# ISS-Lyapunov Functions for Output Feedback Sliding Modes

Andrea Aparicio Martínez[1], Fernando Castaños[2] and Leonid Fridman[3]

*Abstract*— **In this paper we address the problem of establishing conditions for global asymptotic stability in output feedback sliding-mode control. The proposed methodology introduces a linear term in a first-order sliding-mode controller. This allows to characterize the closed loop with the input-to-state stability property. Also, a Lyapunov-based methodology to find the correct gains for this controller is presented.**

## I. INTRODUCTION

Sliding modes (SM) have proven to be one of the most effective control methods to deal with systems that present unknown inputs and disturbances, since they are capable of theoretically rejecting them exactly when they are matched to the control input. Another important feature of the SM controllers is that they provide finite-time and exact convergence of the states to a sliding surface that can be designed in any convenient way for the system [17].

Most of the times, in order to implement a control law, a complete measure of the states of a system is needed. Unfortunately, in real life systems it is not always the case that a measure of the complete state is available, due sometimes to the high cost of some sensors or to the fact that in some systems it is simply impossible to measure a state, no matter how accurate a sensor can be. This is the reason why many efforts have been dedicated to study the problem of control with only output information, which has derived in two main branches: the design of state observers and output feedback (OF) control strategies. One advantage of the latter approach is that in most cases the separation principle does not hold and it is not possible to design separately a controller and an observer that feeds it. Also, the robustness of a controller against perturbations is usually lost when it is connected directly to an observer. The problem of Output Feedback Sliding Modes has been addressed in a number of works, for example [3], [4], [5], [7], with different approaches in terms of the (un)matchedness of the perturbations, the relative degree of the output and the strong or weak observability they require.

Another approach to study systems under disturbances or noise is through the use of the input-to-state stability (ISS) property, which can provide conditions to assure that a system's state remains bounded when its inputs are bounded, and tends to equilibrium when the inputs tend to zero [16]. The input-to-state stability property has been studied since the eighties and many applications have been found. This technique has also proven to be very helpful when studying the stability of interconnected systems, and many efforts have been dedicated to that matter [9], [11]. Another advantage of the ISS literature is that there exists many works dedicated to the relation of the ISS property and the existence of ISS-Lyapunov functions [12], which can be guessed from a Lyapunov function for the unforced system.

Recently, the ISS theory has been extended to a new concept, the integral-input-to-state stability (iISS) property, which can characterize a wider class of systems than the ISS one. The reason for this is that, while the ISS concept requires that the state of a system and its input are bounded by the same norm, the iISS allows for inputs bounded by an integral norm and states bounded by a supremum one [2]. This is the case of sliding-mode controllers with constant gains, for example. Along with the iISS theory, some tools have been developed for the study of the stability of interconnected systems [8], [10]. A useful resource to study the iISS and ISS properties of nonlinear systems through the use of weighted homogeneity is [6], which can be applied to the sliding-mode controllers due to their homogeneous nature. While the iISS concept is an important advance in the control theory, its applications are still not as well explored as the ISS ones.

Regular forms were introduced to the control literature since the early eighties [1], [17] and have been widely used ever since, mainly because of to the simple visualization of a system's characteristics that they offer, which facilitates greatly the control design. A similar technique arose ten years later, with the development of the backstepping theory [13]. One of the main contributions of [4] and [7] is the introduction of an *output-based regular form* for relative degree one for the first work, and a generalization for arbitrary relative degree for the latter. This form is used to design a virtual control that leads to a dynamic sliding surface to which the states of the system converge in finite time with the aid of a sliding-mode controller of order one and two, respectively.

While all the works on output feedback sliding-mode control that we have mentioned offer good solutions and methodologies for systems with different kinds of perturbations and different relative degrees, the common

[1]Andrea Aparicio Martínez is in the Electrical Engineering PhD Program of the Faculty of Engineering, Universidad Nacional Autónoma de México
[2]Fernando Castaños is with Departamento de Control Automático, Cinvestav, IPN, México.
[3]Leonid Fridman is with the Faculty of Engineering, Universidad Nacional Autónoma de México

denominator is that the stability that is ensured is only semi global and depends on the characteristics of the zero dynamics and the initial conditions of the complete system. In particular, in [7] and [4], the way of choosing adequate gains for the controller, in order to ensure global stability of the solution, remained open.

The main contribution of this paper is to propose a sliding-mode controller with an added linear term, in such a way that it can be characterized as ISS i.e. not only iISS. Also, and a methodology to find the correct gains for this controller when applied to the solution described in [7]. Using the methodology proposed in this paper, global and asymptotic stability can be achieved.

## II. PRELIMINARIES

### A. Notation

In this paper we use the following notation: $\|\cdot\|$ denotes the euclidean norm of a signal while $|\cdot|$ denotes the absolute value of a scalar, $\lambda_{max}(A)$ denotes the maximal eigenvalue of a matrix $A$ and $\lambda_{min}(A)$ denotes its minimal eigenvalue. An identity matrix of dimension $m$ is represented by $I_m$.

### B. ISS

*Definition 1:* [14] A continuous function $\alpha : [0,a) \to [0,\infty)$ belongs to class $\mathscr{K}$ if it is strictly increasing and $\alpha(0)=0$. It belongs to class $\mathscr{K}_\infty$ if $a=\infty$ and $\alpha(r)\to\infty$ as $r\to\infty$.

*Definition 2:* [14] A continuous function $\beta : [0,a) \times [0,\infty) \to [0,\infty)$ belongs to class $\mathscr{KL}$ if, for each fixed $s$, the mapping $\beta(r,s)$ belongs to class $\mathscr{K}$ with respect to $r$ and, for each fixed $r$, $\beta(r,s)$ is decreasing with respect to $s$, and $\beta(r,s)\to 0$ as $s\to\infty$.

*Definition 3:* [14] A system $\dot{x} = f(t,x,u)$ is said to be input-to-state stable if there exists a function $\beta \in \mathscr{KL}$ and a function $\gamma \in \mathscr{K}$ such that for any initial state $x(t_0)$, and any bounded input $u(t)$, the solution $x(t)$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0) + \gamma\left(\sup_{t_0\leq\tau\leq t}\|u(\tau)\|\right). \tag{1}$$

*Definition 4:* [12] A smooth function $V$ is said to be an ISS-Lyapunov function for a system $\dot{x} = f(t,x,u)$ if $V$ is proper, positive definite, i.e., there exists functions $\psi_1, \psi_2 \in \mathscr{K}_\infty$ such that

$$\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|),$$

and there exist functions $a \in \mathscr{K}_\infty$ and $\theta \in \mathscr{K}$ such that

$$\nabla V(x) f(x,u) \leq -a(V(x)) + \theta(\|u\|).$$

*Lemma 1:* From Young's inequality it can be proved that if $a$ and $b$ are nonnegative real numbers, then

$$ab \leq \frac{a^2}{2}\gamma^2 + \frac{b^2}{2}\gamma^{-2}$$

for any $\gamma > 0$.

*Theorem 1:* [16] The following properties are equivalent for any system

a) It is ISS
b) It admits an ISS-Lyapunov function
c) There exist a $\mathscr{KL}$ function $\beta$, and a $\mathscr{K}$ function $\gamma$ such that (1) holds

*Theorem 2:* [12] If, for interconnected systems

$$\dot{x}_1 = f_1(x_1,x_2,u_1) \tag{2}$$
$$\dot{x}_2 = f_2(x_1,x_2,u_2), \tag{3}$$

there exist an ISS-Lyapunov function $V_i$, for the $x_i$ subsystem, $i=\{1,2\}$, such that with functions $\alpha_i \in \mathscr{K}_\infty$, $\chi_i$, $\gamma_i \in \mathscr{K}$ the following holds:

$$V_i(x_i) \geq \max\{\chi_i(V_j(x_j)),\, \gamma_i(\|u_i\|)\} \Rightarrow$$
$$\nabla V_i(x_i) f_i(x_i,x_j,u_i) \leq -\alpha_i(V_i),$$

with $j=\{2,1\}$, and

$$\chi_1 \circ \chi_2 < r \quad \forall\, r>0, \tag{4}$$

then the interconnected system (2), (3) is ISS and the zero solution of (2), (3), with $u=0$, is globally asymptotically stable.

*Corollary 1:* If $V_i$ are ISS-Lyapunov functions for (2), (3), and

$$\nabla V_i(x_i) f_i(x_i,x_j,u_i) \leq -a_i(V_i(x_i)) + \theta_i^x(V_j(x_j)) + \theta_i^u(\|u_i\|)$$

with

$$\theta_i^x(s) = \kappa_i a_j(s),$$

for some $\kappa_i > 0$, then the condition (4) is satisfied if $\kappa_1\kappa_2 < 1$.

### C. Output Feedback Sliding-Modes

As mentioned in Section I, in [7] a solution for the output feedback problem with unmatched disturbances was presented, which considers an uncertain system of the form

$$\begin{array}{rcl} \dot{z} &=& Az + Dw + Bu \\ y &=& Cz \end{array} \tag{5}$$

where $z\in\mathbb{R}^n$ is the state variable, $w\in\mathbb{R}^m$ is an unknown matched input and it is assumed that $|w|\leq\bar{w}$ for a known $\bar{w}$, $y\in\mathbb{R}^m$ is the measured output of relative degree one, and $u\in\mathbb{R}^m$ is the control input. The pair $(A,B)$ is assumed to be controllable and the pair $(A,C)$ is assumed to be observable.

Any linear system with the characteristics of (5) can be taken to an *output regular form*

$$\begin{aligned} \dot{z}_1 &= A_{11}z_1 + A_{12}z_2 \\ \dot{z}_2 &= A_{21}z_1 + A_{22}z_2 + D_2 w + u \\ y &= z_2, \end{aligned} \tag{6}$$

where $z_1 \in \mathbb{R}^p$ and $z_2 \in \mathbb{R}^m$, with $p+m=n$, via a state transformation introduced in the above cited work.

The steps involved in the procedure described in [7] are, briefly, (for details consult the cited work):

a) Transform an uncertain linear system (5) to its *output regular form* (6)
b) It can be shown that the reduced-order system

$$\begin{aligned} \dot{z}_1 &= A_{11}z_1 + A_{12}u_v \\ y_v &= A_{21}z_1 + D_2 w, \end{aligned} \tag{7}$$

maintains the controllability and observability of (5), where $u_v = z_2$ is the virtual control and $y_v$ represents a virtual output.
c) Design a dynamic virtual control for (7) of the form $u_v = F\eta$, where the dynamics of $\eta$ is given by

$$\dot{\eta} = \hat{A}\eta + Ly_v. \tag{8}$$

This virtual control should be able to deal with the noise present in the virtual output i.e. $D_2 w$. In [7] this control law is an $H_\infty$ controller.
d) Design a dynamic sliding variable $s = z_2 - F\eta$ such that, when the trajectories of the system are in sliding mode, it is satisfied that $z_2 = -F\eta$.
e) Design a discontinuous control law that enforces the sliding modes, allowing to make the sliding variable converge to zero in finite time. The resulting dynamics is given by

$$\dot{s} = D_2 w + B_2 z_1 - k\,\text{sign}(s). \tag{9}$$

f) Choose the discontinuous controller constant gain as

$$k > |D_2 \bar{w}|. \tag{10}$$

This choice of gain would provide stability for (5) only on a locality where the unmeasured state $z_1$, that affects the dynamics of the sliding variable, is sufficiently small and converges to a neighborhood of the origin fast enough. With a choice of a controller as in (9), if the initial conditions are unknown and the convergence of the state $z_1$ cannot be assured, there is no way of proving stability of the overall system (11) below. Even more, when the conditions are satisfied, only a local stability proof can be obtained. A similar case is found in [3] and [5], where only semi-global asymptotic stabilization is achieved with their respective methodologies. Even more, none of this works provide for an explicit way of choosing gains for the designed controller.

## III. PROBLEM STATEMENT

Once that the procedure of the previous section is carried out up to step d), for a system like (5), and defining a control signal $u = -u_{eq} + v$, where $u_{eq}$ is the equivalent control and $v$ is a new control law, the closed-loop of system (6) with the virtual control defined in c), and the dynamics of the sliding variable (9), with $x^T = [z^T \quad \eta^T]$, can be represented by

$$\begin{aligned} \dot{x} &= f(x,s) \\ \dot{s} &= g(x_1, v, w) \end{aligned} \tag{11}$$

where $f(x,s) = A_x x + B_x s$ and $g(x_1,v,w) = Dw + B_s x_1 + v$.

The closed-loop (11) can be viewed as a feedback interconnection of two systems, as shown in Figure 1. Under controllability and observability of $(A,B)$ and $(A,C)$, matrix $A_x$ can always be chosen Hurwitz [7]. Then, there exists a Lyapunov function for the unforced system $\dot{x} = f(x,0)$

$$V_1(x) = x^T P x, \tag{12}$$

where $P > 0$ satisfies the Lyapunov equation $PA + AP = -I_n$. It then holds that

$$\lambda_{min}(P)\|x\|^2 \le V_1(x) \le \lambda_{max}(P)\|x\|^2$$
$$\dot{V}_1(x) f(x,0) \le -\|x\|^2.$$

For subsystem $\dot{s} = g(0,v,0)$, a Lyapunov function candidate is defined as

$$V_2(s) = \frac{1}{2}s^2. \tag{13}$$

It is easy to check that for $g(0,v,w)$, a choice of the controller $v = -k\,\text{sign}(s)$, as in (9), with the choice of gain (10) is enough for achieving global asymptotical stability but, being this subsystem fed by an unknown, although stable linear system, this choice of controller only achieves local stability.

The problem of (11), also represented by Figure 1, can be summarized as one of choosing a suitable control law for a sliding-mode subsystem connected in feedback to a stable, yet unknown linear subsystem.

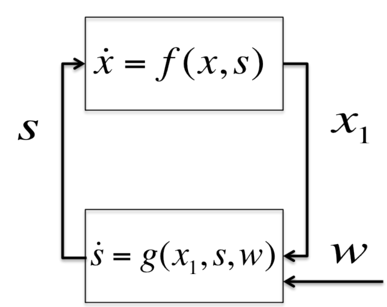


Fig. 1. System (11) in feedback form

The main contributions of this paper are:

- The introduction of a modified first-order sliding-mode controller that includes a linear gain, and that can be characterized as ISS.
- An ISS-Lyapunov-based method of choosing the gains of the controller described above, which assures global stability of the closed-loop (11).

## IV. MAIN RESULT

The following theorem summarizes the main result of this paper.

*Theorem 3:* The feedback interconnection (11), is globally and asymptotically stable with a control law

$$v = -k_1 \operatorname{sign}(s) - k_2 s,$$

and a choice of gains

$$k_1 > 2\|PB_x\|\|B_s\|\sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}} \tag{14}$$

$$k_2 > |D\bar{w}|.$$

*Proof:* The derivative of the Lyapunov function (12) over the trajectories of $f(x,s)$ is

$$\dot{V}_1(x) = \nabla V_1 f(x,s) = -\|x\|^2 + 2PB_x^T xs \tag{15}$$

Using the inequality of Lemma 1, with $a = 2\|B_x^T P\|\|x\|$ and $b = s$, (15) can always be bounded by

$$\dot{V}_1(x) \leq -\left(1 - 2\|B_x^T P\|^2\gamma_1^2\right)\|x\|^2 + \frac{1}{2\gamma_1^2}s^2.$$

Choosing $\gamma_1^2 = \dfrac{1}{4\|B_x^T P\|^2}$, the derivative (15) can be expressed as

$$\begin{aligned} \dot{V}_1(x) &\leq -\frac{1}{2}\|x\|^2 + 2\|B_x^T P\|^2 s^2 \\ &\leq -\frac{1}{2\lambda_{max}(P)}x^T Px + 2\|B_x^T P\|^2 s^2. \end{aligned} \tag{16}$$

The derivative of (13) over the trajectories of $g(x,s,w)$ is

$$\begin{aligned} \dot{V}_2(s) = \nabla V_2(s) g(x,s,w) &= B_s xs + Dws - k_1 s^2 - k_2|s| \\ &\leq B_s xs - k_1 s^2 + (D\bar{w} - k_2)|s| \end{aligned} \tag{17}$$

The last term of the second line of (17) can easily be made negative by choosing the gain $k_2$ as

$$k_2 > |D\bar{w}|,$$

yielding

$$\dot{V}_2(s) \leq B_s xs - k_1 s^2.$$

Again, using the inequality on Lemma 1, with $a = \|B_s\|\|x\|$ and $b = s$, one can further bound (17) as

$$\dot{V}_2(s) \leq -\left(k_1 - \frac{1}{2\gamma_2^2}\right)s^2 + \frac{\|B_s\|^2}{2}\gamma_2^2\|x\|^2.$$

Choosing $\gamma_2^2 = \dfrac{2l_2\lambda_{min}(P)}{\|B_s\|^2}$, where $l_2 > 0$, one gets

$$\begin{aligned} \dot{V}_2(s) &\leq -\left(k_1 - \frac{1}{4l_2\lambda_{min}(P)}\|B_s\|^2\right)s^2 + l_2\lambda_{min}(P)\|x\|^2 \\ &\leq -\left(k_1 - \frac{1}{4l_2\lambda_{min}(P)}\|B_s\|^2\right)s^2 + l_2 x^T Px \end{aligned} \tag{18}$$

Defining functions $a_1, a_2, \theta_1, \theta_2$ of class $\mathscr{K}_\infty$,

$$a_1(r) := \left(\frac{1}{2\lambda_{max}(P)}\right)r$$

$$\theta_1(r) := \left(2\|B_x^T P\|^2\right)r$$

$$a_2(r) := \left(k_1 - \frac{1}{4l_2\lambda_{min}(P)}\|B_s\|^2\right)r$$

$$\theta_2(r) := l_2 r,$$

one can express the derivatives (16) and (18) as

$$\dot{V}_1(x) \leq -a_1(V_1) + \theta_1(V_2)$$

$$\dot{V}_2(s) \leq -a_2(V_2) + \theta_2(V_1),$$

which are, according to Definition 1, ISS-Lyapunov functions for each one of the subsystems of (11).

Making $\kappa_i = a_j^{-1} \circ \theta_i(r)$, constants $\kappa_i > 0$ of Theorem 2, for (11), are

$$\kappa_1 = \frac{2\|B_x^T P\|^2}{\left(k_1 - \dfrac{1}{4l_2\lambda_{min}(P)}\|B_s\|^2\right)},$$

$$\kappa_2 = 2l_2\lambda_{max}(P).$$

From Corollary 1, to achieve stability of (11), it must hold

$$k_1 > 4l_2\|B_x^T P\|^2\lambda_{max}(P) + \frac{\|B_s\|^2}{4l_2\lambda_{min}(P)}. \tag{19}$$

Choosing an optimal $l_2$ that minimizes the right-hand side of (19) as

$$l_2 = \frac{\|B_s\|}{4\|B_x^T P\|\lambda_{min}(P)\lambda_{max}(P)},$$

one obtains condition (14). ■

## V. EXAMPLES

### *A.*

First we will verify the trivial case of a second-order system

$$\begin{array}{rcl} \dot{x} & = & -ax+bs \\ \dot{s} & = & cx+dw-k_1 s-k_2\,\mathrm{sign}(s) \end{array} \tag{20}$$

where $a>0$, $b>0$, $c>0$ and $d>0$ are known, $x\in\mathbb{R}$, $s\in\mathbb{R}$, and $|w|\leq\bar{w}\in\mathbb{R}$ for a known $\bar{w}$.

The Lyapunov equation for the upper part of system (20) is

$$p(-a)+(-a)p=-1$$

and the constant $p>0$ that satisfies it is $p=\dfrac{1}{2a}$. Condition (14), with $B_x=b$ and $B_s=c$ determines that, to achieve global stability of (20), the gains $k_1$, $k_2$ must satisfy

$$k_1>\frac{bc}{a}\ \text{and}\ k_2>d\bar{w}. \tag{21}$$

For a low-order system such as (20), the necessary and sufficient conditions for its stability can also be obtained by the characteristic polynomial of the nominal state matrix, $P(\lambda)=\det(\lambda I_2)-A=\lambda^2+(a+k_1)\lambda+ak_1-cb$. All the coefficients of this polynomial must be positive, in other words, the linear gain must satisfy

$$k_1>\frac{bc}{a} \tag{22}$$

When (22) is satisfied, it is easy to check that the first time-derivative of the Lyapunov function $V_1=px^2$ can be made negative definite by choosing the discontinuous gain as

$$k_2>d\bar{w}. \tag{23}$$

We have shown that, with the method described in this paper, the necessary and sufficient conditions for the global asymptotic stability of (20) are recovered. The characteristic polynomial could be found for a system of any order, but the conditions for the positive definiteness of its coefficients grow in complexity and number as the order grows, whereas condition (14) offers always a simple way of choosing gains that achieve global asymptotical stability, regardless of the order of the system.

### *B.*

Now we will analyze a case of an unstable second-order system, and show how our choice of gains are able to stabilize it.

Consider the following unstable second-order system, which is already in the form (6):

$$\begin{array}{rcl} \dot{x}_1 & = & -3x_1+8x_2 \\ \dot{x}_2 & = & 2x_1-6x_2+w+u \\ y & = & x_2 \end{array} \tag{24}$$

The closed-loop of the system when an $H_\infty$ criterion is used to design the dynamic sliding surface is

$$\begin{array}{rcl} \dot{x}_1 & = & -3x_1-36.0344\eta+8s \\ \dot{\eta} & = & -x_1-23.0172\eta \\ \dot{s} & = & -k_1 s-k_2\,\mathrm{sign}(s)-1.2521w+2.4043x_1 \end{array} \tag{25}$$

The matrix $P>0\in\mathbb{R}^{2\times 2}$ that satisfies the Lyapunov equation $PA^T+AP=-I_2$ is

$$P=\begin{bmatrix} 0.1834 & -0.0014 \\ -0.0014 & 0.0026 \end{bmatrix}$$

From condition (14), the gains of (25) must satisfy

$$k_1>59.72\ \text{and}\ k_2>1.2521\bar{w}. \tag{26}$$

With a disturbance $w=1+.4\sin(2t)$ and initial conditions $x_1(0)=30$ and $x_2(0)=10$, choosing gains as $k_1=60$ and $k_2=1.5$ we get the simulation results of Figure 2.

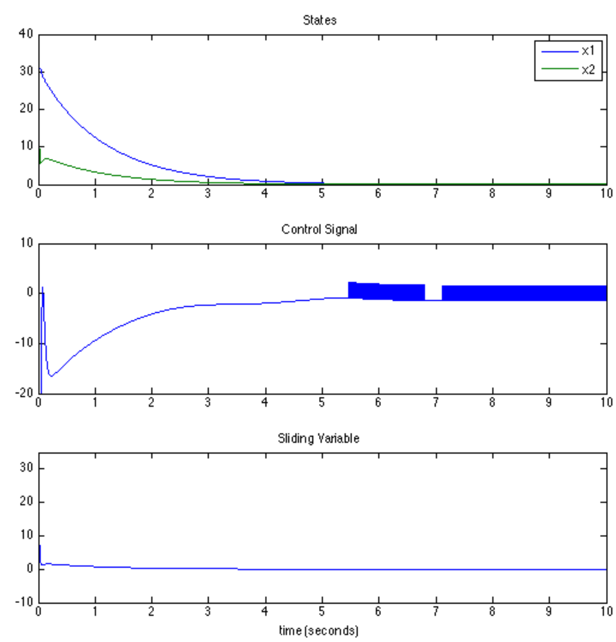


Fig. 2. Simulation results with $k_1=60$ and $k_2=1.5$

Choosing a gain $k_1$ that does not satisfy condition (26), we can verify that the trajectories of the system tend to infinity. For example, with $k_1=10$ and $k_2=1.5$, and the same initial conditions, we get the simulation results of Figure 3.

Of course, with small enough initial conditions, $x_1(0)=0.5$ and $x_2(0)=0.5$, with the small linear gain $k_1=10$, convergence of the states to the origin can still be achieved as shown in Figure 4.

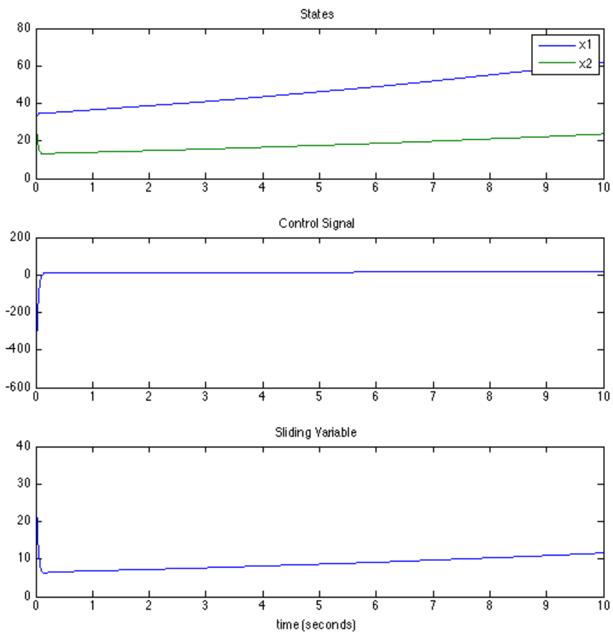


Fig. 3. Simulation results with $k_1 = 10$ and $k_2 = 1.5$

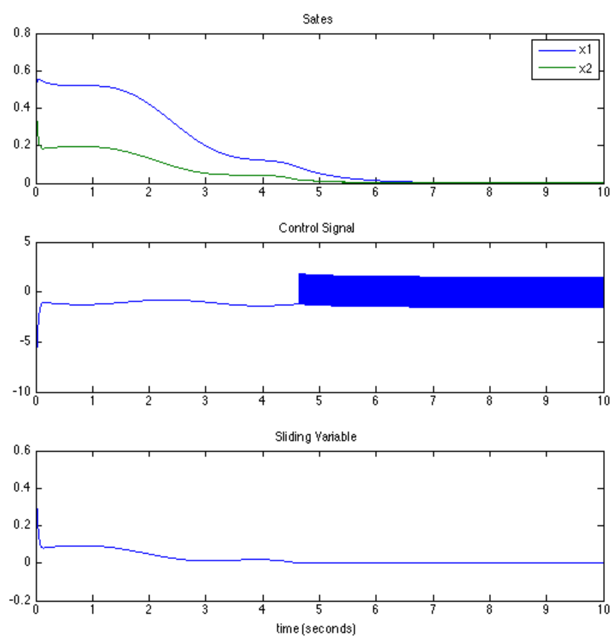


Fig. 4. Simulation results with $k_1 = 10$, $k_2 = 1.5$ and small initial conditions

## VI. CONCLUSIONS

We have introduced linear terms to a first-order sliding-mode controller which allows it to be characterized as ISS, instead of only iISS. Also, we have provided a methodology to choose the gains of such controller in a way that, when applied to an output feedback sliding-mode control solution as the one presented in [7], global and asymptotic stability is achieved.

The simulation results presented in this paper show clearly how, when the initial conditions of the system cannot be known, the linear gain is necessary to assure the convergence of the states to the origin. Moreover, the condition for choosing the linear and discontinuous gains established in the main theorem is sufficient to achieve global asymptotic stability. When this conditions are not met, for unknown initial conditions, not only global stability is not assured, but the trajectories of the system can tend to infinity.

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