# Asymptotic Stabilization via Control by Interconnection of Port-Hamiltonian Systems

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**Abstract:** We study the asymptotic properties of *control by interconnection*, a passivity-based controller design methodology for stabilization of port-Hamiltonian systems. It is well-known that the method, in its basic form, imposes some unnatural controller initialization to yield asymptotic stability of the desired equilibrium. We propose two different ways to overcome this restriction, one based on adaptation ideas, and the other one adding an extra damping injection to the controller. The analysis and design principles are illustrated through an academic example.

Keywords: Nonlinear control systems; Port-Hamiltonian systems; Asymptotic stability; Passivity-based control.

## 1. INTRODUCTION

Recently, port-Hamiltonian (PH) models [van der Schaft, 2000] have been a focus of attention in the control community (e.g. Wang et al. [2007], Cheng et al. [2005], Fujimoto et al. [2003], Ortega et al. [2002]). There are, at least, two reasons for their appeal: first, that they describe a wide class of physical systems, there included (but not limited to), systems described by Euler-Lagrange equations. Second, that PH models directly reveal the fundamental role of the physical concepts of energy, dissipation and interconnection—making passivity-based control (PBC) [Ortega and Spong, 1989, van der Schaft, 2000] a suitable candidate to regulate the behavior of PH systems.

In this paper, we are interested in stabilization of PH systems using control by interconnection (CbI) [Ortega et al., 2001, 2002]. Similarly to other PBC techniques, the objective in CbI is to render the closed-loop passive with respect to a desired energy (storage) function. This is accomplished in CbI selecting the controller to be also a PH system, which connected to the plant through a power-preserving interconnection, results in a closed-loop that is again PH with energy function equal to the sum of the plant's and the controller's energies.

In its original formulation, applicability of CbI is stymied by the so-called dissipation obstacle [Ortega et al., 2001], a problem that appears when the dissipation of the open-loop is different from zero at the desired equilibrium. In Ortega et al. [2008], this problem has been solved generating different passive outputs giving rise to the so-called power shaping CbI. Both methods, standard and power shaping CbI, rely on the creation of functions, called Casimirs, which are independent of the energy function. The existence of these invariants presents an obstruction to the *asymptotic* stabilization of the desired equilibrium. The main contribution of this paper is to propose two modifications to the existing CbI to overcome this problem. The first modification is motivated by adaptation principles, while the second one is based on the addition of an extra damping injection to the controller. As an additional by-product of the analysis performed, the two versions of CbI are unified.

To make the paper self-contained, we begin the following section with a brief description of CbI and refer the reader to Ortega et al. [2008] for more details. Section 3 contains specific guidelines to apply CbI for equilibrium stabilization. The modifications to achieve asymptotic stability are then presented in Section 4. Finally, we state some concluding remarks in Section 5.

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*Notation* The arguments of the functions are omitted once they are defined and there is no possibility of confusion. All vectors defined in the paper are column vectors, even the gradient of a scalar function, denoted with the operator $\nabla \triangleq \frac{\partial}{\partial x}$. We also define $\nabla^2 \triangleq \frac{\partial^2}{\partial x^2}$. Given a vector $x$ and a matrix $K = K^\top > 0$, $\|x\|$ denotes the Euclidean norm and $\|x\|_K$ the norm $x^\top K x$.

## 2. PRELIMINARIES

Although this note deals with PH systems [van der Schaft, 2000] only, it will be useful to consider first a general nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \quad , \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^m$ is the output, with $m \le n$. The functions $f$, $g$ and $h$ are smooth and of appropriate dimensions and the matrix $g$ is full rank, uniformly in $x$.

### 2.1 Cyclo-passivity

*Definition 1.* System (1) is said *cyclo-passive* if there exists a differentiable function $H : \mathbb{R}^n \to \mathbb{R}$ (called the storage function) that satisfies the power balance inequality

$$\dot{H} \le y^\top u \tag{2}$$

when evaluated along the trajectories of (1).

Recall that a system is passive if (2) holds and $H$ is bounded from below. Because of this additional restriction, every passive system is cyclo-passive but the converse is not true. In terms of energy exchange, cyclo-passive systems exhibit a net absorption of energy along *closed* trajectories [Hill and Moylan, 1980], while passive systems absorb energy along *any* trajectory that starts from a state of minimal energy $x(0) = \arg\min H(x)$.

According to Hill-Moylan's Theorem [Hill and Moylan, 1980], system (1) is cyclo-passive (with storage function $H(x)$) if and only if, for some $q \in \mathbb{N}$, there exists a function $l : \mathbb{R}^n \to \mathbb{R}^q$ such that

$$\nabla H^\top f = -\|l\|^2 \tag{3a}$$

$$h = g^\top \nabla H \ . \tag{3b}$$

Setting the dissipation $d \triangleq \|l\|^2$ and differentiating $H$ leads to the power balance

$$\dot{H} = y^\top u - d \ . \tag{4}$$

We now focus on PH systems

$$\Sigma : \begin{cases} \dot{x} = F\nabla H + gu \\ y = g^\top \nabla H \end{cases} \tag{5}$$

where $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, with $F + F^\top \le 0$. It can be easily verified that (5) is cyclo-passive with storage function $H$ and dissipation $d \triangleq -\nabla H^\top F \nabla H$.

For future reference let us compute the assignable equilibria of (5) as the elements of the set

$$\mathcal{E}_x \triangleq \left\{ x \mid g^\perp F \nabla H = 0 \right\} \ , \tag{6}$$

with $g^\perp : \mathbb{R}^n \to \mathbb{R}^{(n-m)\times n}$ a full rank left-annihilator of $g$, that is, $g^\perp g = 0$ and $\operatorname{rank} g = n - m$. Associated to each $x_\star \in \mathcal{E}_x$ there is a uniquely defined constant control given by

$$u_\star \triangleq -g^+(x_\star) F(x_\star) \nabla H(x_\star) \ , \tag{7}$$

where $g^+$ is the Moore-Penrose pseudo-inverse of $g$, that is, $g^+ \triangleq [g^\top g]^{-1} g^\top$. Note that $g^+$ is well-defined since $g$ is assumed full rank, implying that the inverse of $g^\top g$ always exists.

### 2.2 Example

The system described by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_1 + x_2 \\ -x_2^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} - x_2^2 \\ x_2^3 \end{pmatrix} u \tag{8}$$

can be written in the PH form (5) with

$$F = \begin{pmatrix} -\frac{1}{2} & x_2 \\ 0 & -x_2^2 \end{pmatrix} \ , \quad H = \frac{1}{2}x_1^2 + x_2 \ , \quad g = \begin{pmatrix} \frac{1}{2} - x_2^2 \\ x_2^3 \end{pmatrix} \tag{9}$$

and output

$$y = g^\top \nabla H = x_1 \left( \frac{1}{2} - x_2^2 \right) + x_2^3 \ .$$

Notice that equation (4) does not yield any information about the stability of the open-loop equilibrium $(0,0)$, since $H$ is not bounded from below. Actually, it can be readily seen that with $u = 0$ the equilibrium is unstable and that the trajectories of the open-loop system exhibit finite escape time. Moreover, the origin can not be stabilized by any continuous feedback.

The set of assignable equilibria for this system is

$$\mathcal{E}_x = \left\{ (x_1, x_2) \mid x_2^2(1 - x_1 x_2) = 0 \right\} \ . \tag{10}$$

### 2.3 Control by interconnection

In CbI a PH controller of the form

$$\Sigma_{\mathrm{c}} : \begin{cases} \dot{\xi} = u_{\mathrm{c}} \\ y_{\mathrm{c}} = \nabla H_{\mathrm{c}}(\xi) \end{cases} \tag{11}$$

is proposed, where $\xi \in \mathbb{R}^m$ is the state of the controller, $u_{\mathrm{c}}, y_{\mathrm{c}}$ are the input and the output of the controller, respectively, and $H_{\mathrm{c}} : \mathbb{R}^m \to \mathbb{R}$ is a to-be-designed controller storage function. See Ortega et al. [2008], van der Schaft [2000] for a justification of this choice of controller structure.

Control by interconnection comes in two basic variants. In the standard version, $\Sigma$ and $\Sigma_{\mathrm{c}}$ are coupled using the classical unitary feedback power-preserving interconnection

$$\Sigma_I : \left\{ \begin{pmatrix} u \\ u_{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ y_{\mathrm{c}} \end{pmatrix} + \begin{pmatrix} v \\ 0 \end{pmatrix} \right. \ , \tag{12}$$

where $v$ is a new virtual input.[1] It is well-known [van der Schaft, 2000] that the PH structure is invariant under power-preserving interconnection; this pattern leading to the interconnected PH system

$$\Sigma_{\mathtt{Ts}} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} F & -g \\ g^\top & 0 \end{pmatrix} \nabla H_{\mathtt{T}} + \begin{pmatrix} g \\ 0 \end{pmatrix} v \\ y_{\mathtt{Ts}} = \begin{pmatrix} g^\top & 0 \end{pmatrix} \nabla H_{\mathtt{T}} \end{cases} \tag{13}$$

[1] We recall that an interconnection of PH systems is power preserving if it satisfies $y^\top u + y_{\mathrm{c}}^\top u_{\mathrm{c}} = y^\top v$.

with

$$H_{\mathrm{T}}(x,\xi) \triangleq H(x) + H_{\mathrm{c}}(\xi) \tag{14}$$

the new total energy.

A new version of CbI has been recently introduced in Ortega et al. [2008] that, being related to the power shaping procedure of Ortega et al. [2003], is called power shaping CbI. In this case, $F$ is assumed to be non-singular and a modified PH system with a new passive output is generated as

$$\Sigma_{\mathrm{ps}} : \begin{cases} \dot{x} = F\nabla H + gu \\ y_{\mathrm{ps}} = -g^\top F^{-\top}(F\nabla H + gu) \end{cases} \tag{15}$$

Noticing that $y_{\mathrm{ps}} = -g^\top F^{-\top}\dot{x}$ it is easy to show [Ortega et al., 2003] that (15) satisfies $\dot{H} \leq u^\top y_{\mathrm{ps}}$. The interconnection is then given by

$$\Sigma_{Ips} : \left\{ \begin{pmatrix} u \\ u_{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{\mathrm{ps}} \\ y_{\mathrm{c}} \end{pmatrix} + \begin{pmatrix} v \\ 0 \end{pmatrix} \right. , \tag{16}$$

that yields the PH closed-loop system [2]

$$\Sigma_{\mathrm{T\,ps}} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} F & -g \\ -g^\top F^{-\top} F & g^\top F^{-\top} g \end{pmatrix} \nabla H_{\mathrm{T}} + \\ \qquad + \begin{pmatrix} g \\ -g^\top F^{-\top} g \end{pmatrix} v \\ y_{\mathrm{T\,ps}} = \begin{pmatrix} g^\top & -g^\top F^{-1} g \end{pmatrix} \nabla H_{\mathrm{T}} \end{cases} \tag{17}$$

So far, we have constructed interconnected systems which are cyclo-passive with storage function $H_{\mathrm{T}}$. Since $H_{\mathrm{c}}$ can be modified at will, it seems reasonable to use it to "shape" the total storage function. We are interested in shaping $H_{\mathrm{T}}$ along the $x$ coordinates, but unfortunately, $H_{\mathrm{c}}$ is a function of $\xi$, so this idea cannot be applied directly. One way to get around this, is to relate $x$ and $\xi$ in the following way.

*Assumption 2.* There exist a differentiable mapping $C : \mathbb{R}^n \to \mathbb{R}^m$, the Jacobian of which has rank $m$ and at least one of the following conditions is satisfied.

(1) (Standard CbI)

$$\begin{pmatrix} g^\top \\ F \end{pmatrix} \nabla C = -\begin{pmatrix} 0 \\ g \end{pmatrix} . \tag{18}$$

(2) (Power shaping CbI) $\det F(x) \neq 0$ and

$$F\nabla C = -g \tag{19}$$

Assumption 2 is made throughout the paper. That is, it is assumed that, for the given $F$ and $g$, a solution of the partial differential equations (18) or (19) is known. Also, to simplify the presentation, it is assumed that $F$ is full rank. The power shaping CbI presented above is called "Basic CbI-PS" in Ortega et al. [2008], in that paper we present another version of CbI that generates a new, full rank, matrix to replace $F$.

In Ortega et al. [2008] it is shown that condition 1 (resp., 2) of Assumption 2 ensures that, for any $\kappa \in \mathbb{R}^m$, the manifolds $\mathcal{M}_\kappa = \{(x,\xi) \,|\, C(x) - \xi = \kappa\}$ are invariant [3] under the flow of the system (13) (resp., (17)). As discussed in van der Schaft [2000], Ortega et al. [2001, 2008], and also shown below, the construction of this, so-called, *Casimir function* $C(x)-\xi$ is the key step of CbI that allows to shape the storage function in the state coordinates $x$. In order to reveal this property and, at the same time, provide a unified framework to study both versions of CbI, we find it convenient to define the PH system

$$\Sigma_{\mathrm{T}} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = F_{\mathrm{T}} \nabla H_{\mathrm{T}} + g_{\mathrm{T}} v \\ y_{\mathrm{T}} = g_{\mathrm{T}}^\top \nabla H_{\mathrm{T}} \end{cases} \tag{20}$$

where

$$F_{\mathrm{T}} \triangleq \begin{pmatrix} I \\ \nabla C^\top \end{pmatrix} \begin{pmatrix} F & -g \end{pmatrix}, \; g_{\mathrm{T}} \triangleq \begin{pmatrix} I \\ \nabla C^\top \end{pmatrix} g. \tag{21}$$

Notice that (20) describes the behavior of both closed-loop systems, (13) and (18), or (17) and (19). In the sequel we deal only with (20) in the understanding that, depending on which condition of Assumption 2 is satisfied, we are referring to either one of the CbI controllers.

The proposition below opens the possibility of creating appropriate storage functions that can be shaped along $x$.

*Proposition 3.* The PH system (20) is cyclo-passive with storage function

$$W(x,\xi) \triangleq H_{\mathrm{T}}(x,\xi) + \Phi(C(x) - \xi), \tag{22}$$

for any differentiable $\Phi : \mathbb{R}^m \to \mathbb{R}$.

**Proof.** Compute $\dot{W} = \dot{H}_{\mathrm{T}} + \dot{\Phi}$. Since $\Sigma_{\mathrm{T}}$ is cyclo-passive with storage function $H_{\mathrm{T}}$ and dissipation $d_{\mathrm{T}} \triangleq -\nabla H_{\mathrm{T}}^\top F_{\mathrm{T}} \nabla H_{\mathrm{T}}$, we have

$$\begin{aligned} \dot{W} &= v^\top y_{\mathrm{T}} - d_{\mathrm{T}} + \nabla^\top \Phi \begin{pmatrix} \nabla C^\top & -I \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} \\ &= v^\top y_{\mathrm{T}} - d_{\mathrm{T}} \end{aligned}$$

where the last equality follows from (20), (21) and

$$\begin{pmatrix} \nabla C^\top & -I \end{pmatrix} \begin{pmatrix} I \\ \nabla C^\top \end{pmatrix} = 0 .$$

□

## 3. STABILIZATION

In this section we show how Proposition 3 can be used for stabilization of an arbitrary element of the assignable equilibrium set $\mathcal{E}_x$, defined in (6). We propose functions $H_{\mathrm{c}}$ and $\Phi$ and give conditions on $C$ that ensure the stabilization requirement.

As a first step, define the set of admissible equilibria $\mathcal{E}$ for the system (20) in open-loop (i.e., with $v = 0$). According to (20) and (21)

$$\mathcal{E} = \{(x,\xi) \mid F\nabla H - g\nabla H_{\mathrm{c}} = 0\} . \tag{23}$$

In the previous section it has been shown that $W$ satisfies

$$\dot{W} = y_{\mathrm{T}}^\top v - d_{\mathrm{T}} . \tag{24}$$

with $d_{\mathrm{T}} \geq 0$. It follows from standard Lyapunov theory that if $W$ has a strict minimum at a point $(x_\star, \xi_\star) \in \mathcal{E}$ and we set $v = 0$, then $(x_\star, \xi_\star)$ is stable. Our goal is thus,

[2] To verify that it is indeed PH, notice that for any $x$ and $\xi$,

$$\begin{pmatrix} x^\top & \xi^\top \end{pmatrix} \begin{pmatrix} F & -g \\ -g^\top F^{-\top} F & g^\top F^{-\top} g \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \\ = (x - F^{-1} g\xi)^\top F (x - F^{-1} g\xi) \leq 0 .$$

[3] That is, $C(x(t)) - \xi(t) = C(x_0) - \xi_0 \;\; \forall \; t$, where $(x_0,\xi_0) \triangleq (x(0),\xi(0))$.

to find appropriate $\Phi$ and $H_{\rm c}$, and impose conditions on $C$, such that

$$(x_\star, \xi_\star) = \arg\min W(x,\xi)\ . \tag{25}$$

Clearly, negativity of $\dot W$ can be reinforced by setting

$$v = -K_{\rm v} y_{\rm T}\ , \quad K_{\rm v} = K_{\rm v}^\top > 0\ . \tag{26}$$

This damping injection (also called $L_g V$) approach is usually adopted in PBC to try make the equilibrium *asymptotically* stable, which is the case if $y_{\rm T}$ is a detectable output [van der Schaft, 2000]. Unfortunately, we will show below that the latter condition is not satisfied for CbI and we must adopt another strategy, which will be presented in Section 4. But first we propose a solution to the problem of stabilization of an arbitrary element of $\mathcal{E}_x$.

### *3.1 Stabilization of assignable equilibria*

*Proposition 4.* Consider $\Sigma_{\rm T}$ given by (20) with $v = 0$. Fix any point $x_\star \in \mathcal{E}_x$ and compute the corresponding $u_\star$ via (7). Let

$$H_{\rm c} = \frac{1}{2}\left\|\xi - K_{\rm c}^{-1}u_\star\right\|^2_{K_{\rm c}}\ , \tag{27}$$

where $K_{\rm c} = K_{\rm c}^\top > 0$ and select

$$\Phi(\eta) = -u_\star^\top \eta\ .$$

Then $(x_\star, 0)$ is an equilibrium of the closed-loop system (20), that is, $(x_\star, 0) \in \mathcal{E}$.[4] Furthermore, $(x_\star, 0)$ is a stable equilibrium if

$$\nabla^2 H(x_\star) - \sum_{i=1}^m u_{\star i}\nabla^2 C_i(x_\star) > 0\ . \tag{28}$$

**Proof.** First we prove that $(x_\star, 0) \in \mathcal{E}$. From Assumption (2) we have that

$$F\nabla C = -g\ . \tag{29}$$

Consequently, $\nabla C^\perp = g^\perp F$ (recall that $\det F \neq 0$) and (6) can be written as

$$\mathcal{E}_x = \left\{x \mid g^\perp F\nabla H = 0\right\} = \left\{x \mid \nabla C^\perp \nabla H = 0\right\}\ ,$$

while the set of admissible equilibria for the closed-loop system (20), given in (23), can be written as

$$\begin{aligned}\mathcal{E} &= \{(x,\xi) \mid F\nabla H - g\nabla H_{\rm c} = 0\}\\ &= \{(x,\xi) \mid \nabla H + \nabla C\nabla H_{\rm c} = 0\}\\ &= \left\{(x,\xi) \mid \nabla C^\perp \nabla H = 0,\ \nabla H_{\rm c} = -\nabla C^+\nabla H\right\}\ ,\end{aligned} \tag{30}$$

where we have used (29) and $\det F \neq 0$ in the second identity and Lemma 2 of Ortega et al. [2008] to establish the last identity. Now, from (29) it can be seen that $\nabla C^+ = -(g^\top g)^{-1} g^\top F$, therefore

$$\mathcal{E} = \left\{(x,\xi) \mid \nabla C^\perp \nabla H = 0,\ \nabla H_{\rm c} = (g^\top g)^{-1} g^\top F\nabla H\right\}.$$

To prove that $(x_\star, 0) \in \mathcal{E}$ for any $x_\star \in \mathcal{E}_x$ we note, from the definition of $H_{\rm c}$, that $\nabla H_{\rm c}(0) = -u_\star$, with $u_\star$ given in (7). The implication

$$g^\perp F\nabla H = 0 \quad \text{and} \quad g\nabla H_{\rm c} = gg^+ F\nabla H \implies F\nabla H - g\nabla H_{\rm c} = 0$$

is then easily established.

We now prove that $(x_\star, 0) = \arg\min W(x,\xi)$ by verifying the conditions $\nabla W(x_\star, 0) = 0$ and $\nabla^2 W(x_\star, 0) > 0$. Let $\mathcal{A} \triangleq \{(x,\xi) \mid \nabla W = 0\}$ be the set of extrema of $W$. From (22) and (14) one obtains

$$\mathcal{A} = \{(x,\xi) \mid \nabla H + \nabla C\nabla H_{\rm c} = 0,\ \nabla H_{\rm c} = \nabla\Phi\}\ .$$

Using the definitions of $\Phi$ and $H_{\rm c}$ and the second equation in (30) we conclude that $(x_\star, 0) \in \mathcal{A}$ is equivalent to $(x_\star, 0) \in \mathcal{E}$.

It has been shown above that for all $x_\star \in \mathcal{E}_x$, $(x_\star, 0) \in \mathcal{A}$. We now give conditions under which they are minimum points. Some simple calculations proceeding from $W(x,\xi) = H(x) + \frac{1}{2}\left\|\xi - K_{\rm c}^{-1}u_\star\right\|^2_{K_{\rm c}} - u_\star^\top[C(x) - \xi]$, yield the Hessian

$$\nabla^2 W = \begin{pmatrix} \nabla^2 H - \sum_{i=1}^m u_{\star i}\nabla^2 C_i & 0 \\ 0 & K_{\rm c}\end{pmatrix}\ ,$$

from which we conclude that the equilibrium $(x_\star, 0)$ is stable if (28) holds.[5] □

### *3.2 Example (continued)*

The function $C(x) = x_1 + \frac{1}{2}x_2^2$ satisfies (19) for system (5), (9), that is,

$$F\nabla C = \begin{pmatrix} -\frac{1}{2} & x_2 \\ 0 & -x_2^2\end{pmatrix}\begin{pmatrix} 1 \\ x_2\end{pmatrix} = \begin{pmatrix} -\frac{1}{2} + x_2^2 \\ -x_2^3\end{pmatrix} = -g\ .$$

The matrix $F$ is non-singular everywhere except at the line $x_2 = 0$, that will be ruled out of the analysis. Since Condition 2 of Assumption 2 is satisfied we apply power shaping CbI.

Because of the assignable equilibria set (10), we consider equilibria of the form $x_\star = \mathrm{col}(x_{1\star}, \frac{1}{x_{1\star}})$, with $x_{1\star} \in \mathbb{R}\setminus\{0\}$. Remark that $u_\star = x_{1\star}$.

Since the Hessians of $H$ and $C$ are

$$\nabla^2 H = \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix} \quad \text{and} \quad \nabla^2 C = \begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\ ,$$

condition (28) is satisfied if and only if $u_\star < 0$. Then, applying Proposition 4, any point of the form $(x_{1\star}, \frac{1}{x_{1\star}})$, $x_{1\star} < 0$, is stabilized by the controller

$$\begin{aligned}\dot\xi &= -\nabla C^\top g\nabla H_{\rm c} + \nabla C^\top F\nabla H\\ u &= -\nabla H_{\rm c}\ .\end{aligned}$$

## 4. MAIN RESULT: ASYMPTOTIC STABILITY

In Subsection 2.3 we have proposed to shape the storage function (along the state $x$) via generation of the invariant manifolds $\mathcal{M}_\kappa = \{(x,\xi) \,|\, C(x) - \xi = \kappa\}$. Unfortunately, the latter poses the following problem. Suppose the system starts at an arbitrary initial condition $(x_0, \xi_0)$. There is no reason why the desired equilibrium $(x_\star, \xi_\star)$ should satisfy

$$C(x_\star) - \xi_\star = C(x_0) - \xi_0\ . \tag{31}$$

One way to fulfill (31) is to initialize the controller at the value $\xi_0$ that puts the system in the proper invariant manifold. This approach is simple but the dependence on the initial conditions makes it highly non-robust. In

[4] Later on, we will exploit the possibility of setting the equilibrium at points other that $(x_\star, 0)$.

[5] Sure, $\nabla^2 W > 0$ is only a sufficient condition for local optimality. Nevertheless, it is not a very conservative one in the sense that given (27), a necessary condition is $\nabla^2 W \geq 0$.

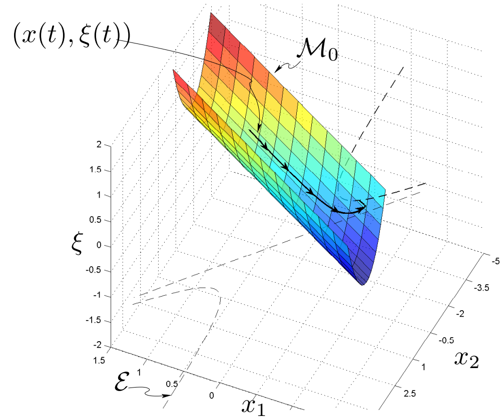


Fig. 1. The invariant manifold $\mathcal{M}_0$, the equilibria locus $\mathcal{E}$ and the simulated response.

general, $(x_\star, \xi_\star)$ does not belong to the orbit of the solution starting at $(x_0, \xi_0)$, hence the output $y_\mathrm{T}$ is not detectable, and the desired equilibrium might be stable but not asymptotically stable even with the damping injection (26).

Our main contribution is to present two alternative solutions to the problem. Before giving these results let us take a closer look at our example to get a clearer picture of the role of the Casimir function.

### *4.1 Example (continued)*

Suppose that we want to stabilize the point $(-1, -1, 0)$, so that $u_\star = x_{1\star} = -1$. By setting $K_\mathrm{c} = 1$, the Lyapunov function is

$$\begin{aligned} W(x, \xi) &= H(x) - u_\star^\top (C(x) - \xi) + H_\mathrm{c}(\xi) \\ &= \frac{1}{2}\left[(x_1+1)^2 + (x_2+1)^2 + \xi^2\right] - \frac{1}{2}\ , \end{aligned}$$

the level sets of which are spheres centered at $(-1, -1, 0)$.

Suppose, further, that the system is initially at $(x_0, \xi_0) = (\frac{3}{2}, -\frac{1}{2}, \frac{13}{8})$, so that $C(x_0) - \xi_0 = \frac{3}{2} + \frac{1}{2}\frac{1}{4} - \frac{13}{8} = 0$. Since $C(x_\star) - \xi_\star = -1 + \frac{1}{2} + 0 \neq 0$, the trajectory does not reach the desired value. The trajectories cannot diverge either, since $W$ is radially unbounded. Instead, the trajectory reaches an invariant set contained in the invariant manifold $\mathcal{M}_0 = \{(x, \xi) \,|\, C(x) - \xi = 0\}$. The set $\mathcal{E}$ is the union of the sets described by the parametrized curves $q_1(\bar{x}_1) = \mathrm{col}(\bar{x}_1, \frac{1}{\bar{x}_1}, -\bar{x}_1 - 1)$, $\bar{x}_1 \in \mathbb{R} \setminus \{0\}$ and $q_2(\bar{x}_1) = \mathrm{col}(\bar{x}_1, 0, -\bar{x}_1 - 1)$, $\bar{x}_1 \in \mathbb{R}$ (see Appendix A for details). Note that $\mathcal{E} \cap \mathcal{M}_0 = \{(-0.85, -1.18, -0.15), (-0.5, 0, -0.5)\}$. Figure 1 shows $\mathcal{M}_0$, $\mathcal{E}$ and the trajectory starting at $(x_0, \xi_0) = (\frac{3}{2}, -\frac{1}{2}, \frac{13}{8})$ and converging to $(-0.85, -1.18, -0.15)$. Figure 2 shows the intersection of $\mathcal{M}_0$ and the level sets of $W$ with the planes $x_2 = x_{2\star} = -1$ and $\xi = \xi_\star = 0$. The projections of $\mathcal{E}$ and the trajectory are also shown.

### *4.2 Adaptive CbI*

It is clear that another way to satisfy the constraint (31) is by shifting away from zero the desired value of $\xi$ to the new value

$$\xi_\star = C(x_\star) - C(x_0) + \xi_0\ . \tag{32}$$

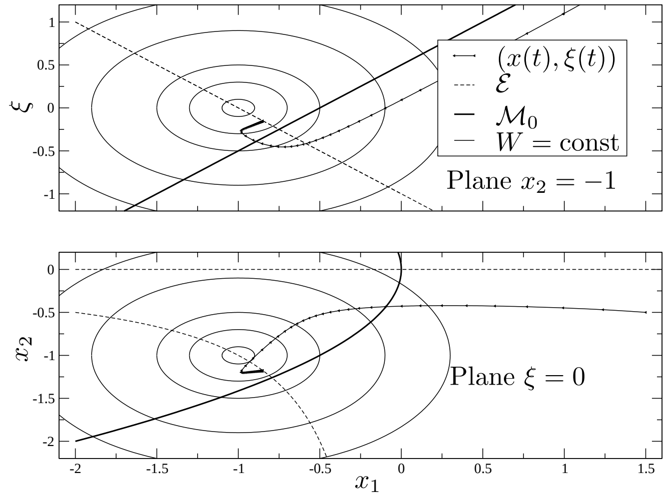


Fig. 2. Level sets of $W$ and invariant manifold $\mathcal{M}_0$, equilibria locus $\mathcal{E}$ and simulated response, projected into the planes $x_2 = -1$ (above) and $\xi = 0$ (below).

This amounts to changing $H_\mathrm{c}$ to

$$H_\mathrm{c}(\xi) = \frac{1}{2}\left\|\xi - \xi_\star - K_\mathrm{c}^{-1} u_\star\right\|_{K_\mathrm{c}}^2\ , \tag{33}$$

so that $\nabla H_\mathrm{c}(\xi_\star) = -u_\star$. Geometrically, we are shifting the equilibrium locus $\mathcal{E}$ along $\xi$, so that it intersects the manifold where the trajectory starts, that is, $\mathcal{M}_{\kappa_0}$, with

$$\kappa_0 \triangleq C(x_0) - \xi_0\ , \tag{34}$$

at the desired $x_\star$.

In principle, this scheme still hinges on knowledge of the initial condition, but this issue can be removed by reformulating it as a parameter estimation problem. We try first a classical certainty-equivalence adaptive control approach viewing $\xi_\star$ as the unknown parameter. This is indeed possible because the plant is linear in $u$ and, for quadratic $H_\mathrm{c}$, $\xi_\star$ enters also linearly in $u$. Define a new storage function for the controller (11) as

$$\bar{H}_\mathrm{c}(\xi, \hat{\xi}_\star) \triangleq \frac{1}{2}\left\|\xi - \hat{\xi}_\star - K_\mathrm{c}^{-1} u_\star\right\|_{K_\mathrm{c}}^2\ ,$$

where $\hat{\xi}_\star$ denotes the estimate of $\xi_\star$. Let us compute

$$\begin{aligned} \nabla_\xi \bar{H}_\mathrm{c} &= K_c(\xi - \hat{\xi}_\star) - u_\star = K_c(\xi - \xi_\star) - u_\star - K_c\tilde{\xi}_\star \\ &= \nabla H_\mathrm{c} - K_c\tilde{\xi}_\star, \end{aligned}$$

where we have defined the parameter error $\tilde{\xi}_\star \triangleq \hat{\xi}_\star - \xi_\star$. The control signal then becomes $u = -\nabla_\xi \bar{H}_\mathrm{c} = -\nabla H_\mathrm{c} + K_c\tilde{\xi}_\star$. The closed-loop system is still of the form (20) with $v$ replaced by $v + K_c\tilde{\xi}_\star$. Since the invariance of the manifolds $\mathcal{M}_\kappa$ is preserved, the power balance equation (24) is still satisfied with the "new $v$". Proceeding with the classical adaptive control design we would propose a candidate Lyapunov function $V(x, \xi, \tilde{\xi}_\star) = W(x, \xi) + \frac{1}{2}\|\tilde{\xi}_\star\|_{\Gamma^{-1}}$, $\Gamma = \Gamma^\top > 0$, and an estimation law of the form $\dot{\hat{\xi}}_\star = -\Gamma K_c y_\mathrm{T}$, which would make $\dot{V} = \dot{W} \leq 0$. Unfortunately, this simple scheme will not solve our problem. Indeed, since the derivative of the new Lyapunov function has not changed, the lack of detectability problem is still present. The only way to achieve the desired objective is to ensure parameter convergence, that is, $\lim_{t\to\infty} \tilde{\xi}_\star(t) = 0$, which is not satisfied due to existence of a manifold of equilibria.

It turns out that if we estimate the parameter $\kappa_0$ (instead of $\xi_\star$) and use the invariance of the manifold $\mathcal{M}_{\kappa_0}$ we can design a scheme that ensures parameter convergence. The result is summarized in the proposition below, which is the adaptive version of Proposition 4.

*Proposition 5.* Consider the PH system $\Sigma$ (resp., $\Sigma_{\rm ps}$) given in (5) (resp., (15)) interconnected through $\Sigma_I$ (12) (resp., $\Sigma_{I\rm ps}$ (16)) with the adaptive controller

$$\hat{\Sigma}_{\rm c} : \begin{cases} \dot{\xi} = u_{\rm c} \\ \dot{\hat{\kappa}}_0 = -\Gamma\left(\hat{\kappa}_0 - C(x) + \xi\right) \\ y_{\rm c} = \nabla_\xi \hat{H}_{\rm c}(\xi, \hat{\kappa}_0) \end{cases}$$

where $\hat{H}_{\rm c}(\xi,\hat{\kappa}_0) \triangleq \frac{1}{2}\left\|\xi - C(x_\star) + \hat{\kappa}_0 - K_{\rm c}^{-1}u_\star\right\|^2_{K_{\rm c}}$, $u_\star$ is defined in (7) and $v = -K_v y_{\rm T}$.

- (i) Exponential parameter convergence is ensured, more precisely $\|\hat{\kappa}_0(t) - \kappa_0\| \le e^{\lambda_{min}\{\Gamma\}t}\|\hat{\kappa}_0(0) - \kappa_0\| = 0$ for all $t \ge 0$.
- (ii) For any $x_\star \in \mathcal{E}_x$ the point $(x_\star, \xi_\star, 0)$, where $\xi_\star$ is given in (32), is a stable equilibrium if (28) holds.
- (iii) The orbits of the residual dynamics are confined to the set $\mathcal{Z} \times \{\xi = \bar{\xi}\}$, where $\bar{\xi}$ is a constant and
$$\mathcal{Z} \triangleq \left\{ x \,\middle|\, \begin{pmatrix} \nabla H^\top \\ \nabla C^\top \end{pmatrix} \cdot \right.$$
$$\left. \cdot \left[F\nabla H - g\left(K_{\rm c}(C(x) - C(x_\star)) - u_\star\right)\right] = 0 \right\} .$$
- (iv) Suppose no trajectory $x(t)$ can stay identically in $\mathcal{Z}$, other than isolated points. Then, $(x_\star, \xi_\star, 0)$ is an asymptotically stable equilibrium. It will be globally asymptotically stable if it is the only point in $\mathcal{Z}$ and if $W$ is radially unbounded.

**Proof.** Define $\tilde{\kappa}_0 \triangleq \hat{\kappa}_0 - \kappa_0$. From invariance of the manifold $\mathcal{M}_{\kappa_0}$ we have that $\kappa_0 = C(x_0) - \xi_0 = C(x(t)) - \xi(t)$. Consequently, $\dot{\tilde{\kappa}}_0 = -\Gamma\tilde{\kappa}_0$, from which claim (i) follows immediately.

Proceeding as done for the standard adaptive controller above one has that $\nabla_\xi \hat{H}_{\rm c} = \nabla H_{\rm c} - K_c\tilde{\kappa}_0$, $u = -\nabla H_{\rm c} + K_c\tilde{\kappa}_0$, and the power balance equation becomes

$$\dot{W} = y_{\rm T}^\top(v - K_c\tilde{\kappa}_0) - d_{\rm T} . \tag{35}$$

Consider the Lyapunov function candidate $V(x,\xi,\tilde{\kappa}_0) = W + \frac{1}{2}\|\tilde{\kappa}_0\|^2_{\mu\Gamma^{-1}}$, with $\mu > 0$. Differentiation with respect to time and some standard bounding shows that, for all $K_v, K_c, \Gamma$, there exists $\mu$ such that

$$\dot{V} \le -d_{\rm T} - \epsilon(\|y_{\rm T}\|^2 + \|\tilde{\kappa}_0\|^2) \tag{36}$$

holds for some $\epsilon > 0$, which shows that $V$ is a Lyapunov function, so the equilibrium is stable establishing (ii).

Now, we apply LaSalle's Theorem [La Salle and Lefschetz, 1961] and conclude from (36) that $d_{\rm T}$ and $y_{\rm T}$ tend to zero as $t \to \infty$. The residual dynamics are obtained imposing to the system the restrictions $d_{\rm T} = 0$, $y_{\rm T} = 0$ and $\tilde{\kappa}_0 = 0$. First, note that with $\tilde{\kappa}_0 = 0$ the dynamics reduce to $\Sigma_{\rm T}$. Second, $y_{\rm T} = 0$ implies $v = 0$ and $\dot{\xi} = 0$, consequently $\xi = \bar{\xi}$. Furthermore, from the equation of $\dot{\xi}$, we have

$$0 = \dot{\xi} = \nabla C^\top\left[F\nabla H - g\nabla H_{\rm c}(\bar{\xi})\right] . \tag{37}$$

Now, recall that the dissipation is

$$\begin{aligned} 0 = d_{\rm T} &= -\nabla H_{\rm T}^\top F_{\rm T}\nabla H_{\rm T} \\ &= -\left(\nabla H^\top \;\; \nabla H_{\rm c}^\top\right)\begin{pmatrix} I \\ \nabla C^\top \end{pmatrix}\left(F \;\; -g\right)\begin{pmatrix} \nabla H \\ \nabla H_{\rm c} \end{pmatrix} \\ &= (\nabla H^\top + \nabla H_{\rm c}^\top\nabla C^\top)(F\nabla H - g\nabla H_{\rm c}) , \end{aligned} \tag{38}$$

which combined with (37) yields,

$$\nabla H^\top\left[F(x)\nabla H(x) - g(x)\nabla H_{\rm c}(\bar{\xi})\right] = 0 . \tag{39}$$

The proof of (iii) is completed noting that $C(x) - \bar{\xi} = \kappa_0$ and evaluating $\nabla H_{\rm c}$ at $\bar{\xi}$.

The proof of (iv) is a direct consequence of the celebrated theorem by Barbashin and Krasovskii [1952]. □

### 4.3 Example (continued)

We now apply adaptive CbI to the example. Except for points on the hyperbola $x_1x_2 = 1$, the matrix

$$\begin{pmatrix} \nabla H^\top \\ \nabla C^\top \end{pmatrix} = \begin{pmatrix} x_1 & 1 \\ 1 & x_2 \end{pmatrix} \tag{40}$$

is non-singular, so the orbits of the residual dynamics are confined to equilibrium points $\bar{x} \in \mathcal{E}$ satisfying

$$F(\bar{x})\nabla H(\bar{x}) - g(\bar{x})(C(\bar{x}) - C(x_\star) + u_\star) = 0 .$$

For all $x_1^\star < -\frac{1}{2}$ the only solutions of the above equation are[6] $\bar{x}^I = \mathrm{col}(x_{1\star}, x_{2\star})$ and $\bar{x}^{II} = \mathrm{col}(x_{1\star} + \frac{1}{4}x_{2\star}^2, 0)$. When $x_1x_2 = 1$, the vector $\mathrm{col}(x_2, -1)$ is an eigenvector associated to the zero eigenvalue of the matrix (40), so points $\bar{x}$ satisfying

$$F(\bar{x})\nabla H(\bar{x}) - g(\bar{x})(C(\bar{x}) - C(x_\star) + u_\star) = \begin{pmatrix} \bar{x}_2 \\ -1 \end{pmatrix}\psi(\bar{x})$$

for some function $\psi : \mathbb{R}^2 \to \mathbb{R}$ can also contain the orbits of the residual dynamics. Since $\bar{x}_1\bar{x}_2 = 1$ implies $g^\perp(\bar{x})F(\bar{x})\nabla H(\bar{x}) = 0$ (see Appendix A for details), then one obtains $g^\perp(\bar{x})\,\mathrm{col}(\bar{x}_2, -1) = 0$. The solution set of the previous equation is empty, which implies that

$$\mathcal{Z} = \left\{\mathrm{col}(x_{1\star}, x_{2\star}), \mathrm{col}\left(x_{1\star} + \frac{1}{4}x_{2\star}^2, 0\right)\right\} .$$

Figure 3 shows that now $\mathcal{M}_0$ and $\mathcal{E}$ intersect at the desired $x_\star$. Convergence towards the desired value is achieved with the adaptive scheme.

### 4.4 Controller damping injection CbI

Another possible way to achieve convergence, is to destroy the invariance of the Casimirs adding a damping injection to the controller. The idea is to go back to the previous controller storage function (27), that we repeat here for ease of reference

$$H_{\rm c}(\xi) = \frac{1}{2}\left\|\xi - K_{\rm c}^{-1}u_\star\right\|^2_{K_{\rm c}} , \tag{41}$$

but add an extra virtual input $w \in \mathbb{R}^m$ to the controller through the interconnection, that is,

$$\Sigma_{I\,\rm w} : \left\{ \begin{pmatrix} u \\ u_{\rm c} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y_{\rm T} \\ y_{\rm c} \end{pmatrix} + \begin{pmatrix} v \\ w \end{pmatrix} \right. . \tag{42}$$

[6] The details are not shown, but this fact can be verified by looking at the discriminant of the resulting cubic polynomial.

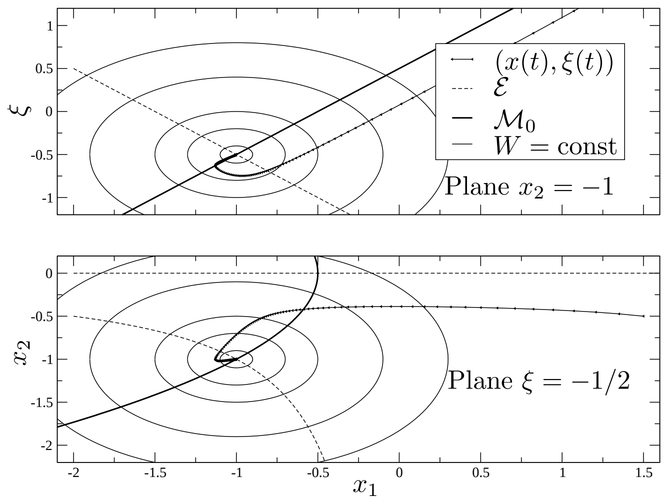


Fig. 3. Level sets of $W$ and invariant manifold $\mathcal{M}_0$, all intersected with the planes $x_2 = -1$ (above) and $\xi = -\frac{1}{2}$ (below). Equilibrium set $\mathcal{E}$ and simulated response, both projected into the planes $x_2 = -1$ and $\xi = -\frac{1}{2}$.

The interconnected system takes the form

$$\Sigma_{\mathrm{T\,w}} : \begin{cases} \begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = F_{\mathrm{T}} \nabla H_{\mathrm{T}} + g_{\mathrm{T}} v + \begin{pmatrix} 0 \\ I \end{pmatrix} w \\ y_{\mathrm{T}} = g_{\mathrm{T}}^\top \nabla H_{\mathrm{T}} \\ z = (0 \;\; I) \nabla W \end{cases} . \tag{43}$$

where we have defined the corresponding conjugate output $z$. Notice that, for all $w \neq 0$, the invariance of the manifolds $\mathcal{M}_\kappa$ has been destroyed because $\dot{C} - \dot{\xi} = -w$. However, the time derivative of $W$ is

$$\dot{W} = -d_{\mathrm{T}} + y_{\mathrm{T}}^\top v + z^\top w \,, \tag{44}$$

so the new system is also cyclo-passive with the same storage function $W$ and port variables $((y_{\mathrm{T}}, z), (v, w))$.

*Proposition 6.* Consider $\Sigma_{\mathrm{T\,w}}$ with $H_{\mathrm{c}}$ given by (41), with $u_\star$ defined in (7), $v$ by (26) and

$$w = -K_{\mathrm{w}} z \,, \quad K_{\mathrm{w}} = K_{\mathrm{w}}^\top > 0 \,. \tag{45}$$

(i) For any $x_\star \in \mathcal{E}_x$ the point $(x_\star, 0)$ is a stable equilibrium if (28) holds.
(ii) The orbits of the residual dynamics are confined to the set $\mathcal{Z}_{\mathrm{w}} \times \{\xi = 0\}$, where

$$\mathcal{Z}_{\mathrm{w}} = \left\{ x \,\middle|\, \begin{pmatrix} \nabla H^\top \\ \nabla C^\top \end{pmatrix} [F \nabla H - g u_\star] = 0 \right\} .$$

(iii) If no trajectory $x(t)$ can stay identically in $\mathcal{Z}_{\mathrm{w}}$, other than isolated points, $(x_\star, 0)$ is an asymptotically stable equilibrium. It will be globally asymptotically stable if it is the only point in $\mathcal{Z}_{\mathrm{w}}$ and if $W$ is radially unbounded.

**Proof.** Take $W$ as a candidate Lyapunov function. Equations (44), (26) and (45) imply that it is a Lyapunov function and (i) follows. Applying LaSalle's Theorem gives that $d_{\mathrm{T}}$, $y_{\mathrm{T}}$ and $z$ tend to zero as $t \to \infty$. The residual dynamics are those of $\Sigma_{\mathrm{T}w}$ with the restrictions $d_{\mathrm{T}} = 0$, $y_{\mathrm{T}} = 0$ and $z = 0$. From the latter it follows that $\nabla_\xi W = 0$, which implies $\nabla H_{\mathrm{c}} = \nabla \Phi(C(x) - \xi) = u_\star$, which in turn implies $\xi = 0$. From the equation of $\dot{\xi}$, with $\xi = v = w = 0$, we get $0 = \dot{\xi} = \nabla C^\top(x) \, [F(x) \nabla H(x) - g(x) u_\star] = 0$, which is the second row in $\mathcal{Z}_{\mathrm{w}}$. From this equation and (38) one is lead to conclude that $\nabla H^\top(x) \, [F(x) \nabla H(x) - g(x) u_\star] = 0$, that gives the first row, and completes the proof of (ii).

Point (iii) follows from Barbashin-Krasovskii's Theorem. □

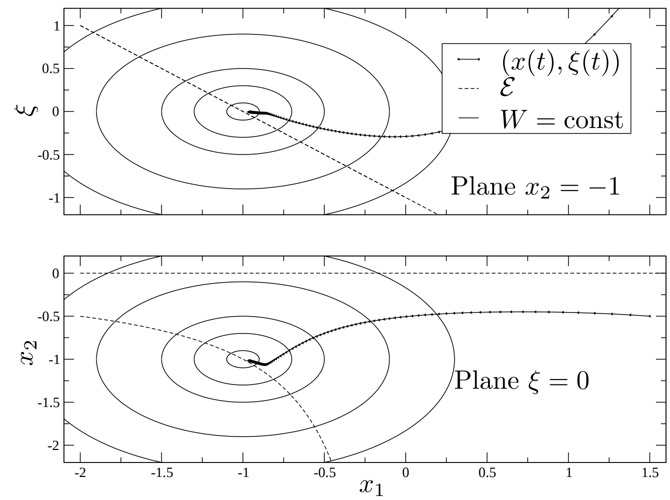


Fig. 4. Level sets of $W$ intersected with the planes $x_2 = -1$ (above) and $\xi = 0$ (below). Equilibria locus $\mathcal{E}$ and simulated response, both projected into the planes $x_2 = -1$ and $\xi = 0$.

### *4.5 Example (continued)*

We now apply controller damping injection CbI to the system of the example. The analysis follows along the same lines as in the adaptive CbI scenario. In this case $\mathcal{Z}_{\mathrm{w}} = \{\mathrm{col}(x_{1\star}, x_{2\star}), \mathrm{col}(x_{1\star}, 0)\}$. Figure 4 shows the trajectories of the system for $K_{\mathrm{w}} = 2$. These are no longer restricted to $\mathcal{M}_0$. Again, convergence to $x_\star$ is achieved.

Simulations show that for the initial condition $(x_0, \xi_0) = (-1/2, 1/2, 0)$, convergence of the state of $\Sigma_{\mathrm{T}}$ is towards $(x_{1\star} + x_{2\star}^2/4, 0) = (-3/4, 0)$ for the adaptive CbI and towards $(x_{1\star}, 0) = (-1, 0)$ for the controller damping injection CbI. Indeed, since $\mathcal{Z}$ and $\mathcal{Z}_{\mathrm{w}}$ contain more than one point, stability is global but convergence is not. Notice, however, that in the controller damping injection scenario, the exact value of the unwanted equilibrium is known. This, together with the fact that the Lyapunov function $W$ is non-decreasing over time, allows to obtain an estimate of the region of attraction: the open ball centered at $\mathrm{col}(x_{1\star}, x_{2\star}, 0)$ and of radius $\| \mathrm{col}(x_{1\star}, x_{2\star}, 0) - \mathrm{col}(x_{1\star}, 0, 0) \| = |x_{2\star}|$.

## 5. CONCLUSIONS

We have shown that the existence of the Casimir functions, inherent in the CbI design methodology, present an obstacle for asymptotic convergence of the state towards a desired equilibrium. In order to surmount this obstacle, two variations of the method have been developed. Paradoxically, once the modified versions are used, the same Casimir functions narrow the possible limit sets, thus contributing to the desired asymptotic behaviour. The Casimir functions also simplify the analysis of such limit sets, as they provide $m$ algebraic constraints that, as shown in the example, can sometimes obviate the need to differentiate the output to obtain the residual dynamics. Interestingly, each method generates a different limit set.

It is clear that the selection of a quadratic function for $H_c$ renders the controller linear, more precisely, a linear PI (for a suitably defined plant output). The results in the paper may be then interpreted as identification of a class of nonlinear PH systems that are asymptotically stabilizable via linear PI. Although the choice of a linear PI may be restrictive for some academic examples it is certainly a family of controllers of practical interest. It should be, furthermore, pointed out that the general framework of CbI does not impose this restriction on $H_c$, and it is made here to obtain easily interpretable general results. We are currently exploring other controller structures for which similar results can be established.

## Appendix A. APPENDIX: THE SET $\mathcal{E}$

Consider an arbitrary point $(\bar{x}, \bar{\xi}) \in \mathcal{E}$. From Ortega et al. [2008, Lemma 2], we know that the conditions that define the set (23) are equivalent to

$$g^{\perp}(\bar{x})F(\bar{x})\nabla H(\bar{x}) = 0 \tag{A.1a}$$

$$\nabla H_c(\bar{\xi}) = g^{+}(\bar{x})F(\bar{x})\nabla H(\bar{x}) \; . \tag{A.1b}$$

From (A.1a) we get that $\frac{\bar{x}_2^2}{2}(1 - \bar{x}_1\bar{x}_2) = 0$. In other words, if a given $\bar{x}$ is in $\mathcal{E}$, then it must satisfy

$$\bar{x} \in \{(\bar{x}_1, 1/\bar{x}_1) \,|\, \bar{x}_1 \neq 0\} \cup \{(\bar{x}_1, 0) \,|\, \bar{x}_1 \in \mathbb{R}\} \; . \tag{A.2}$$

Note that

$$g^{+}(\bar{x})F(\bar{x})\nabla H(\bar{x}) = \frac{(\frac{1}{2}\bar{x}_1 - \bar{x}_2)(\bar{x}_2^2 - \frac{1}{2}) + \bar{x}_2^5}{(\bar{x}_2^2 - \frac{1}{2})^2 + \bar{x}_2^6} \; ,$$

hence, because of (A.2),

$$g^{+}(\bar{x})F(\bar{x})\nabla H(\bar{x}) = \begin{cases} -\dfrac{1}{\bar{x}_2}\dfrac{(\bar{x}_2^2 - \frac{1}{2})^2 + \bar{x}_2^6}{(\bar{x}_2^2 - \frac{1}{2})^2 + \bar{x}_2^6} = -\dfrac{1}{\bar{x}_2} & \bar{x}_2 \neq 0 \\ -\bar{x}_1 & \bar{x}_2 = 0 \end{cases}$$

In any case, $g^{+}(\bar{x})F(\bar{x})\nabla H(\bar{x}) = -\bar{x}_1$. Finally, from (A.1b) and the fact that $u_\star = -1$ we get that $\nabla H_c(\bar{\xi}) = \bar{\xi} + 1 = -\bar{x}_1$ or, equivalently, $\bar{\xi} = -\bar{x}_1 - 1$, so

$$\mathcal{E} = \left\{ \mathrm{col}(\bar{x}_1, \frac{1}{\bar{x}_1}, -\bar{x}_1 - 1) \,|\, \bar{x}_1 \neq 0 \right\} \cup \{\mathrm{col}(\bar{x}_1, 0, -\bar{x}_1 - 1) \,|\, \bar{x}_1 \in \mathbb{R}\} \; . \tag{A.3}$$

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