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CONTROL BY (STATE–MODULATED) INTERCONNECTION OF PORT–HAMILTONIAN SYSTEMS [1]

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Abstract: It is well known that the dynamics of many physical processes can be suitably described by Port–Hamiltonian (PH) models. In this paper we consider the Passivity–Based Control (PBC) technique of Control by Interconnection (CbI), where the controller is another PH system connected to the plant to add up their energy functions. We propose two extensions to this method, first, we exploit the non–uniqueness of the PH representation of the system to generate new cyclo–passive outputs. Applying CbI through these new port variables overcomes the so–called dissipation obstacle. Second, when the plant state variables are measurable, we show that the conditions for applicability of the method can be relaxed replacing the simple unitary feedback by a state–modulated interconnection. A central contribution of the paper is the proof that the conditions for energy shaping via CbI are equivalent to those imposed in Interconnection and Damping Assignment PBC, providing in this way a nice geometric interpretation to this successful controller design technique.

Keywords: Passivity, nonlinear systems, stabilization, interconnection, passivity–based control, Hamiltonian systems

1. INTRODUCTION

In the last few years we have witnessed in the control literature, both theoretical and applied, an ever increasing predominance of control techniques that respect, and effectively exploit, the structure of the system over the more classical techniques that try to impose some predetermined dynamic behavior—usually through nonlinearity cancelation and high gain (some times euphemistically called "nonlinearity domination" (Krstic *et al.*, 1995)). The property of passivity plays a central role in most of these developments. Passivity–Based Control (PBC) is a generic name, introduced in (Ortega and Spong, 1989), to define a controller design methodology which achieves the

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control objective, e.g., stabilization, by rendering the system passive with respect to a desired storage function and injecting damping. There are many variations of the basic PBC idea, and we refer the interested reader to (Ortega *et al.*, 1998; Ortega and García-Canseco, 2004; van der Schaft, 2000) for further details and a list of references.

In this paper we are interested in the control of dynamical systems endowed with a special geometric structure, called a Port–Hamiltonian (PH) model. As shown in (van der Schaft, 2000), PH models provide a suitable representation of many physical processes and have the essential feature of underscoring the importance of the energy function, the interconnection pattern and the dissipation of the system. There are many possible representations of PH models, here we will consider the so–called input–state–output form, where the state is assumed finite dimensional and the port variables are the input and output vectors, which satisfy a cyclo–passivity inequality. (The distinction between cyclo–passivity and the more standard passivity property will be discussed later.) To regulate the behavior of PH systems it is natural then to adopt a PBC perspective.

We consider in this paper the PBC technique of Control by Interconnection (CbI) (Dalsmo and van der Schaft, 1999; Ortega *et al.*, 2001), where the controller is another PH system with its own state variables and energy function. The regulator and the plant are interconnected in a power–preserving way, that is, through a loss–less subsystem. A straightforward application of the passivity theorem shows that the overall system is still cyclo–passive with new energy storage function the sum of the energy functions of the plant and the controller. To assign to the overall energy function a desired shape, it is necessary to "relate" the states of the plant and the controller via the generation of invariant manifolds—defined by, so–called, Casimir functions. In its basic formulation, CbI assumes that only the plant output is measurable and considers the classical output feedback interconnection. In this case, the Casimir functions are fully determined by the plant, which imposes a severe restriction on the plant dissipation structure. It has been shown in (Ortega *et al.*, 2001) that, roughly speaking, "dissipation cannot be present on the coordinates to be shaped". This, so–called, dissipation obstacle stymies the use of CbI for applications other than mechanical systems where the coordinates to be shaped are typically positions, which are unaffected by friction.

To overcome the dissipation obstacle and increase the domain of applicability of CbI we introduce here two extensions to the method. First, exploiting the non–uniqueness of the PH representation of the system, we propose a procedure to generate new cyclo–passive outputs (with new storage functions). The procedure is inspired by the power–shaping stabilization technique recently introduced for RLC circuits in (Ortega *et al.*, 2003) and later extended to general nonlinear systems in (García-Canseco *et al.*, 2006). Applying CbI through these new port variables overcomes the dissipation obstacle, but still rules out several interesting physical examples.

Given the fact that CbI is an output feedback control strategy it is not surprising that there are some limitations for its successful application. Our second, and key modification, assumes that the plant state variables are available for measurement, and proposes to replace the simple output feedback by a suitably defined state–modulated interconnection. In this way, the conditions for existence of Casimir functions can be further relaxed. Our main contribution is the proof that the latter conditions are necessary and sufficient for the solution of the matching equations of Basic Interconnection and Damping Assignment (IDA) PBC introduced in (Ortega *et al.*, 2002*a*), see also (Fujimoto and Sugie, 2001; Ortega *et al.*, 2001; Ortega and García-Canseco, 2004; van der Schaft, 2000) for more recent developments and (Bloch *et al.*, 2002; Ortega and Spong, 1989; Ortega *et al.*, 2002*b*) for the particular case of mechanical systems. Actually, it is shown in the paper that the (static state feedback) IDA–PBC law is the projection of the (dynamic state feedback) state–modulated CbI on the invariant manifold defined by the Casimir functions. Similarly, the closed–loop dynamics resulting from application of IDA–PBC is the reduction of the dynamics of the CbI controlled system to this invariant subspace.

The importance of establishing the equivalence between CbI and IDA–PBC can hardly be overestimated. On one hand, it provides a nice geometric interpretation to this highly successful controller design technique, which has been previously presented from a uninspiring and non–intuitive model matching perspective. (At a more fundamental level, viewing IDA–PBC as (a projection of) interconnected subsystems is consistent with the behavioral framework, which claims that the classical input–to–output assignment perspective is unsuitable to deal, at an appropriately general level, with the basic tenets of systems theory.) On the other hand, the experience gained in the design of Lyapunov–based stabilizing state feedbacks paves the way for new extensions of CbI, which is by far the most natural procedure for controller design.

Notation All vectors defined in the paper are *column* vectors, even the gradient of a scalar function that we denote with the operator $\nabla_x = \frac{\partial}{\partial x}$. For vector functions $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$, we define the (transposed) Hessian matrix $\nabla_x \mathcal{C}(x) \triangleq [\nabla_x \mathcal{C}_1(x), \ldots, \nabla_x \mathcal{C}_m(x)]$. When clear from the context the subindex of the operator ∇ and the arguments of the functions will be omitted.

Caveat This is an abridged version of the paper where all proofs have been omitted. The full version of the paper is available upon request to the authors.

2. CONTROL BY (OUTPUT FEEDBACK) INTERCONNECTION OF PH SYSTEMS

In order to make this paper self–contained we briefly review in this section the basic version of the CbI method. Also, we discuss its relation with the EBC technique of (Ortega *et al.*, 2001) and with Basic IDA–PBC.

2.1 Cyclo–Passivity of Port–Hamiltonian Systems

PH models of power–conserving physical systems were introduced in (Bernhard Maschke and Bredveld, 1992), see (van der Schaft, 2000) for a review. The input–state–output representation of PH systems is of the form

$$\Sigma_{(u,y)} \begin{cases} \dot{x} = [\mathcal{J}(x) - \mathcal{R}(x)]\nabla H(x) + g(x)u \\ y = g^\top(x)\nabla H(x), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$, $m \leq n$, is the control action, $H : \mathbb{R}^n \to \mathbb{R}$ is the total stored energy, $\mathcal{J}, \mathcal{R} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, with $\mathcal{J} = -\mathcal{J}^\top$ and $\mathcal{R} = \mathcal{R}^\top \geq 0$, are the natural interconnection and damping matrices, respectively, u, $y \in \mathbb{R}^m$, are conjugated variables whose product has units of power and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is assumed full rank. We bring to the readers attention the important fact that H is not assumed to be positive semi–definite (nor bounded from below). Also, to simplify the notation in the sequel we define the matrix

$$F(x) \triangleq \mathcal{J}(x) - \mathcal{R}(x),$$

which clearly satisfies $F + F^\top = -2\mathcal{R} \leq 0$.

The power conservation property of PH systems is captured by the power–balance equation

$$\dot{H} = -(\nabla H)^\top \mathcal{R}\nabla H + u^\top y. \tag{2}$$

Using the fact that $\mathcal{R} \geq 0$ we obtain the bound

$$\dot{H} \leq u^\top y, \tag{3}$$

that, following the original denomination of (Willems, 1972), we refer as cyclo–passivity inequality. Systems satisfying such an inequality are called cyclo–passive, which should be distinguished from passive systems where H is positive semi–definite. In words, a system is cyclo–passive when it cannot create energy over closed paths in the state–space. It might, however, produce energy along some initial portion of such a trajectory; if so, it would not be passive. On the other hand, every passive system is cyclo–passive. It has been shown in (Hill and Moylan, 1980) that, similarly to passive systems, one can use storage functions and passivity inequalities to characterize cyclo–passivity provided we eliminate the restriction that these storage functions be non–negative.

2.2 Energy Shaping via Control by Interconnection with $\Sigma_{(u,y)}$

As indicated above, in PBC the control objective is achieved rendering the system passive with respect to a desired storage function and injecting damping. For the basic problem of stabilization, the desired energy function should have a minimum at the equilibrium and the damping injection insures that the function is non–increasing. In this way, the energy function qualifies as a Lyapunov function. We now briefly review the PBC method of CbI for stabilization of PH systems, we refer the reader to (van der Schaft, 2000) for further details and extensions. The configuration used for CbI is shown in Fig. 1, where the controller, Σ_c, is a PH system, coupled with the plant, $\Sigma_{(u,y)}$, via the interconnection subsystem, Σ_I, that we select to be power–preserving. That is, such that, for all $t \geq 0$,

$$y^\top(t)u(t) + y_c^\top(t)u_c(t) = y^\top(t)v(t), \tag{4}$$

where v is an external signal that we introduce to define the port variables of the interconnected system and (possibly) inject additional damping.

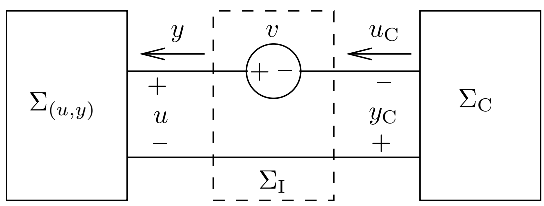

Fig. 1. Block diagram of the CbI scheme.

We choose the dynamics of the controller to be a simple set of (possibly nonlinear) integrators, that is,

$$\Sigma_c : \begin{cases} \dot{\zeta} = u_c \\ y_c = \nabla_\zeta H_c(\zeta), \end{cases} \tag{5}$$

where $\zeta, u_c, y_c \in \mathbb{R}^m$, and $H_c : \mathbb{R}^m \to \mathbb{R}$ is the controllers energy function—to be defined later. [2] From

[2] For ease of notation, and without loss of generality, we have taken the order of Σ_c to be equal to the number of

$$\dot{H}_c = u_c^\top y_c, \tag{6}$$

we see that Σ_c is cyclo–passive. In its simplest formulation, CbI assumes that we measure only the plant output and fixes Σ_I to be the standard negative feedback interconnection

$$\Sigma_I : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}, \right. \tag{7}$$

which clearly satisfies (4). Combining (3), (4) and (6), we obtain that the interconnected system is also cyclo–passive with port variables (v, y) and energy function the sum of the energy functions of the plant and the controller, that is

$$\dot{H} + \dot{H}_c \leq v^\top y. \tag{8}$$

To complete the shaping of the energy function CbI invokes the Energy–Casimir method—well–known in Hamiltonian systems analysis, see e.g. (Dalsmo and van der Schaft, 1999), (Marsden and Ratiu, 1999)—and looks for conserved quantities (dynamical invariants) of the overall system. If such quantities can be found we can generate Lyapunov function candidates combining the conserved quantities and the energy function. We will look, in particular, for functions that are conserved for all energy functions H and H_c—such functions are called Casimir.

The application of the Energy–Casimir method for stability analysis of (output feedback) CbI is summarized below.

Proposition 1. Consider the PH system $\Sigma_{(u,y)}$ (1) coupled with the PH controller Σ_c (5) through the power–preserving interconnection subsystem Σ_I (7). Assume there exists a vector function $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$F^\top(x)\nabla\mathcal{C}(x) = g(x), \quad g^\top(x)\nabla\mathcal{C}(x) = 0 \tag{9}$$

Then, for all functions $\Phi : \mathbb{R}^m \to \mathbb{R}$, the following cyclo–passivity inequality is satisfied

$$\dot{W} \leq v^\top y, \tag{10}$$

where the storage function $W : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is defined as

$$W(x,\zeta) \stackrel{\triangle}{=} H(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta). \tag{11}$$

Remark 1. In (Ortega *et al.*, 2001) the energy shaping action of CbI was viewed from an alternative perspective—geometric instead of Lyapunov–based—that proceeds as follows. First, we notice that the level sets of the Casimir functions, $\zeta - \mathcal{C}(x)$, are invariant sets for the interconnected system. That is, the manifolds

$$\Omega_\kappa \stackrel{\triangle}{=} \{(x,\zeta) \in \mathbb{R}^n \times \mathbb{R}^m | \, \zeta = \mathcal{C}(x) + \kappa\}, \ \kappa \in \mathbb{R}$$

are invariant for the overall dynamics.[3] Then, projecting the system on Ω_κ yields the reduced dynamics $\dot{x} = F\nabla H_s$, where $H_s(x) \stackrel{\triangle}{=} H(x) + H_c[\mathcal{C}(x) + \kappa]$ plays the role of shaped energy function. Even though with a proper selection of the initial conditions of the controller we can set $\kappa = 0$, the fact that the shaped energy function depends on this constant is rather unnatural, thus we have presented the result using a Lyapunov viewpoint.

2.3 CbI with $\Sigma_{(u,y)} \Rightarrow$ Energy Balancing Control $\Rightarrow$ Basic IDA–PBC

In this subsection we prove that, when the state is available for measurement, the conditions of Proposition 1 ensure the existence of a static state–feedback that shapes the energy function without modifying the interconnection and damping structures. That is, that yields the closed–loop dynamics $\dot{x} = F\nabla H_d + gv$, where the storage function $H_d : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$H_d(x) \stackrel{\triangle}{=} H(x) + H_a(x) \tag{12}$$

for some $H_a : \mathbb{R}^n \to \mathbb{R}$. Furthermore, we prove that the shaped energy function equals the difference between the systems stored energy and the energy extracted from the environment—which was called an Energy Balancing Controller (EBC) in (Ortega *et al.*, 2001).

Before presenting the main result, it is useful to recall the distinction between EBC and a controller that simply shapes the energy, without modifying the interconnection and damping structures, that was called Basic IDA–PBC in (Ortega *et al.*, 2002*a*). In the latter, we require the existence of a state feedback control $\hat{u}_{\mathtt{BIDA}} : \mathbb{R}^n \to \mathbb{R}^m$ so that the matching condition

$$F(x)\nabla H(x) + g(x)\hat{u}_{\mathtt{BIDA}}(x) = F(x)\nabla H_d(x)$$
$$\Leftrightarrow$$
$$g(x)\hat{u}_{\mathtt{BIDA}}(x) = F(x)\nabla H_a(x)$$

is verified. Pre–multiplying the right hand side equation by the full rank square matrix $\begin{bmatrix} g^\top(x) \\ g^\perp(x) \end{bmatrix}$, where $g^\perp : \mathbb{R}^n \to \mathbb{R}^{(n-m)\times n}$, is a full–rank left annihilator of g, that is, $g^\perp g = 0$ and rank $g^\perp = n - m$, we obtain that all solutions of the matching problem above are given by the solutions of the PDE

$$g^\perp F\nabla H_a = 0, \tag{13}$$

together with the (uniquely defined) control

$$\hat{u}_{\mathtt{BIDA}} = (g^\top g)^{-1} g^\top F\nabla H_a.$$

inputs. A discussion on this issue may be found in (van der Schaft, 2000).

[3] We recall that a manifold $\Omega_\kappa \subset \mathbb{R}^n \times \mathbb{R}^m$ is invariant if $(x(0),\zeta(0)) \in \Omega_\kappa \Rightarrow (x(t),\zeta(t)) \in \Omega_\kappa$ for all $t \geq 0$.

In an EBC, besides satisfying the matching condition, we additionally require that the control action, called $\hat{u}_{\tt EB} : \mathbb{R}^n \to \mathbb{R}^m$, and the added energy function satisfy

$$H_a(x(t)) = -\int_0^t \hat{u}_{\tt EB}^\top(x(s)) \underbrace{g^\top(x(s))\nabla H(x(s))}_{y(s)} ds + \\ + H_a(x(0)). \quad (14)$$

Hence, H_a has the interpretation of the energy extracted from the controller. The condition for energy balancing (14) is equivalent to the PDE

$$\nabla H_a^\top (F\nabla H + g\hat{u}_{\tt EB}) = -\hat{u}_{\tt EB}^\top g^\top \nabla H. \qquad (15)$$

Adding $\dot{H}$ to both sides of the previous equation shows that, for an EBC,

$$\dot{H}_d = \nabla H^\top F \nabla H + v^\top y.$$

Remark that the dissipation term appears with the open–loop energy function, in contrast with the dissipation for the (more general) Basic IDA–PBC where it takes the form $\nabla H_d^\top F \nabla H_d$. Clearly, not all controllers satisfying the matching equation of Basic IDA–PBC are EBC. In the next subsection we investigate the implications of the energy balancing condition on the systems natural dissipation.

The following proposition shows that, projecting the CbI on the manifold $\zeta = \mathcal{C}(x)$, yields an EBC.

Proposition 2. Assume the PDEs (9) admit a solution. Then, for all functions $H_c : \mathbb{R}^m \to \mathbb{R}$, the PH system $\Sigma_{(u,y)}$ (1) in closed–loop with the static state–feedback control $u = \hat{u}_{\tt EB}(x) + v$, where

$$\hat{u}_{\tt EB}(x) = -\nabla_{\mathcal{C}} H_c(\mathcal{C}(x)),$$

satisfies the cyclo–passivity inequality

$$\dot{H}_d \le v^\top y, \qquad (16)$$

where the storage function H_d is defined by (12) with

$$H_a(x) \stackrel{\triangle}{=} H_c(\mathcal{C}(x)). \qquad (17)$$

Furthermore, the controller is an EBC that satisfies (14).

2.4 The Dissipation Obstacle

Proposition 1 shows that, via the selection of H_c and Φ, it is possible to shape the energy function of the interconnected system—provided we can generate Casimir functions. That is, if we can solve the PDEs (9). Unfortunately, the solvability of the latter imposes a serious constraint on the dissipation structure of the system. Indeed, it is possible to show (see (van der Schaft, 2000)) that (9) are equivalent to

$$\mathcal{J}\nabla\mathcal{C} = -g, \quad \mathcal{R}\nabla\mathcal{C} = 0. \qquad (18)$$

The second condition clearly implies that

$$\mathcal{R}\nabla_x \Phi(\mathcal{C}(x) - \zeta) = 0, \qquad (19)$$

that, together with (11), indicates that energy cannot be shaped for those coordinates that are affected by physical damping. In (Ortega *et al.*, 2001) we referred to this restriction as dissipation obstacle. This obstruction is intrinsic, in the sense that it is determined only by the damping interconnection structure and is independent of the actual value of the damping elements.

From the first equation in (9), assuming for simplicity that F is full rank, we get $\nabla\mathcal{C} = F^{-\top} g$, which replaced in the second equation of (18) gives $\mathcal{R}F^{-\top} g = 0$. This is a necessary condition for the existence of Casimirs, hence if the system does not satisfy it their energy function cannot be shaped via CbI. In order to relate with forthcoming derivations it is convenient to obtain another necessary condition for Casimirs. For, we consider the second equation in (18) for which we have

$$\mathcal{R}\nabla\mathcal{C} = 0 \;\Leftrightarrow F\,\nabla\mathcal{C} = -F^\top \nabla\mathcal{C},$$

which combined with the second equation in (9) and the second equation of (18) yields

$$\mathcal{R}F^{-1} g = 0. \qquad (20)$$

EBC also imposes a restriction on the dissipation. Indeed, evaluating (15) at $x = x_\star$ we conclude that $\hat{u}_{\tt EB}^\top(x_\star) g^\top(x_\star)\nabla H(x_\star) = 0$. This means that the power extracted from the source should be zero at the equilibrium. On the other hand, evaluating the power–balance equation (2) at $x = x_\star$ we conclude that $\mathcal{R}(x_\star)\nabla H(x_\star) = 0$ must be satisfied. Similarly to CbI, the latter condition restricts the set of assignable Lyapunov–energy functions. Indeed, if we impose the stability condition $x_\star = \arg\min H_d(x)$, which implies that $\nabla H_a(x_\star) = -\nabla H(x_\star)$, we have that the Lyapunov–energy functions assignable via EBC must satisfy

$$\mathcal{R}(x_\star)\nabla H_a(x_\star) = 0, \qquad (21)$$

which should be compared to (19) (that, besides being evaluated for all x, applies to all assignable energy functions W, whether or not they qualify as Lyapunov functions.)

Basic IDA–PBC, on the other hand, does not impose any a priori restriction on the dissipation. Indeed, from the derivations of the previous subsection we concluded that EBC had the additional constraint $\nabla H^\top F \nabla H = \nabla H_d^\top F \nabla H_d$—that is, the dissipated power should remain invariant under the action of the control. Some simple calculations prove that the latter is equivalent to

$$(2\nabla H + \nabla H_a)^\top \mathcal{R} \nabla H_a = 0.$$

Evaluating the expression at the equilibrium and enforcing the stability requirement we recover (21).

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3. GENERATING NEW CYCLO–PASSIVITY PROPERTIES

To overcome the dissipation obstacle we propose in this section to exploit the non–uniqueness of the PH representation to generate new cyclo–passive outputs. More precisely, we will look for full rank matrices $F_d : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, with

$$F_d(x) + F_d^\top(x) \leq 0, \qquad (22)$$

and storage functions $H_{\mathsf{PS}} : \mathbb{R}^n \to \mathbb{R}$ such that

$$F(x)\nabla H(x) = F_d(x)\nabla H_{\mathsf{PS}}(x). \qquad (23)$$

It is clear that, if (22) and (23) hold, then the PH system with port variables $(u, g^\top \nabla H_{\mathsf{PS}})$ will be cyclo–passive with storage function H_{PS}. It turns out that $g^\top \nabla H_{\mathsf{PS}}$ is not adequate to overcome the dissipation obstacle and another cyclo–passive output—that, being related with the power shaping procedure of (Ortega *et al.*, 2003), we call y_{PS}—must be generated. Interestingly, we also prove that in the single input case a necessary and sufficient condition for the new cyclo–passive output y_{PS} to be equal to the "natural" output $g^\top \nabla H_{\mathsf{PS}}$ is precisely the absence of the dissipation obstacle.

3.1 Construction of y_{PS}

The procedure to identify the new passive outputs is contained in the following proposition, which requires F_d to be full rank and relies on a direct application of Poincare's Lemma. [4]

Proposition 3. For all solutions F_d of the PDE

$$\nabla\left(F_d^{-1}F\nabla H\right) = \left[\nabla\left(F_d^{-1}F\nabla H\right)\right]^\top, \qquad (24)$$

verifying (22) there exists a storage function H_{PS} such that the PH system

$$\Sigma_{(u,y_{\mathsf{PS}})} \begin{cases} \dot{x} &= F(x)\nabla H(x) + g(x)u \\ y_{\mathsf{PS}} &= -g^\top(x)F_d^{-\top}(x)[F(x)\nabla H(x) + g(x)u] \end{cases} \qquad (25)$$

satisfies the cyclo–passivity inequality

$$\dot{H}_{\mathsf{PS}} \leq u^\top y_{\mathsf{PS}} \qquad (26)$$

Remark 2. Under the assumption that F is full rank we obtain a trivial solution of (24) setting $F_d = F$. In this case, $H_d = H$ and we obtain the new power–balance equation

$$\dot{H} = \dot{x}^\top F^{-1}\dot{x} + u^\top y_{\mathsf{PS}}.$$

Comparing with (2) we see that the new passive output is obtained swapping the damping—as first observed in (Jeltsema *et al.*, 2004).

[4] Poincare's Lemma: Given $f : \mathbb{R}^n \to \mathbb{R}^n$, $f \in \mathcal{C}^1$. There exists $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla\psi = f$ if and only if $\nabla f = (\nabla f)^\top$.

Remark 3. The construction proposed in (Ortega *et al.*, 2003) for power–shaping can be used also here to provide solutions of (24), provided F is full rank. Namely, it is easy to show that for all matrices $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, with $M(x) = M^\top(x)$ and all $\lambda \in \mathbb{R}$, such that

$$\tilde{M}(x) \triangleq \frac{1}{2}[(\nabla^2 H(x))M(x) + \nabla(M(x)\nabla H(x)) + 2\lambda I_n]$$

is full rank, $F_d^{-1} = \tilde{M}F^{-1}$ solves (24). The resulting storage function being $H_{\mathsf{PS}} = \lambda H + (\nabla H)^\top M \nabla H$.

4. CONTROL BY INTERCONNECTION WITH $\Sigma_{(U,Y_{\mathsf{PS}})}$ AND IDA–PBC

In this section we apply CbI to the new PH system $\Sigma_{(u,y_{\mathsf{PS}})}$ and show that, in this way, we can shape even the coordinates where dissipation is present. More precisely, we will remove the second condition for existence of Casimirs in (9), obviating the dissipation obstacle (19). Additionally, we will show that these new conditions for CbI ensures a solution to the matching equation of IDA–PBC (with modified interconnection and damping structure).

4.1 CbI with $\Sigma_{(u,y_{\mathsf{PS}})}$ Overcomes the Dissipation Obstacle

Proposition 4. Assume the PDE (24) admits a solution F_d verifying (22) and such that

$$F_d \nabla\mathcal{C} = -g, \qquad (27)$$

for some vector function $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$. Consider the PH system (25) coupled with the PH controller Σ_c (5) through the power–preserving interconnection subsystem

$$\Sigma_I^{\mathsf{PS}} : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y_{\mathsf{PS}} \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}. \right. \qquad (28)$$

Then, for all functions $\Phi : \mathbb{R}^m \to \mathbb{R}$, the following cyclo–passivity inequality is satisfied

$$\dot{W}_{\mathsf{PS}} \leq v^\top y, \qquad (29)$$

where the storage function $W_{\mathsf{PS}} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is defined as

$$W_{\mathsf{PS}}(x,\zeta) \triangleq H_{\mathsf{PS}}(x) + H_c(\zeta) + \Phi(\mathcal{C}(x) - \zeta), \qquad (30)$$

with $H_{\mathsf{PS}} = \int(F_d^{-1}F\nabla H)dx$.

Remark 4. The key difference between Propositions 1 and 4 is that the second condition for generation of Casimirs in the former, namely $g^\top\nabla\mathcal{C} = 0$, is conspicuously absent in the latter. As pointed out in Subsection 2.4 if both conditions in (9) are satisfied then the dissipation obstacle condition for CbI with $\Sigma_{(u,y)}$ (19) appears—see also (20). This restriction is not imposed in CbI with $\Sigma_{(u,y_{\mathsf{PS}})}$.

4.2 CbI with $\Sigma_{(u,y_{\mathtt{PS}})} \Rightarrow$ IDA–PBC

Similarly to CbI with $\Sigma_{(u,y)}$, CbI with $\Sigma_{(u,y_{\mathtt{PS}})}$ also admits a static state feedback realization. Now, the resulting control law and storage function are solutions of the matching equation of IDA–PBC. More precisely, the conditions (24) and (27) of Proposition 4 ensure the existence of a static state–feedback, $\hat{u}_{\mathtt{IDA}} : \mathbb{R}^n \to \mathbb{R}^m$ and a storage function H_d, such that the matching condition of IDA–PBC (Ortega and García-Canseco, 2004)

$$F\nabla H + g\hat{u}_{\mathtt{IDA}} = F_d \nabla H_d \tag{31}$$

is satisfied. To prove this fact note that, using (24) (which implies (23)) and (27) we can rewrite (31) as

$$F_d(\nabla H_{\mathtt{PS}} - \nabla \mathcal{C}\hat{u}_{\mathtt{IDA}}) = F_d \nabla H_d.$$

For all functions $H_c : \mathbb{R}^m \to \mathbb{R}$, this equation is satisfied selecting

$$\hat{u}_{\mathtt{IDA}}(x) = -\nabla_{\mathcal{C}} H_c(\mathcal{C}(x)), \quad H_a(x) = H_c(\mathcal{C}(x)),$$
$$H_d(x) = H_{\mathtt{PS}}(x) + H_a(x).$$

As for the case of CbI with $\Sigma_{(u,y)}$, these expressions result from the projection of the overall system on the manifold $\zeta = \mathcal{C}(x)$.

Remark 5. We have shown above that the conditions for CbI with $\Sigma_{(u,y_{\mathtt{PS}})}$ ensures a solution to (31), however this does not imply that it generates all solutions of this equation. Indeed, it is easy to see that (31) may have solutions even though $F_d^{-1} F\nabla H$ is not a gradient of some function—as required by (24). In other words CbI with $\Sigma_{(u,y_{\mathtt{PS}})} \Rightarrow$ IDA–PBC but the converse is not true. In the next section we will establish the equivalence with Basic IDA–PBC using a state–modulated interconnection.

5. STATE–MODULATED CBI WITH $\Sigma_{(U,Y_{\mathrm{PS}})} \Leftrightarrow$ BASIC IDA–PBC

In this section we prove that using a state–modulated interconnection, see (van der Schaft, 2000), we can further relax the condition for existence of Casimirs (27). We will, furthermore, prove that the new condition is actually the matching condition of Basic IDA–PBC, establishing in this way the equivalence between the two methods.

Proposition 5. Assume the PDE (24) admits a solution F_d verifying (22) and such that

$$g^{\perp} F_d \nabla \mathcal{C} = 0, \tag{32}$$

for some vector function $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$. Consider the PH system (25) coupled with the PH

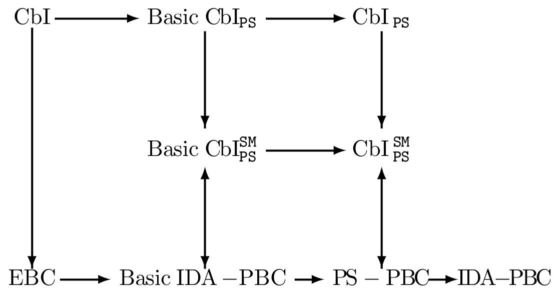

Fig. 2. Relationship between the different control schemes.

controller Σ_c (5) through the state–modulated power–preserving interconnection subsystem

$$\Sigma_I^{SM} : \left\{ \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} 0 & -\alpha(x) \\ \alpha(x) & 0 \end{bmatrix} \begin{bmatrix} y_{\mathtt{PS}} \\ y_c \end{bmatrix} + \begin{bmatrix} v \\ 0 \end{bmatrix}, \right. \tag{33}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is defined as

$$\alpha = -(g^\top g)^{-1} g^\top F_d \nabla \mathcal{C}. \tag{34}$$

Then, for all functions $\Phi : \mathbb{R}^m \to \mathbb{R}$, the cyclo–passivity inequality (29) with storage function (30) is satisfied.

Corollary 1. There exists a function H_a that solves the matching equation of Basic IDA–PBC, i.e., a solution of the PDE (13), if and only if there exists a Casimir function that solves (32) for $F_d = F$.

6. CONCLUDING REMARKS

We have investigated in this paper the relationships between CbI, EBC and the well–known IDA–PBC—in its basic and general forms. We have concentrated our attention on the ability of the methods to shape the energy function and the role of dissipation to fulfill this task. Energy–shaping is, of course, the key step for the successful application of PBC and, similarly to all existing methods for nonlinear systems controller (or observer) design, requires the solution of a set of PDEs. In the case of CbI methods the solutions of the PDEs are the Casimir functions $\mathcal{C}$ and, eventually, F_d. On the other hand, for EBC and IDA–PBC their solution directly provides the “added” energy function H_a, with F_d a free parameter for IDA–PBC. The comparison between the various methods has been done, precisely, on the basis of these PDEs.

To enlarge the domain of application of CbI several variations of the method have been considered—all of them adopting the simple (m–th order) nonlinear integrator controller subsystem Σ_c given in (5)—and they are summarized, together with the PDE to be solved, as follows:

– (CbI) System $\Sigma_{(u,y)}$ (1) with unitary feedback interconnection Σ_I (7). PDEs to be solved

$$F\nabla\mathcal{C} = -g, \quad g^\top \nabla\mathcal{C} = 0.$$

– (Basic CbI$_{\mathtt{PS}}$) System $\Sigma_{(u,y_{\mathtt{PS}})}$ (25) with $F_d = F$ and unitary feedback interconnection $\Sigma_I^{\mathtt{PS}}$ (28). PDE to be solved

$$F\nabla\mathcal{C} = -g.$$

– (CbI$_{\mathtt{PS}}$) System $\Sigma_{(u,y_{\mathtt{PS}})}$ (25) with $F_d \neq F$ and unitary feedback interconnection $\Sigma_I^{\mathtt{PS}}$ (28). PDEs to be solved

$$F_d\nabla\mathcal{C} = -g, \quad \nabla\left(F_d^{-1}F\nabla H\right) = \left[\nabla\left(F_d^{-1}F\nabla H\right)\right]^\top,$$

the latter with the constraint $F_d + F_d^\top \leq 0$.

– (Basic CbI$_{\mathtt{PS}}^{\mathtt{SM}}$) System $\Sigma_{(u,y_{\mathtt{PS}})}$ (25) with $F_d = F$ and state–modulated feedback interconnection $\Sigma_I^{\mathtt{SM}}$ (33). PDE to be solved

$$g^\perp F\nabla\mathcal{C} = 0.$$

– (CbI$_{\mathtt{PS}}^{\mathtt{SM}}$) System $\Sigma_{(u,y_{\mathtt{PS}})}$ (25) with $F_d \neq F$ and state–modulated feedback interconnection $\Sigma_I^{\mathtt{SM}}$ (33). PDEs to be solved

$$g^\perp F_d\nabla\mathcal{C} = 0, \quad \nabla\left(F_d^{-1}F\nabla H\right) = \left[\nabla\left(F_d^{-1}F\nabla H\right)\right]^\top,$$

the latter with the constraint $F_d + F_d^\top \leq 0$.

On the other hand, we have:

– (EBC) PDE to be solved

$$g^\perp F\nabla H_a = 0, \quad \textit{subject to } (2\nabla H + \nabla H_a)^\top \mathcal{R}\nabla H_a = 0.$$

– (Basic IDA–PBC) PDE to be solved

$$g^\perp F\nabla H_a = 0.$$

– (IDA–PBC) PDE to be solved

$$g^\perp[(F_d - F)\nabla H + F_d\nabla H_a] = 0.$$

The relationship between all these schemes is summarized in Fig. 2.

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