}x_\star^\top P x_\star = \frac{1}{2}(x - x_\star)^\top P(x - x_\star) \; ,$$

that is, $H_{\mathrm{t}}(x) = H(x - x_\star)$.

In addition to linear time invariant systems, condition (iii) is satisfied by any port-Hamiltonian system with constant interconnection and dissipation matrices. This is because

$$(\nabla H(x) - \nabla H(x_\star))^\top (J - R) (\nabla H(x) - \nabla H(x_\star)) = \\ - (\nabla H(x) - \nabla H(x_\star))^\top R (\nabla H(x) - \nabla H(x_\star)) \leq 0 \,.$$

Interestingly, mechanical systems with holonomical constraints and the RLC networks discussed in section 3.3.3 are covered by the class of port-Hamiltonian systems with constant matrices.

4.3.1 A large class of electric circuits globally stabilizable by proportional plus integral control

In this subsection we study the problem of global flow regulation of RLC networks described by (3.41). We will exploit their port-Hamiltonian structure to show that, under some suitable additional conditions on their characteristic functions, the problem can be solved with a simple PI controller. We will not use proposition 4.10 because (3.41) contains feedthrough terms, but the idea of relative passivity is the same as before.

It is widely known that RLC networks consisting of passive inductors, capacitors and resistors are passive [14]. For circuits described by (3.41), this can be readily seen using (3.42), $y_{\mathrm{R}}^\top u_{\mathrm{R}} \geq 0$, and the fact that H is non-negative for passive inductors and capacitors. In most practical applications of RLC circuits the control objective is not to drive the flow to zero but to a desired value $y_{1\star} \neq 0$ (cf. the example of p. 71). In this case it is natural to look for passivity relative to $u_{1\star}$ and $y_{1\star}$, where $u_{1\star}$, $y_{1\star}$ and $x_\star$ satisfy

$$0 = J\nabla H_\star + g_1 u_{1\star} + g_2 u_{\mathrm{R}}(y_{\mathrm{R}\star}) \tag{4.38a}$$

$$y_{1\star} = g_1 \nabla H_\star + h_{11} u_{1\star} + h_{12} u_{\mathrm{R}}(y_{\mathrm{R}\star}) \tag{4.38b}$$

$$y_{\mathrm{R}\star} = -g_2 \nabla H_\star + h_{12}^\top u_{1\star} + h_{22} u_{\mathrm{R}}(y_{\mathrm{R}\star}) \tag{4.38c}$$

for some $y_{\mathrm{R}\star}$.

To establish relative passivity, we need to strengthen assumption 3.17, which was made for modelling purposes only.

Assumption 4.11. *The characteristics of the inductors and capacitors are strictly increasing and continuously differentiable. Those of the resistors are monotone non-decreasing*[1]*.*

Remark 4.12. Since $\nabla H = \mathrm{col}(\hat{i}_\phi(\phi), \hat{v}_{\mathrm{q}}(q))$ and the inductors and capacitors characteristics are strictly increasing, we know that H, and consequently H_{t} (4.31), are strictly convex [23, p.185].

[1] Recall that $f : \mathbb{R}^n \to \mathbb{R}^n$ is monotone non-decreasing if $(a-b)^\top (f(a) - f(b)) \geq 0$, $\forall\, a, b \in \mathbb{R}^n$. It is strictly increasing if the inequality is strict whenever $a \neq b$.

Define $\tilde{u}_\mathrm{R} = u_\mathrm{R} - u_{\mathrm{R}\star}$ and $\tilde{y}_\mathrm{R} = y_\mathrm{R} - y_{\mathrm{R}\star}$. The constant controller $\Pi_\mathrm{c} = (\mathbb{W} \times \mathbb{W}_\mathrm{t}, \mathcal{B}_\mathrm{c})$, with

$$\mathcal{B}_\mathrm{c} = \left\{ (u_\mathrm{c}, y_\mathrm{c}, u_\mathrm{t}, y_\mathrm{t}) \,\middle|\, u_\mathrm{c} = u_{1\star} + u_\mathrm{t},\ y_\mathrm{t} = -y_{1\star} - y_\mathrm{c} \right\} ,$$

produces a $\hat{\Pi}_\mathrm{t}$ governed by the equations

$$\begin{aligned} \dot{x} &= J(\nabla H - \nabla H_\star) + g_1 u_\mathrm{t} + g_2 \tilde{u}_\mathrm{R} && (4.39a) \\ y_\mathrm{t} &= g_1^\top (\nabla H - \nabla H_\star) + h_{11} u_\mathrm{t} + h_{12} \tilde{u}_\mathrm{R} && (4.39b) \\ \tilde{y}_\mathrm{R} &= -g_2^\top (\nabla H - \nabla H_\star) + h_{12}^\top u_\mathrm{t} + h_{22} \tilde{u}_\mathrm{R} , && (4.39c) \end{aligned}$$

where the first line is due to (4.38a). The derivative of H_t (4.31) is then:

$$\begin{aligned} \dot{H}_\mathrm{t}(x) &= (\nabla H - \nabla H_\star)^\top (J(\nabla H - \nabla H_\star) + g_1 u_\mathrm{t} + g_2 \tilde{u}_\mathrm{R}) \\ &= y_\mathrm{t}^\top u_\mathrm{t} - \tilde{y}_\mathrm{R}^\top \tilde{u}_\mathrm{R} . && (4.40) \end{aligned}$$

Finally, the monotonicity of the resistors leads to $\dot{H}_\mathrm{t} \leq y_\mathrm{t}^\top u_\mathrm{t}$.

Recall from section 3.2.1 that the flow of a passive system can be regulated to zero with a proportional feedback. In the present context, this suggests that we close the port with $u_\mathrm{t} = -K_\mathrm{P} y_\mathrm{t}$. Its implementation $u_\mathrm{t} = -K_\mathrm{P}(y_1 - y_{1\star}) + u_{1\star}$ clearly requires the exact value of the feed-through term $u_{1\star}$. The latter, obtained from the solution of (4.38), requires the precise knowledge of the system parameters, rendering the controller highly non-robust. This problem can be surmounted by the use of an integral action, as shown in the following theorem.

Theorem 4.13. *Consider network (3.41). Under Assumption 4.11, the PI controller*

$$\begin{aligned} \dot{\xi} &= -(y_1 - y_{1\star}) && (4.41a) \\ u_1 &= K_\mathrm{I} \xi - K_\mathrm{P}(y_1 - y_{1\star}) && (4.41b) \end{aligned}$$

with $K_\mathrm{I} = K_\mathrm{I}^\top > 0$ and $K_\mathrm{P} = K_\mathrm{P}^\top > 0$ ensures that for all initial conditions, y_1 converges asymptotically to $y_{1\star}$, that is,

$$\lim_{t \to \infty} y_1 = y_{1\star} . \qquad (4.42)$$

Moreover, it globally stabilizes the equilibrium point $(x_\star, \xi_\star)$[2], where $\xi_\star \triangleq K_\mathrm{I}^{-1} u_{1\star}$. If, in addition, the closed-loop system (3.41), (4.41) satisfies the detectability condition

$$y_1 - y_{1\star} \equiv 0 \quad \Longrightarrow \quad \lim_{t \to \infty} (x, \xi) = (x_\star, \xi_\star) , \qquad (4.43)$$

then the equilibrium is globally asymptotically stable.

Proof: It is straightforward to see that the (shifted) candidate Lyapunov function

$$V(x, \xi) = H_\mathrm{t}(x) + \frac{1}{2}(\xi - \xi_\star)^\top K_\mathrm{I} (\xi - \xi_\star) \qquad (4.44)$$

[2] By global stability we mean stability in the sense of Lyapunov plus boundedness of the solutions for every initial condition.

is indeed a Lyapunov function:

$$\begin{aligned}\dot{V}(x,\xi) &= \dot{H}_{\mathrm{t}}(x) + \dot{\xi}^{\top} K_{\mathrm{I}}(\xi - \xi_\star) \\ &= y_{\mathrm{t}}^{\top} u_{\mathrm{t}} - \tilde{y}_{\mathrm{R}}^{\top}\tilde{u}_{\mathrm{R}} - y_{\mathrm{t}}^{\top} K_{\mathrm{I}}(\xi - \xi_\star) \\ &= y_{\mathrm{t}}^{\top} u_{\mathrm{t}} - \tilde{y}_{\mathrm{R}}^{\top}\tilde{u}_{\mathrm{R}} - y_{\mathrm{t}}^{\top}(u_{\mathrm{t}} + K_{\mathrm{P}} y_{\mathrm{t}}) = -\tilde{y}_{\mathrm{R}}^{\top}\tilde{u}_{\mathrm{R}} - y_{\mathrm{t}}^{\top} K_{\mathrm{I}} y_{\mathrm{t}} \le 0 \,. \end{aligned} \tag{4.45}$$

The second equation is due to (4.40) and (4.41a), while the third is due to (4.41b) and the definition of $\xi_\star$. Non-positivity is due to the monotonicity of the resistors and the positive definiteness of K_{I}. It follows then from standard Lyapunov theory that $(x_\star, \xi_\star)$ is stable. Since V is also radially unbounded (see proposition 4.10), the solutions remain bounded for any initial condition [28, p. 124]. From LaSalle's invariance principle [29, p. 66] we conclude that y_1 converges to $y_{1\star}$ and that the detectability condition (4.43) leads to asymptotic stability. ■

Example, a nonlinear RLC circuit (continued)

Consider again the example of p. 56. The derivatives of the inductor and capacitor characteristics are, respectively,

$$\frac{I_{\mathrm{s}_\phi}}{\delta_\phi}\,\mathrm{sech}^2\left(\frac{\phi}{\delta_\phi}\right), \quad \frac{1}{C_1} \quad \text{and} \quad \frac{1}{C_2} \,.$$

Those of the resistive elements are

$$R, \quad \frac{I_{\mathrm{s}_1}}{V_{\mathrm{T}}}\exp\left(\frac{v_{\mathrm{vc}_1}}{V_{\mathrm{T}}}\right) \quad \text{and} \quad G \,.$$

Assumption 4.11 is verified, since strict positivity of the derivatives implies strict monotonicity.

Computation of the equilibrium effort-flow pairs requires to solve (4.38)

$$\begin{aligned} y_{2\star} &= \begin{pmatrix} i_{\mathrm{ic}\,\star} \\ v_{\mathrm{vc}_1\,\star} \\ v_{\mathrm{vc}_2\,\star} \end{pmatrix} = \begin{pmatrix} y_{1\,\star} \\ V_{\mathrm{T}} \ln\left(y_{1\,\star}/I_{\mathrm{s}_1} + 1\right) \\ y_{1\,\star}/G \end{pmatrix} \\ x^\star &= \begin{pmatrix} \phi_\star \\ q_{1\,\star} \\ q_{2\,\star} \end{pmatrix} = \begin{pmatrix} \delta_\phi \,\mathrm{artanh}\left(y_{1\,\star}/I_{\mathrm{s}_\phi}\right) \\ y_{1\,\star} C_1/G \\ y_{2\,\star} C_2/G \end{pmatrix} \\ u_{1\,\star} &= v_{\mathrm{vs}} = v_{\mathrm{vc}_1\,\star} + y_{1\,\star}\left(\frac{1}{G} + R\right) \,, \end{aligned}$$

but we will not need to, since we will use the PI controller (4.41).

Suppose that the system parameters are

$$I_{\mathrm{s}_\phi} = 30\,\mathrm{mA}, \quad \delta_\phi = 250\,\mu\mathrm{Wb}, \quad C_1 = C_2 = 2\,\mu\mathrm{F} \tag{4.46}$$

and

$$R = 100\,\Omega, \quad I_{\mathrm{s}_1} = 0.1\,\mu\mathrm{A}, \quad G = 1\,\mathrm{mA/V} \,. \tag{4.47}$$

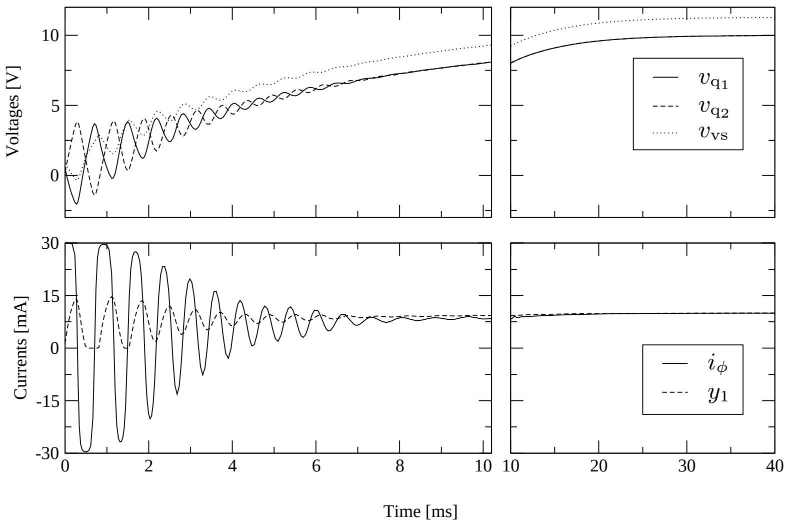

Figure 4.5: Simulated response of the regulated circuit.

Suppose, further, that we want to stabilize the diode's current at $y_{1\star} = 10$mA. For an initial condition

$$x(0) = \begin{pmatrix} 1\,\text{mWb} \\ 1\,\mu\text{C} \\ 0.5\,\mu\text{C} \end{pmatrix} , \tag{4.48}$$

the PI controller (4.41), with $K_\text{I} = 500\,\text{V/mA}\cdot\text{s}$ and $K_P = 100\,\Omega$, produces the set of currents and voltages shown in Fig. 4.5. It can be seen that both, y_1 and u_1 converge to their preset values of 10 mA and 11.3 V respectively.

Counter example

To underscore the importance of the monotonicity of the resistors (as opposed to the weaker condition $y_\text{R}^\top u_\text{R} \geq 0$), consider the circuit shown in Fig. 4.6. The components within the dashed box are modeled as a single voltage controlled resistive port. The remaining resistor is also considered to be voltage controlled. The graph of the circuit is thus partitioned as

$$v_\mathcal{T} = \begin{pmatrix} v_\text{q} \\ v_\text{vs} \end{pmatrix}, \quad v_\mathcal{L} = \begin{pmatrix} v_{\text{vc}_1} \\ v_{\text{vc}_2} \end{pmatrix} \tag{4.49}$$

and the equations of the circuit are

$$\dot{q} = \hat{i}_{\text{vc}_1}(v_{\text{vc}_1}) - \hat{i}_{\text{vc}_2}(v_{\text{vc}_2}) \tag{4.50a}$$

$$i_\text{vs} = \hat{i}_{\text{vc}_1}(v_{\text{vc}_1}) \tag{4.50b}$$

$$\begin{pmatrix} v_{\text{vc}_1} \\ v_{\text{vc}_2} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \hat{v}_\text{q}(q) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_\text{vs} \,. \tag{4.50c}$$

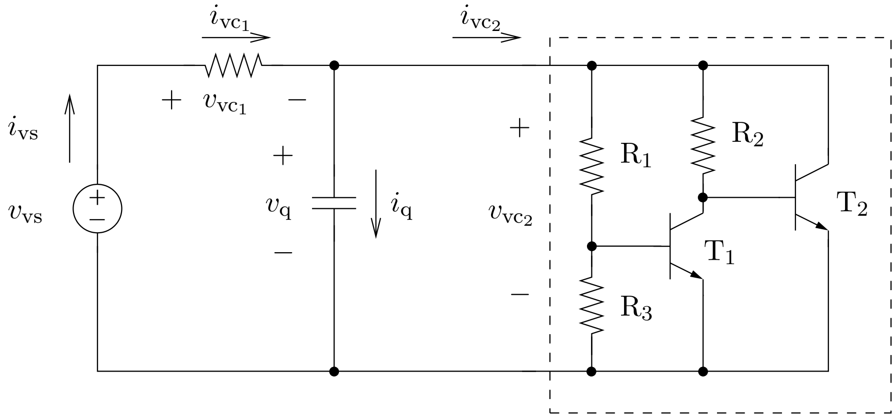

Figure 4.6: A nonlinear RLC circuit with a non-monotonic resistance.

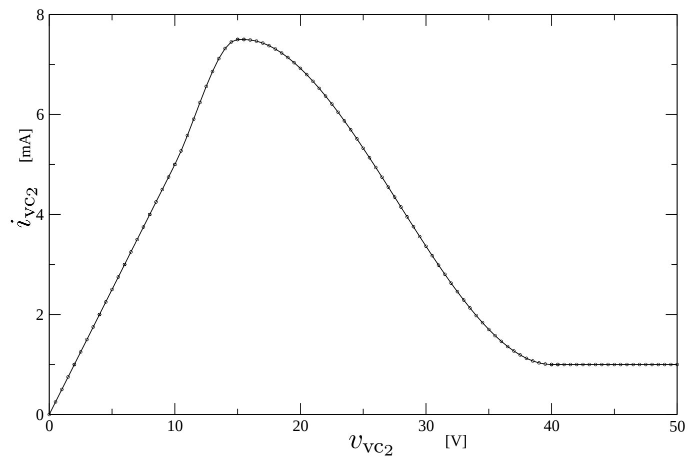

Figure 4.7: Characteristic of the N-type negative differential resistance.

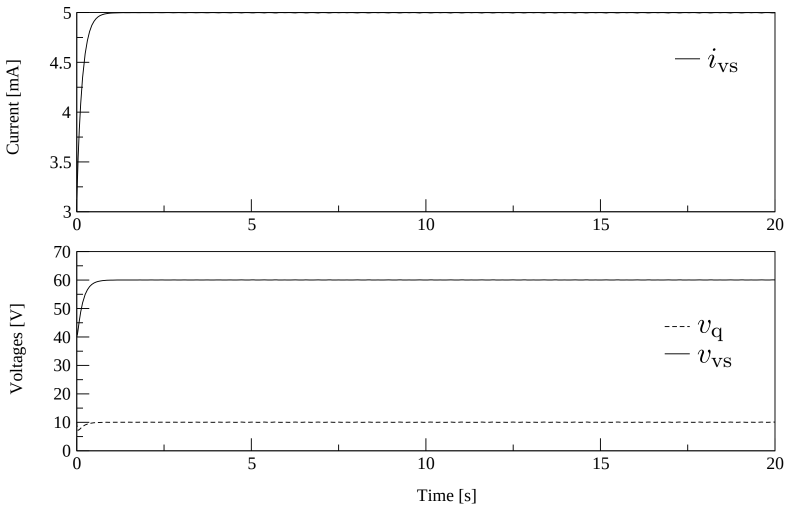

Figure 4.8: Simulated response with initial condition $\hat{v}_q(q(0)) = 10$ V

The single resistor and the capacitor are assumed to be linear with characteristics

$$i_{vc_1} = Gv_{vc_1} \quad \text{and} \quad v_q = \frac{1}{C}q \; . \tag{4.51}$$

According to [56], the nonlinear port exhibits an N-type negative differential resistance (NDR), like the one shown in Fig. 4.7. Suppose that

$$G = 10\,\text{mA/V}, \quad C = 10\,\mu\text{F} \tag{4.52}$$

and that the transistors and resistors are such that measurements produce the set of points shown in Fig. 4.7. Suppose that we want to set the flow $y_{1\star} = i_{vs\star}$ at 5 mA, so we construct the controller (4.41) with

$$K_P = 2\frac{1}{G} \quad \text{and} \quad K_I = 2C \; . \tag{4.53}$$

For an initial condition of 0.1 mC, the voltage of the capacitor equals 10 V. Since at this voltage the differential resistance is positive, we can expect the state of the circuit to converge to the desired value (see Fig. 4.8). On the other hand, an initial condition of 0.5 mC sets the voltage of the capacitor within the NDR zone. It can be seen (Fig. 4.9) that the flow of the circuit does not converge to the desired value. Moreover, the state and the control diverge. Thus, the closed-loop circuit is locally stable, but not globally.

4.4 Interconnection and damping assignment

It is clear that dissipation should be modified to stabilize, with passivity-based controllers, systems that dissipate energy at the equilibrium. A candidate dissipation function d_t which is compatible with the requirement $d_{t\star} = 0$ (and thus overcomes the

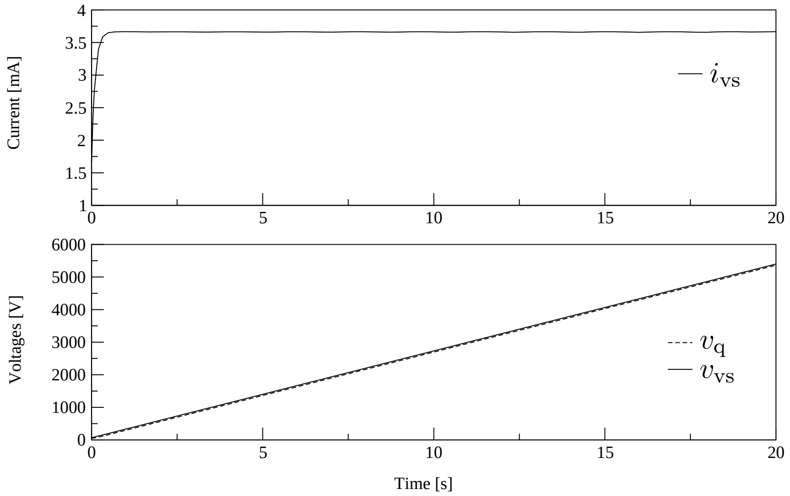

Figure 4.9: Simulated response with initial condition $\hat{v}_{\mathrm{q}}(q(0)) = 50$ V

dissipation obstacle) is given in the following proposition, where the well-known *interconnection and damping assignment* PBC [45] is re-derived.

Proposition 4.14. *Consider a controller* $\hat{\Pi}_{\mathrm{c}}$ *satisfying assumption 4.2 and a plant* $\hat{\Pi}_{\mathrm{p}}$ *satisfying assumption 4.1. Fix* H_{t} *and the dissipation function*

$$d_{\mathrm{t}}(x) = \nabla H_{\mathrm{t}}^{\top}(x) R_{\mathrm{t}}(x) \nabla H_{\mathrm{t}}(x) \tag{4.54}$$

with $R_{\mathrm{t}}(x) \in \mathbb{R}^{n \times n}$, $R_{\mathrm{t}}(x) = R_{\mathrm{t}}^{\top}(x) \geq 0$. *Then,*

(i) $\hat{\Pi}_{\mathrm{c}} \in$ *PBC if, and only if,*

$$\mathring{y}_{\mathrm{t}}(x) = g^{\top}(x) \nabla H_{\mathrm{a}}(x) \tag{4.55}$$

and

$$g(x)\mathring{u}_{\mathrm{c}}(x) = -f(x) - R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) + \gamma(x) \tag{4.56}$$

for some vector field γ *such that* $\nabla H_{\mathrm{t}}^{\top}(x)\gamma(x)$ *is identically zero.*

(ii) If $x_\star \in \mathcal{E}_x$ *satisfies (4.25), then* $\gamma_\star \triangleq \gamma(x_\star) = 0$.

(iii) For any skew-symmetric matrix $J_{\mathrm{t}}(x) \in \mathbb{R}^{n \times n}$, *the vector field*

$$\gamma(x) = J_{\mathrm{t}}(x) \nabla H_{\mathrm{t}}(x) \tag{4.57}$$

satisfies both restrictions: $\gamma_\star = 0$ *and* $\nabla H_{\mathrm{t}}^{\top}(x)\gamma(x) = 0$. *Furthermore,* $\hat{\Pi}_{\mathrm{t}}$ *is port-Hamiltonian with* $(u_{\mathrm{t}}, y_{\mathrm{t}}, x) \in \hat{\mathcal{B}}_{\mathrm{t}}$ *equivalent to*

$$\begin{aligned} \dot{x} &= F_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) + g(x)u_{\mathrm{t}} , \quad F_{\mathrm{t}}(x) \triangleq J_{\mathrm{t}}(x) - R_{\mathrm{t}}(x) \\ y_{\mathrm{t}} &= g^{\top}(x) \nabla H_{\mathrm{t}}(x) . \end{aligned}$$

Proof: For sufficiency of (i), assume (4.56) and premultiply by $\nabla H_{\mathrm{t}}^{\top}(x)$:

$$\nabla H_{\mathrm{t}}^{\top}(x)g(x)\mathring{u}_{\mathrm{c}}(x) = -\nabla H_{\mathrm{t}}^{\top}(x)f(x) - \nabla H_{\mathrm{t}}^{\top}(x)R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) \ ,$$

which is equivalent to

$$(\nabla H(x) + \nabla H_{\mathrm{a}}(x))^{\top}g(x)\mathring{u}_{\mathrm{c}}(x) = -\nabla H_{\mathrm{a}}^{\top}(x)f(x) - \nabla H^{\top}(x)f(x) - d_{\mathrm{t}}(x) \ .$$

By reordering terms one obtains

$$\nabla H^{\top}(x)g(x)\mathring{u}_{\mathrm{c}}(x) = -\nabla H_{\mathrm{a}}^{\top}(x)[f(x) + g(x)\mathring{u}_{\mathrm{c}}(x)] + d(x) - d_{\mathrm{t}}(x) \ ,$$

or

$$h^{\top}(x)\mathring{u}_{\mathrm{c}}(x) = -\nabla H_{\mathrm{a}}^{\top}(x)[f(x) + g(x)\mathring{u}_{\mathrm{c}}(x)] - d_{\mathrm{a}}(x) \ .$$

According to theorem 4.5, $\hat{\Pi}_{\mathrm{c}} \in$ PBC.

For necessity, assume that $\hat{\Pi}_{\mathrm{c}} \in$ PBC (so (4.9a) holds) and notice that the aggregated dissipation is

$$d_{\mathrm{a}}(x) = \nabla H_{\mathrm{t}}^{\top}(x)R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) + \nabla H(x)f(x) \ , \tag{4.59}$$

Then, substituting (4.59) in (4.9a) yields

$$\begin{aligned}&\nabla H^{\top}(x)g(x)\mathring{u}_{\mathrm{c}}(x) = \\ &\qquad - \nabla H_{\mathrm{a}}^{\top}(x)[f(x) + g(x)\mathring{u}_{\mathrm{c}}(x)] - \nabla H^{\top}(x)f(x) - \nabla H_{\mathrm{t}}^{\top}(x)R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) \ .\end{aligned}$$

The latter can be simplified as

$$0 = \nabla H_{\mathrm{t}}^{\top}(x)[g(x)\mathring{u}_{\mathrm{c}}(x) + f(x) + R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x)] \ ,$$

which implies the existence of a vector field γ satisfying (4.56) and

$$\nabla H_{\mathrm{t}}^{\top}(x)\gamma(x) = 0$$

for all $x \in \mathbb{X}$.

Regarding (ii), notice that for a control that satisfies (4.56), the drift $f_{\mathrm{t}}(x)$ of the controlled system is

$$f_{\mathrm{t}}(x) = f(x) - f(x) - R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) + \gamma(x) = -R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) + \gamma(x) \ . \tag{4.60}$$

If $x_\star$ is an equilibrium of $\hat{\Pi}_{\mathrm{t}}$, then $f_{\mathrm{t}}(x_\star) = 0$. This equation, together with $\nabla H_{\mathrm{t}\star} = 0$ and (4.60), implies that $\gamma_\star = 0$.

The first assertion of (iii) is proved by substituting $\nabla H_{\mathrm{t}_\star} = 0$ directly in (4.57),

$$\gamma_\star = J_{\mathrm{t}}(x_\star)\nabla H_{\mathrm{t}\star} = 0 \ .$$

Orthogonality follows from the fact that $\nabla H_{\mathrm{t}}^{\top}(x)J_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) = 0$ for any skew-symmetric matrix $J_{\mathrm{t}}(x)$. The second assertion of (iii) can be verified by replacing $\gamma(x) = J_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x)$ in (4.60) to obtain

$$\begin{aligned} f_{\mathrm{t}}(x) &= -R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) + J_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) \\ &= F_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) \ . \end{aligned}$$

■

Remark 4.15. Statement (iii) corresponds to the interconnection and damping assignment (IDA) strategy mentioned above. It amounts to finding a $\mathring{u}_{\mathrm{c}}$ and suitable R_{t}, J_{t} and H_{t} that satisfy

$$g(x)\mathring{u}_{\mathrm{c}}(x) = -f(x) + (J_{\mathrm{t}}(x) - R_{\mathrm{t}}(x))\nabla H_{\mathrm{t}}(x) \ ,$$

which is equivalent to

$$g^{\perp}(x)\left[(J_{\mathrm{t}}(x) - R_{\mathrm{t}}(x))\nabla H_{\mathrm{t}}(x) - f(x)\right] = 0 \tag{4.61}$$

and

$$\mathring{u}_{\mathrm{c}}(x) = g^{+}(x)\left[(J_{\mathrm{t}}(x) - R_{\mathrm{t}}(x))\nabla H_{\mathrm{t}}(x) - f(x)\right] \ .$$

There are several approaches to solving (4.61) [39]. One possibility is to fix a desired H_{t} and solve (4.61) as an algebraic equation on J_{t} and R_{t} [18]. Another option is to fix J_{t} and R_{t} and solve (4.61) as a PDE with H_{t} the unknown [45]. A third approach is to restrict the desired energy function to a certain class. For instance, for mechanical systems the sum of a potential and kinetic energy. Fixing the structure of the energy function yields a new PDE for its unknown terms and, at the same time, imposes some constraints on the interconnection and damping matrices [1].

Remark 4.16. For port-Hamiltonian systems, equation (4.61) translates into

$$g^{\perp}(x)F_{\mathrm{t}}(x)\nabla H_{\mathrm{a}}(x) = g^{\perp}(x)\left[F(x) - F_{\mathrm{t}}(x)\right]\nabla H(x) \ .$$

4.5 Explicit solutions of the algebraic equations

The content of this section is motivated by statement (i) of proposition 4.14. Since

$$\nabla H_{\mathrm{t}}^{\top}(x)\gamma(x) = 0 \ ,$$

the vector field γ does not play any role in the power balance

$$\dot{H}_{\mathrm{t}}(x) = y^{\top}u - \nabla H_{\mathrm{t}}^{\top}(x)R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}} \ ,$$

so we say that γ is *workless*. On the other hand, γ plays an important role in making (4.56) consistent, that is, it can be chosen so that

$$g^{\perp}(x)\left[f(x) + R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) - \gamma(x)\right] = 0 \ .$$

We pose the following algebraic problem.

Problem 4.17. Given f, g, R_{t} and H_{t}, find a continuous γ such that

$$\begin{pmatrix} g^{\perp}(x) \\ \nabla H_{\mathrm{t}}^{\top}(x) \end{pmatrix}\gamma(x) = \begin{pmatrix} g^{\perp}(x)[f(x) + R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x)] \\ 0 \end{pmatrix} \ . \tag{4.62}$$

The purpose of solving this problem is two-fold: To find explicit solutions for certain control problems and at the same time establish necessary conditions on H_{t} and R_{t}.

4.5.1 A set-valued approach

Let (X, d_X) and (Y, d_Y) be metric spaces and let 2^Y denote the set of all subsets of Y. By a set valued map Γ of X into Y we mean a map $\Gamma : X \to 2^Y$.

To address problem 4.17, define

$$A(x) \triangleq \begin{pmatrix} g^{\perp}(x) \\ \nabla H_{\mathrm{t}}^{\top}(x) \end{pmatrix} \quad \text{and} \quad \psi(x) \triangleq \begin{pmatrix} g^{\perp}(x)[f(x) + R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x)] \\ 0 \end{pmatrix} , \tag{4.63}$$

so that (4.62) can be written as

$$A(x)\gamma(x) = \psi(x) . \tag{4.64}$$

Notice that, if we look at the problem pointwise, we are confronted with a *linear* equation. Indeed, at each x there exists a (possibly empty) set of solutions

$$\Gamma(x) = \left\{ \gamma' \mid A(x)\gamma' = \psi(x) \right\} .$$

It is thus natural to take a set-valued approach to the problem, in which case the real challenge is to find a continuous map $\gamma(x)$ that satisfies

$$\gamma(x) \in \Gamma(x) \quad \forall\, x ,$$

that is, such that γ is a *continuous selection* of Γ.

4.5.2 Pointwise solutions of the linear equation

Since we will first look at the problem pointwise, we will temporarily drop the dependency on x.

Our goal is to find a g-inverse of A, defined as in (4.63). Notice that $A \in \mathbb{R}^{o \times n}$, with $o = n - m + 1$. Regarding the rank of A, we consider two possible cases:

(i) $\nabla H_{\mathrm{t}}^{\top}$ is independent of $g^{\perp}$. This implies $\mathrm{rank}(A) = \mathrm{rank}(g^{\perp}) + 1 = n - m + 1$.

(ii) $\nabla H_{\mathrm{t}}^{\top}$ depends linearly on $g^{\perp}$. This implies $\mathrm{rank}(A) = \mathrm{rank}(g^{\perp}) = n - m$.

Remark 4.18. Case (ii) implies that the closed-loop passive flow $y_{\mathrm{t}} = g^{\top}\nabla H_{\mathrm{t}}$ equals zero. This case covers of course the event $\nabla H_{\mathrm{t}} = 0$.

Lemma 4.19. *Consider case (i), i.e., $\nabla H_{\mathrm{t}}^{\top}$ is independent of $g^{\perp}$. A g-inverse of A is*

$$A^{-} = \begin{pmatrix} g_{+}^{\perp} - \frac{gg^{+}\nabla H_{\mathrm{t}}\nabla H_{\mathrm{t}}^{\top} g_{+}^{\perp}}{\nabla H_{\mathrm{t}}^{\top} gg^{+}\nabla H_{\mathrm{t}}} & \frac{gg^{+}\nabla H_{\mathrm{t}}}{\nabla H_{\mathrm{t}}^{\top} gg^{+}\nabla H_{\mathrm{t}}} \end{pmatrix} . \tag{4.65}$$

(Notice that A^{-} is well defined, since the independence between $\nabla H_{\mathrm{t}}^{\top}$ and $g^{\perp}$ prevents the product $\nabla H_{\mathrm{t}}^{\top} gg^{+}\nabla H_{\mathrm{t}}$ from being equal to zero.)

Proof: Direct computation gives

$$\begin{aligned} A^- A &= g_+^\perp g^\perp + \frac{-gg^+ \nabla H_\mathrm{t} \nabla H_\mathrm{t}^\top g_+^\perp g^\perp + gg^+ \nabla H_\mathrm{t} \nabla H_\mathrm{t}^\top}{\nabla H_\mathrm{t}^\top gg^+ \nabla H_\mathrm{t}} \\ &= g_+^\perp g^\perp + \frac{gg^+ \nabla H_\mathrm{t} \nabla H_\mathrm{t}^\top gg^+}{\nabla H_\mathrm{t}^\top gg^+ \nabla H_\mathrm{t}} \end{aligned}$$

(recall that $gg^+ = I - g_+^\perp g^\perp$). Finally,

$$\begin{aligned} AA^- A &= \begin{pmatrix} g^\perp g_+^\perp g^\perp + \frac{g^\perp gg^+ \nabla H_\mathrm{t}}{\nabla H_\mathrm{t}^\top gg^+ \nabla H_\mathrm{t}} \nabla H_\mathrm{t}^\top gg^+ \\ \nabla H_\mathrm{t}^\top g_+^\perp g^\perp + \frac{\nabla H_\mathrm{t}^\top gg^+ \nabla H_\mathrm{t}}{\nabla H_\mathrm{t}^\top gg^+ \nabla H_\mathrm{t}} \nabla H_\mathrm{t}^\top gg^+ \end{pmatrix} \\ &= \begin{pmatrix} g^\perp \\ \nabla H_\mathrm{t}^\top \end{pmatrix} = A \,, \end{aligned}$$

since $g^\perp g_+^\perp g^\perp = g^\perp$, $g^\perp g = 0$ and $I = gg^+ + g_+^\perp g^\perp$. ∎

Lemma 4.20. *Consider case (ii), i.e., $\nabla H_\mathrm{t}^\top$ is linearly independent on $g^\perp$. A g-inverse of A is*

$$A^- = \begin{pmatrix} g_+^\perp & 0 \end{pmatrix} \,.$$

Proof: Since $\nabla H_\mathrm{t}^\top$ depends linearly on $g^\perp$, there exists a row vector $\delta \in \mathbb{R}^{n-m}$ such that $\nabla H_\mathrm{t}^\top = \delta g^\perp$. Direct computation gives

$$A^- A = g_+^\perp g^\perp \,.$$

Finally,

$$AA^- A = \begin{pmatrix} g^\perp \\ \delta g^\perp \end{pmatrix} g_+^\perp g^\perp = \begin{pmatrix} g^\perp g_+^\perp g^\perp \\ \delta g^\perp g_+^\perp g^\perp \end{pmatrix} = \begin{pmatrix} g^\perp \\ \delta g^\perp \end{pmatrix} = \begin{pmatrix} g^\perp \\ \nabla H_\mathrm{t}^\top \end{pmatrix} = A \,.$$

∎

Now we can use theorem A.6 to investigate the consistency of (4.64).

Proposition 4.21. *Equation (4.64) is consistent if, and only if,*

$$y_\mathrm{t} = g^\top \nabla H_\mathrm{t} = 0 \quad \Longrightarrow \quad \nabla H_\mathrm{t}^\top f = -\nabla H_\mathrm{t}^\top R_\mathrm{t} \nabla H_\mathrm{t}^\top \leq 0 \,. \tag{4.66}$$

Proof: Let us begin with case (i). According to theorem A.6, equation (4.64) is consistent if, and only if, equation (A.4) holds. From lemma 4.19 one obtains

$$\begin{aligned} AA^- &= \begin{pmatrix} g^\perp \\ \nabla H_\mathrm{t}^\top \end{pmatrix} \begin{pmatrix} g_+^\perp - \frac{gg^+ \nabla H_\mathrm{t} \nabla H_\mathrm{t}^\top g_+^\perp}{\nabla H_\mathrm{t}^\top gg^+ \nabla H_\mathrm{t}} & \frac{gg^+ \nabla H_\mathrm{t}}{\nabla H_\mathrm{t}^\top gg^+ \nabla H_\mathrm{t}} \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix} \,, \end{aligned}$$

so $AA^{-}\psi = \psi$ for any ψ.

Consider now the case $y_{\rm t} = 0$, i.e., case (ii). For ease of notation, define

$$f_{\rm c} \triangleq f + R_{\rm t}\nabla H_{\rm t} \ ,$$

so that ψ in (4.63) can be written as

$$\psi = \begin{pmatrix} g^{\perp} f_{\rm c} \\ 0 \end{pmatrix} \ .$$

Using lemma 4.20 and verifying (A.4) we obtain

$$\begin{aligned} AA^{-} &= \begin{pmatrix} g^{\perp} \\ \delta g^{\perp} \end{pmatrix} \begin{pmatrix} g^{\perp}_{+} & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ \delta & 0 \end{pmatrix} \\ AA^{-}\psi &= \begin{pmatrix} I & 0 \\ \delta & 0 \end{pmatrix} \begin{pmatrix} g^{\perp} f_{\rm c} \\ 0 \end{pmatrix} = \begin{pmatrix} g^{\perp} f_{\rm c} \\ \delta g^{\perp} f_{\rm c} \end{pmatrix} = \begin{pmatrix} g^{\perp} f_{\rm c} \\ \nabla H_{\rm t}^{\top} f_{\rm c} \end{pmatrix} \ . \end{aligned}$$

Clearly, $AA^{-}\psi = \psi$ if, and only if,

$$\nabla H_{\rm t}^{\top} f_{\rm c} = 0 \ . \tag{4.67}$$

It follows then, from the definition of $f_{\rm c}$, that $\nabla H_{\rm t}^{\top} f = -\nabla H_{\rm t}^{\top} R_{\rm t} \nabla H_{\rm t}$. ■

If $H_{\rm t}$ is smooth, then the condition

$$y_{\rm t} = 0 \quad \Longrightarrow \quad \nabla H_{\rm t}^{\top} f < 0 \tag{4.68}$$

is sufficient for asymptotic stabilizability and an explicit controller can be constructed [52]. Although this result is undoubtedly strong, the search for an $H_{\rm t}$ that satisfies (4.68) with the *strict* inequality[3] can be rather difficult, even if physical information about the system is at hand. Take for instance the completely actuated mechanical systems discussed in p. 69. The derivative of $H_{\rm t}$ is

$$\dot{H}_{\rm t} = \nabla H_{\rm t}^{\top} f + \nabla H_{\rm t}^{\top} g u = -\dot{q}^{\top} R \dot{q} \leq 0 \ ,$$

which shows that

$$y_{\rm t} = 0 \quad \Longrightarrow \quad \nabla H_{\rm t}^{\top} f \leq 0 \ ,$$

so $H_{\rm t}$ is a *weak* control Lyapunov function, even if the system is asymptotically stable (this can be shown using a routine LaSalle argument).

On the other hand, if one allows non-smooth CLFs and one admits the use of relaxed controls, then condition (4.68) can be weakened to a non-strict inequality and can be shown to be also necessary (in this case we only talk about Lyapunov stability without asymptotic convergence). Unfortunately, there is no explicit construction of the controller (see [53] for details).

The consistency of (4.64) is only necessary for stabilizability, since we still need to find a continuous solution of the equation. In section 4.5.3 we provide an extra condition which

[3] In that case $H_{\rm t}$ is called a strict control Lyapunov function (CLF).

is sufficient for stabilizability and for writing an explicit solution (see theorem 4.24). In this respect, the contents of this section lies in between the results presented in [52] and [53].

Let us apply theorem A.6 in order to construct the set-valued solution of (4.64).

Assumption 4.22.

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} , \quad g = \begin{pmatrix} 0 \\ I \end{pmatrix} , \quad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \quad \textit{and} \quad f_{\mathrm{c}} = \begin{pmatrix} f_{\mathrm{c}\,1}(x) \\ f_{\mathrm{c}\,2}(x) \end{pmatrix} .$$

This assumption is made only to alleviate notation, the general case can be carried out in a similar fashion. The g-inverses and annihilator are then

$$g^{+} = \begin{pmatrix} 0 & I \end{pmatrix} , \quad g^{\perp} = \begin{pmatrix} I & 0 \end{pmatrix} \quad \text{and} \quad g^{\perp}_{+} = \begin{pmatrix} I \\ 0 \end{pmatrix} .$$

Define the set-valued functions

$$\Gamma_1(x) \triangleq \left\{ \begin{pmatrix} f_{\mathrm{c}\,1} \\ -\frac{\nabla_{x_2} H_{\mathrm{t}} \nabla_{x_1}^{\top} H_{\mathrm{t}}}{\|\nabla_{x_2} H_{\mathrm{t}}\|^2} f_{\mathrm{c}\,1} \end{pmatrix} + \begin{pmatrix} 0 \\ \left(I - \frac{\nabla_{x_2} H_{\mathrm{t}} \nabla_{x_2}^{\top} H_{\mathrm{t}}}{\|\nabla_{x_2} H_{\mathrm{t}}\|^2} \right) \zeta_2 \end{pmatrix} \;\middle|\; \zeta_2 \in \mathbb{R}^m \right\}$$

and

$$\Gamma_2(x) \triangleq \left\{ \begin{pmatrix} f_{\mathrm{c}\,1} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \zeta_2 \end{pmatrix} \;\middle|\; \zeta_2 \in \mathbb{R}^m \right\} .$$

Proposition 4.23. *Consider a system satisfying assumption 4.22 and suppose that (4.64) is consistent (cf. proposition 4.21) and that Assumption 4.22 holds. Then,* the *set-valued function*

$$\Gamma(x) = \begin{cases} \Gamma_1(x) & \textit{if} \quad y_{\mathrm{t}} \neq 0 \\ \Gamma_2(x) & \textit{if} \quad y_{\mathrm{t}} = 0 \end{cases} \tag{4.69}$$

is the *solution set of (4.64).*

Proof: The proposition follows directly from theorem A.6 and lemmas 4.19 and 4.20. Indeed, when $y_{\mathrm{t}} \neq 0$ we have that

$$I - A^{-}A = gg^{+} - \frac{gg^{+}\nabla H_{\mathrm{t}} \nabla H_{\mathrm{t}}^{\top} gg^{+}}{\nabla H_{\mathrm{t}}^{\top} gg^{+} \nabla H_{\mathrm{t}}} = \begin{pmatrix} 0 & 0 \\ 0 & I - \frac{\nabla_{x_2} H_{\mathrm{t}} \nabla_{x_2}^{\top} H_{\mathrm{t}}}{\|\nabla_{x_2} H_{\mathrm{t}}\|^2} \end{pmatrix}$$

and

$$A^{-}\psi = g^{\perp}_{+} g^{\perp} f_{\mathrm{c}} - \frac{gg^{+}\nabla H_{\mathrm{t}} \nabla H_{\mathrm{t}}^{\top} g^{\perp}_{+} g^{\perp}}{\nabla H_{\mathrm{t}}^{\top} gg^{+} \nabla H_{\mathrm{t}}} f_{\mathrm{c}} = \begin{pmatrix} f_{\mathrm{c}\,1} \\ -\frac{\nabla_{x_2} H_{\mathrm{t}} \nabla_{x_1}^{\top} H_{\mathrm{t}}}{\|\nabla_{x_2} H_{\mathrm{t}}\|^2} f_{\mathrm{c}\,1} \end{pmatrix} .$$

From the last two equations and theorem A.6 we derive Γ_1. When $y_{\mathrm{t}} = 0$ we have that

$$I - A^{-}A = gg^{+} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

$$A^{-}\psi = g^{\perp}_{+} g^{\perp} f_{\mathrm{c}} = \begin{pmatrix} f_{\mathrm{c}\,1} \\ 0 \end{pmatrix} .$$

As before, from the last two equations and theorem A.6 we derive Γ_2. ■

4.5.3 Continuous selections

Now that we have the set-valued solution $\Gamma(x)$ of (4.64), we present sufficient conditions for the existence of a continuous selection $\gamma(x) \in \Gamma(x)$. We address the problem by making use of the algebraic structure of $\Gamma(x)$.

Theorem 4.24. *Consider a system satisfying assumption 4.22 and suppose there exists a continuous map $\phi : \mathbb{X} \to \mathbb{R}^{n-m}$ such that*

$$\nabla_{x_1} H_\mathrm{t}^\top(x) f_{\mathrm{c}\,1}(x) = \nabla_{x_2} H_\mathrm{t}^\top(x)\phi(x) \ . \tag{4.70}$$

Then, equation (4.66) holds. Moreover, there exists a continuous selection $\gamma \in \Gamma$.

Proof: Notice that for a system satisfying assumption 4.22,

$$\begin{aligned} y_\mathrm{t} &= \nabla_{x_2} H_\mathrm{t}(x) && \text{(4.71a)} \\ f_{\mathrm{c}\,1}(x) &= f_1(x) + R_{\mathrm{t}\,11}(x)\nabla_{x_1} H_\mathrm{t}(x) + R_{\mathrm{t}\,12}(x)\nabla_{x_2} H_\mathrm{t}(x) \ . && \text{(4.71b)} \end{aligned}$$

From (4.71a), one can see that $y_\mathrm{t} = 0$ is equivalent to $\nabla_{x_2} H_\mathrm{t}(x) = 0$, so

$$y_\mathrm{t} = 0 \quad \Longrightarrow \quad \nabla H_\mathrm{t}^\top(x) f(x) = \nabla_{x_1} H_\mathrm{t}^\top(x) f_1(x) \tag{4.72}$$

and

$$y_\mathrm{t} = 0 \quad \Longrightarrow \quad \nabla_{x_1} H_\mathrm{t}^\top(x) R_{\mathrm{t}\,11}(x) \nabla_{x_1} H_\mathrm{t}(x) = \nabla H_\mathrm{t}^\top(x) R_\mathrm{t}(x) \nabla H_\mathrm{t}(x) \ . \tag{4.73}$$

On the other hand, equation (4.70) shows that

$$y_\mathrm{t} = 0 \quad \Longrightarrow \quad \nabla_{x_1} H_\mathrm{t}^\top(x) f_{\mathrm{c}\,1}(x) = 0 \ ,$$

so according to (4.71b), we have the implication

$$y_\mathrm{t} = 0 \quad \Longrightarrow \quad \nabla_{x_1} H_\mathrm{t}^\top(x) f_1(x) = -\nabla_{x_1} H_\mathrm{t}^\top(x) R_{\mathrm{t}\,11}(x) \nabla_{x_1} H_\mathrm{t}(x) \ . \tag{4.74}$$

Equations (4.72), (4.74) and (4.73) show that

$$y_\mathrm{t} = 0 \quad \Longrightarrow \quad \nabla H_\mathrm{t}^\top(x) f(x) = -\nabla H_\mathrm{t}^\top(x) R_\mathrm{t}(x) \nabla H_\mathrm{t}(x) \ ,$$

which is (4.66).

Taking $\zeta_2(x) = -\phi(x)$ in $\Gamma_1(x)$ yields

$$\gamma(x) = \begin{pmatrix} f_{\mathrm{c}\,1}(x) \\ -\phi(x) \end{pmatrix} \tag{4.75}$$

which is in $\Gamma_1(x)$ by construction. Similarly, by taking $\zeta_2(x) = -\phi(x)$ in $\Gamma_2(x)$ one recovers (4.75), which proves that $\gamma(x) \in \Gamma(x)$. Since $f_{\mathrm{c}\,1}$ and ϕ are continuous, the map (4.75) is continuous as well. ■

Corollary 4.25. *A system satisfying the properties of theorem 4.24 can be controlled with a controller defined by*

$$\begin{aligned}\mathring{u}_\mathrm{c}(x) &= \begin{pmatrix}0 & I\end{pmatrix}\left[\begin{pmatrix}f_{\mathrm{c}\,1}(x)\\ -\phi(x)\end{pmatrix} - \begin{pmatrix}f_1(x)\\ f_2(x)\end{pmatrix} - \begin{pmatrix}R_{\mathrm{t}\,11}(x)\nabla_{x_1}H_\mathrm{t}(x) + R_{\mathrm{t}\,12}(x)\nabla_{x_2}H_\mathrm{t}(x)\\ R_{\mathrm{t}\,21}(x)\nabla_{x_1}H_\mathrm{t}(x) + R_{\mathrm{t}\,22}(x)\nabla_{x_2}H_\mathrm{t}(x)\end{pmatrix}\right]\\ &= \begin{pmatrix}0 & I\end{pmatrix}\begin{pmatrix}0\\ -\phi(x) - f_2(x) - R_{\mathrm{t}\,21}(x)\nabla_{x_1}H_\mathrm{t}(x) - R_{\mathrm{t}\,22}(x)\nabla_{x_2}H_\mathrm{t}(x)\end{pmatrix}\\ &= -\phi(x) - f_2(x) - R_{\mathrm{t}\,21}(x)\nabla_{x_1}H_\mathrm{t}(x) - R_{\mathrm{t}\,22}(x)\nabla_{x_2}H_\mathrm{t}(x)\end{aligned}$$

and

$$\mathring{y}_\mathrm{t}(x) = \nabla_{x_2}H_\mathrm{t}(x) - \nabla_{x_2}H(x)\ .$$

Example: Systems in backstepping form

As in [28], consider a system in the backstepping form

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 \tag{4.76a}$$
$$\dot{x}_2 = f_2(x_1, x_2) + u \tag{4.76b}$$
$$y = h(x)\ . \tag{4.76c}$$

Suppose that the component described by (4.76a) can be stabilized by a smooth state feedback $x_2 = s(x_1)$, with $s(0) = 0$. Suppose, further, that we know a storage function $H_1 : \mathbb{R}^{(n-m)} \to \mathbb{R}_+$ that satisfies

$$\nabla H_1^\top(x_1)(f_1(x_1) + g_1(x_1)s(x_1)) = -\nabla H_1^\top(x_1)R_{11}(x_1)\nabla H_1(x_1) \tag{4.77}$$

for some non-negative $R_{11}(x_1)$.

By applying the change of coordinates

$$z(x) = \begin{pmatrix}x_1\\ x_2 - s(x_1)\end{pmatrix} \tag{4.78}$$

one obtains the system

$$\dot{z}_1 = f_1(z_1) + g_1(z_1)s(z_1) + g_1(z_1)z_2 \tag{4.79a}$$
$$\dot{z}_2 = f_2(z_1, z_2 + s(z_1)) - \nabla s^\top(z_1)\dot{z}_1 + u\ . \tag{4.79b}$$

Consider the candidate storage function

$$H_\mathrm{t}(z) = H_1(z_1) + \frac{1}{2}\|z_2\|^2\ . \tag{4.80}$$

By setting $R_{12} = 0$, equation (4.70) becomes

$$\nabla H_1^\top(z_1)(f_1(z_1) + g_1(z_1)s(z_1) + g_1(z_1)z_2 + R_{11}(z_1)\nabla H_1(z_1)) = z_2^\top\phi(z) \tag{4.81}$$

Substituting (4.77) in (4.81) gives

$$\nabla H_1^\top(z_1)g_1(z_1)z_2 = z_2^\top\phi(z)\ .$$

Then, $\phi(z_1) = g_1^\top(z_1)\nabla H_1(z_1)$ (which is the passive flow of the first subsystem), satisfies (4.70) and according to theorem 4.24,

$$\gamma(z) = \begin{pmatrix} f_1(z_1) + g_1(z_1)s(z_1) + g_1(z_1)z_2 + R_{11}(z_1)\nabla H_1(z_1) \\ -g_1^\top(z_1)\nabla H_1(z_1) \end{pmatrix}$$

satisfies the equation

$$g^\perp \left[\begin{pmatrix} f_1(z_1) + g_1(z_1)s(z_1) + g_1(z_1)z_2 \\ f_2(z_1, z_2 + s(z_1)) - \nabla s^\top(z_1)\dot{z}_1 \end{pmatrix} + \begin{pmatrix} R_{11}(z_1) & 0 \\ 0 & R_{22} \end{pmatrix} \nabla H_\mathrm{t}(z) - \gamma(z) \right] = 0$$

for any $R_{22} \in \mathbb{R}^{m \times m}$, which can be verified immediately by recalling that $g^\perp = \begin{pmatrix} I & 0 \end{pmatrix}$. This method produces the controller $\hat{\Pi}_\mathrm{c}$ defined by

$$\mathring{u}_\mathrm{c}(z) = -f_2(z_1, z_2 + s(z_1)) + \nabla s^\top(z_1)\dot{z}_1 - R_{22}z_2 - h_1(z_1)$$

and

$$\mathring{y}_\mathrm{t}(z) = z_2 - h(z_1, z_2 + s(z_1)) \ ,$$

where $h_1(z_1) \triangleq g_1^\top(z_1)\nabla H_1(z_1)$. Confront [38].

Example: Generalized backstepping

Before presenting this example, let us state a lemma.

Lemma 4.26. *Let $p : \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{C}^1$ and satisfy $p(0) = 0$. There exists a continuous vector field $\pi : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$p(x) = x^\top \pi(x) \ .$$

Proof: Take

$$\pi(x) = \int_0^1 \nabla_x p(\lambda x) \mathrm{d}\lambda \ .$$

The assertion is readily verified:

$$x^\top \pi(x) = \int_0^1 x^\top \nabla_x p(\lambda x) \mathrm{d}\lambda = p(\lambda x)\Big|_{\lambda x = 0}^{\lambda x = x} = p(x) \ .$$

■

Consider a system in regular form

$$\dot{x}_1 = f_1(x_1, x_2) \quad (4.82a)$$

$$\dot{x}_2 = f_2(x_1, x_2) + u \quad (4.82b)$$

$$y = h(x) \ . \quad (4.82c)$$

As in the previous example, suppose that the component (4.82a) can be stabilized by a smooth state feedback $x_2 = s(x_1)$, with $s(0) = 0$, and that we know a storage function H_1 that satisfies

$$\nabla H_1^\top(x_1) f_1(x_1, s(x_1)) = -\nabla H_1^\top(x_1) R_{11}(x_1) \nabla H_1(x_1) \tag{4.83}$$

for some non-negative R_{11}. By applying (4.78) again, we obtain the system

$$\dot{z}_1 = f_1(z_1, z_2 + s(z_1)) \tag{4.84a}$$
$$\dot{z}_2 = f_2(z_1, z_2 + s(z_1)) - \nabla s^\top(z_1) f_1(z_1, z_2 + s(z_1)) + u \; . \tag{4.84b}$$

Taking again (4.80) and setting $R_{12} = 0$, turns (4.70) into

$$\nabla H_1^\top(z_1)[f_1(z_1, z_2 + s(z_1)) + R_{11} \nabla H_1(z_1)] = z_2^\top \phi(z) \; .$$

Define

$$p(z) \triangleq \nabla H_1^\top(z_1)(f_1(z_1, z_2 + s(z_1)) + R_{11} \nabla H_1(z_1))$$

and suppose that p is differentiable with respect to z_2 (p can be regarded as a supply rate for H_1). Equation (4.83) implies that $p(z_1, 0) = 0$, so according to lemma 4.26,

$$\phi(z) = \int_0^1 \nabla_{z_2} p(z_1, \lambda z_2) \mathrm{d}\lambda$$

and

$$\gamma(z) = \begin{pmatrix} f_1(z_1, z_2 + s(z_1)) + R_{11} \nabla H_1(z_1) \\ -\int_0^1 \nabla_{z_2} p(z_1, \lambda z_2) \mathrm{d}\lambda \end{pmatrix} \; .$$

Once we have $\gamma(z)$, the controller is easily computed as

$$\begin{aligned} \mathring{u}_\mathrm{c}(z) &= -f_2(z_1, z_2 + s(z_1)) + \nabla s^\top(z_1) \dot{z}_1 - \int_0^1 \nabla_{z_2} p(z_1, \lambda z_2) \mathrm{d}\lambda \\ \mathring{y}_\mathrm{t}(z) &= z_2 - h(z_1, z_2 + s(z_1)) \; . \end{aligned}$$

Example: Swing up an inverted pendulum

Consider an inverted pendulum made up by a cart driven by a belt which exerts a force upon it and constraints its movement to the horizontal axis. Attached to the cart is a free pendulum as shown in Fig. 4.10. Suppose that friction is negligible. By applying the methods outlined in section 3.3.1 (or by directly consulting [17]) one arrives at the following model:

$$\begin{aligned} \begin{pmatrix} \dot{q}_\theta \\ \dot{q}_x \\ \dot{p}_\theta \\ \dot{p}_x \end{pmatrix} &= \begin{pmatrix} \nabla_{p_\theta} H(q,p) \\ \nabla_{p_x} H(q,p) \\ -\nabla_{q_\theta} H(q,p) \\ -\nabla_{q_x} H(q,p) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u \\ y &= \nabla_{p_x} H(q,p) \; , \end{aligned}$$

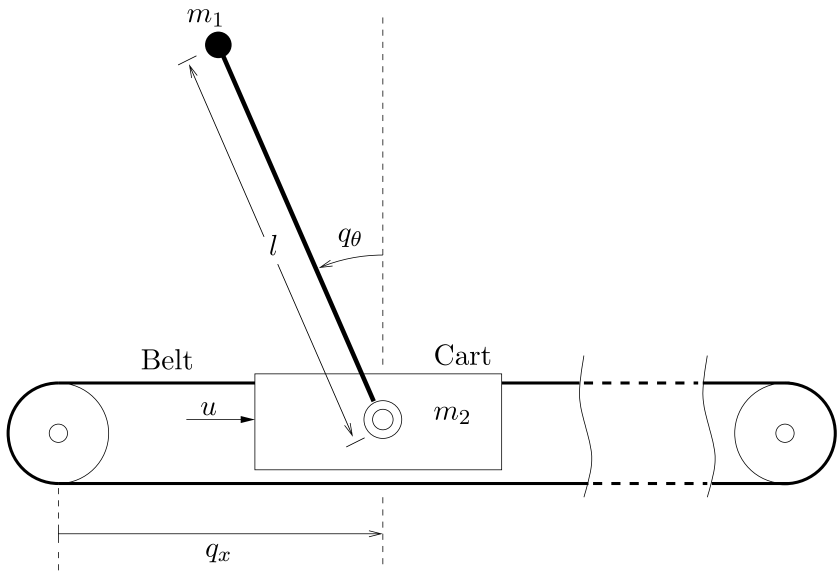

Figure 4.10: An inverted pendulum.

where

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q_\theta) p + V(q_\theta)$$

with

$$M(q_\theta) = \begin{pmatrix} \Theta + ml^2 & -m_1 l \cos q_\theta \\ -m_1 l \cos q_\theta & m_1 + m_2 \end{pmatrix} , \quad V(q_\theta) = mgl \cos q_\theta .$$

The system parameters and variables described in table 4.1.

Suppose we want to swing-up the pendulum by forcing its energy to converge to a desired value H_0 [3]. To do so, consider the desired energy function [51, 30]

$$H_{\mathrm{t}}(q,p) = \frac{1}{2} k_{\mathrm{H}} (H(q_\theta,p) - H_0)^2 + \frac{1}{2} k_{\mathrm{v}} \left(\nabla_{p_x} H(q_\theta,p) \right)^2 + \frac{1}{2}(q_x - a)^2 ,$$

where a is a desired position for q_x and $k_{\mathrm{H}}, k_{\mathrm{v}} > 0$ are design parameters. The function H_{t} attains its minimum when

$$H(q_\theta,p) = H_0 , \quad q_x = a \quad \text{and} \quad \nabla_{p_x} H(q_\theta,p) = \dot{q}_x = 0 .$$

Figs. 4.11 and 4.12 show a (q_θ, p_θ)-slice of H_{t}.

With the objective of applying theorem 4.24, let us compute

$$\begin{aligned}
\nabla_{q_\theta} H_{\mathrm{t}}(q,p) &= k_{\mathrm{H}} (H(q_\theta,p) - H_0) \nabla_{q_\theta} H(q_\theta,p) + k_{\mathrm{v}} \dot{q}_x \nabla^2_{q_\theta p_x} H(q_\theta,p) \\
\nabla_{q_x} H_{\mathrm{t}}(q,p) &= q_x - a \\
\nabla_{p_\theta} H_{\mathrm{t}}(q,p) &= k_{\mathrm{H}} (H(q_\theta,p) - H_0) \nabla_{p_\theta} H(q_\theta,p) + k_{\mathrm{v}} \dot{q}_x \nabla^2_{p_\theta p_x} H(q_\theta,p)
\end{aligned}$$

Parameter	Description
m_1	Mass of the pendulum
m_2	Mass of the cart
l	Distance from the pivot point to the center of gravity of the pendulum
Θ	Inertia of the pendulum about its center of gravity
g	Acceleration due to gravity
Variable	
q_x	Distance from the center of mass of the cart to a fixed reference point
q_θ	Angle that the pendulum makes with the vertical
u	Force applied on the cart

Table 4.1: Parameters and variables of the inverted pendulum

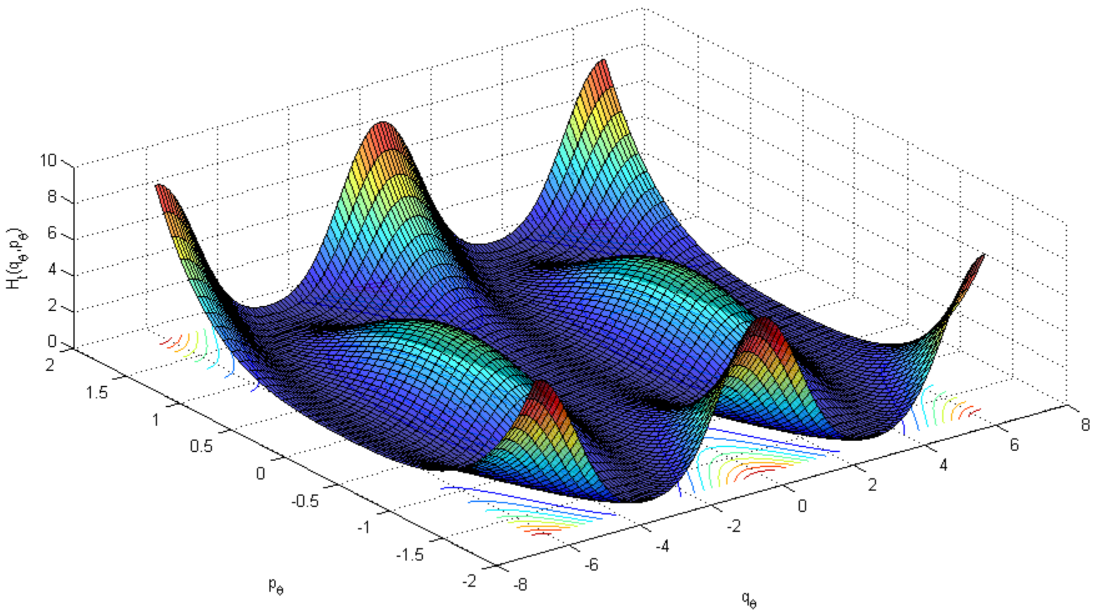

Figure 4.11: (q_θ, p_θ)-slice of H_{t}. Surface plot.

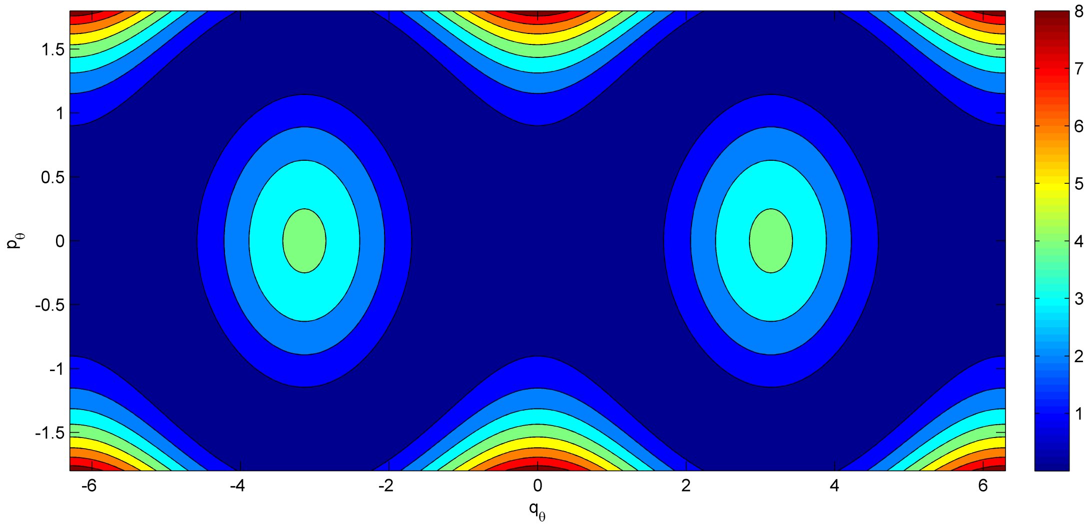

Figure 4.12: (q_θ, p_θ)-slice of H_t. Contour plot.

and

$$\begin{aligned}\nabla_{p_x} H_\mathrm{t}(q,p) &= k_\mathrm{H}(H(q_\theta,p)-H_0)\nabla_{p_x}H(q_\theta,p) + k_\mathrm{v}\dot{q}_x\nabla^2_{p_xp_x}H(q_\theta,p)\\ &= k_\mathrm{u}(q_\theta,p)\dot{q}_x \end{aligned} \tag{4.85}$$

where $k_\mathrm{u}(q_\theta,p) \triangleq k_\mathrm{H}(H(q_\theta,p)-H_0) + k_\mathrm{v}\nabla^2_{p_xp_x}H(q_\theta,p)$. Setting $R_\mathrm{t}(x)=0$ gives

$$f_{\mathrm{c}\,1}(q,p) = \begin{pmatrix} \nabla_{p_\theta}H(q_\theta,p)\\ \nabla_{p_x}H(q_\theta,p)\\ -\nabla_{q_\theta}H(q_\theta,p)\end{pmatrix} ,$$

so (4.70) is

$$\begin{pmatrix}\nabla_{q_\theta}H_\mathrm{t}(q,p) & \nabla_{q_x}H_\mathrm{t}(q,p) & \nabla_{p_\theta}H_\mathrm{t}(q,p)\end{pmatrix}\begin{pmatrix} \nabla_{p_\theta}H(q_\theta,p)\\ \nabla_{p_x}H(q_\theta,p)\\ -\nabla_{q_\theta}H(q_\theta,p)\end{pmatrix} = \nabla_{p_x}H(q_\theta,p)\phi(q,p)$$

with ϕ the unknown. Straightforward computations give

$$\left[k_\mathrm{v}\left[\nabla^2_{q_\theta p_x}H(q_\theta,p)\nabla_{p_\theta}H(q_\theta,p) - \nabla^2_{p_\theta p_x}H(q_\theta,p)\nabla_{q_\theta}H(q_\theta,p)\right] + (q_x-a)\right]\dot{q}_x = \nabla_{p_x}H(q_\theta,p)\phi(q,p)\ . \tag{4.86}$$

Since for any set of parameters we can always find an ϵ such that $\nabla^2_{p_xp_x}H(q_\theta,p) \geq \epsilon > 0$, and since H is bounded from below, there always exists a k_v that makes $k_\mathrm{u}(q_\theta,p)$ strictly positive for all q_θ and all p. Thus, we can write (4.85) as

$$\dot{q}_x = \frac{\nabla_{p_x}H(q_\theta,p)}{k_\mathrm{u}(q_\theta,p)}\ .$$

The solution of (4.86) is now evident:

$$\phi(q,p) = \frac{1}{k_{\mathrm{u}}(q_\theta,p)}\big[k_{\mathrm{v}}\left[\nabla^2_{q_\theta p_x} H(q_\theta,p)\nabla_{p_\theta} H(q_\theta,p) - \nabla^2_{p_\theta p_x} H(q_\theta,p)\nabla_{q_\theta} H(q_\theta,p)\right] + \\ + (q_x - a)\big] \quad (4.87)$$

and the controller is given by

$$\begin{aligned} \mathring{u}_{\mathrm{c}}(q,p) &= \nabla_{q_x} H(q_\theta,p) - \phi(q,p) = -\phi(q,p) \\ \mathring{y}_{\mathrm{t}}(q,p) &= (k_{\mathrm{u}}(q_\theta,p) - 1)\nabla_{p_x} H(q_\theta,p) \; . \end{aligned}$$

4.6 Basic interconnection and damping assignment

Although in some cases the choice of the matrices J_{t} and R_{t} in IDA may be motivated by physical considerations (see, e.g., the magnetic levitation example in [45]), besides the requirement of the solvability of the matching equations, there are no general guidelines. If the original system is already port-Hamiltonian, one natural first choice of F_{t} is simply

$$F_{\mathrm{t}}(x) = F(x) \; .$$

In this case the controller is called basic IDA (BIDA) and the equation to solve is, according to (4.56),

$$\begin{aligned} g(x)\mathring{u}_{\mathrm{c}}(x) &= -F(x)\nabla H(x) - R_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) + J_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x) \\ &= F(x)\nabla H_{\mathrm{a}}(x) \; . \end{aligned} \quad (4.88)$$

Regarding the flow, we have that

$$\mathring{y}_{\mathrm{t}}(x) = g^\top(x)\nabla H_{\mathrm{a}}(x) \; .$$

Remark 4.27. Equation (4.88) is equivalent to

$$g^\perp(x)F(x)\nabla H_{\mathrm{a}}(x) = 0$$

and $\mathring{u}_{\mathrm{c}}(x) = g^+(x)F(x)\nabla H_{\mathrm{a}}(x)$.

Notice that in BIDA the dissipation is modified from

$$d(x) = -\nabla H^\top(x)f(x) = -\nabla H^\top(x)F(x)\nabla H(x) = \nabla H^\top(x)R(x)\nabla H(x)$$

to

$$d_{\mathrm{t}}(x) = -\nabla H_{\mathrm{t}}^\top(x)f_{\mathrm{t}}(x) = -\nabla H_{\mathrm{t}}^\top(x)F(x)\nabla H_{\mathrm{t}}(x) = \nabla H_{\mathrm{t}}^\top(x)R(x)\nabla H_{\mathrm{t}}(x) \; .$$

BIDA controllers posses an interesting property relating flow and dissipation invariance.

Proposition 4.28. *A BIDA controller that is flow-preserving is necessarily dissipation-preserving. Consequently, it is energy balancing.*

Proof: Premultiply (4.88) by $\nabla H_{\mathrm{a}}^{\top}(x)$ to obtain

$$\nabla H_{\mathrm{a}}^{\top}(x)g(x)\mathring{u}_{\mathrm{c}}(x) = \nabla H_{\mathrm{a}}^{\top}(x)F(x)\nabla H_{\mathrm{a}}(x) = -\nabla H_{\mathrm{a}}^{\top}(x)R(x)\nabla H_{\mathrm{a}}(x)\ . \tag{4.89}$$

Under the flow-preserving assumption $\nabla H_{\mathrm{a}}^{\top}(x)g(x) = 0$, equation (4.89) shows that

$$\nabla H_{\mathrm{a}}^{\top}(x)R(x)\nabla H_{\mathrm{a}}(x) = 0\ . \tag{4.90}$$

Since $R(x)$ is symmetric and non-negative, $R(x)\nabla H_{\mathrm{a}}(x) = 0$. This means that dissipation is preserved:

$$d_{\mathrm{t}}(x) = (\nabla H(x) + \nabla H_{\mathrm{a}}(x))^{\top}R(x)(\nabla H(x) + \nabla H_{\mathrm{a}}(x)) = \nabla H(x)R(x)\nabla H(x) = d(x)\ .$$

■

Remark 4.29. A relative-passivity controller modifies the energy function only, so when dealing with port-Hamiltonian systems, it is a particular case of BIDA.

4.6.1 Any basic IDA is energy-balancing (with the appropriate flow function)

In the preceding sections we used corollary 3.10, which is Hill-Moylan's theorem for systems without feedthrough terms. In this section we show that the incorporation of a feedthrough component allows to generate new cyclo-passive flows. In particular, to identify one which is invariant to the action of BIDA. It turns out that the dissipation associated to the new flow is also invariant under BIDA. Output and dissipation invariance then establish that BIDA is energy balancing (with respect to the definition of the new flow) and as such, can be realized by a controller like the one in Fig. 4.3.

The first step will be to relax assumption 4.1.

Assumption 4.30. *The plants under consideration are cyclo-passive m-ports*

$$\hat{\Pi}_{\mathrm{p}}^{j} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{p}})$$

with $\mathbb{W} = \mathbb{U} \times \mathbb{Y}$ *and*

$$\hat{\mathcal{B}}_{\mathrm{p}} = \big\{(u, y, x) \;\big|\; \textit{Eq. (3.1) holds}\big\} \tag{4.91}$$

(notice that we no longer have $j(x) = 0$*).*

The full version of theorem 3.9 can be used to *construct* new cyclo-passive flows. Indeed, it provides a means to parametrize the flow function h and the dissipation function d in terms of the free square matrix j (hence the notation). The construction of cyclo-passive flows can be carried out as follows. Start with a cyclo-passive system having $j = 0$ (that we denote by $\hat{\Pi}_{\mathrm{p}}^{j} = \hat{\Pi}_{\mathrm{p}}$). From the cyclo-passivity assumption, equation (3.7a) must hold for some l — hence, l is fixed. Now, for any j whose symmetric part is non-negative, there always exist a w satisfying (3.7c). The matrix w can then be used to re-define, via (3.7b) and (3.8), h and d, respectively.

Although theorem 3.9 is applicable to general nonlinear systems, our interest in this section is restricted to the case when $\hat{\Pi}_{\mathrm{p}}$ is port Hamiltonian. For this class of systems, we construct a new cyclo-passive flow which is an extension of the power-shaping flow introduced in [40] to the case when F is not full rank.

Assumption 4.31. $\hat{\Pi}_{\mathrm{p}} = (\mathbb{U} \times \mathbb{Y}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{p}})$ *is port-Hamiltonian with* $\hat{\mathcal{B}}_{\mathrm{p}}$ *given by (3.34) and satisfies*

$$F^\top(x)(F^-)^\top(x)F(x) = F(x) \tag{4.92}$$

and

$$\text{span } g(x) \subseteq \text{span } F(x) \; . \tag{4.93}$$

It is important to mention that equation (4.92) does not depend on the particular choice of $F^-(x)$[4]. Furthermore, if $F(x)$ is non-singular, then (4.92) and (4.93) are trivially satisfied.

Lemma 4.32. *The equation*

$$F^\top(x)Z(x)F(x) = -F(x) \; , \tag{4.94}$$

with unknown $Z(x) \in \mathbb{R}^{n\times n}$*, is consistent if, and only if, equation (4.92) is satisfied.*

Proof: Equation (4.94) is a special case of the linear matrix equation

$$AXB = C \; , \tag{4.95}$$

where X is the unknown. According to theorem A.6, equation (4.95) is consistent if, and only if,

$$AA^-CB^-B = C \; . \tag{4.96}$$

By matching the terms in (4.94) and (4.95) we get

$$A = F^\top(x) \; , \quad X = Z(x) \; , \quad B = F(x) \quad \text{and} \quad C = -F(x) \; .$$

By substituting these in (4.96) we obtain

$$-F^\top(x)(F^\top)^-(x)F(x)F^-(x)F(x) = -F(x) \; ,$$

which is equivalent to

$$F^\top(x)(F^-)^\top(x)F(x) = F(x)$$

(recall that $F(x)F^-(x)F(x) = F(x)$ and that a possible generalized inverse of $F^\top(x)$ is $(F^-)^\top(x)$). ■

Lemma 4.33. *Equations (4.94) and (4.93) imply that*

$$F^\top(x)Z(x)g(x) = -g(x) \; . \tag{4.97}$$

Proof: Equation (4.93) implies the existence of a map $\beta : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ such that

$$g(x) = F(x)\beta(x) \; .$$

On the other hand, equation (4.94) implies that

$$F^\top(x)Z(x)F(x)\beta(x) = -F(x)\beta(x)$$

for any β. Combining the last two equations yields (4.97). ■

[4] Recall that, in general, A^- is not unique, but it always exists (see theorem A.5).

Proposition 4.34. *Consider an m-port $\hat{\Pi}_\mathrm{p}$ satisfying assumption 4.31 and define*

$$Z(x) \triangleq -(F^-)^\top(x)F(x)F^-(x) \; . \tag{4.98}$$

The m-port $\hat{\Pi}_\mathrm{p}^j$ described by the equations

$$\dot{x} = F(x)\nabla H(x) + g(x)u \tag{4.99a}$$
$$y = g^\top(x)Z(x)\left[F(x)\nabla H(x) + g(x)u\right] \tag{4.99b}$$

is cyclo-passive with storage function H.

Proof: The proof is established verifying the conditions of theorem 3.9. Notice that for system (4.99) we have

$$j(x) = g^\top(x)Z(x)g(x) \tag{4.100}$$

and

$$h(x) = g^\top(x)Z(x)F(x)\nabla H(x) \; . \tag{4.101}$$

We will show that there exists maps l and w such that (3.7) is satisfied. Because of (4.92) and lemma 4.32, equation (4.94) is consistent. Under assumption 4.31, $Z(x)$ defined as in (4.98) is a particular non-negative solution of (4.94). Equation (4.94) then implies that

$$\nabla H^\top(x)F^\top(x)Z(x)F(x)\nabla H(x) = -\nabla H^\top(x)F(x)\nabla H(x) \; .$$

Given $Z(x)$ compute $Y(x) \in \mathbb{R}^{n\times n}$ as

$$Y^\top(x)Y(x) = \frac{1}{2}\left(Z^\top(x) + Z(x)\right) \; , \tag{4.102}$$

which can always be obtained since $(Z(x) + Z^\top(x)) \geq 0$. It is then easy to see that

$$l(x) \triangleq Y(x)F(x)\nabla H(x) \tag{4.103}$$

satisfies (3.7a). Furthermore,

$$w(x) \triangleq Y(x)g(x) \tag{4.104}$$

satisfies

$$w^\top(x)w(x) = \frac{1}{2}g^\top(x)\left(Z^\top(x) + Z(x)\right)g(x) = \frac{1}{2}(j^\top(x) + j(x)) \; .$$

Substituting $l(x)$ and $w(x)$ into (3.7b) gives

$$\begin{aligned} h(x) &= g^\top(x)\nabla H(x) + 2g^\top(x)Y^\top(x)Y(x)F(x)\nabla H(x) \\ &= g^\top(x)\nabla H(x) + g^\top(x)(Z^\top(x) + Z(x))F(x)\nabla H(x) \\ &= g^\top(x)\nabla H(x) - g^\top(x)\nabla H(x) + g^\top(x)Z(x)F(x)\nabla H(x) \\ &= g^\top(x)Z(x)F(x)\nabla H(x) \; , \end{aligned}$$

where (4.102) is used to obtain the second identity, while (4.94) and lemma 4.33 are invoked in the third one. ■

Remark 4.35. When $F(x)$ is non-singular, the new cyclo-passive flow y coincides with the power-shaping flow of [40]. It is shown in [26] that the generation of the new flow, for a class of electrical and electromechanical systems, is tantamount to the application of the classical Thévenin-Norton transformation of electrical circuits. Additional connections with power-shaping may be found in these two papers.

As one might expect, in the zero-relative-degree case there is also a connection between energy-balancing and flow and dissipation invariance.

Proposition 4.36. *Consider a controller* $\hat{\Pi}_\mathrm{c}$ *satisfying assumption 4.2 and a plant* $\hat{\Pi}_\mathrm{p}^j$ *satisfying assumption 4.30. The controller* $\hat{\Pi}_\mathrm{c}$ *is energy balancing if the flow and the dissipation remain invariant, that is, if*

$$\dot{H}_\mathrm{t}(x) = y_\mathrm{t}^\top u_\mathrm{t} - d_\mathrm{t}(x, u_\mathrm{t}) \tag{4.105}$$

holds with

$$y = y_\mathrm{t} \quad \text{and} \quad d(x, u) = d_\mathrm{t}(x, u)\ . \tag{4.106}$$

Proof: By substracting (3.9) from (4.105) it is readily seen that

$$\dot{H}_\mathrm{a}(x) = y_\mathrm{t}^\top u_\mathrm{t} - y^\top u + d_\mathrm{t}(x, u_\mathrm{t}) - d(x, u)\ , \tag{4.107}$$

with H_a defined as in (4.12). Substitution of the hypothesis (4.106) into (4.107) yields

$$\dot{H}_\mathrm{a}(x) = y^\top (u_\mathrm{t} - u)\ .$$

Since $u = \mathring{u}_\mathrm{c}(x) + u_\mathrm{t}$,

$$\dot{H}_\mathrm{a}(x) = -y^\top \mathring{u}_\mathrm{c}(x)\ . \tag{4.108}$$

■

The next proposition shows that y and d are invariant under BIDA control (a similar result, using different arguments, was obtained in [26] for the case when $F(x)$ is non-singular).

Proposition 4.37. *Consider a cyclo-passive m-port described by (4.98) and (4.99). Suppose that assumption 4.31 holds. The BIDA controller (4.5), with* $\mathring{u}_\mathrm{c}(x)$ *as in (4.88) and*

$$\mathring{y}_\mathrm{t}(x) = 0\ , \tag{4.109}$$

is passivity-based. The controller is flow and dissipation-preserving; therefore, it is energy balancing.

Proof: Since $\mathring{y}_\mathrm{t}(x) = 0$, we have that $y_\mathrm{t} = y$ and the controller is flow-preserving. Substituting $u = \mathring{u}_\mathrm{c}(x) + u_\mathrm{t}$ in (4.99) gives a $\hat{\Pi}_\mathrm{t}$ defined by the equations

$$\dot{x} = F(x)\nabla H(x) + g(x)(\mathring{u}_\mathrm{c}(x) + u_\mathrm{t}) \tag{4.110a}$$
$$y_\mathrm{t} = g^\top(x)Z(x)[F(x)\nabla H(x) + g(x)(\mathring{u}_\mathrm{c}(x) + u_\mathrm{t})]\ . \tag{4.110b}$$

By using (4.88) we can write (4.110) as

$$\begin{aligned} \dot{x} &= F(x)\nabla H_{\mathrm{t}}(x) + g(x)u_{\mathrm{t}} && (4.111\mathrm{a}) \\ y_{\mathrm{t}} &= g^{\top}(x)Z(x)[F(x)\nabla H_{\mathrm{t}}(x) + g(x)u_{\mathrm{t}}] && (4.111\mathrm{b}) \end{aligned}$$

We will show that $\hat{\Pi}_{\mathrm{t}}$, is cyclo-passive with storage function H_{t}. To this effect, we will prove that there exists an l_{t} satisfying the conditions of Hill-Moylan's theorem. Indeed, equation (4.94) implies that

$$\nabla H_{\mathrm{t}}^{\top}(x)F^{\top}(x)Z(x)F(x)\nabla H_{\mathrm{t}}(x) = -\nabla H_{\mathrm{t}}^{\top}(x)F(x)\nabla H_{\mathrm{t}}(x)\ ,$$

so

$$l_{\mathrm{t}}(x) = Y(x)F(x)\nabla H_{\mathrm{t}}(x)\ ,$$

with $Y(x)$ defined as in (4.102), satisfies the first condition of Hill Moylan's theorem. Selecting $w_{\mathrm{t}}(x) = w(x)$ with $w(x)$ as in (4.104) and substituting into the second condition of Hill-Moylan's theorem gives

$$\begin{aligned} h_{\mathrm{t}}(x) &= g^{\top}(x)\nabla H_{\mathrm{t}}(x) + g^{\top}(x)(Z^{\top}(x) + Z(x))F(x)\nabla H_{\mathrm{t}}(x) \\ &= g^{\top}(x)\nabla H_{\mathrm{t}}(x) - g^{\top}(x)\nabla H_{\mathrm{t}}(x) + g^{\top}(x)Z(x)F(x)\nabla H_{\mathrm{t}}(x) \\ &= g^{\top}(x)Z(x)F(x)\nabla H_{\mathrm{t}}(x)\ . && (4.112) \end{aligned}$$

This proves our cyclo-passivity claim.

Regarding dissipation, we will prove that $d(x,u) = d_{\mathrm{t}}(x,u_{\mathrm{t}})$, that is, that

$$|l(x) + w(x)\mathring{u}_{\mathrm{c}}(x) + w(x)u_{\mathrm{t}}|^2 = |l_{\mathrm{t}}(x) + w(x)u_{\mathrm{t}}|^2\ . \qquad (4.113)$$

Direct substitution of the expressions of $l(x)$ and $w(x)$ gives

$$\begin{aligned} l(x) + w(x)\mathring{u}_{\mathrm{c}}(x) &= Y(x)F(x)\nabla H(x) + Y(x)g(x)\mathring{u}_{\mathrm{c}}(x) \\ &= Y(x)F(x)\nabla H(x) + Y(x)F(x)\nabla H_{\mathrm{a}}(x) \\ &= Y(x)F(x)\nabla H_{\mathrm{t}}(x)\ . \end{aligned}$$

Since $l_{\mathrm{t}}(x) = F(x)\nabla H_{\mathrm{t}}(x)$, we conclude (4.113). ■

Remark 4.38. Notice that the property of energy-balancing for BIDA is established with respect to the definition of the new passive flow (cf. (4.108)), which is not the same used in (4.18).

4.7 Summary

Table 4.2 contains a summary of some of the controllers described in this chapter. To be able to compare their corresponding PDE's, we have assumed that $\hat{\Pi}_{\mathrm{p}}$ is port-Hamiltonian. From the table it is easy to establish, in terms of the solutions H_{a} of the PDE's, the following implications:

$$\text{EB} \implies \text{BIDA} \implies \text{IDA}\ .$$

Controller	$\hat{\Pi}_t$'s flow	$\hat{\Pi}_t$'s dissipation	PDE
EB (as in remark 4.9)	Invariant	Invariant	$\begin{pmatrix} g^\perp F \\ g^\top \end{pmatrix} \nabla H_a = 0$
BIDA	$g^\top \nabla H_t$	$\nabla H_t^\top R \nabla H_t$	$g^\perp F \nabla H_a = 0$
BIDA with invariant flow	Invariant	Invariant	$g^\perp F \nabla H_a = 0$
IDA	$g^\top \nabla H_t$	$\nabla H_t^\top R_t \nabla H_t$	$g^\perp F_t \nabla H_a = g^\perp (F - F_t) \nabla H$
$\hat{\Pi}_t$'s storage function $H_t(x) = H(x) + H_a(x)$			

Table 4.2: Passivity-based controllers and their corresponding PDE when $\hat{\Pi}_p$ is port-Hamiltonian.

5 Dynamic Passivation

In this chapter we discus the stabilization of port-Hamiltonian systems using control by interconnection [13, 60, 44, 45]. In control by interconnection (CbI), the energy-shaping process is accomplished by selecting the controller to be also port-Hamiltonian, which connected to the plant through a power-preserving interconnection, results in a compound m-port that is again port-Hamiltonian with energy function equal to the sum of the plant and the controller energies.

5.1 Control by interconnection

The basic building block is a *flow*-driven port-Hamiltonian

$$\hat{\Pi}^{\mathrm{f}}_{\mathrm{c}\,1} = (\mathbb{W}, \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}\,1}) \;, \tag{5.1}$$

where $\mathbb{X}_{\mathrm{c}} = \mathbb{R}^m$. The equations

$$\begin{aligned} \dot{\xi} &= -y_{\mathrm{c}\,1} && (5.2\mathrm{a}) \\ u_{\mathrm{c}\,1} &= -\nabla H_{\mathrm{c}}(\xi) && (5.2\mathrm{b}) \end{aligned}$$

define its behavior as

$$\hat{\mathcal{B}}_{\mathrm{c}\,1} = \left\{ (u_{\mathrm{c}\,1}, y_{\mathrm{c}\,1}, \xi) \;\middle|\; \text{Eq. (5.2) holds} \right\} \;. \tag{5.3}$$

In (5.2), ξ is the state of the controller and $H_{\mathrm{c}} : \mathbb{X}_{\mathrm{c}} \to \mathbb{R}$ is a to-be-designed controller storage function (see [43, 60] for a justification of this choice of controller structure).

The building block $\hat{\Pi}_{\mathrm{c}\,1}$ is connected in series with the external terminals of the controller as shown in Fig. 5.1, yielding the dynamic controller

$$\hat{\Pi}_{\mathrm{c}} = (\mathbb{W} \times \mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}}) \tag{5.4}$$

with behavior

$$\hat{\mathcal{B}}_{\mathrm{c}} = \left\{ (u_{\mathrm{c}}, y_{\mathrm{c}}, u_{\mathrm{t}}, y_{\mathrm{t}}, x, \xi) \;\middle|\; (u_{\mathrm{c}} - u_{\mathrm{t}}, y_{\mathrm{c}}, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,1},\ y_{\mathrm{c}} = -y_{\mathrm{t}} \right\} \;. \tag{5.5}$$

Lemma 5.1. *Consider a port-Hamiltonian* $\hat{\Pi}_{\mathrm{p}} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{p}})$*, with* $\hat{\mathcal{B}}_{\mathrm{p}}$ *given by (3.34). The controller (5.4) with behavior (5.5) is passivity-based with*

$$H_{\mathrm{t}}(x, \xi) = H(x) + H_{\mathrm{c}}(\xi) \;. \tag{5.6}$$

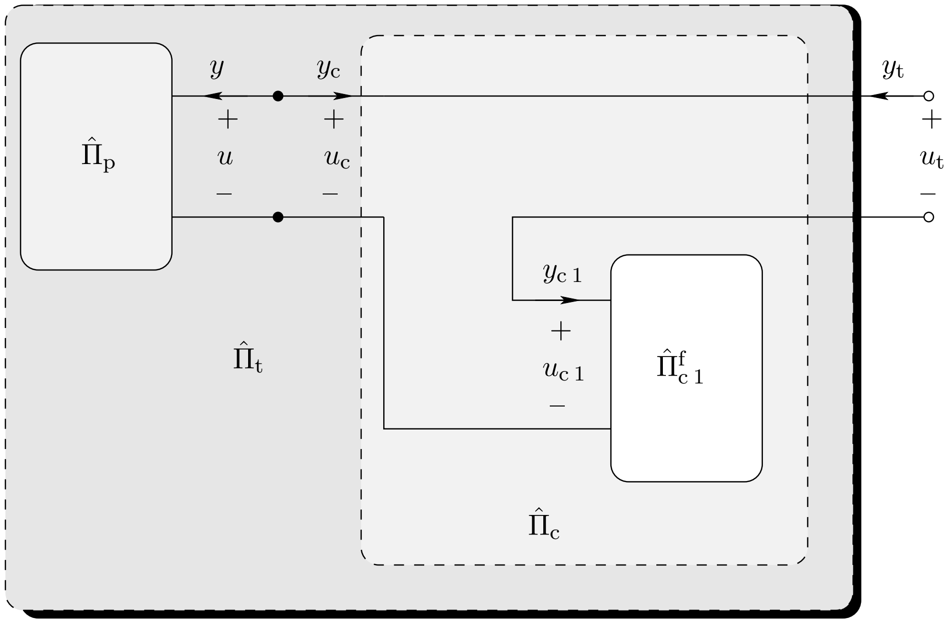

Figure 5.1: Classical control by interconnection.

Proof: The compound m-port $\hat{\Pi}_t = \hat{\Pi}_p \wedge \hat{\Pi}_c = (\mathbb{W}_t, \mathbb{X} \times \mathbb{X}_c, \hat{\mathcal{B}}_t)$ has the behavior

$$\hat{\mathcal{B}}_t = \Big\{ (u_t, y_t, x, \xi) \,\Big|\, \exists\, (u, y) \text{ s.t. } (u, y, x) \in \hat{\mathcal{B}}_p,\ (u, -y, u_t, y_t, x, \xi) \in \hat{\mathcal{B}}_c \Big\} \ .$$

Substituting (5.5) in $\hat{\mathcal{B}}_t$ gives

$$\hat{\mathcal{B}}_t = \Big\{ (u_t, y_t, x, \xi) \,\Big|\, \exists\, (u, y) \text{ s.t. } (u, y, x) \in \hat{\mathcal{B}}_p,\ (u - u_t, -y, \xi) \in \hat{\mathcal{B}}_{c\,1},\ y_t = y \Big\} \ . \quad (5.7)$$

From $(u, y, x) \in \hat{\mathcal{B}}_p$ and $(u - u_t, -y, \xi) \in \hat{\mathcal{B}}_{c\,1}$ we have that

$$\begin{aligned} \dot{x} &= F(x)\nabla H(x) + g(x)u && (5.8a)\\ y &= g^\top(x)\nabla H(x) && (5.8b)\\ \dot{\xi} &= y && (5.8c)\\ u - u_t &= -\nabla H_c(\xi)\ . && (5.8d) \end{aligned}$$

By eliminating u and y from (5.8), and by using $y_t = y$ one obtains

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} F(x) & -g(x) \\ g^\top(x) & 0 \end{pmatrix} \begin{pmatrix} \nabla H(x) \\ \nabla H_c(\xi) \end{pmatrix} + \begin{pmatrix} g(x) \\ 0 \end{pmatrix} u_t \quad (5.9a)$$

$$y_t = \begin{pmatrix} g^\top(x) & 0 \end{pmatrix} \begin{pmatrix} \nabla H(x) \\ \nabla H_c(\xi) \end{pmatrix} \ . \quad (5.9b)$$

If we denote

$$F_{\mathrm{t\,s}}(x) \triangleq \begin{pmatrix} F(x) & -g(x) \\ g^\top(x) & 0 \end{pmatrix} \quad \text{and} \quad g_{\mathrm{t\,s}}(x) \triangleq \begin{pmatrix} g(x) \\ 0 \end{pmatrix} ,$$

we can write (5.9) more compactly as

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = F_{\mathrm{t\,s}}(x)\nabla H_{\mathrm{t}}(x,\xi) + g_{\mathrm{t\,s}}(x)u_{\mathrm{t}} \tag{5.10a}$$

$$y_{\mathrm{t}} = g_{\mathrm{t\,s}}(x)\nabla H_{\mathrm{t}}(x,\xi) . \tag{5.10b}$$

The total behavior is then

$$\hat{\mathcal{B}}_{\mathrm{t}} = \left\{(u_{\mathrm{t}}, y_{\mathrm{t}}, x, \xi) \,\middle|\, \text{Eq. (5.10) holds}\right\} . \tag{5.11}$$

To prove that $\hat{\Pi}_{\mathrm{t}}$ is port-Hamiltonian (and hence cyclo-passive with storage function H_{t}), it suffices to show that $F_{\mathrm{t\,s}}(x) + F_{\mathrm{t\,s}}^\top(x) \leq 0$. Indeed,

$$F_{\mathrm{t\,s}}(x) + F_{\mathrm{t\,s}}^\top(x) = \begin{pmatrix} -2R(x) & 0 \\ 0 & 0 \end{pmatrix} \leq 0 .$$

■

So far, we have constructed an interconnected system which is cyclo-passive with storage function H_{t}. Since H_{c} can be modified at will, it seems reasonable to use it to shape the total storage function and solve the stabilization problem.

We require to shape H_{t} along the x coordinates but unfortunately H_{c} is a function of ξ only, so the energy-shaping idea cannot be applied directly. For example, suppose we have $H(x) = x$. Clearly, the function $H_{\mathrm{t}}(x,\xi) = x + H_{\mathrm{c}}(\xi)$ will never have a lower bound, no matter how we choose H_{c}. On the other hand, if given a fixed $C(x)$ we are allowed to shape

$$W(x,\xi) \triangleq H(x) + \Phi(C(x) - \xi) + H_{\mathrm{c}}(\xi) \tag{5.12}$$

via the functions Φ and H_{c}, then the energy-shaping problem might have a solution. To fix the idea, take the previous example and suppose that $C(x) = x$. Then we can set $\phi(z) = z^2 - z$ and $H_{\mathrm{c}}(\xi) = \xi^2 - \xi$ so that

$$W(x,\xi) = x + (x-\xi)^2 - (x-\xi) + \xi^2 - \xi = (x-\xi)^2 + \xi^2 ,$$

which has a unique, isolated minimum at $x = \xi = 0$.

The objective is then to make W (not only H_{t}) a storage function of $\hat{\Pi}_{\mathrm{t}}$. To achieve it, we will look for conditions on C under which $\dot{\Phi}(C(x) - \xi) = 0$.

Assumption 5.2. *There exists a differentiable map $C : \mathbb{X} \to \mathbb{X}_{\mathrm{c}}$ such that*

$$\begin{pmatrix} F^\top(x) \\ g^\top(x) \end{pmatrix} \nabla C(x) = \begin{pmatrix} g(x) \\ 0 \end{pmatrix} . \tag{5.13}$$

Lemma 5.3. *Under assumption 5.2, the port-Hamiltonian* $\hat{\Pi}_{\mathrm{t}}$ *with* $\hat{\mathcal{B}}_{\mathrm{t}}$ *given by (5.11) is cyclo-passive with storage function (5.12), where* $\Phi : \mathbb{X}_{\mathrm{c}} \to \mathbb{R}$ *is any differentiable map.*

Proof: Notice that, because of (5.13),

$$F_{\mathrm{t\,s}}(x) = \begin{pmatrix} I \\ \nabla C^\top(x) \end{pmatrix} \begin{pmatrix} F(x) & -g(x) \end{pmatrix} .$$

and

$$g_{\mathrm{t\,s}}(x) = \begin{pmatrix} I \\ \nabla C^\top(x) \end{pmatrix} g(x) .$$

The time derivative of Φ is

$$\dot{\Phi}(C(x) - \xi) = \nabla^\top \Phi(C(x) - \xi) \begin{pmatrix} \nabla C^\top(x) & -I \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} .$$

Since

$$\begin{pmatrix} \nabla C^\top(x) & -I \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} =$$

$$\begin{pmatrix} \nabla C^\top(x) & -I \end{pmatrix} \begin{pmatrix} I \\ \nabla C^\top(x) \end{pmatrix} \left[\begin{pmatrix} F(x) & -g(x) \end{pmatrix} \nabla H_{\mathrm{t}}(x) + g(x) u_{\mathrm{t}} \right] = 0 ,$$

we have that $\dot{\Phi}(C(x) - \xi) = 0$, for any Φ.

Now, since $\hat{\Pi}_{\mathrm{t}}$ is cyclo-passive with storage function H_{t} and $\dot{\Phi}(C(x) - \xi) = 0$, we know that it is also cyclo-passive with storage function W, that is,

$$\dot{W}(x, \xi) = \dot{H}_{\mathrm{t}}(x) + \dot{\Phi}(C(x) - \xi) \le y_{\mathrm{t}}^\top u_{\mathrm{t}} .$$

■

Remark 5.4. The function $\Phi(C(x) - \xi)$ is invariant for any Hamiltonian H_{t} (these functions are called Casimirs in the literature of analytical mechanics [31]). This property allows us to select H_{c} and Φ independently.

Example, a simple pendulum (continued)

Consider the pendulum of the example on p. 51. The pendulum with friction is described by

$$\begin{aligned} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -1 & -R \end{pmatrix} \nabla H(q,p) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \dot{q} \end{aligned}$$

with Hamiltonian $H(q,p) = \frac{1}{2}(ml^2)^{-1} p^2 + mgl(1 + \cos(q))$.

The function $C(q,p) = q$ satisfies

$$\begin{pmatrix} 0 & -1 \\ 1 & -R \end{pmatrix} \nabla C(q,p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & I \end{pmatrix} \nabla C(q,p) = 0 ,$$

so according to lemma 5.3, any function of the form

$$W(q,p,\xi) = H(q,p) + \Phi(q-\xi) + H_{\mathrm{c}}(\xi)$$

is a storage function for $\hat{\Pi}_{\mathrm{t}}$. It can be readily verified that for $\Phi(z) = mglz^2$ and $H_{\mathrm{c}}(\xi) = mgl\xi^2$, the storage function

$$W(q,p,\xi) = mgl\left[\cos q + 1 + (q-\xi)^2 + \xi^2\right] + \frac{1}{2}(ml^2)^{-1}p^2$$

is convex and minimal at $q = p = \xi = 0$.

5.1.1 Using other passive flows

Notice that, by construction, CbI preserves the flow (see Fig. 5.1) and the dissipation (recall that $\hat{\Pi}^{\mathrm{f}}_{\mathrm{c}\,1}$ is lossless). We can then expect the condition (5.13) to be at least as restrictive as those imposed by energy balancing control. This idea is formalized latter in section 5.7, but before getting there, notice that

$$(5.13) \quad \Longrightarrow \quad \nabla C^\top(x) F^\top(x) \nabla C(x) = 0 \quad \Longrightarrow \quad \nabla C^\top(x) R(x) \nabla C(x) = 0\,.$$

Since $R(x)$ is symmetric and non-negative, the latter implies

$$R(x)\nabla C(x) = 0\,. \tag{5.14}$$

From (5.14) we can see that the aggregated Hamiltonian

$$H_{\mathrm{a}}(x,\xi) = W(x,\xi) - H(x) = \Phi(C(x)-\xi) + H_{\mathrm{c}}(\xi)$$

satisfies

$$R(x)\nabla_x H_{\mathrm{a}}(x,\xi) = R(x)\nabla C(x)\nabla\Phi(C(x)-\xi) = 0\,,$$

which is reminiscent to the necessary condition for dissipation preservation in BIDA (cf. (4.90)). This suggest that we apply the CbI strategy to port-Hamiltonian systems with the flow function constructed in subsection 4.6.1, which assures flow and dissipation preservation without additional constraints.

Assumption 5.5. *There exists a differentiable map $C : \mathbb{X} \to \mathbb{X}_{\mathrm{c}}$ such that*

$$F(x)\nabla C(x) = -g(x) \tag{5.15}$$

The following lemma relates assumptions 5.2 and 5.5.

Lemma 5.6. *Equation (5.13) is equivalent to*

$$\begin{pmatrix} F(x) \\ g^\top(x) \end{pmatrix} \nabla C(x) = -\begin{pmatrix} g(x) \\ 0 \end{pmatrix}\,. \tag{5.16}$$

The proof is given in appendix B.2. From (5.16), it can be readily seen that assumption 5.5 is strictly weaker than assumption 5.2, since the second line in (5.16) is absent in (5.15).

Lemma 5.7. *Consider a port-Hamiltonian* $\hat{\Pi}_{\mathrm{p}}^{j} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{p}})$ *with*

$$\hat{\mathcal{B}}_{\mathrm{p}} = \big\{(u, y, x) \,\big|\, \text{Eq. (4.99) holds}\big\} \tag{5.17}$$

and the controller (5.5), (5.4). Under assumptions 5.5 and 4.31, the compound system $\hat{\Pi}_{\mathrm{t}}$ *is cyclo-passive with storage function*

$$W(x, \xi) = H(x) + \Phi(C(x) - \xi) + H_{\mathrm{c}}(\xi) \ ,$$

where $\Phi : \mathbb{X}_{\mathrm{c}} \to \mathbb{R}$ *is any differentiable map.*

As expected, the invariant flow allows to suitably shape the energy under less stringent conditions.

Proof: The proof follows along the same lines of the proofs for lemmas 5.1 and 5.3. Let us show that

$$\hat{\Pi}_{\mathrm{t}} = \hat{\Pi}_{\mathrm{p}}^{j} \wedge \hat{\Pi}_{\mathrm{c}} = (\mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{t}})$$

is port-Hamiltonian with Hamiltonian $H_{\mathrm{t}}(x, \xi) = H(x) + H_{\mathrm{c}}(\xi)$. The behavior of $\hat{\Pi}_{\mathrm{t}}$ is

$$\hat{\mathcal{B}}_{\mathrm{t}} = \Big\{(u_{\mathrm{t}}, y_{\mathrm{t}}, x, \xi) \,\Big|\, \exists\, (u, y) \text{ s.t. } (u, y, x) \in \hat{\mathcal{B}}_{\mathrm{p}},\ (u - u_{\mathrm{t}}, -y, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,1},\ y_{\mathrm{t}} = y\Big\} \ . \tag{5.18}$$

From $(u, y, x) \in \hat{\mathcal{B}}_{\mathrm{p}}$ and $(u - u_{\mathrm{t}}, -y, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,1}$ we have that

$$\begin{aligned}
\dot{x} &= F(x)\nabla H(x) + g(x)u && \text{(5.19a)}\\
y &= g^{\top}(x)Z(x)F(x)\nabla H(x) + g^{\top}(x)Z(x)g(x)u && \text{(5.19b)}\\
\dot{\xi} &= y && \text{(5.19c)}\\
u - u_{\mathrm{t}} &= -\nabla H_{\mathrm{c}}(\xi) \ . && \text{(5.19d)}
\end{aligned}$$

By eliminating u and y from (5.19) and by using $y_{\mathrm{t}} = y$ in (5.7), one obtains

$$\begin{aligned}
\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} &= F_{\mathrm{t\,ps}}(x)\nabla H_{\mathrm{t}}(x, \xi) + g_{\mathrm{t\,ps}}(x)u_{\mathrm{t}} && \text{(5.20a)}\\
y_{\mathrm{t}} &= g_{\mathrm{t\,ps}}(x)\nabla H_{\mathrm{t}}(x, \xi) \ , && \text{(5.20b)}
\end{aligned}$$

where

$$F_{\mathrm{t\,ps}}(x) \triangleq \begin{pmatrix} F(x) & -g(x) \\ g^{\top}(x)Z(x)F(x) & -g^{\top}(x)Z(x)g(x) \end{pmatrix} \quad \text{and} \quad g_{\mathrm{t\,ps}}(x) \triangleq \begin{pmatrix} g(x) \\ g^{\top}(x)Z(x)g(x) \end{pmatrix} \ .$$

The total behavior is then

$$\hat{\mathcal{B}}_{\mathrm{t}} = \big\{(u_{\mathrm{t}}, y_{\mathrm{t}}, x, \xi) \,\big|\, \text{Eq. (5.20) holds}\big\} \ . \tag{5.21}$$

To prove that $\hat{\Pi}_{\mathrm{t}}$ is port-Hamiltonian, it remains to show that $F_{\mathrm{t\,ps}}(x) + F_{\mathrm{t\,ps}}^{\top}(x) \le 0$. After some lengthy, but straightforward calculations, one can verify that for any $\alpha_1 \in \mathbb{X}$ and $\alpha_2 \in \mathbb{X}_{\mathrm{c}}$, one has

$$\begin{pmatrix}\alpha_1^{\top} & \alpha_2^{\top}\end{pmatrix}\begin{pmatrix}F(x) & -g(x) \\ g^{\top}(x)Z(x)F(x) & -g^{\top}(x)Z(x)g(x)\end{pmatrix}\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} = \\ \left[\alpha_1 - F^{-}(x)g(x)\alpha_2\right]^{\top} F(x)\left[\alpha_1 - F^{-}(x)g(x)\alpha_2\right] \le 0\ .$$

To prove that $W(x,\xi)$ is also a storage function, we will show (again) that

$$\dot{\Phi}(C(x) - \xi) = 0\ .$$

Equations (5.15) and (4.94) imply that

$$g^{\top}(x)Z(x)F(x) = -\nabla C^{\top}(x)F^{\top}(x)Z(x)F(x) = \nabla C^{\top}(x)F(x)\ ,$$

and equations (5.15) and (4.97) imply

$$g^{\top}(x)Z(x)g(x) = -\nabla C^{\top}(x)F^{\top}(x)Z(x)g(x) = \nabla C^{\top}(x)g(x)\ ,$$

so

$$F_{\mathrm{t\,ps}}(x) = \begin{pmatrix}I \\ \nabla C^{\top}(x)\end{pmatrix}\begin{pmatrix}F(x) & -g(x)\end{pmatrix} \quad \text{and} \quad g_{\mathrm{t\,ps}}(x) = \begin{pmatrix}I \\ \nabla C^{\top}(x)\end{pmatrix} g(x)\ .$$

It is now clear that

$$\begin{pmatrix}\nabla C^{\top}(x) & -I\end{pmatrix} F_{\mathrm{t\,ps}}(x) = 0 \quad \text{and} \quad \begin{pmatrix}\nabla C^{\top}(x) & -I\end{pmatrix} g_{\mathrm{t\,ps}}(x) = 0\ ,$$

so $\dot{\Phi}(C(x) - \xi) = 0$. ■

5.2 Stabilization

Notice that, whether we use the standard flow together with assumption 5.2, or whether we use the modified flow together with assumptions 5.5 and 4.31, the behavior of $\hat{\Pi}_{\mathrm{t}}$ is determined by the equations

$$\begin{pmatrix}\dot{x} \\ \dot{\xi}\end{pmatrix} = F_{\mathrm{t}}(x)\nabla H_{\mathrm{t}}(x,\xi) + g_{\mathrm{t}}(x)u_{\mathrm{t}} \tag{5.22a}$$

$$y_{\mathrm{t}} = g_{\mathrm{t}}^{\top}(x)\nabla H_{\mathrm{t}}(x)\ , \tag{5.22b}$$

where

$$F_{\mathrm{t}}(x) \triangleq \begin{pmatrix}I \\ \nabla C^{\top}(x)\end{pmatrix}\begin{pmatrix}F(x) & -g(x)\end{pmatrix} \quad \text{and} \quad g_{\mathrm{t}} \triangleq \begin{pmatrix}I \\ \nabla C^{\top}(x)\end{pmatrix} g(x)\ . \tag{5.23}$$

In the sequel, we deal only with (5.22) in the understanding that, depending on which assumptions are satisfied, we are referring to either one of the flow functions.

Assumption 5.8. *The plant is port-Hamiltonian given by, either*

(a) $\hat{\Pi}_{\mathrm{p}} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{p}})$ *with (3.34) and satisfies (5.16) for some differentiable* $C : \mathbb{X} \to \mathbb{X}_{\mathrm{c}}$*,*

or by

(b) $\hat{\Pi}_{\mathrm{p}}^{j} = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_{\mathrm{p}})$ *with (5.17) and satisfies (5.15) for some differentiable* $C : \mathbb{X} \to \mathbb{X}_{\mathrm{c}}$

The matrix $F(x)$ *is non-singular.*

Now we can unify lemmas 5.3 and 5.7.

Theorem 5.9. *Under assumption 5.8, the controller (5.4), (5.5) produces the port-Hamiltonian* $\hat{\Pi}_{\mathrm{t}} = (\mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{t}})$ *with behavior*

$$\hat{\mathcal{B}}_{\mathrm{t}} = \left\{ (u_{\mathrm{t}}, y_{\mathrm{t}}, x) \,\middle|\, \textit{Eq. (5.22) holds} \right\} \;, \tag{5.24}$$

For any differentiable $\Phi : \mathbb{X}_{\mathrm{c}} \to \mathbb{R}$*,* $\hat{\Pi}_{\mathrm{t}}$ *is cyclo-passive with storage function*

$$W(x, \xi) = H(x) + \Phi(C(x) - \xi) + H_{\mathrm{c}}(\xi) \;. \tag{5.25}$$

In what follows, we provide general guidelines on how to choose H_{c} and Φ in theorem 5.9 in order to stabilize an arbitrary element of the assignable equilibrium set $\mathcal{E}_x$ (defined in (3.56)).

As a first step, define the set of equilibria $\mathcal{E}$ for $\hat{\Pi}_{\mathrm{t}}$ with zero effort (that is, with $u_{\mathrm{t}} = 0$). According to (5.22) and (5.23),

$$\mathcal{E} = \left\{ (x, \xi) \,\middle|\, F(x)\nabla H(x) - g(x)\nabla H_{\mathrm{c}}(\xi) = 0 \right\} \;. \tag{5.26}$$

According to theorem 5.9, W satisfies

$$\dot{W}(x, \xi) \leq y_{\mathrm{t}}^{\top} u_{\mathrm{t}} \;. \tag{5.27}$$

It follows from standard Lyapunov theory that if W has a strict minimum at a point $(x_\star, \xi_\star) \in \mathcal{E}$ and we set $u_{\mathrm{t}} = 0$, then $(x_\star, \xi_\star)$ is stable. Our goal is then to find appropriate Φ and H_{c}, while imposing conditions on C, such that

$$(x_\star, \xi_\star) = \arg\min W(x, \xi) \;.$$

Of course, negativity of $\dot{W}(x, \xi)$ can be reinforced by setting

$$u_{\mathrm{t}} = -K_{\mathrm{v}} y_{\mathrm{t}} \;, \quad K_{\mathrm{v}} = K_{\mathrm{v}}^{\top} > 0 \;. \tag{5.28}$$

As we have already mentioned, this damping injection is usually adopted to try to make the equilibrium *asymptotically* stable, which is the case if y_{t} is a detectable flow. Unfortunately, we will show below that the latter condition is not satisfied for CbI and we must adopt other strategies, which will be presented in section 5.3. But first we propose a solution to the problem of stabilization of an arbitrary element of $\mathcal{E}_x$.

Proposition 5.10. *Consider the m-port $\hat{\Pi}_t$ with behavior (5.24) and $u_t = 0$. Fix any point $x_\star \in \mathcal{E}_x$ and compute the corresponding $u_\star$ via (3.57). Let*

$$H_c(\xi) = \frac{1}{2}\left\|\xi - K_c^{-1}u_\star\right\|_{K_c}^2 , \tag{5.29}$$

where $K_c = K_c^\top > 0$ and select

$$\Phi(z) = -u_\star^\top z . \tag{5.30}$$

Then $(x_\star, 0)$ is an equilibrium of $\hat{\Pi}_t$, that is, $(x_\star, 0) \in \mathcal{E}$[1]*. Furthermore, $(x_\star, 0)$ is a stable equilibrium if*

$$\nabla^2 H(x_\star) - \sum_{i=1}^{m} u_{\star i}\nabla^2 C_i(x_\star) > 0 . \tag{5.31}$$

Proof: First we prove that $(x_\star, 0) \in \mathcal{E}$ for any $x_\star \in \mathcal{E}_x$. Notice that (5.26) can be written equivalently as

$$\mathcal{E} = \left\{(x,\xi) \,\middle|\, x \in \mathcal{E}_x,\ \nabla H_c(\xi) = g^+(x)F(x)\nabla H(x)\right\} .$$

From the definition of H_c, we get that $\nabla H_c(\xi) = K_c\xi - u_\star$, from which we conclude that

$$\mathcal{E} = \left\{(x_\star, 0) \,\middle|\, x_\star \in \mathcal{E}_x\right\} .$$

We now prove that $(x_\star, 0) = \arg\min W(x,\xi)$ by verifying the conditions

$$\nabla W(x_\star, 0) = 0 \quad \text{and} \quad \nabla^2 W(x_\star, 0) > 0 .$$

Let $\mathcal{A} \triangleq \left\{(x,\xi) \,\middle|\, \nabla W(x,\xi) = 0\right\}$ be the set of critical points of W. From (5.25) one obtains

$$\mathcal{A} = \left\{(x,\xi) \,\middle|\, \nabla H(x) + \nabla C(x)\nabla H_c(\xi) = 0,\ \nabla H_c(\xi) = \nabla\Phi(C(x) - \xi)\right\} . \tag{5.32}$$

Since $\nabla\Phi(C(x) - \xi) = -u_\star$ and $\nabla H_c(\xi) = K_c\xi - u_\star$, the second equation in (5.32) is satisfied by any point of the form $(x, 0)$, with x arbitrary. On the other hand, from assumption 5.2 or 5.5, we have that

$$F(x)\nabla C(x) = -g(x) ,$$

so the set of equilibria (5.26) can be written as

$$\mathcal{E} = \left\{(x,\xi) \,\middle|\, F(x)\left[\nabla H(x) + \nabla C(x)\nabla H_c(\xi)\right] = 0\right\} .$$

Since $F(x)$ is non-singular, we have that

$$\mathcal{E} = \left\{(x,\xi) \,\middle|\, \nabla H(x) + \nabla C(x)\nabla H_c(\xi) = 0\right\} ,$$

which shows that $\mathcal{A} = \mathcal{E}$.

[1] Later on, we will exploit the possibility of setting the equilibrium at points other that $(x_\star, 0)$.

We now give conditions under which the points in $\mathcal{A}$ are minima. Some simple calculations proceeding from

$$W(x,\xi) = H(x) + \frac{1}{2}\left\|\xi - K_c^{-1}u_\star\right\|_{K_c}^2 - u_\star^\top [C(x) - \xi] \ ,$$

yield the Hessian

$$\nabla^2 W(x,\xi) = \begin{pmatrix} \nabla^2 H(x) - \sum_{i=1}^m u_{\star i}\nabla^2 C_i(x) & 0 \\ 0 & K_c \end{pmatrix} \ ,$$

from which it is easy to see that the equilibrium $(x_\star, 0)$ is stable if (5.31) holds. ■

Remark 5.11. By similar arguments, the non-singularity of $F(x)$ can be replaced by the weaker condition

$$\left[I - F^-(x_\star)F(x_\star)\right]\left[\nabla H_\star + \nabla C(x_\star)\nabla H_c(0)\right] = 0 \ .$$

Example, a system affine in the effort

The system described by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_1 + x_2 \\ -x_2^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} - x_2^2 \\ x_2^3 \end{pmatrix} u \tag{5.33}$$

can be written in the port-Hamiltonian form (3.33) with

$$F(x) = \begin{pmatrix} -\frac{1}{2} & x_2 \\ 0 & -x_2^2 \end{pmatrix} \ , \quad H(x) = \frac{1}{2}x_1^2 + x_2 \ , \quad g(x) = \begin{pmatrix} \frac{1}{2} - x_2^2 \\ x_2^3 \end{pmatrix} \tag{5.34}$$

and natural flow

$$y = g^\top(x)\nabla H(x) = x_1\left(\frac{1}{2} - x_2^2\right) + x_2^3 \ .$$

Notice that the port-Hamiltonian structure does not yield any information about the stability of the open-loop equilibrium $(0,0)$, since H is not bounded from below. Actually, it can be readily seen that with $u = 0$ the equilibrium is unstable and that the trajectories of the open-loop system exhibit finite escape time. Moreover, the origin can not be stabilized by any continuous feedback.

The function $C(x) = x_1 + \frac{1}{2}x_2^2$ satisfies (5.15) for the system given by (5.34), that is,

$$F(x)\nabla C(x) = \begin{pmatrix} -\frac{1}{2} & x_2 \\ 0 & -x_2^2 \end{pmatrix}\begin{pmatrix} 1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} + x_2^2 \\ -x_2^3 \end{pmatrix} = -g(x) \ .$$

The matrix $F(x)$ is non-singular everywhere except at the line $x_2 = 0$, that will be ruled out of the analysis. Since assumption 5.5 is satisfied, we apply CbI on the system with the modified flow.

Because of the assignable equilibria set (3.56), we consider equilibria of the form

$$x_\star = \operatorname{col}\left(x_{1\star}, \frac{1}{x_{1\star}}\right) \ ,$$

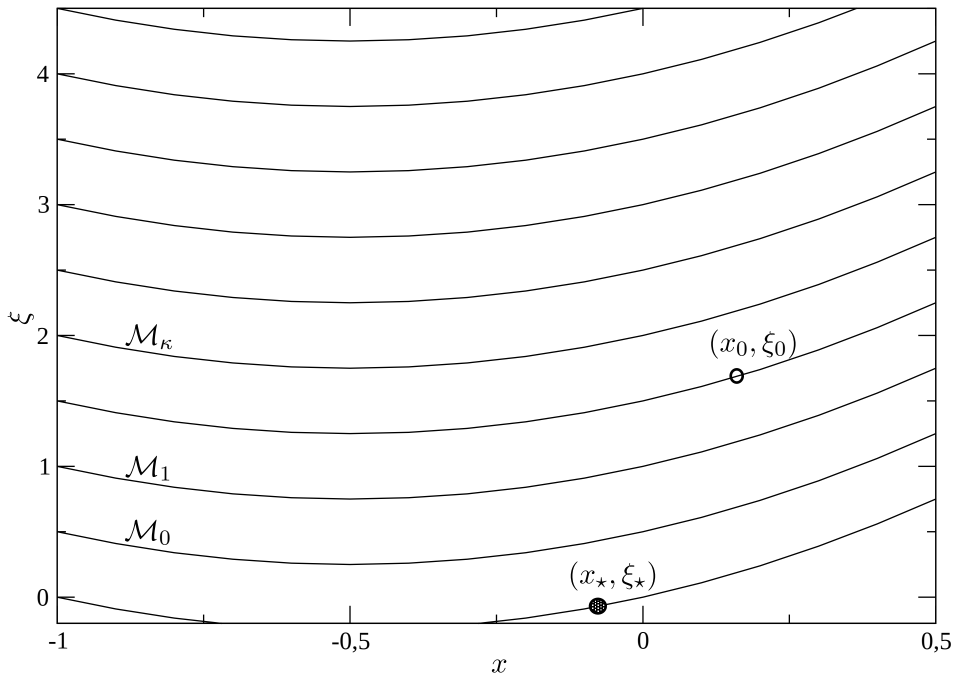

Figure 5.2: The extended state-space is partitioned by a family of invariant manifolds.

$x_{1\star} \in \mathbb{R} \setminus \{0\}$. Remark that $u_\star = x_{1\star}$.

Since the Hessians of H and C are

$$\nabla^2 H(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \nabla^2 C(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ,$$

condition (5.31) is satisfied if, and only if, $u_\star < 0$. Then, by redefining the flow as

$$\begin{aligned} y &= -g^\top(x) F^{-\top}(x)[F(x)\nabla H(x) + g(x)u \\ &= \nabla C^\top(x)\,[F(x)\nabla H(x) + g(x)u] \end{aligned}$$

and applying proposition 5.10, any point of the form $(x_{1\star}, \frac{1}{x_{1\star}})$, $x_{1\star} < 0$, is stabilized by a controller with behavior

$$\hat{\mathcal{B}}_\mathrm{c} = \left\{ (u_\mathrm{t}, y_\mathrm{t}, u_\mathrm{c}, y_\mathrm{c}, x, \xi) \,\middle|\, \dot{\xi} = -y_\mathrm{c},\; u_\mathrm{c} - u_\mathrm{t} = -K_\mathrm{c}\xi + u_\star,\; y_\mathrm{t} = -y_\mathrm{c} \right\} .$$

5.3 Asymptotic stability

Notice that the key property that allows us to shape the storage function along x is the invariance of $C(x) - \xi$. In other words, the key step in successfully shaping the energy is the generation of the invariant manifolds

$$\mathcal{M}_\kappa = \left\{ (x, \xi) \,\middle|\, C(x) - \xi = \kappa \right\} .$$

Unfortunately, the latter poses the following problem. Suppose the system starts at an arbitrary initial condition (x_0, ξ_0). There is no reason why the desired equilibrium $(x_\star, \xi_\star)$ should satisfy

$$C(x_\star) - \xi_\star = C(x_0) - \xi_0 \tag{5.35}$$

(see Fig. 5.2). One way to fulfill (5.35) is to initialize the controller at the value ξ_0 that puts the system in the proper invariant manifold. This approach is simple but the dependence on the initial conditions makes it highly non-robust. In general, $(x_\star, \xi_\star)$ does not belong to the orbit of the solution starting at (x_0, ξ_0), hence the flow y_t is not detectable and the desired equilibrium might be stable but not asymptotically stable, even with the damping injection (5.28).

In what follows we present two alternative solutions to the problem. Before giving these results let us take a closer look at our example to get a clearer picture of the role of the Casimir function.

Example, a system affine in the effort (continued)

Suppose that we want to stabilize the point $(-1, -1, 0)$, so that $u_\star = x_{1\star} = -1$. By setting $K_c = 1$, the Lyapunov function is

$$\begin{aligned} W(x,\xi) &= H(x) - u_\star^\top (C(x) - \xi) + H_c(\xi) \\ &= \frac{1}{2}x_1^2 + x_2 + \frac{1}{2}x_2^2 + x_1 - \xi + \frac{1}{2}\xi^2 + \xi + \frac{1}{2} \\ &= \frac{1}{2}\left[(x_1+1)^2 + (x_2+1)^2 + \xi^2\right] - \frac{1}{2} , \end{aligned}$$

the level sets of which are spheres centered at $(-1, -1, 0)$.

Suppose, further, that the system is initially at $(x_0, \xi_0) = (\frac{3}{2}, -\frac{1}{2}, \frac{13}{8})$, so that

$$C(x_0) - \xi_0 = \frac{3}{2} + \frac{1}{2}\frac{1}{4} - \frac{13}{8} = 0 .$$

Since $C(x_\star) - \xi_\star = -1 + \frac{1}{2} + 0 \neq 0$, the trajectory does not reach the desired value. The trajectories cannot diverge either, since W is radially unbounded. Instead, the trajectory reaches an invariant set contained in the invariant manifold

$$\mathcal{M}_0 = \{(x,\xi) \,|\, C(x) - \xi = 0\} .$$

The set $\mathcal{E}$ is the union of the sets described by the parametrized curves

$$q_1(x_1) = \mathrm{col}\left(x_1, \frac{1}{x_1}, -x_1 - 1\right) , \quad x_1 \in \mathbb{R} \setminus \{0\}$$

and

$$q_2(x_1) = \mathrm{col}(x_1, 0, -x_1 - 1) , \quad x_1 \in \mathbb{R}$$

(see appendix B.3 for details). Note that

$$\mathcal{E} \cap \mathcal{M}_0 = \{(-0.85, -1.18, -0.15), (-0.5, 0, -0.5)\} .$$

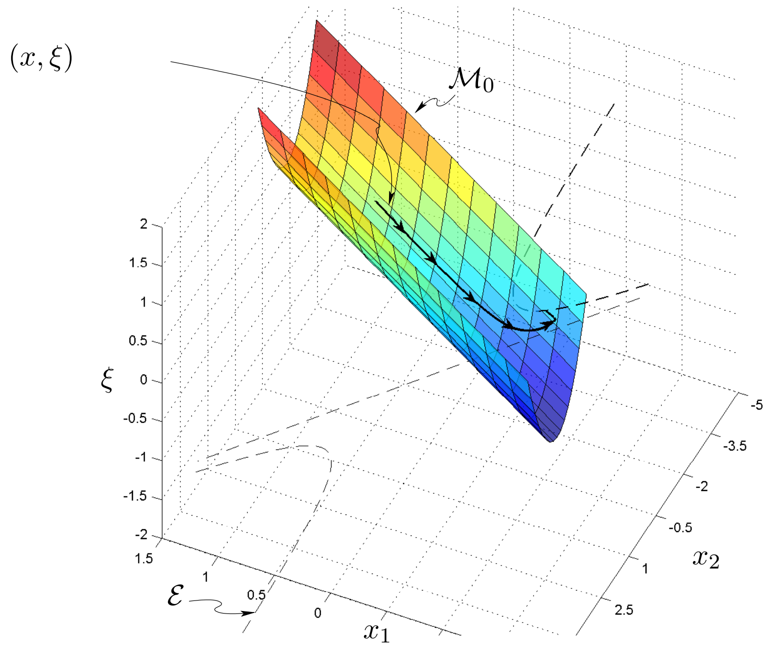

Figure 5.3: The invariant manifold $\mathcal{M}_0$, the equilibria locus $\mathcal{E}$ and the simulated response.

Fig. 5.3 shows $\mathcal{M}_0$, $\mathcal{E}$ and the trajectory starting at $(x_0, \xi_0) = (\frac{3}{2}, -\frac{1}{2}, \frac{13}{8})$ and converging to $(-0.85, -1.18, -0.15)$. Fig. 5.4 shows the intersection of $\mathcal{M}_0$ and the level sets of W with the planes $x_2 = x_{2\star} = -1$ and $\xi = \xi_\star = 0$. The projections of $\mathcal{E}$ and the trajectory are also shown.

5.3.1 Adaptive control by interconnection

It is clear that another way to satisfy the constraint (5.35) is by shifting away from zero the desired value of ξ to the new value

$$\xi_\star = C(x_\star) - C(x_0) + \xi_0 \ . \tag{5.36}$$

This amounts to changing H_c to

$$H_\mathrm{c}(\xi) = \frac{1}{2} \left\| \xi - \xi_\star - K_\mathrm{c}^{-1} u_\star \right\|_{K_\mathrm{c}}^2 \ , \tag{5.37}$$

so that $\nabla H_\mathrm{c}(\xi_\star) = -u_\star$. Geometrically, we are shifting the equilibrium locus $\mathcal{E}$ along ξ, so that the intersection between $\mathcal{E}$ and the manifold where the trajectory starts (i.e., $\mathcal{M}_{\kappa_0}$ with $\kappa_0 \triangleq C(x_0) - \xi_0$) occurs at the desired $x_\star$.

In principle, this scheme still hinges on knowledge of the initial condition, but this issue can be removed by reformulating it as a parameter estimation problem. We try first a classical certainty-equivalence adaptive control approach viewing $\xi_\star$ as the unknown

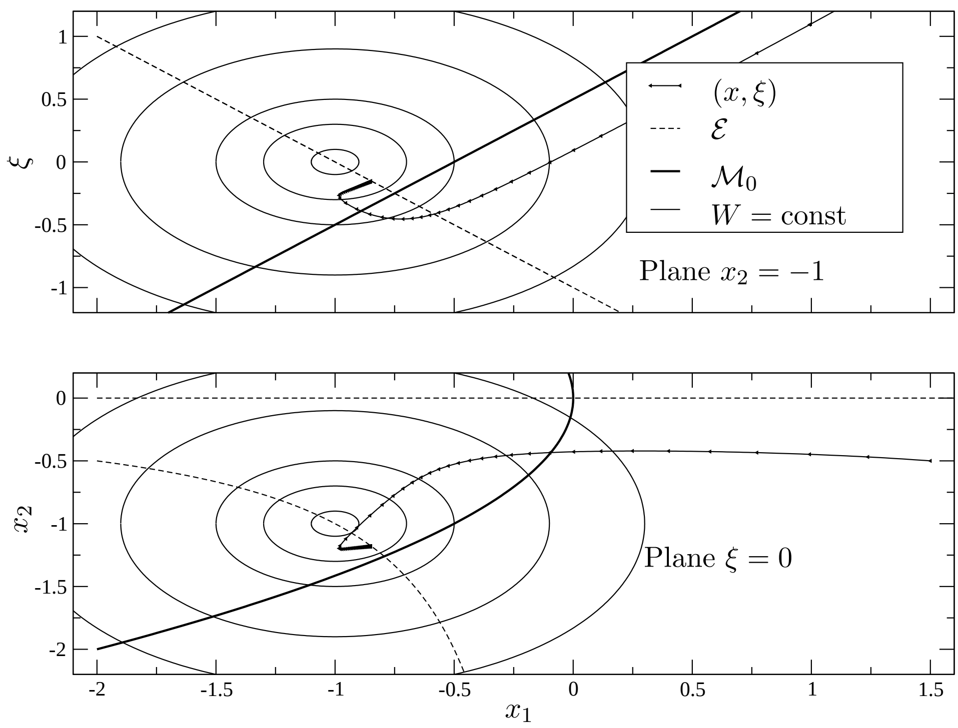

Figure 5.4: Level sets of W and invariant manifold $\mathcal{M}_0$, equilibria locus $\mathcal{E}$ and simulated response, projected into the planes $x_2 = -1$ (above) and $\xi = 0$ (below).

parameter. This is indeed possible because the plant is linear in u and, for quadratic H_c, $\xi_\star$ enters also linearly in u. Define a new storage function for the building block (5.2) as

$$\bar{H}_c(\xi, \hat{\xi}_\star) \triangleq \frac{1}{2} \left\| \xi - \hat{\xi}_\star - K_c^{-1} u_\star \right\|_{K_c}^2 ,$$

where $\hat{\xi}_\star$ denotes the estimate of $\xi_\star$. Let us compute

$$\begin{aligned} \nabla_\xi \bar{H}_c(\xi, \hat{\xi}_\star) &= K_c(\xi - \hat{\xi}_\star) - u_\star = K_c(\xi - \xi_\star) - u_\star - K_c\tilde{\xi}_\star \\ &= \nabla H_c(\xi) - K_c\tilde{\xi}_\star , \end{aligned}$$

where we have defined the parameter error $\tilde{\xi}_\star \triangleq \hat{\xi}_\star - \xi_\star$. The building block is then redefined as

$$\begin{aligned} \dot{\xi} &= -y_{c\,1} \\ u_{c\,1} &= -\nabla_\xi \bar{H}_c(\xi, \hat{\xi}_\star) \\ &= -\nabla H_c(\xi) + K_c\tilde{\xi}_\star . \end{aligned}$$

The compound m-port is still of the form (5.22), but with u_t replaced by $u_t + K_c\tilde{\xi}_\star$, that is,

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} &= F_t(x)\nabla H_t(x, \xi) + g_t(x)(u_t + K_c\tilde{\xi}_\star) \\ y_t &= g_t^\top(x)\nabla H_t(x) . \end{aligned}$$

Since the invariance of the manifolds $\mathcal{M}_\kappa$ is preserved, the power balance (5.27) is still satisfied, but with the 'new u_t':

$$\dot{W}(x, \xi) \leq y_t^\top(u_t + K_c\tilde{\xi}_\star) .$$

Proceeding with the classical adaptive control design, we would propose a candidate Lyapunov function

$$V(x, \xi, \tilde{\xi}_\star) = W(x, \xi) + \frac{1}{2}\|\tilde{\xi}_\star\|_{\Gamma^{-1}} , \quad \Gamma = \Gamma^\top > 0 ,$$

and an estimation law of the form $\dot{\hat{\xi}}_\star = -\Gamma K_c y_t$, which would make

$$\begin{aligned} \dot{V}(x, \xi, \tilde{\xi}_\star) = \dot{W}(x, \xi) - \tilde{\xi}_\star^\top \Gamma^{-1}\Gamma K_c y_t &= \dot{W}(x, \xi) - \tilde{\xi}_\star^\top K_c y_t \\ &\leq y_t^\top u_t + y_t^\top K_c\tilde{\xi}_\star - \tilde{\xi}_\star^\top K_c y_t = y_t^\top u_t . \end{aligned}$$

Unfortunately, this simple scheme will not solve our problem. Indeed, the zero-level set of $\dot{V}$ in the adaptive scenario and the zero-level set of $\dot{W}$ without the adaptive scheme are the same, so the lack of detectability problem is still present. The only way to achieve the desired objective is to ensure parameter convergence, that is, $\lim_{t\to\infty} \tilde{\xi}_\star = 0$, which is not satisfied due to existence of a manifold of equilibria.

It turns out that if we estimate the parameter κ_0 (instead of $\xi_\star$) and use the invariance of the manifold $\mathcal{M}_{\kappa_0}$, we can design a scheme that ensures parameter convergence. To formulate the precise statement, consider the set of equations

$$\begin{aligned} \dot{\xi} &= -y_{\mathrm{c}\,1} && (5.38\mathrm{a}) \\ \dot{\hat{\kappa}}_0 &= -\Gamma\left(\hat{\kappa}_0 - C(x) + \xi\right) && (5.38\mathrm{b}) \\ u_{\mathrm{c}\,1} &= -\nabla_\xi \hat{H}_{\mathrm{c}}(\xi, \hat{\kappa}_0) && (5.38\mathrm{c}) \end{aligned}$$

with $\hat{H}_{\mathrm{c}}(\xi,\hat{\kappa}_0) \triangleq \frac{1}{2}\left\|\xi - C(x_\star) + \hat{\kappa}_0 - K_{\mathrm{c}}^{-1}u_\star\right\|_{K_{\mathrm{c}}}^2$. Define the adaptive building block $\hat{\Pi}_{\mathrm{c}\,1}^{\mathrm{f}} = (\mathbb{W}, \mathbb{X}_{\mathrm{c}} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}\,1})$ with behavior

$$\hat{\mathcal{B}}_{\mathrm{c}\,1} = \left\{(u_{\mathrm{c}\,1}, y_{\mathrm{c}\,1}, \xi, \hat{\kappa}_0) \,\middle|\, \text{Eq. (5.38) holds}\right\}$$

and construct the controller

$$\hat{\Pi}_{\mathrm{c}} = (\mathbb{W} \times \mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}}) \tag{5.39}$$

with

$$\hat{\mathcal{B}}_{\mathrm{c}} = \left\{(u_{\mathrm{c}}, y_{\mathrm{c}}, u_{\mathrm{t}}, y_{\mathrm{t}}, x, \xi, \hat{\kappa}_0) \,\middle|\, (u_{\mathrm{c}} - u_{\mathrm{t}}, y_{\mathrm{c}}, \xi, \hat{\kappa}_0) \in \hat{\mathcal{B}}_{\mathrm{c}\,1},\ y_{\mathrm{c}} = -y_{\mathrm{t}}\right\} . \tag{5.40}$$

Proposition 5.12. *Consider a $\hat{\Pi}_{\mathrm{p}}$ satisfying assumption 5.8. The compound m-port $\hat{\Pi}_{\mathrm{t}}$, resulting from the controller (5.39), (5.40) satisfies the following:*

(i) Exponential parameter convergence is ensured, more precisely

$$\|\hat{\kappa}_0 - \kappa_0\| \leq e^{-\lambda_{min}\{\Gamma\}t}\|\hat{\kappa}_0(0) - \kappa_0\|$$

for all $t \geq 0$.

Close the port with $u_{\mathrm{t}} = -K_v y_{\mathrm{t}}$. Then,

(ii) For any $x_\star \in \mathcal{E}_x$ the point $(x_\star, \xi_\star, \kappa_0)$, where $\xi_\star$ is given in (5.36), is a stable equilibrium if (5.31) holds.

(iii) The orbits of the residual dynamics are confined to the set $\mathcal{Z} \times \{\xi = \bar{\xi}\}$, where $\bar{\xi}$ is a constant and

$$\mathcal{Z} \triangleq \left\{x \,\middle|\, \begin{pmatrix}\nabla H^\top(x) \\ \nabla C^\top(x)\end{pmatrix}[F(x)\nabla H(x) - g(x)\left(K_{\mathrm{c}}(C(x) - C(x_\star)) - u_\star\right)] = 0\right\} .$$

(iv) Suppose no trajectory x can stay identically in $\mathcal{Z}$, other than isolated points. Then, $(x_\star, \xi_\star, \kappa_0)$ is an asymptotically stable equilibrium. It will be globally asymptotically stable if it is the only point in $\mathcal{Z}$ and if W is radially unbounded.

Proof: Define $\tilde{\kappa}_0 \triangleq \hat{\kappa}_0 - \kappa_0$. From invariance of $\mathcal{M}_{\kappa_0}$ we have that

$$\kappa_0 = C(x_0) - \xi_0 = C(x) - \xi \; .$$

Consequently, $\dot{\tilde{\kappa}}_0 = -\Gamma\tilde{\kappa}_0$, from which claim (i) follows immediately.

Proceeding as we did for the standard adaptive controller above, one has that

$$\nabla_\xi \hat{H}_{\mathrm{c}}(\xi, \hat{\kappa}_0) = \nabla H_{\mathrm{c}}(\xi) - K_c\tilde{\kappa}_0 = -\nabla H_{\mathrm{c}}(\xi) + K_c\tilde{\kappa}_0$$

and we have the power balance

$$\dot{W}(x,\xi) = y_{\mathrm{t}}^\top(u_{\mathrm{t}} - K_c\tilde{\kappa}_0) - d_{\mathrm{t}}(x,\xi) \; , \tag{5.41}$$

where

$$d_{\mathrm{t}}(x,\xi) = -\nabla H_{\mathrm{t}}^\top(x,\xi) F_{\mathrm{t}}(x) \nabla H_{\mathrm{t}}(x,\xi) \; . \tag{5.42}$$

Consider the candidate Lyapunov function

$$V(x,\xi,\tilde{\kappa}_0) = W(x,\xi) + \frac{1}{2}\|\tilde{\kappa}_0\|^2_{\mu\Gamma^{-1}} \; ,$$

with $\mu > 0$. Differentiation with respect to time and some standard bounding shows that, for all K_v, K_c, Γ, there exists μ such that

$$\dot{V}(x,\xi,\tilde{\kappa}_0) \le -d_{\mathrm{t}}(x,\xi) - \epsilon(\|y_{\mathrm{t}}\|^2 + \|\tilde{\kappa}_0\|^2) \tag{5.43}$$

holds for some $\epsilon > 0$, which shows that V is a Lyapunov function, so the equilibrium is stable and (ii) is established.

Now we apply LaSalle's theorem [29] and conclude from (5.43) that $d_{\mathrm{t}}(x,\xi)$ and y_{t} tend to zero as $t \to \infty$. The residual dynamics are obtained imposing the restrictions $d_{\mathrm{t}}(x,\xi) = 0$, $y_{\mathrm{t}} = 0$ and $\tilde{\kappa}_0 = 0$. First, note that with $\tilde{\kappa}_0 = 0$ the dynamics reduce to those of (5.22). Second, $y_{\mathrm{t}} = 0$ implies $u_{\mathrm{t}} = 0$ and $\dot{\xi} = 0$, consequently $\xi = \bar{\xi}$. Furthermore, from the equation of $\dot{\xi}$ in (5.22), we have

$$0 = \dot{\xi} = \nabla C^\top(x)\left[F(x)\nabla H - g\nabla H_{\mathrm{c}}(\bar{\xi})\right] \; . \tag{5.44}$$

By developing (5.42) one obtains

$$\begin{aligned} d_{\mathrm{t}}(x,\xi) &= -\begin{pmatrix}\nabla H^\top(x) & \nabla H_{\mathrm{c}}^\top(\xi)\end{pmatrix}\begin{pmatrix} I \\ \nabla C^\top(x)\end{pmatrix}\begin{pmatrix}F(x) & -g(x)\end{pmatrix}\begin{pmatrix}\nabla H(x) \\ \nabla H_{\mathrm{c}}(\xi)\end{pmatrix} \\ &= -\left[\nabla H^\top(x) + \nabla H_{\mathrm{c}}^\top(\xi)\nabla C^\top(x)\right]\left[F(x)\nabla H(\xi) - g(x)\nabla H_{\mathrm{c}}(\xi)\right] \; . \end{aligned} \tag{5.45}$$

From $d_{\mathrm{t}}(x,\xi) = 0$ and (5.44) one concludes that

$$\nabla H^\top(x)\left[F(x)\nabla H(x) - g(x)\nabla H_{\mathrm{c}}(\bar{\xi})\right] = 0 \; . \tag{5.46}$$

The proof of (iii)) is completed noting that $C(x) - \bar{\xi} = \kappa_0$ and evaluating $\nabla H_{\mathrm{c}}(\xi)$ at $\bar{\xi}$.

The proof of (iv) is a direct consequence of the celebrated theorem by Barbashin and Krasovskii [4]. ∎

Example, a system affine in the effort (continued)

We now apply adaptive CbI to the example. Except for points on the hyperbola $x_1 x_2 = 1$, the matrix

$$\begin{pmatrix} \nabla H^\top(x) \\ \nabla C^\top(x) \end{pmatrix} = \begin{pmatrix} x_1 & 1 \\ 1 & x_2 \end{pmatrix} \tag{5.47}$$

is non-singular, so the orbits of the residual dynamics are confined to equilibrium points $(x, \xi) \in \mathcal{E}$ satisfying

$$F(x)\nabla H(x) - g(x)(C(x) - C(x_\star) + u_\star) = 0 \; .$$

For all $x_{1\star} < -\frac{1}{2}$ the only solutions of the above equation are[2] $\bar{x}^{\mathrm{I}} = \mathrm{col}(x_{1\star}, x_{2\star})$ and $\bar{x}^{\mathrm{II}} = \mathrm{col}(x_{1\star} + \frac{1}{4}x_{2\star}^2, 0)$. When $x_1 x_2 = 1$, the vector $\mathrm{col}(x_2, -1)$ is an eigenvector associated to the zero eigenvalue of the matrix (5.47), so points x satisfying

$$F(x)\nabla H(x) - g(x)(C(x) - C(x_\star) + u_\star) = \begin{pmatrix} x_2 \\ -1 \end{pmatrix} \psi(x) \tag{5.48}$$

for some function $\psi : \mathbb{R}^2 \to \mathbb{R}$ can also contain the orbits of the residual dynamics. Since $x_1 x_2 = 1$ implies $g^\perp(x)F(x)\nabla H(x) = 0$ (see appendix B.3 for details), premultiplication of (5.48) by $g^\perp(x)$ gives

$$g^\perp(x) \begin{pmatrix} x_2 \\ 1 \end{pmatrix} = 0 \; .$$

The solution set of the previous equation is empty, which implies that

$$\mathcal{Z} = \left\{ \mathrm{col}(x_{1\star}, x_{2\star}), \mathrm{col}\left(x_{1\star} + \frac{1}{4}x_{2\star}^2, 0 \right) \right\} \; .$$

Fig. 5.5 shows that now $\mathcal{M}_0$ and $\mathcal{E}$ intersect at the desired $x_\star$. Convergence towards the desired value is achieved with the adaptive scheme.

5.3.2 Adding extra damping to the controller

Another possible way to achieve convergence is to destroy the invariance of the Casimirs by adding a damping injection to the controller. The idea is to go back to the previous storage function (5.29) (repeated for here for ease of reference)

$$H_{\mathrm{c}}(\xi) = \frac{1}{2} \left\| \xi - K_{\mathrm{c}}^{-1} u_\star \right\|_{K_{\mathrm{c}}}^2 \; , \tag{5.49}$$

but add an extra pair $(w, z) \in \mathbb{W}$ of port variables to $\hat{\Pi}^{\mathrm{f}}_{\mathrm{c}\,1}$. This is accomplished in the following way. Redefine the building block as

$$\hat{\Pi}^{\mathrm{f}}_{\mathrm{c}\,1} = (\mathbb{W} \times \mathbb{W}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}\,1}) \; .$$

[2] The details are not shown, but this fact can be verified by looking at the discriminant of the resulting cubic polynomial.

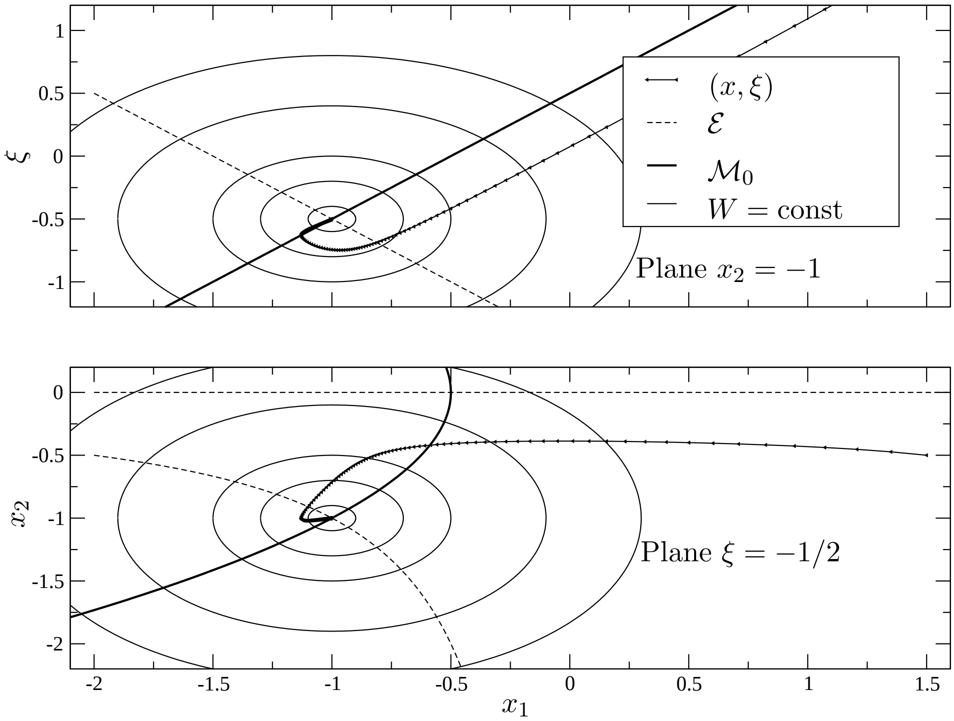

Figure 5.5: Level sets of W and invariant manifold $\mathcal{M}_0$, all intersected with the planes $x_2 = -1$ (above) and $\xi = -\frac{1}{2}$ (below). Equilibrium set $\mathcal{E}$ and simulated response, both projected into the planes $x_2 = -1$ and $\xi = -\frac{1}{2}$.

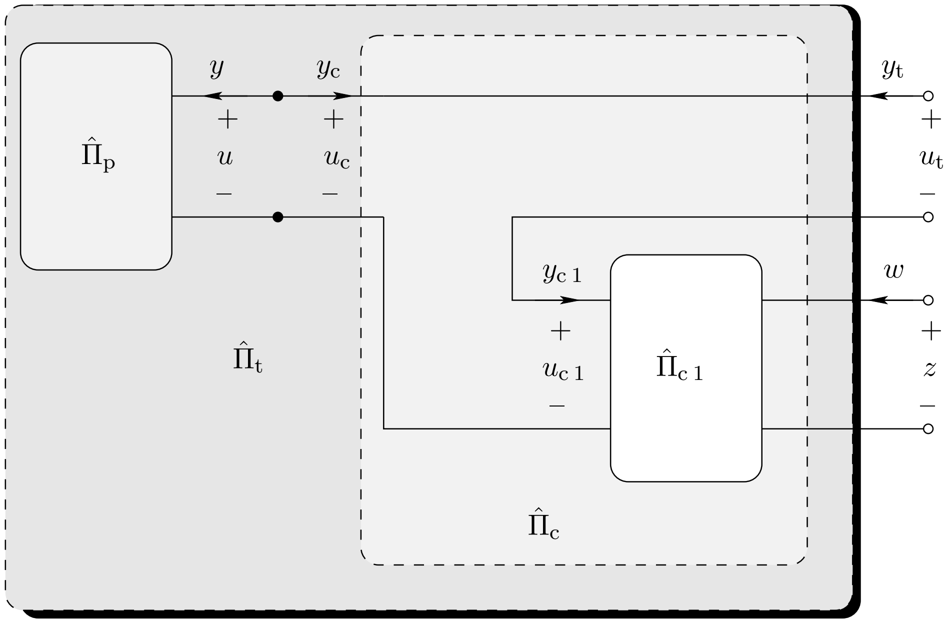

Figure 5.6: The control by interconnection scheme is modified by adding an extra port. Further damping can be injected, destroying the Casimirs.

The equations

$$\begin{aligned} \dot{\xi} &= -y_{c\,1} + w && (5.50a) \\ u_{c\,1} &= -\nabla H_c(\xi) && (5.50b) \\ z &= \nabla_\xi W(x, \xi) \ , && (5.50c) \end{aligned}$$

with W as in (5.25), define the behavior of $\hat{\Pi}^f_{c\,1}$ as

$$\hat{\mathcal{B}}_{c\,1} = \left\{ (u_{c\,1}, y_{c\,1}, w, z, x, \xi) \,\middle|\, \text{Eq. (5.50) holds} \right\} \ .$$

The controller $\hat{\Pi}_c = (\mathbb{W} \times \mathbb{W}_t \times \mathbb{W}_t, \mathbb{X} \times \mathbb{X}_c, \hat{\mathcal{B}}_c)$ is then constructed as

$$\hat{\mathcal{B}}_c = \left\{ (u, y, u_c, y_c, z, w, x, \xi \,\middle|\, (u - u_t, y_c, w, z, \xi) \in \hat{\mathcal{B}}_{c\,1}, \ y_c = -y_t \right\}$$

(see Fig. 5.6). It is readily verified that $\hat{\Pi}_t$ is determined by

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} &= F_t \nabla H_t + g_t u_t + \begin{pmatrix} 0 \\ I \end{pmatrix} w && (5.51a) \\ y_t &= g_t^\top \nabla H_t && (5.51b) \\ z &= \begin{pmatrix} 0 & I \end{pmatrix} \nabla W \ . && (5.51c) \end{aligned}$$

Notice that for all $w \neq 0$, the invariance of the manifolds $\mathcal{M}_\kappa$ is destroyed because $\dot{C}(x) - \dot{\xi} = -w$. However, the time derivative of W is

$$\dot{W}(x, \xi) = y_{\mathrm{t}}^\top u_{\mathrm{t}} + w^\top z - d_{\mathrm{t}}(x, \xi) \ , \tag{5.52}$$

so the new system is also cyclo-passive with storage function W and port variables $((u_{\mathrm{t}}, z), (y_{\mathrm{t}}, w))$.

Regarding stability, we have the homologue of proposition 5.10.

Proposition 5.13. *Consider the 2m-port* $\hat{\Pi}_{\mathrm{t}} = \left\{ \mathbb{W}_{\mathrm{t}} \times \mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{t}} \right\}$*, with behavior given by (5.51),* H_{c} *given by (5.49) and* Φ *by (5.30). Close the ports with*

$$u_{\mathrm{t}} = -K_{\mathrm{v}} y_{\mathrm{t}} \ , \quad K_{\mathrm{v}} = K_{\mathrm{v}}^\top > 0 \tag{5.53}$$

and

$$w = -K_{\mathrm{w}} z \ , \quad K_{\mathrm{w}} = K_{\mathrm{w}}^\top > 0 \ . \tag{5.54}$$

(i) For any $x_\star \in \mathcal{E}_x$*, the point* $(x_\star, 0)$ *is a stable equilibrium if (5.31) holds.*

(ii) The orbits of the residual dynamics are confined to the set $\mathcal{Z}_{\mathrm{w}} \times \{\xi = 0\}$*, where*

$$\mathcal{Z}_{\mathrm{w}} = \left\{ x \mid \begin{pmatrix} \nabla H^\top(x) \\ \nabla C^\top(x) \end{pmatrix} [F(x) \nabla H(x) - g(x) u_\star] = 0 \right\} \ .$$

(iii) If no trajectory x *can stay identically in* $\mathcal{Z}_{\mathrm{w}}$*, other than isolated points,* $(x_\star, 0)$ *is an asymptotically stable equilibrium. It will be globally asymptotically stable if it is the only point in* $\mathcal{Z}_{\mathrm{w}}$ *and if* W *is radially unbounded.*

Proof: Take W as a candidate Lyapunov function. Equations (5.52), (5.53) and (5.54) imply that it is a Lyapunov function and (i) follows.

Applying LaSalle's theorem gives that $d_{\mathrm{t}}(x, \xi)$, y_{t} and z tend to zero as $t \to \infty$. The residual dynamics are those of $\hat{\Pi}_{\mathrm{t}}$ with the restrictions $d_{\mathrm{t}}(x, \xi) = 0$, $y_{\mathrm{t}} = 0$ and $z = 0$. From the latter it follows that $\nabla_\xi W(x, \xi) = 0$, which implies

$$\nabla H_{\mathrm{c}}(\xi) = \nabla \Phi(C(x) - \xi) = u_\star \ ,$$

which in turn implies $\xi = 0$. From the equation of $\dot{\xi}$, with ξ, u_{t} and w equal to zero, we get

$$0 = \dot{\xi} = \nabla C^\top(x) \, [F(x) \nabla H(x) - g(x) u_\star] = 0 \ ,$$

which is the second row in $\mathcal{Z}_{\mathrm{w}}$. From this equation and (5.45) one is lead to conclude that $\nabla H^\top(x) \, [F(x) \nabla H(x) - g(x) u_\star] = 0$, which gives the first row and completes the proof of (ii).

Point (iii) follows from Barbashin-Krasovskii's theorem. ■

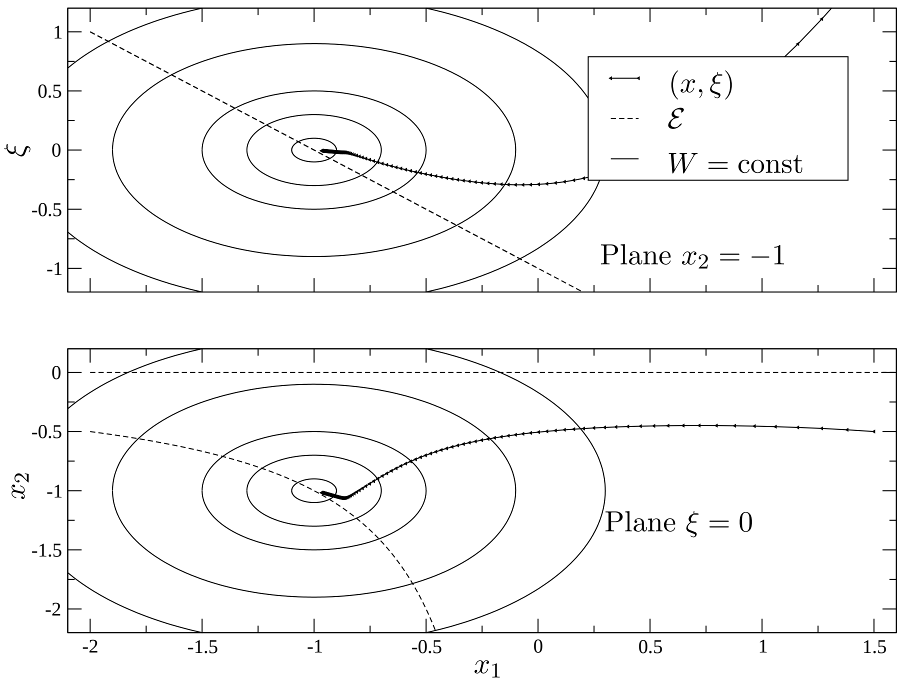

Figure 5.7: Level sets of W intersected with the planes $x_2 = -1$ (above) and $\xi = 0$ (below). Equilibria locus $\mathcal{E}$ and simulated response, both projected into the planes $x_2 = -1$ and $\xi = 0$.

Example, a system affine in the effort (continued)

We now apply CbI with extra damping in the controller. The analysis follows along the same lines as in the adaptive CbI scenario. In this case

$$\mathcal{Z}_\text{w} = \{\text{col}(x_{1\star}, x_{2\star}), \text{col}(x_{1\star}, 0)\} \ .$$

Fig. 5.7 shows the trajectories of the system for $K_\text{w} = 2$, which are no longer restricted to $\mathcal{M}_0$. Again, convergence to $x_\star$ is achieved.

Fig. 5.8 shows that for the initial condition $(x_0, \xi_0) = (-1/2, 1/2, 0)$, convergence of x is towards $(x_{1\star} + x_{2\star}^2/4, 0) = (-3/4, 0)$ for the adaptive CbI and towards $(x_{1\star}, 0) = (-1, 0)$ for the CbI with extra damping injection. Indeed, since $\mathcal{Z}$ and $\mathcal{Z}_\text{w}$ contain more than one point, stability is global but convergence is not. Notice, however, that in the controller damping injection scenario, the exact value of the unwanted equilibrium is known. This, together with the fact that the Lyapunov function W is non-decreasing over time, allows to obtain an estimate of the region of attraction: the open ball centered at $\text{col}(x_{1\star}, x_{2\star}, 0)$ and of radius $|\,\text{col}(x_{1\star}, x_{2\star}, 0) - \text{col}(x_{1\star}, 0, 0)| = |x_{2\star}|$.

The existence of the Casimir functions, inherent in the CbI design methodology, present an obstacle for asymptotic convergence of the state towards a desired equilibrium. To surmount this obstacle, we have presented two variations of the method. Paradoxi-

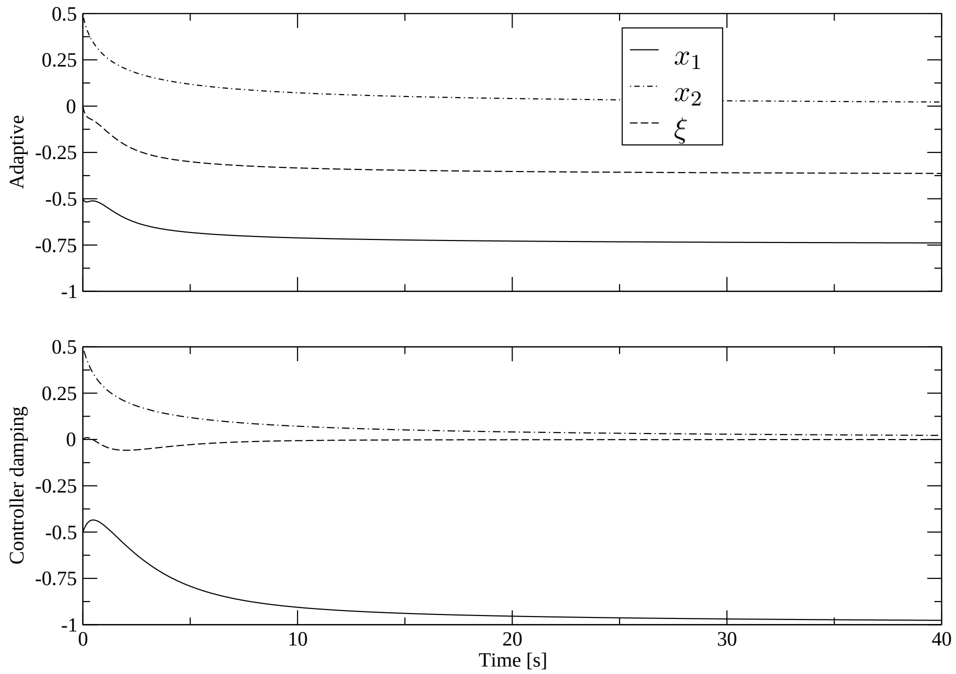

Figure 5.8: Response of the system when it is set at the initial condition $(-\frac{1}{2}, \frac{1}{2}, 0)$. Adaptive CbI (above) and controller damping CbI (below).

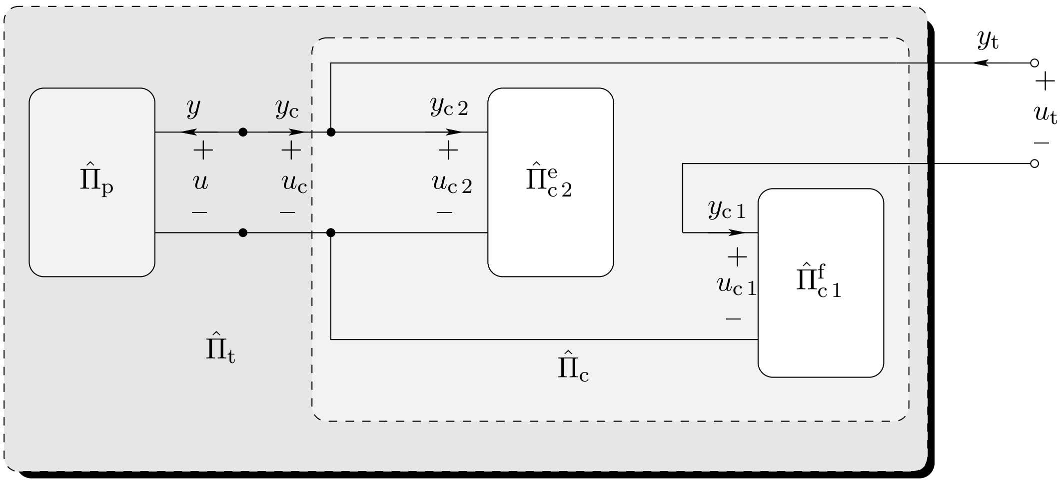

Figure 5.9: Flow and dissipation invariance are destroyed by adding $\hat{\Pi}^e_{c\,2}$.

cally, once the modified versions are used, the same Casimir functions narrow the possible limit sets, thus contributing to the desired asymptotic behavior. The Casimir functions also simplify the analysis of such limit sets, as they provide m algebraic constraints that, as shown in the example, can sometimes obviate the need to differentiate the output to obtain the residual dynamics. Interestingly, each method generates a different limit set.

5.4 Breaking up the flow and dissipation invariance

In the previous sections we showed that, in its standard formulation, CbI imposes strong restrictions arising from the flow and dissipation preservation. We also showed that the restriction

$$\nabla C^\top(x)g(x) = 0 \tag{5.55}$$

can be removed by using the invariant flow (4.99b), which naturally preserves flow and dissipation under BIDA.

In this section, we show that by intentionally building a non-preserving controller, it is also possible to remove (5.55) without using the invariant flow.

Assumption 5.14. $\hat{\Pi}_p = (\mathbb{W}, \mathbb{X}, \hat{\mathcal{B}}_p)$ *is port-Hamiltonian with (3.34) and satisfies (5.15) for some differentiable map* $C : \mathbb{X} \to \mathbb{X}_c$

Consider again the flow-driven m-port $\hat{\Pi}^f_{c\,1}$ described by (5.1) and (5.3). Set W as in (5.25) and build an additional effort-driven m-port as

$$\hat{\Pi}^e_{c\,2} = (\mathbb{W}, \mathbb{X} \times \mathbb{X}_c, \hat{\mathcal{B}}_{c\,2})$$

with behavior

$$\hat{\mathcal{B}}_{c\,2} = \left\{ (u_{c\,2}, y_{c\,2}, x, \xi) \,\middle|\, y_{c\,2} = g^\top(x)\nabla C(x)\nabla\Phi(C(x) - \xi) \right\} .$$

Use $\hat{\Pi}^{\mathrm{f}}_{\mathrm{c}\,1}$ and $\hat{\Pi}^{\mathrm{e}}_{\mathrm{c}\,2}$ to construct the dynamic controller $\hat{\Pi}_{\mathrm{c}} = (\mathbb{W} \times \mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{c}})$ with

$$\hat{\mathcal{B}}_{\mathrm{c}} = \Big\{ (u_{\mathrm{c}}, y_{\mathrm{c}}, u_{\mathrm{t}}, y_{\mathrm{t}}, x, \xi) \;\big|\; (u_{\mathrm{c}} - u_{\mathrm{t}}, y_{\mathrm{t}}, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,1},\ (u_{\mathrm{c}}, y_{\mathrm{c}} + y_{\mathrm{t}}, x, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,2} \Big\} \tag{5.56}$$

(see Fig. 5.9 and confront Fig. 4.1).

Proposition 5.15. *Under assumption 5.14,*

(i) the controller (5.56) is passivity-based. Moreover, $\hat{\Pi}_{\mathrm{t}}$ is port-Hamiltonian with Hamiltonian function

$$W(x, \xi) = H(x) + \Phi(C(x) - \xi) + H_{\mathrm{c}}(\xi) \; .$$

Set $H_{\mathrm{c}}(\xi) = \frac{1}{2} \left\| \xi - K_{\mathrm{c}}^{-1} u_\star \right\|^2_{K_{\mathrm{c}}}$ and $\Phi(z) = -u_\star^\top z$, then,

(ii) for any $x_\star \in \mathcal{E}_x$, the point $(x_\star, 0)$ is a stable equilibrium if (5.31) holds.

Proof: The compound m-port $\hat{\Pi}_{\mathrm{t}} = \hat{\Pi}_{\mathrm{p}} \wedge \hat{\Pi}_{\mathrm{c}} = (\mathbb{W}_{\mathrm{t}}, \mathbb{X} \times \mathbb{X}_{\mathrm{c}}, \hat{\mathcal{B}}_{\mathrm{t}})$ has the behavior

$$\hat{\mathcal{B}}_{\mathrm{t}} = \Big\{ (u_{\mathrm{t}}, y_{\mathrm{t}}, x, \xi) \;\big|\; \exists\, (u, y) \text{ s.t. } (u, y, x) \in \hat{\mathcal{B}}_{\mathrm{p}},\ (u, -y, u_{\mathrm{t}}, y_{\mathrm{t}}, x, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}} \Big\} \; .$$

Substituting (5.56) in $\hat{\mathcal{B}}_{\mathrm{t}}$ gives

$$\hat{\mathcal{B}}_{\mathrm{t}} = \Big\{ (u_{\mathrm{t}}, y_{\mathrm{t}}, x, \xi) \;\big|\; \exists\, (u, y) \text{ s.t. } (u, y, x) \in \hat{\mathcal{B}}_{\mathrm{p}},\ (u - u_{\mathrm{t}}, -y_{\mathrm{t}}, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,1}, \\ (u, y_{\mathrm{t}} - y, x, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,2} \Big\} \; . \tag{5.57}$$

From $(u, y, x) \in \hat{\mathcal{B}}_{\mathrm{p}}$ and $(u, y_{\mathrm{t}} - y, x, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,2}$ we have that

$$\begin{aligned} \dot{x} &= F(x)\nabla H(x) + g(x)u && (5.58a) \\ y_{\mathrm{t}} &= g^\top \nabla H(x) + g^\top(x)\nabla C(x)\nabla\Phi(C(x) - \xi) \\ &= g^\top(x)\nabla_x W(x, \xi) \; , && (5.58b) \end{aligned}$$

and from $(u - u_{\mathrm{t}}, -y_{\mathrm{t}}, \xi) \in \hat{\mathcal{B}}_{\mathrm{c}\,1}$ we have that

$$\begin{aligned} \dot{\xi} &= y_{\mathrm{t}} && (5.59a) \\ u - u_{\mathrm{t}} &= -\nabla H_{\mathrm{c}}(\xi) \; . && (5.59b) \end{aligned}$$

By eliminating u from (5.58) and (5.59) one obtains

$$\begin{aligned} \dot{x} &= F(x)\nabla H(x) - g(x)\nabla H_{\mathrm{c}}(\xi) + g(x)u_{\mathrm{t}} && (5.60a) \\ \dot{\xi} &= g^\top(x)\nabla_x W(x, \xi) && (5.60b) \\ y_{\mathrm{t}} &= g^\top(x)\nabla_x W(x, \xi) \; . && (5.60c) \end{aligned}$$

Notice that

$$\nabla H_{\mathrm{c}}(\xi) = \nabla_\xi W(x, \xi) + \nabla\Phi(C(x) - \xi) \; ,$$

so one can write (5.60a) as

$$\begin{aligned}\dot{x} &= F(x)\nabla H(x) - g(x)\nabla_\xi W(x,\xi) - g(x)\nabla\Phi(C(x)-\xi) + g(x)u_\mathrm{t}\\ &= F(x)\nabla_x W(x,\xi) - g(x)\nabla_\xi W(x,\xi) + g(x)u_\mathrm{t}\ ,\end{aligned}$$

where the second line follows from (5.15). Then we can write the set of equations (5.60) as the port-Hamiltonian

$$\begin{pmatrix}\dot{x}\\ \dot{\xi}\end{pmatrix} = \begin{pmatrix}F(x) & -g(x)\\ g^\top(x) & 0\end{pmatrix}\nabla W(x,\xi) + \begin{pmatrix}g(x)\\ 0\end{pmatrix}u_\mathrm{t} \tag{5.61a}$$

$$y_\mathrm{t} = \begin{pmatrix}g^\top(x) & 0\end{pmatrix}\nabla W(x,\xi)\ . \tag{5.61b}$$

This proves (i) — and (ii), since the strict convexity of W has already been established. ■

Remark 5.16. Even without the existence of invariants, the controller (5.56) suffers from the detectability problem of standard CbI. This problem can be removed by adding an extra port, like we did in subsection 5.3.2.

5.5 Further extensions

Whether we use CbI with the flow described in section 5.1.1 or with the extra building block of the previous section, energy-shaping is accomplished by means of a map C satisfying (5.15). The restriction (5.15) on C can be relaxed to

$$F(x)\nabla C(x) = -g(x)\alpha(x)$$

for some $\alpha : \mathbb{X} \to \mathbb{X}_\mathrm{c}$ (this is of course equivalent to $g^\perp(x)F(x)\nabla C(x) = 0$). Under this weaker assumption, energy shaping is achieved by replacing (5.2) with

$$\dot{\xi} = -\alpha(x)y_{\mathrm{c}\,1} \tag{5.62a}$$

$$u_{\mathrm{c}\,1} = -\alpha^\top(x)\nabla H_\mathrm{c}(\xi) \tag{5.62b}$$

(which still satisfies $\dot{H}_\mathrm{c}(x) = y_{\mathrm{c}\,1}^\top u_{\mathrm{c}\,1}$). Then, theorem 5.9, propositions 5.10, 5.12, 5.13 and 5.15, and their respective proofs, follow *mutatis-mutandis*. This type of control, is called state-modulated control by interconnection ($\mathrm{CbI_{sm}}$).

Suppose now that there exists $\bar{F}(x) \in \mathbb{R}^{n\times n}$ and $\bar{H} : \mathbb{X} \to \mathbb{R}$ such that

$$F(x)\nabla H(x) = \bar{F}(x)\nabla\bar{H}(x)\ . \tag{5.63}$$

Clearly, $W(x,\xi) = \bar{H}(x) + \Phi(C(x)-\xi) + H_\mathrm{c}(\xi)$ will be a storage function of $\hat{\Pi}_\mathrm{t}$ if

$$g^\perp(x)\bar{F}(x)\nabla C(x) = 0\ ,$$

further enlarging the range of application of CbI. By assuming $\bar{F}$ non-singular and applying Poincaré's lemma [24, p. 30], one can see that there exists an $\bar{H}$ satisfying (5.63) if, and only if

$$\nabla\left(\bar{F}^{-1}(x)F(x)\nabla H(x)\right) = \left[\nabla\left(\bar{F}^{-1}(x)F(x)\nabla H(x)\right)\right]^\top\ . \tag{5.64}$$

Controller	$\hat{\Pi}_{\mathrm{t}}$'s flow	$\hat{\Pi}_{\mathrm{t}}$'s dissipation	PDE
$\mathrm{CbI_{sm}}$	Invariant	Invariant	$\begin{pmatrix} g^{\perp}F \\ g^{\top} \end{pmatrix} \nabla C = 0$
$\mathrm{CbI_{sm}}$ with invariant flow	Invariant	Invariant	$g^{\perp}F\nabla C = 0$
$\mathrm{CbI_{sm}}$ with $\hat{\Pi}_{\mathrm{c}\,2}$	$g^{\top}\nabla_x W$	$\nabla_x^{\top} W R \nabla_x W$	$g^{\perp}F\nabla C = 0$

$\hat{\Pi}_{\mathrm{t}}$'s storage function
$W(x) = H(x) + \Phi(C(x) - \xi) + H_{\mathrm{c}}(\xi)$

Table 5.1: Passivity-based controllers and their corresponding PDE.

The construction proposed in [40] for power shaping can be used also here to provide solutions of (5.64). Namely, it is easy to show that for all matrices $T(x) \in \mathbb{R}^{n\times n}$, with $T(x) = T^{\top}(x)$ and all $\lambda \in \mathbb{R}$ such that

$$\bar{T}(x) \triangleq \frac{1}{2}\left[\left(\nabla^2 H(x)\right) T(x) + \nabla\left(T(x)\nabla H(x)\right) + 2\lambda I\right]$$

is non-singular, $\bar{F}^{-1}(x) = \bar{T}(x)F^{-1}(x)$ solves (5.64). The resulting storage function being $\bar{H}(x) = \lambda H(x) + \nabla H^{\top}(x)T(x)\nabla H(x)$.

Remark 5.17. This procedure can also be applied to the static controllers of chapter 4.

5.6 Summary

Table 5.1 contains a summary of some of the controllers described in this chapter. From the table it is easy to establish, in terms of the solutions C of the PDE's, the following implication:

$$\mathrm{CbI_{sm}} \quad \Longrightarrow \quad \mathrm{CbI_{sm}}\text{ with invariant flow} \quad \Longleftrightarrow \quad \mathrm{CbI_{sm}}\text{ with } \hat{\Pi}_{\mathrm{c}\,2}\ .$$

It is clear that the selection of a quadratic function for H_{c} renders the controller linear, more precisely, a linear PI (for a suitably defined plant flow). The results in the paper may be then interpreted as identification of a class of nonlinear PH systems that are asymptotically stabilizable via linear PI. Although the choice of a linear PI may be restrictive for some academic examples it is certainly a family of controllers of practical interest. It should be, furthermore, pointed out that the general framework of CbI does not impose this restriction on H_{c}, and it is made here to obtain easily interpretable general results. We are currently exploring other controller structures for which similar results can be established.

5.7 Comparison between static and dynamic

From tables 5.1 and 4.2, it is clear that CbI and IDA are connected in some way. The following proposition links static and dynamic passivation in the context of stabilization.

Proposition 5.18. *Consider a triple (F, g, H) and a point $x_\star \in \mathcal{E}_x$. The following statements are equivalent:*

1. *There exists Φ, H_c and C such that*

$$\begin{pmatrix} g^\perp(x)F(x) \\ g^\top(x) \end{pmatrix} \nabla C(x) = 0 \,, \tag{5.65}$$

   $W(x,\xi) = H(x) + \Phi(C(x) - \xi) + H_\mathrm{c}(\xi)$ is strictly convex and minimal at $(x_\star, \xi_\star)$, with $\xi_\star = C(x_\star)$).

2. *There exists an H_a such that*

$$\begin{pmatrix} g^\perp(x)F(x) \\ g^\top(x) \end{pmatrix} \nabla H_\mathrm{a}(x) = 0 \,, \tag{5.66}$$

   $H_\mathrm{t}(x) = H(x) + H_\mathrm{a}(x)$ is strictly convex and minimal at $x_\star$.

Proof: To prove ($1 \Rightarrow 2$), suppose there exists a set of functions $C_1(x), \ldots, C_m(x)$ satisfying (5.65). Then, for sure there exists one function $H_\mathrm{a}(x)$ satisfying (5.66). Indeed, any function of the form

$$H_\mathrm{a}(x) = \beta(C(x)) \,, \tag{5.67}$$

where $\beta : \mathbb{X}_\mathrm{c} \to \mathbb{R}$ is arbitrary, will be a solution of (5.66).

Now, suppose that W is strictly convex. Then, for any pair of distinct points (x', ξ') and (x'', ξ''),

$$W\left(\lambda(x',\xi') + (1-\lambda)(x'',\xi'')\right) < \lambda W(x',\xi') + (1-\lambda)W(x'',\xi'') \,, \quad 0 < \lambda < 1 \,.$$

For any given C, the inequality will hold (as a particular case) for pairs of points of the form $(x', C(x'))$ and $(x'', C(x''))$, so the function

$$\bar{W}(x) \triangleq W(x, C(x)) = H(x) + \Phi(0) + H_\mathrm{c}(C(x))$$

is also strictly convex. In other words, setting

$$H_\mathrm{a}(x) = H_\mathrm{c}(C(x))$$

ensures (5.66) and the strict convexity of $H_\mathrm{t}(x) = H(x) + H_\mathrm{a}(x)$.

Since W is minimal at $(x_\star, \xi_\star)$, it statisfies

$$\nabla W(x_\star, \xi_\star) = \begin{pmatrix} \nabla H(x_\star) + \nabla C(x_\star)\Phi\left(C(x_\star) - \xi_\star\right) \\ \nabla H_\mathrm{c}(\xi_\star) - \Phi\left(C(x_\star) - \xi_\star\right) \end{pmatrix} = 0 \,,$$

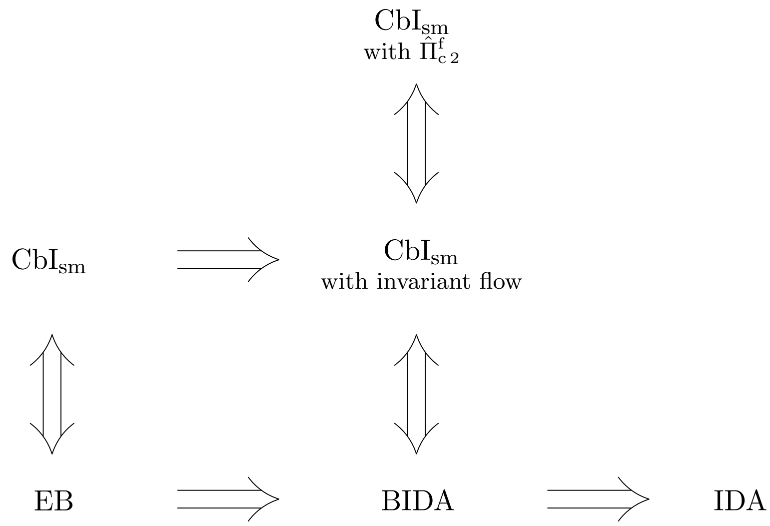

Figure 5.10: Relationships between the different control schemes from the point of view of solvability of the stabilization problem. The target storage function is assumed to be strictly convex.

which implies that $\nabla H(x_\star) + \nabla C(x_\star)\nabla H_{\mathrm{c}}(\xi_\star) = 0$. It then follows that

$$\nabla H_{\mathrm{t}}(x_\star) = \nabla H(x_\star) + \nabla C(x_\star)\nabla H_{\mathrm{c}}(C(x_\star)) = 0 ,$$

which, together with the strict convexity of H_{t}, implies that H_{t} is minimal at $x_\star$.

The converse result is readily proved by setting, e.g.,

$$C_1(x) = H_{\mathrm{a}}(x) , \quad C_2(x) = C_3(x) = \cdots = C_m(x) = 0 , \quad \Phi(z) = z_1$$

and

$$H_{\mathrm{c}}(\xi) = \xi_1 + \frac{1}{2}|\xi - \xi_\star|^2 .$$

This gives

$$W(x,\xi) = H(x) + C_1(x) - \xi_1 + \xi_1 + \frac{1}{2}|\xi - \xi_\star|^2 = H(x) + C_1(x) + \frac{1}{2}|\xi - \xi_\star|^2 ,$$

which is strictly convex since $H(x) + H_{\mathrm{a}}(x)$ and $\frac{1}{2}|\xi - \xi_\star|^2$ are. The function $W(x,\xi)$ is minimal at $(x_\star, \xi_\star)$ because

$$\nabla W(x_\star, \xi_\star) = \begin{pmatrix} \nabla H(x_\star) + \nabla H_{\mathrm{a}}(x_\star) \\ \xi_\star - \xi_\star \end{pmatrix} = 0 .$$

■

Remark 5.19. Using the same arguments, the term

$$\begin{pmatrix} g^\perp(x)F(x) \\ g^\top(x) \end{pmatrix}$$

can be replaced by $g^\perp(x)F(x)$ in the previous proposition.

Figure 5.10 shows the different relationships between CbI and IDA when the target storage functions are strictly convex.

6 Conclusions and further work

The results presented in this work suggest a discussion on two levels, one mostly theoretical and the other more technical, but both deeply interrelated. From a theoretical point of view, we are interested in understanding control as the act of interconnecting systems with the purpose of obtaining a desired behavior. We claim that energy plays a fundamental role in modelling and that energy-shaping should be incorporated as a design principle. *A priori*, this paradigm rests on the premise that, for the purposes of control, interconnection of systems and energy exchange is more intuitive, conceptually appealing and *natural* than signal processing and input–output thinking.

From a technical point of view, we simply want to solve the control problem. We want to formulate the correct equations, establish if they admit solutions and, if they do, determine them, so we can effectively implement the controller. Finally, we want to see if our technical results actually support, *a posteriori*, our claims on energy-shaping and systems interconnection.

Since the early papers on passivity-based control [42], the idea of energy-shaping has been present and has led to static controller design techniques like energy-balancing and IDA. The importance of relying on a physical interpretation of the controller (which in a sense relates to the idea of control as interconnection of systems), has also been there since the beginning [41]. In particular, energy-balancing controllers establish a power balance of the form

$$\dot{H}_{\mathrm{c}}(x) = y_{\mathrm{c}\,1}^{\top} u_{\mathrm{c}\,1}$$

(cf. Fig. 4.3), which suggests that behind the state feedback control law there is an abstract or virtual cyclo-lossless system at work. The application of this idea is stymied by the dissipation obstacle, which motivates the use of more general methods like IDA. Although IDA is applicable to a wider range of problems, it is closer to the idea of control as a mapping from states to inputs and, on a first approach, it does not admit a power balance and is incompatible with the notion of control as interconnection.

In a certain way, the material introduced in chapter 3 was developed so that, besides energy-balancing, other passivity-based controllers could fit in the control as interconnection framework (see chapter 4). This was done with the purpose of better understanding the existing methods, rather than to artificially justify the new paradigm — otherwise it would be bad science. As a result of this theoretical effort, we gained a better understanding on the role of flow and dissipation invariance, which set a baseline for classifying the existing methods. On a technical level, we obtained an algebraic characterization of all static passivity-based controllers, we identified a particular flow which is invariant to the action of BIDA and we came with an extended version of IDA, together with an explicit solution to its algebraic equations and a necessary condition on the desired dissipation and energy function.

Control *by* interconnection, on the other hand, is a particular instance of the control *as* interconnection approach[1]. The controller is no longer interpreted as a virtual system: it is an actual, real, dynamical system satisfying its own power balance. For developing the material of chapter 5 we started with a controller design strategy which is directly derived from the theoretical claims and worked out its technical details, using some of the results of chapter 4 to widen its domain of applicability. Opposite to the static feedback case, we started from theory and headed towards application.

These lines of research converge in section 5.7, which states that: as long as we do not attempt to modify the structure matrix F, a static passivity-based controller and a CbI controller are equivalent in terms of the solvability of the PDE's. Recall that in order to effectively shape H, it is sometimes necessary to modify F as well (e.g., the case of under-actuated mechanical systems [1]). The main open question is then:

> Is our inability to change F a fundamental limitation of control as interconnection, or is it merely a technical, temporary obstacle?

On another fundamental level but with a longer term perspective, there is the problem of formulating everything on a purely behavioral setting, without resorting to particular state-space representations (see the *caveat lector* on the introduction). The property of passivity has been intrinsically characterized in [71], but the problem of defining cyclo-passivity without using the notion of state is still open. The reader is referred to [11] for a treatment on energy-shaping in an intrinsic-oriented setting.

[1] Quite unfortunately, the only way we have to distinguish the particular method presented in chapter 5 from the general perspective proposed by Willems are the prepositions 'by' and 'are', respectively.

A Mathematical Background

A.1 Ordinary differential equations

This material is mainly based on the course notes developed by Laurent Praly for his lectures on Lyapunov functions and stablilization of equilibria. Another good reference is [20].

Definition A.1 ($\mathcal{L}_\infty^{\mathrm{loc}}(\mathbb{R},\mathbb{U})$)**.** The set of essentialy bounded measurable functions $u : \mathbb{R} \to \mathbb{U}$ such that for every compact K in $\mathbb{R}$, there exists a $c \in \mathbb{R}$ such that

$$|u(t)| \le c \quad \text{for almost all } t \in K \ .$$

Definition A.2 (Lipshitz)**.** A function $f : \mathbb{X} \mapsto \mathbb{R}^n$ is said to be *locally Lipschitz* if, for every $x \in \mathbb{X}$, there exists a neighborhood $\mathcal{X}_x$ of x and a real number L_x such that

$$|f(x') - f(x'')| \le L_x |x' - x''| \quad \forall\, x', x'' \in \mathcal{X}_x \ . \tag{A.1}$$

It is said to be *Lipschitz* if L_x in (A.1) is independent of x.

A function $f : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^n$ is said to be *locally Lipschitz uniformly on every compact* if, for all points $x \in \mathbb{X}$ and every compact K_{u} of $\mathbb{U}$, there exists a neighborhood $\mathcal{X}_x$ of x and a real number L_x such that

$$|f(x', u) - f(x'', u)| \le L_x |x' - x''| \quad \forall\, x', x'' \in \mathcal{X}_x \ , \quad \forall u \ \in K_{\mathrm{u}} \ .$$

Theorem A.3. *Let $\mathcal{O}$ be an open set in $\mathbb{X}$, $f : \mathcal{O} \times \mathbb{U} \to \mathbb{R}^n$ a continuous function and u a function in $\mathcal{L}_\infty^{\mathrm{loc}}(\mathbb{R},\mathbb{U})$. For every x_0 in $\mathcal{O}$, there exists a neighborhood J of 0 in $\mathbb{R}$ and a function $t \in J \mapsto X(x_0, t; u) \in \mathcal{O}$ solution of*

$$\dot{x} = f(x, u(t))$$

from x_0. This function is locally Lipschitz in general and $\mathcal{C}^1$ if u is continuous. It verifies

$$\frac{\partial X}{\partial t} = f(X(x_0, t; u), u(t)) \quad \textit{for almost all } t \in J \ .$$

Moreover, J can be extended to an open interval (a, b) ($-\infty \le a < 0 < b \le \infty$), called maximal interval of existence such that if a (respectively b) is finite, we have

$$\lim_{t \to a} \left(\textit{respectively } \lim_{t \to b} \right) |X(x_0, t; u)| + \frac{1}{d(X(x_0, t; u), \partial\mathcal{O})} = +\infty$$

where $d(x, \partial\mathcal{O})$ represents the distance from a point x to the boundary $\partial\mathcal{O}$ of $\mathcal{O}$.

Finally, if the function f is locally Lipschitz in x, uniformly on every compact, then the solutions are unique.

A.2 Generalized inverses

Definition A.4. Let A be an $o \times n$ matrix of arbitrary rank. A *generalized inverse* (g-inverse) of A is an $n \times o$ matrix A^- such that

$$AA^-A = A\ .$$

Theorem A.5. *[48] Let A be an $o \times n$ matrix. Then A^- exists. The entire class of g-inverses is generated from any given A^- by the formula*

$$A^- + U - AA^-UAA^- \tag{A.2}$$

where U is arbitrary, or by the formula

$$A^- + V(I - A^-A) + (I - A^-A)W \tag{A.3}$$

where V and W are arbitrary. Further, a matrix is uniquely determined by the class (A.2) or (A.3).

Concerning linear equations, we have

Theorem A.6. *[47, 48] Let A be of order $o \times n$ and A^- be any g-inverse of A. Then the following hold:*

1. *A necessary and sufficient condition that $A\xi = \psi$ is consistent (i.e., that there exists a ξ that satisfies the equation), is that*

$$AA^-\psi = \psi\ . \tag{A.4}$$

2. *Suppose $A\xi = \psi$ is consistent, then the class of all solutions is provided by*

$$A^-\psi + (I - A^-A)\zeta\ ,$$

ζ arbitrary.

Definition A.7. Let g by an $n \times m$ matrix. The *Moore-Penrose pseudo inverse* of g is the (unique) $m \times n$ matrix g^+ such that

1. $gg^+g = g$ (it is a particular g-inverse)
2. $g^+gg^+ = g^+$ (it is reflexive)
3. $(gg^+)^\top = gg^+$ (gg^+ is symmetric)
4. $(g^+g)^\top = g^+g$ (g^+g is also symmetric).

Remark A.8. It is well known that if $n \geq m$ and the columns of g are linearly independent, then $g^\top g$ is invertible. In this case, an explicit formula is:

$$g^+ = (g^\top g)^{-1} g^\top \ .$$

It follows that g^+ is a left inverse of g, i.e., $g^+ g = I$. The other product, gg^+, is the orthogonal projector on the range of g.

Definition A.9. Let g by an $n \times m$ matrix. A *left-annihilator* of g is a full rank $p \times n$ matrix $g^\perp$ such that

$$g^\perp g = 0 \ .$$

Remark A.10. Suppose that $n \geq m$ and the columns of g are linearly independent, then $g^\perp$ is an $(n-m) \times m$ matrix. Since the rows of $g^\perp$ are linearly independent, Remark A.8 holds *mutatis mutandis*:

- $g^\perp_+ = {g^\perp}^\top (g^\perp {g^\perp}^\top)^{-1}$.
- $g^\perp g^\perp_+ = I$ (it is a right inverse).
- The other product, $g^\perp_+ g^\perp$, is the orthogonal projector on the range of $g^\perp$.

Moreover, the identity $gg^+ + g^\perp_+ g^\perp \equiv I$ holds.

B Proofs and details

B.1 Lemma 4.4

Proof: By definition,

$$\hat{\mathcal{B}}_{\mathrm{t}} = \Big\{ (w_{\mathrm{t}}, x) \in \mathbb{W}_{\mathrm{t}} \times \mathbb{X} \,\Big|\, \exists\, w \text{ s.t. } (w, x) \in \hat{\mathcal{B}}_{\mathrm{p}} \,, (\bar{w}, w_{\mathrm{t}}, x) \in \hat{\mathcal{B}}_{\mathrm{c}} \Big\} \ . \tag{B.1}$$

For $\hat{\mathcal{B}}_{\mathrm{c}}$ as in (4.5), it is clear that $(\bar{w}, w_{\mathrm{t}}, x) \in \hat{\mathcal{B}}_{\mathrm{c}}$ is equivalent to

$$(u, y, x) = (\mathring{u}_{\mathrm{c}}(x) + u_{\mathrm{t}}, y_{\mathrm{t}} - \mathring{y}_{\mathrm{t}}(x), x) \ ,$$

which implies that $(w_{\mathrm{t}}, x) \in \hat{\mathcal{B}}_{\mathrm{t}}$ if, and only if,

$$(\mathring{u}_{\mathrm{c}}(x) + u_{\mathrm{t}}, y_{\mathrm{t}} - \mathring{y}_{\mathrm{t}}(x), x) \in \hat{\mathcal{B}}_{\mathrm{p}} \ . \tag{B.2}$$

According to (3.10), equation (B.2) is equivalent to

$$\begin{aligned} \dot{x} &= f(x) + g(x)(\mathring{u}_{\mathrm{c}}(x) + u_{\mathrm{t}}) \\ y_{\mathrm{t}} - \mathring{y}_{\mathrm{t}}(x) &= h(x) \end{aligned}$$

and the claim follows immediately. ■

B.2 Lemma 5.6

Proof: We wish to show that

$$F(x)\nabla C(x) = -g(x) \tag{B.3a}$$
$$g^{\top}(x)\nabla C(x) = 0 \tag{B.3b}$$

is equivalent to

$$F^{\top}(x)\nabla C(x) = g(x) \tag{B.4a}$$
$$g^{\top}(x)\nabla C(x) = 0 \ . \tag{B.4b}$$

First we will show that (B.3) implies (B.4a). Indeed, we have the following chain of implications:

$$\text{(B.3)} \implies \nabla C^{\top}(x)F^{\top}(x)\nabla C(x) = 0 \implies \nabla C^{\top}(x)R(x)\nabla C(x) = 0$$
$$\implies R(x)\nabla C(x) = 0 \implies -F(x)\nabla C(x) = F^{\top}(x)\nabla C(x) \implies \text{(B.4a)} \ .$$

Likewise, we have that

$$\text{(B.4)} \implies \nabla C^\top(x)F(x)\nabla C(x) = 0 \implies \nabla C^\top(x)R(x)\nabla C(x) = 0$$
$$\implies R(x)\nabla C(x) = 0 \implies F^\top(x)\nabla C(x) = -F(x)\nabla C(x) \implies \text{(B.3a)} .$$

■

B.3 The set $\mathcal{E}$

Consider an arbitrary point $(x, \xi) \in \mathcal{E}$. The conditions that define the set (5.26) are equivalent to

$$g^\perp(x)F(x)\nabla H(x) = 0 \tag{B.5a}$$
$$\nabla H_c(\xi) = g^+(x)F(x)\nabla H(x) . \tag{B.5b}$$

From (B.5a) we get that

$$\begin{aligned} g^\perp(x)F(x)\nabla H(x) &= \begin{pmatrix} x_2^3 & x_2^2 - \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2}x_1 + x_2 \\ -x_2^2 \end{pmatrix} \\ &= \frac{x_2^3}{2}(1 - x_1 x_2) = 0 . \end{aligned}$$

In other words, if a given (x, ξ) is in $\mathcal{E}$, then it must satisfy

$$x \in \left\{ \left(x_1, \frac{1}{x_1} \right) \,\middle|\, x_1 \neq 0 \right\} \cup \left\{ (x_1, 0) \,\middle|\, x_1 \in \mathbb{R} \right\} . \tag{B.6}$$

Notice that

$$\begin{aligned} g^+(x)F(x)\nabla H(x) &= \frac{1}{(x_2^2 - \frac{1}{2})^2 + x_2^6} \begin{pmatrix} \frac{1}{2} - x_2^2 & x_2^3 \end{pmatrix} \begin{pmatrix} -\frac{1}{2}x_1 + x_2 \\ -x_2^2 \end{pmatrix} \\ &= \frac{(\frac{1}{2}x_1 - x_2)(x_2^2 - \frac{1}{2}) + x_2^5}{(x_2^2 - \frac{1}{2})^2 + x_2^6} . \end{aligned}$$

Because of (B.6),

$$g^+(x)F(x)\nabla H(x) = \begin{cases} -\frac{1}{x_2}\frac{(x_2^2 - \frac{1}{2})^2 + x_2^6}{(x_2^2 - \frac{1}{2})^2 + x_2^6} = -\frac{1}{x_2} & x_2 \neq 0 \\ -x_1 & x_2 = 0 \end{cases}$$

In any case, $g^+(x)F(x)\nabla H(x) = -x_1$. Finally, from (B.5b) and the fact that $u_\star = -1$, we get that $\nabla H_c(\xi) = \xi + 1 = -x_1$ or, equivalently, $\xi = -x_1 - 1$, so

$$\mathcal{E} = \left\{ \operatorname{col}\left(x_1, \frac{1}{x_1}, -x_1 - 1 \right) \,\middle|\, x_1 \neq 0 \right\} \cup \left\{ \operatorname{col}(x_1, 0, -x_1 - 1) \,\middle|\, x_1 \in \mathbb{R} \right\} .$$

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Nomenclature

Symbols

$|\cdot|$ The Euclidean norm

$\mathcal{B}$ The behavior of a system, page 41

$\mathcal{H}(\mathbb{R}, \mathbb{R}^m)$ The class of all measurable functions $s : \mathbb{R} \to \mathbb{R}^m$

$\mathbb{L}$ The set of latent variables, page 42

$\mathcal{L}_\infty^{\text{loc}}(\mathbb{R}, \mathbb{R}^m)$ The class of all essentially bounded measurable functions, page 133

Φ A general operator in an abstract space

$D(\Phi)$ The domain of definition of the operator Φ

$\hat{\Pi}$ An m-port, page 42

$\hat{\Pi}_\text{a} \wedge \hat{\Pi}_\text{b}$ The interconnection of $\hat{\Pi}_\text{a}$ and $\hat{\Pi}_\text{b}$, page 60

$\mathbb{U}$ The set of effort values, page 42

$\mathbb{Y}$ The set of flow values, page 42

Σ A system, page 41

$\mathbb{T}$ The time axis, page 41

$\mathbb{W}$ The signal space or set of manifest variables, page 42

$\mathbb{X}$ The state space, page 42

Acronyms and abbreviations

BIDA Interconnection and damping assignment, page 94

CbI Control by interconnection, page 101

EB Energy-balancing, page 66

ED Effort-driven, page 42

FD Flow-driven, page 42

g-inverse Generalized inverse, page 134

IDA	Interconnection and damping assignment, page 81
ODE	Ordinary differential equation
PBC	Passivity-based control, page 32
PDE	Partial differential equation
s.t.	Such that

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