Preprints of the 20th World Congress
The International Federation of Automatic Control
Toulouse, France, July 9-14, 2017

# The geometric structure of interconnected thermo-mechanical systems. ⋆

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**Abstract:** This contribution reports on an ongoing research project aimed in developing a unified theoretical framework for the description of interconnected thermo-mechanical systems with a particular emphasis on thermodynamic engines. We analyse from the geometrical viewpoint the structure of thermodynamic and mechanical interconnection and propose an approach to the unified description of thermo-mechanical systems. The theoretical results are illustrated by a physical example.

*Keywords:* Hamiltonian systems, Mathematical modelling, Equilibrium thermodynamics, Contact geometry, Interconnected Systems

## 1. INTRODUCTION

During the last decades there have been a growing interest in geometrical description and interpretation of thermodynamic systems. We refer the interested reader to the works Mrugała et al. (1991); Eberard et al. (2007); Merker and Krüger (2013); Delvenne and Sandberg (2014); Gromov and Caines (2015) and references therein for an overview of different directions of research within this broad field.

One particularly important application of thermodynamics is the design and optimisation of thermodynamic (heat) engines, that is systems that transform heat energy into the mechanic energy. By now, dozens of heat engines have been developed working according to different schemes (i.e., implementing different thermodynamic cycles). However, there is one aspect common for any heat engine: the interaction between the thermodynamic subsystem and the mechanical one. We believe that the geometrical analysis of the interconnection structure of these two systems may allow us to better understand and optimise the overall system in order to achieve maximal possible efficiency.

The main obstacle in developing this programme is that thermodynamic and mechanical systems "live in different worlds": a mechanical system evolves on an even-dimensional symplectic manifold while a thermodynamic system evolves on an odd-dimensional submanifold of a contact manifold (often referred to as the thermodynamic phase space). Recently, there have been several attempts to reconcile these representations. In particular, it was shown that contact vector fields can be used to describe the evolution of dissipative Hamiltonian systems (see Bravetti et al. (2016) and references therein). On the other hand, there are a number of results that attempt to describe thermodynamic systems using the Hamiltonian (symplectic) framework (see, e.g., Morrison (1998); Öttinger (2005)). However, despite many theoretical advances there have not been substantial progress in the geometrical description of interconnected thermo-mechanical systems so far.

In this contribution we use the approach based upon the symplectification of the thermodynamic evolution. It is shown formally that the thermodynamic evolutionary equations can be obtained in the same way as the ones generated by a mechanical Hamiltonian.This approach leads to certain loss of information. However, we argue that this does not restrict the applicability of the approach as we retain most important information. The developed approach is illustrated by a simple, but physically relevant example.

The paper is organized as follows. Section 2 gives a brief overview of modelling Hamiltonian systems with constraints. Section 3 presents necessary facts about the description of thermodynamic evolution and discusses in detail the bundle isomorphism induced by the thermodynamic contact 1-form. In Sec. 4 we discuss different approaches to the description of interconnected systems while Sec. 5 presents an example.

## 2. HAMILTONIAN SYSTEMS WITH CONSTRAINTS

Consider a controlled mechanical system with the Hamiltonian $H(q,p) : T^*Q \to \mathbb{R}$, where $Q$ is the configuration space which we assume to be equal to $\mathbb{R}^n$. Let there be a number of, generally, non-holonomic constraints expressed as a distribution $C(q) \in T_qQ$ restricting the evolution of the system. We assume that the distribution $C$ is generated by a set of linearly independent 1-forms $\phi^i(q) \in T^*Q$, $i = 1, \dots, k$. This implies that the admissible velocity vectors $\dot{q}$ belong to the kernel of a smooth $k$-dimensional co-distribution $C^* \subset T^*Q$, i.e. $\dot{q} \in \ker C^*$, which is expressed as $\langle \dot{q}, \sigma \rangle = 0$ with $\sigma \in C^*$ and $\langle \cdot, \cdot \rangle : TQ \times T^*Q \to \mathbb{R}$

⋆ The first author acknowledges the research grant 17-11-01093 from the Russian Science Foundation.

the standard pairing operation or, in algebraic notation, as $C^T(q)\dot{q} = 0$, where $C(q)$ is an $[n \times k]$ matrix whose columns are the components of the 1-forms spanning $C^*$ written in local coordinates.

The distribution $C$ is said to be *involutive* if $X, Y \in C \Rightarrow [X, Y] \in C$, where the square brackets denote the Lie commutator of two vector fields. By Frobenius theorem an involutive distribution can be integrated to yield $k$ smooth functions $c(q)$ such that $X(c) = 0$. These functions are referred to as the first integrals. In this case we say that the respective constraints (2b) are *holonomic*. Otherwise, the constraints are said to be *non-holonomic*. In practice, the set of constraints include both holonomic and non-holonomic constraints.

An unconstrained Hamiltonian system evolves on the state space manifold $T^*Q$ which is endowed with the canonical symplectic form $\omega = dq^i \wedge dp_i$ (here and henceforth the Einstein summation convention is implied). This symplectic form defines a canonical isomorphism between the tangent and cotangent bundles: $\Omega : T(T^*Q) \to T^*(T^*Q)$ defined by $\Omega(X)(\cdot) = \omega(X, \cdot)$. The vector field, corresponding to the Hamiltonian $H$, is defined as $X_H = \Omega^{-1}(dH)$, i.e. $\omega(X_H, \cdot) = dH$.

When dealing with the constrained system, the Hamiltonian function has to be augmented to take into account the constraints. Thus, we define the constrained vector field as follows:

$$X_{H,\phi} = \Omega^{-1}(dH + \lambda_i \pi_Q^* \phi^i), \tag{1}$$

where $\pi_Q : T^*Q \to Q$ is the projection of the cotangent bundle on its base and $\pi_Q^*$ is the pull-back of $\pi_Q$ which lifts $\phi^i$ to $T^*(T^*Q)$.

In local coordinates, the dynamics of a port-Hamiltonian system with constraints is described by a set of differential-algebraic equations of the form (Neimark and Fufaev, 1972; Arnold et al., 2006; Castaños et al., 2013):

$$\dot{x} = J\nabla H(x) + \hat{C}(x)\lambda + \hat{g}(x)u \tag{2a}$$

$$\mathbf{0} = C^T(q)\nabla_p H(x) \tag{2b}$$

$$y = \nabla^T H(x)\hat{g}(x)\ , \tag{2c}$$

where $H$ is the Hamiltonian (energy) function of the unconstrained system, the state is given by $x^T = \begin{pmatrix} q^T & p^T \end{pmatrix}$ with $r \in Q$ and $p \in T_r^*Q$ the positions and momenta, respectively; $\hat{C}(x) = \begin{pmatrix} \mathbf{0}_{[k\times n]} & C^T(x) \end{pmatrix}^T$, $\lambda \in \mathbb{R}^k$ is the vector of implicit variables that enforce the constraints; $(u, y) \in \mathbb{R}^{*m} \times \mathbb{R}^m$ are the conjugated external port variables, and $\hat{g}(x) = \begin{pmatrix} \mathbf{0}_{[m\times n]} & g^T(x) \end{pmatrix}^T$ is a $(2n \times m)$-matrix such that rank $\hat{g}(x) = m$ for all $x \in \mathbb{R}^n \times \mathbb{R}^{*n}$. The $[2n \times 2n]$-matrix $J$ is the one associated with the canonical symplectic form,

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}\ .$$

Here and forth all functions are assumed to be smooth enough and the gradient is assumed to be a column vector.

The vector field $X \in T(T^*\mathbb{R}^n)$ is written as

$$X = D_H + D_c\lambda + X_g u \tag{3}$$

where

$$D_H = \Omega^{-1}(dH) = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{4}$$

is the Hamiltonian vector field,

$$D_C\lambda = C_i^j(q)\lambda_j \frac{\partial}{\partial p_i} \tag{5}$$

is the vector field of the internal (constraint) forces, and

$$X_g u = g_i^j u_j \frac{\partial}{\partial p_i}$$

is the control vector field.

Equation (2b) constrains the configuration space of (2) and can be written as $D_C(H) = 0$. This is equivalent to saying that the internal forces do not produce work as there is no displacement in the direction of the constraint forces and hence they do not alter the total energy of the system. However, this may not be true in general, when non-holonomic constraints of general form are considered (see, e.g., (Bloch, 2003; Baruh, 1999)).

## 3. THERMODYNAMIC CONTACT VECTOR FIELDS

In this section, we present a brief overview of the geometric approach to the description of thermodynamic systems' evolution. For a more detailed treatment see Callen (1985); Kondepudi and Prigogine (1998) for thermodynamics, Geiges (2008); Arnold (1989) for contact geometry, and Mrugała et al. (1991); Gromov and Caines (2015) for the contact description of thermodynamics.

### *3.1 Contact geometry basics*

In the following, we will consider single phase, single component homogeneous thermodynamic systems that do not undergo any chemical transformations. The state space of such a system can be represented as an embedded manifold in the thermodynamic phase space $\mathcal{M}$. This manifold is shown to be an integral (*Legendre*) manifold corresponding to a specific contact 1-form.

*Definition 1.* Let $(x^0, x^1, \ldots, x^n, y_1, \ldots, y_n)$ be the local coordinates on $\mathcal{M}$. The *canonical thermodynamic contact 1-form* is defined as

$$\alpha = dx^0 - y_i dx^i, \qquad 1 \le i \le n. \tag{6}$$

Each Legendre manifold on $(\mathcal{M}, \alpha)$ is uniquely determined by a particular function.

*Lemma 2.* (Arnold (1989)). Let $\mathcal{N} = \{1, \ldots, n\}$ be the set of indices. Given the contact form (6), a disjoint partitioning $I, J \subset \mathcal{N}$, $I \cap J = \emptyset$, $I \cup J = \mathcal{N}$ with $n_I$ and $n_J$ components, $n_I + n_J = n$, and a smooth function $\zeta(x^i, y_j)$, $i \in I, j \in J$, the following equations define the Legendre manifold $\mathcal{L}_\zeta$ on $(\mathcal{M}, \alpha)$:

$$\lambda^0(x, y) = x^0 - \zeta + y_j \frac{\partial \zeta}{\partial y_j} = 0, \tag{7a}$$

$$\lambda^j(x, y) = x^j + \frac{\partial \zeta}{\partial y_j} = 0, \tag{7b}$$

$$\lambda_i(x, y) = y_i - \frac{\partial \zeta}{\partial x^i} = 0. \tag{7c}$$

The variables $(x^i, y_j)$, $i \in I, j \in J$ can be chosen as local coordinates in some open neighbourhood of $a \in \mathcal{L}_\zeta$. The function $\zeta$ is called the *generating function* of the Legendre manifold $\mathcal{L}_\zeta$.

In Gromov and Caines (2011) it was shown that the choice of $U(S,V,N)$ as a generating function naturally leads to the identification of $x$ and $y$ with the extensive and the intensive thermodynamic variables. It can be shown either, see Gromov and Caines (2011), that any state function obtained as the Legendre transformation of $U(S,V,N)$ generates the same Legendre manifold up to a choice of coordinate basis.

In the following, we will focus on the energy based representation and thus will consider only the Legendre manifolds generated by functions of $n$ arguments $x$, i.e., $\zeta(x^1,\ldots,x^n)$. While not restricting the generality of the approach this will allow us to simplify notation at some points.

Below, we formulate a number of results on the properties of contact forms that will be used later on.

*Proposition 3.* (Libermann and Marle (1987)). Given a contact manifold $(\mathcal{M},\alpha)$, the tangent bundle $T\mathcal{M}$ can be decomposed into

$$T\mathcal{M} = \mathcal{V}(\alpha) \oplus \mathcal{H}(\alpha),$$

where $\mathcal{H}(\alpha) = \ker\alpha$, called the *horizontal bundle*, is of dimension $2n$, and $\mathcal{V}(\alpha) = \ker d\alpha = T\mathcal{M} \setminus \mathcal{H}(\alpha)$, called the *vertical bundle*, is of dimension 1.

The following result shows that a contact 1-form can be used to define a bundle isomorphism similar to that defined by a symplectic form.

*Proposition 4.* (Libermann and Marle (1987)). The mapping $\Omega : T\mathcal{M} \to T^*\mathcal{M}$, defined by

$$\Omega : X \mapsto \imath_X d\alpha,$$

maps any vector field to a *semi-basic 1-form* $\beta \in \mathcal{B}$, where $\mathcal{B} = \{\phi \in T^*\mathcal{M} | \imath_E \phi = 0\}$ and $E$ is the Reeb vector field defined by $\imath_E\alpha = 1$, $\imath_E d\alpha = 0$. Furthermore, the restriction of $\Omega$ to the vector space of horizontal vector fields defines an isomorphism between this vector space and the vector space $\mathcal{B}$. We denote this by $\tilde{\Omega} = \Omega\big|_{\mathcal{H}(\alpha)} : \mathcal{H}(\alpha) \to \mathcal{B}$.

Henceforth, for any $X \in \mathcal{H}(\alpha)$ and $\beta \in \mathcal{B}$ we will employ the "musical" notation and will write $^\flat X \in \mathcal{B}$ to denote $\tilde{\Omega}(X)$ and $^\sharp\beta$ to denote the inverse operation, i.e., $^\sharp\beta = \tilde{\Omega}^{-1}(\beta) \in \mathcal{H}(\alpha)$.

### *3.2 Contact vector fields*

On particular approach to the description of the evolution of a thermodynamic system consists in considering the respective vector field as a contact vector field as discussed below.

*Definition 5.* Let $X$ be a vector field on the contact manifold $(\mathcal{M},\alpha)$. We denote the local flow of $X$ by $\psi_t$. The vector field $X$ is called a *contact vector field* if $(\psi_t)_*\mathcal{H}(\alpha) = \mathcal{H}(\alpha)$, where $\mathcal{H}(\alpha)$ is the horizontal bundle. The vector field $X$ is called a *strict contact vector field* if $\psi_t^*\alpha = \alpha$.

In other words, the flow of a contact vector field preserves the contact structure, i.e., $[X,\mathcal{H}(\alpha)] \subseteq \mathcal{H}(\alpha)$, whereas the flow of a strict contact vector field preserves the contact form.

There is a unique correspondence between a contact vector field and a real-valued function on $\mathcal{M}$, which is sometimes called the *contact Hamiltonian* (see, e.g., Arnold (1989)) as stated in the following theorem.

*Theorem 6.* (Geiges (2008)). Given a contact 1-form $\alpha$ (6), let $X$ be a contact vector field. Its contact Hamiltonian $F : \mathcal{M} \to \mathbb{R}$ is defined as $F = \alpha(X)$. Conversely, given a contact Hamiltonian $F$, the corresponding contact vector field is

$$X = \left(F - y_i\frac{\partial F}{\partial y_i}\right)\frac{\partial}{\partial x^0} - \frac{\partial F}{\partial y_j}\frac{\partial}{\partial x^j} + \left(y_i\frac{\partial F}{\partial x^0} + \frac{\partial F}{\partial x^i}\right)\frac{\partial}{\partial y_i} \quad (8)$$

Contact Hamiltonians differ in some respects from their symplectic counterparts. In particular, contact vector fields are, in general, transverse to the Legendre manifolds as follows from the definition of the contact Hamiltonian. To overcome this, it was proposed in (Mrugała et al., 1991) to consider contact vector fields satisfying an additional property:

*Theorem 7.* Let $\mathcal{L} \subset \mathcal{M}$ be a Legendre manifold. Then $X$ (8) is tangent to $\mathcal{L}$, i.e. $X|_\mathcal{L} \in T\mathcal{L}$, if and only if $F$ vanishes on $\mathcal{L}$, i.e., $\mathcal{L} \subset \ker(F)$.

Thus one defines the class of thermodynamic contact vector fields.

*Definition 8.* A contact vector field $X_c$ is said to be a *thermodynamic contact vector field* on the Legendre manifold $\mathcal{L}$ if the corresponding contact Hamiltonian $F_c$ satisfies the invariance condition $\mathcal{L} \subset \ker(F_c)$.

*Corollary 9.* For the restriction of a thermodynamic contact vector field $X_c$ to the corresponding Legendre manifold $\mathcal{L}$ holds

$$X_c|_\mathcal{L} = y_i\frac{\partial F}{\partial y_i}\frac{\partial}{\partial x^0} + \frac{\partial F}{\partial y_j}\frac{\partial}{\partial x^j} - \left(y_i\frac{\partial F}{\partial x^0} + \frac{\partial F}{\partial x^i}\right)\frac{\partial}{\partial y_i}. \quad (9)$$

Let, furthermore, the contact Hamiltonian be independent of $x^0$, then (9) turns into

$$X_c|_\mathcal{L} = y_i\frac{\partial F}{\partial y_i}\frac{\partial}{\partial x^0} + \frac{\partial F}{\partial y_j}\frac{\partial}{\partial x^j} - \frac{\partial F}{\partial x^i}\frac{\partial}{\partial y_i}. \quad (10)$$

Following Favache et al. (2009), we consider the following contact Hamiltonian:

$$F(x^0, x^i, y_i) = \left(\frac{\partial\zeta}{\partial x^i} - y_i\right)\Phi^i, \quad (11)$$

where $\Phi^i$ are the flow rates (fluxes) of the respective extensive variables $x^i$.

### *3.3 Bundle isomorphism on the thermodynamic phase space*

Consider the thermodynamic phase space $\mathcal{M}$ and the Legendre manifold $\mathcal{L}$. We define the projection $\pi_\mathcal{L} : \mathcal{M} \to \mathcal{L}_\zeta$. The following result follows immediately:

*Lemma 10.* Let $\alpha$ be the thermodynamic contact 1-form and $X_c$ be the contact vector field on the Legendre manifold $\mathcal{L}$. Then the following holds:

$$(\pi_\mathcal{L})_* X_c \in \mathcal{H}(\alpha).$$

**Proof.**

$$\alpha\left((\pi_\mathcal{L})_* X_c\right) = \alpha\left(X_c\right)|_\mathcal{L} = F_c|_\mathcal{L} = 0 \Rightarrow X_c \in \mathcal{H}(\alpha).$$

□

Also, let $\pi^o : \mathcal{M} \to \mathcal{M}^o$ be a canonical projection of a point $z \in \mathcal{M}$ onto its last $2n$ coordinates. This can be seen as a mapping from $\mathcal{M}$ to the equivalence class $\mathcal{M}/x^0$.

Denote by $\mathbf{F}^o$ the set of all contact Hamiltonians that do not depend on $x^0$ and denote the contact vector field generated by $F^o \in \mathbf{F}^o$ by $X_c^o$. It can be easily shown that this vector field has the structure (10). Denote the set of all vector fields $X_c^o$ by $\mathcal{X}^o$. We have the following result:

*Lemma 11.* Let $\alpha$ be the thermodynamic 1-form, $\mathcal{M}$ the thermodynamic phase space and $\pi^0$ the canonical projection as defined above. The following inclusion holds: $\pi_*^o \mathcal{X}^o \subset \pi_*^o T\mathcal{M} \cap \mathcal{H}(\alpha)$.

Furthermore, we have the following characterization for the set of semi-basic 1-forms $\mathcal{B}$:

*Lemma 12.* The set of semi-basic 1-forms $\mathcal{B}$ is identified with the cotangent bundle to $\pi^o\mathcal{M}$:

$$\mathcal{B} = T^*(\pi^o\mathcal{M}).$$

Let $\zeta(x) : \mathcal{M} \to \mathbb{R}$ be a generating function and $\mathcal{L}_\zeta$ be the equilibrium manifold generated by $\zeta(x)$ as defined in Lemma 2. The tangent bundle to $\mathcal{L}_\zeta$ is a span of $n$ linearly independent sections of $T\mathcal{M}$:

$$X_i = y_i \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^i} + \frac{\partial^2 \zeta}{\partial x^i \partial x^i} \frac{\partial}{\partial y_i}, \quad i = 1, \ldots, n. \tag{12}$$

The generic thermodynamic vector field can therefore be written as

$$X = y_i \Phi^{[i]} \frac{\partial}{\partial x^0} + \Phi^{[i]} \frac{\partial}{\partial x^i} + \Phi^{[i]} \frac{\partial^2 \zeta}{\partial x^i \partial x^i} \frac{\partial}{\partial y_i}, \tag{13}$$

where $\Phi^{[i]}$ are the *thermodynamic controls* which can be associated with the fluxes of the respective variables $x^i$.

*Theorem 13.* For control contact Hamiltonians (11) the following holds:

- The differential of any control contact Hamiltonian (11) is a semi-basic 1-form;
- The map $\Omega$ sends vector field (13) to the differential of the respective control contact Hamiltonian (11): ${}^\flat X = dF$.
- Conversely, the inverse transformation maps differentials of the control contact Hamiltonians to the subset of $T\mathcal{M}$ as follows:

$$^\sharp dF = \pi_*^o X \in \pi_*^0 T\mathcal{M} \cap \mathcal{H}(\alpha)$$

The above result states that the differential of the thermodynamic 1-form $\alpha$ can play a role similar to that played by the symplectic 2-form. Namely, when restricted to the $2n$ coordinates $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ it defines an isomorphism between the differential of the contact Hamiltonian and a section in the subspace $\pi_*^0 T\mathcal{M} \cap \mathcal{H}(\alpha)$. This subspace consists of vector fields defining the evolution of the respective $2n$ state variables while neglecting $x^0$.

Therefore, this approach allows to describe the evolution of a thermodynamic system in the same way as the evolution of a mechanical system is described while sacrificing a part of information related to the dynamics of $x^0$.

In the subsequent sections we will use this result to develop a unified framework for the description of thermo-mechanical systems.

## 4. INTERCONNECTION STRUCTURE

### *4.1 Mechanical systems*

When considering interconnection of mechanical systems one may stay with the standard input/output interconnection framework where input of one system is attached to the output of another system and vice versa. However, in many cases this may lead to unnecessary complications. One particular reason for this is that the output of one system can not always serve as an input for another system because the outputs are typically positions and velocities and the inputs are forces. In general, this approach works well when one considers the interconnection between a mechanical system and a controller which transforms the signals in the opposite direction: from positions and velocities to forces.

An alternative approach was developed by Willems in a series of works on behavioural systems (Willems, 1991, 2007). The behavioural approach identifies the dynamics of a system with a family of trajectories, called the behaviour. Within this approach, one does not distinguish between inputs and outputs. The interconnection structure is thus a set of constraints imposed on the system. This approach is similar to that adopted in this paper. Namely, we do not consider the signals as inputs or outputs, but rather as variables which stay in certain functional relations determined by the physical laws.

### *4.2 Thermo-mechanic systems*

When dealing with thermo-mechanical systems, the restrictions associated with the input/output approach becomes more evident. Consider, for instance, the system consisting of two chambers with gas separated by a movable wall. This system was studied in detail in (Gromov and Caines, 2015). Here we consider it from the I/O viewpoint. The system consists of two thermodynamic systems and a mechanical one. The inputs of thermodynamic systems are the flows of extensive variables $\Phi^{(\cdot)}$ while the outputs are the intensive variables. For the mechanical system (the movable wall), the input is the force applied to the wall and the output is the velocity (conjugated variable w.r.t. the system's energy). As was shown in (Gromov and Caines, 2015), the input of the mechanical system depends on the difference of the inputs of the respective thermodynamic systems.

An even more complex interconnection structure occurs when we consider a thermodynamic engine (see, e.g., Mueller-Roemer and Caines (2015)). Such systems exhibit a complex interaction between thermodynamic and mechanical components accompanied by continuous transformation between kinetic and potential energy on the one side and the internal energy of the working body on the other side. This interconnection is particularly difficult to capture within the I/O interconnection framework.

### *4.3 Interconnection structure*

Let $q^i$, $p_i$, $i = 1, \ldots, n$ be the coordinates and momenta describing the mechanical subsystem. We extend the set of variables $(q^i, p_i)$, by new variables $q^{n+j} = x^j$ and $p_{n+j} =$

$y_j$, $j = 1, \ldots, k$, describing the thermodynamic subsystem. Let $\omega$ and $\alpha$ be the symplectic and contact differential forms corresponding to the respective subsystems. We define the extended symplectic form to be $\tilde{\omega} = \omega + d\alpha$. Finally, the extended Hamiltonian function is defined as $\tilde{H} = H(q,p) + F(x,y)$, where $H(q,p)$ and $F(x,y)$ are the (mechanical) Hamiltonian and the contact Hamiltonian. Note that $F(x,y)$ is assumed to be independent of $x^0$. This is a standard assumption which is – to the best of authors' knowledge – not justified formally. An analysis of this issue is a subject of future investigation.

The vector field describing the dynamics of the composite system is given by

$$\tilde{\omega}(\tilde{X}, \cdot) = d\tilde{H}. \tag{14}$$

Note that $\tilde{X}$ evolves on a $2(n+k)$-dimensional manifold and does not reflect the change in the coordinate $x^0$. However, since $x^0$ does not enter $F(x,y)$, this does not influence the result.

Next, we assume that there is a number of (non-)holonomic constraints imposed on the system. In Gromov and Caines (2015), an attempt to classify possible constraints was undertaken. However, we believe that the particular form of the constraints depends on the specific interconnection structure and should be studied on the case by case basis. This is the subject of an ongoing research that will be reported elsewhere. Below we present a couple of general remarks.

The set of constraints is described as a kernel of a co-distribution formed by 1-forms describing individual constraints. Any holonomic constraint can be transformed into a 1-form by taking a differential. Since the internal energy ($x^0$) does not enter the model, any constraints involving the internal energy can be alternatively expressed using $dx^0 = y_i dx^i$. This follows from the fact that the vector field $X_c^o$ generated by $F^o$ belongs to $H(\alpha)$ as stated in Theorem 13.

For a given set of 1-forms $\tilde{\phi}$ the resulting constrained vector field is defined as

$$\tilde{X}_{\tilde{H},\phi} = \tilde{\Omega}^{-1}(d\tilde{H} + \lambda_i \tilde{\phi}^i), \tag{15}$$

where we assume that the respective 1-forms are defined on the cotangent bundle to the extended thermo-mechanic phase space.

## 5. EXAMPLE

Consider a simple system that can be considered as a precursor of the Stirling engine. Its schematic representation is shown in Fig. 1. This system consists of two subsystems: a mechanical and a thermodynamic one.

First consider the thermodynamic subsystem (cylinder with gas) which interacts with its environment through the work exchange (piston) and heat exchange (heater/cooler). The contact Hamiltonian has the form (11) . Written using thermodynamic notation it takes the following form:

$$F(S,V,T,p) = \left(\frac{\partial U}{\partial S} - T\right)\Phi^S + \left(\frac{\partial U}{\partial V} + p\right)\Phi^V.$$

Here we chose the generating function $\zeta$ to be equal to the internal energy $U(S,V)$.

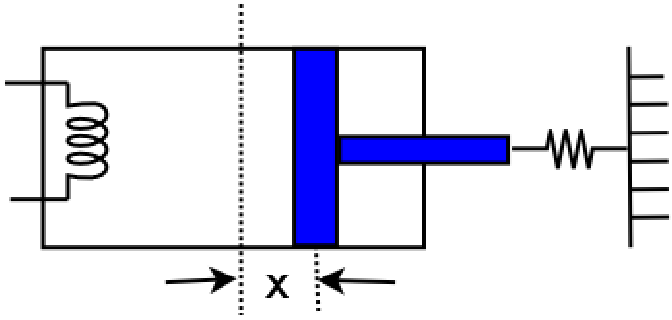


Fig. 1. Thermo-mechanical system consisting of a cylinder with pressurised gas, a heater/cooler and a piston connected to a spring

One can also describe the dynamics of the mechanical part by using the (mechanical) Hamiltonian

$$H = \frac{r^2}{2m} + \frac{k_s x^2}{2},$$

where the first term is the kinetic energy and the second one is the potential energy. Here, $x$ is the displacement of the piston, $r$ is the linear momentum, $m$ is the mass of the piston, and $k_s$ is the spring constant.

Consider an extended symplectic form defined as $\tilde{\omega} = \omega + d\alpha$, where $\omega$ is the symplectic 2-form and $\alpha$ is the thermodynamic contact 1-form. The extended Hamiltonian function is $\tilde{H} = H + F$. As described in Subsection 4.3, the vector field of the composite system $\tilde{X}$ is defined as in (14). The corresponding set of ODEs is

$$\begin{cases} \dot{x} &= \dfrac{r}{m} \\ \dot{r} &= -k_s x + F \\ \dot{S} &= \Phi^S \\ \dot{V} &= \Phi^V \\ \dot{T} &= \dfrac{\partial^2 U}{\partial S \partial S}\Phi^S + \dfrac{\partial^2 U}{\partial V \partial S}\Phi^V \\ -\dot{p} &= \dfrac{\partial^2 U}{\partial S \partial V}\Phi^S + \dfrac{\partial^2 U}{\partial V \partial V}\Phi^V, \end{cases} \tag{16}$$

where $F$ is the force applied to the piston.

The thermodynamic and the mechanical subsystems interact through the piston which is affected both by the pressure within the cylinder and by the force with which the spring pushes the piston. This gives two constraints:

(1) The change of the mechanical energy is equal to the work done by the thermodynamic system:

$$\phi^1 = pdV - \frac{r}{m}dr - k_s x dx.$$

(2) The change of the volume of the gas is related to the velocity of the piston:

$$\phi^2 = dV - Adx.$$

The dynamics of the constraint system is thus

$$\begin{cases} \dot{x} &= \dfrac{r}{m} - \lambda_1 k_s x - \lambda_2 A \\ \dot{r} &= -k_s x + F - \lambda_1 \dfrac{r}{m} \\ \dot{S} &= \Phi^S \\ \dot{V} &= \Phi^V + \lambda_1 p + \lambda_2 \\ \dot{T} &= \dfrac{\partial^2 U}{\partial S \partial S}\Phi^S + \dfrac{\partial^2 U}{\partial V \partial S}\Phi^V \\ -\dot{p} &= \dfrac{\partial^2 U}{\partial S \partial V}\Phi^S + \dfrac{\partial^2 U}{\partial V \partial V}\Phi^V. \end{cases} \tag{17}$$

The system (17) can be approached from different directions. The most obvious way is to use the constraint to restrict the set of admissible fields. In this particular case this would imply that we "fix" $\Phi^V$ and $F$ using the relations $p\Phi^V = \dfrac{r}{m}F$ and $\Phi^V = \dfrac{Ar}{m}$.

Should there be holonomic constraints imposed on the system, it may prove useful to use the approach based on implicit description of constrained Hamiltonian systems developed in (Castaños et al., 2013; Castaños and Gromov, 2016).

## 6. CONCLUSIONS

This contribution presents initial results on the unified description of interconnected thermo-mechanical systems. It is shown that such systems can be described within the framework of constrained Hamiltonian systems. The next step is to extend the obtained results to a thermodynamic engine (the Stirling engine being the first candidate) and apply to the obtained model standard control methods developed for constrained Hamiltonian systems.

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