Dynamic Surface for Output Feedback Sliding Modes, the Case of Relative Degree Two

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***Abstract*— A general transformation that takes linear systems into their regular form, for any relative degree is introduced. A sliding surface where unmatched unknown inputs are attenuated via a reduced order H_∞ control is designed, for the case of relative degree two. By a discontinuous control action, the surface is reached exactly in finite time, guaranteeing the minimization of the unmatched disturbance. Complete state measurements are not necessary.**

I. INTRODUCTION

Output feedback control is a subject that has interested the control community for a very long time, the reason being, overall, that one cannot expect to have a measure of the complete state available at all times when dealing with real-life, physical systems. Numerous works have been dedicated to this particular subject trying to propose more reliable control strategies and that can be applied to a wider class of systems.

Sliding modes (SM) control has proven to be a very convenient way to deal with matched disturbances, as they are capable of rejecting them exactly, bringing the system's state to a sliding surface in finite time. On the downside, a measure of the complete state is usually required in order to implement the control, and also, while the SM are theoretically exact at compensating matched disturbances, they are quite sensitive to unmatched ones.

In [4] the construction of dynamic compensators is proposed in order to add dynamics to systems for which a direct pole assignment cannot be done, so it would hold for an augmented one, all this with an output feedback approach. This work offers good results, but limits the number of cases where the strategy can be applied because the only kind of allowed disturbances are matched ones.

A very well known and popular theory is H_∞, due to its well studied method of implementation and the great number of cases to which it can be adapted, for example, problems where measures of only part of the state are available, in other words, output feedback problems. This is possible due to the observer-like structure of the H_∞ controllers, which is partly inherited from the H_2 observation and control theory. One great advantage of H_∞ is that it offers a minimization criteria, which makes it very suitable to attenuate undesired effects of unknown inputs or disturbances that otherwise are difficult to deal with.

A way of combining SM and H_∞ with the purpose of attenuating unmatched disturbances is proposed in [7] where the existence conditions for a sliding surface are found via LMIs. The disadvantage of this approach is that it increases the computational effort needed. Another, yet more straightforward combination of Sliding Modes and H_∞ is in [6], where a way of obtaining a reduced order H_∞ controller is proposed for an unmeasured state that is affected by unmatched disturbances. That approach is complemented with a sliding surface design and a discontinuous control action and considers also disturbances matched to the control and that affect the measured state. The restriction in this work is that the measured output has to have relative degree $r = 1$. This restrictive condition is also imposed in the other two works cited, as well as in most of the literature. As a consequence, the SM control law proposed in all of them is of first order. Some work has been done in order to do output feedback control with systems that have unknown inputs, regardless of the relative degree of the output, for example [2]. The restriction imposed over the system in this work is that that the disturbances have to be matched to the control necessarily, and the system has to be strongly observable.

Second order sliding modes (SOSM) is a very well explored field that has been implemented in a great number of applications [5], [13], [8]. In this paper we propose an output feedback control strategy via a combination of SOSM and H_∞, for an uncertain system with an output of relative degree $r = 2$ that is affected by both matched and unmatched disturbances or unknown inputs. The objective is to use the sliding modes to assure the minimizing action of H_∞ to attenuate the effect of the unmatched disturbances in finite time. The main contributions of this work are the overcome of the relative degree one restriction found in the output feedback sliding modes control literature, the possibility of considering matched and unmatched disturbances to the system, and the introduction of a transformation to a regular form for systems with relative degree higher that one and of which only part of the state is measurable.

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This is an ongoing work that aims to present, in the future, a complete methodology for outputs of any relative degree.

II. PROBLEM STATEMENT

Consider the following uncertain system

$$\begin{array}{rcl}\dot{z} & = & Az + Bv + Dw \\ y & = & Cz,\end{array} \qquad (1)$$

where $z \in R^n$ is the state vector, $w \in R^p$ is a bounded unknown input, $v \in \mathbb{R}$ is the control signal and $y \in \mathbb{R}$ is the measured output with relative degree r with respect to the control.

Assumption 1: The following items hold:

a. The pair (A, B) is controllable.
b. The pair (A, C) is observable.
c. The relative degree of the output with respect to the unknown input, r_w, is assumed to satisfy $r \leq r_w$.

Assumptions 1.a and 1.b are standard assumptions in control theory. Assumption 1.c means that no disturbances are found in the measured state, but they do affect the state associated to the control (matched disturbances), and/or appear in some derivative of the output, higher than r (unmatched disturbances).

Problem statement: System (3) is an uncertain system with an output y of relative degree $r > 1$, which means that the state associated to the control is not measurable. In this work it is considered the case when a disturbance w may have a matched and also an unmatched component that satisfies Assumption 1.c. The output y measures only part of the state and has no explicit information of the state affected by the unmatched disturbance nor of the state associated to the control. The goal is to obtain a stabilizing closed loop that attenuates the effect of the disturbances in finite time.

III. TRANSFORMATION TO A REGULAR FORM FOR ARBITRARY RELATIVE DEGREE

In [15] it is shown how the sliding surface design is much simpler if the system is in its regular form. A simple example of a system with matched and unmatched disturbances on this form is:

$$\begin{array}{rcl}\dot{x}_1 & = & A_{11}x_1 + A_{12}x_2 + B_{11}w_1 \\ \dot{x}_2 & = & A_{21}x_1 + A_{22}x_2 + B_{12}w_2 + B_2u.\end{array}$$

This arrangement is widely known as the standard regular form, and the existing coordinate transformations that lead to this form normally assume that a complete state measure is available. In [6] a transformation to a regular form, for the output feedback case and relative degree $r = 1$ is introduced. In this section we will present a transformation for systems with output of any relative degree r, that only contains information of a part of the state. This transformation maintains the controllability and the observability of the original system. Both properties are essential to the development of this work.

Consider system (1), with relative degree r and dimension n. From the definition of relative degree it is known that $CA^{i-1}B = 0,\ 1 \leq i < r$ and $CA^{r-1}B \neq 0$. Also, since $B \neq 0$, there exists a matrix $B^{\perp} \in R^{(n-r)\times n}$ with $n - r$ linearly independent rows such that $B^{\perp}B = 0$. One can take the output y and its successive $(r-1)$ derivatives as a set of coordinates $\chi_1, \ldots, \chi_r$ to construct a coordinate transformation with invertible T that brings a linear system with relative degree r to a regular form. This transformation is (2):

$$\begin{bmatrix}\xi \\ \chi\end{bmatrix} = Tz = \begin{bmatrix} B_1^{\perp} \\ \vdots \\ B_{n-r}^{\perp} \\ \hline C \\ \vdots \\ CA^{r-1}\end{bmatrix} z. \qquad (2)$$

Note that there is not an unique $B^{\perp}$ that makes this hold.

This transformation brings (1) to the form

$$\begin{bmatrix}\dot{\xi} \\ \dot{\chi}\end{bmatrix} = T\dot{z} =$$

$$\begin{array}{rcl}\dot{\xi} & = & A_0\xi + B_0\chi + B_{11}w_1 \\ \dot{\chi}_1 & = & \chi_2 \\ \vdots & & \vdots \\ \dot{\chi}_r & = & [A_{n1} \ \ldots \ A_{nn}] \begin{bmatrix}\xi \\ \chi\end{bmatrix} + B_{12}w_2 + u \\ y & = & \chi_1,\end{array} \qquad (3)$$

where $A_0 \in R^{(n-r)\times(n-r)}$, $B_0 \in R^{(n-r)\times r}$. In the absence of disturbances, $w = 0$, χ can be taken exactly to zero and the subsystem ξ would become $\dot{\xi} = A_0\xi$, which represents the zero dynamics of the complete system.

IV. REDUCED ORDER SYSTEM AND H_∞ CONTROLLER

One of the goals of this work is to attenuate the effect of unmatched uncertainty, w_1, and to do so, a dynamic H_∞ controller will be obtained. This kind of controllers provide a minimization of a transfer function T_{Jw} that maps the disturbance w_1 to a penalization variable J, below a real number $\gamma > 0$ [9]. From simple examination of system (3) one can notice that w_1 affects only a very specific part of the state. A reduced order system that maintains the controllability and observability properties of (3), and that contains the unmatched disturbance w_1 will be obtained. The H_∞ controller will be calculated for this reduced order system, finding the solution of two Riccati equations. The order of this equations will be the same of the reduced order system. In order to make this step simpler, the following additional assumption is made.

Assumption 2: For system (1) the following items hold:

a. $r = 2$
b. $A_{r-1} \in \text{span}(B)$ where A_{r-1} represent the last $r-1$ columns of matrix A.

Assumption 2.a. is made to analyze the case when the relative degree of the output equals two. Assumption 2.b. means that there exists a matrix $B^{\perp}$ such that $B^{\perp}B = B^{\perp}A_{r-1} = 0$.

Under assumption 2, system (3) can be rewritten as:

$$\begin{array}{rcl} \dot{\xi} & = & A_0\xi + B_{01}\chi_1 + B_{11}w_1 \\ \dot{\chi}_1 & = & \chi_2 \\ \dot{\chi}_2 & = & A_{n1}\xi + A_{n2}\chi_1 + A_{n3}\chi_2 + B_{12}w_2 + u \\ y & = & \chi_1, \end{array} \tag{4}$$

where B_{01} is the first element of B_0. Figure 1 represents a block diagram for (4). Assumption 2 leaves the system in a quasi-cascade form.

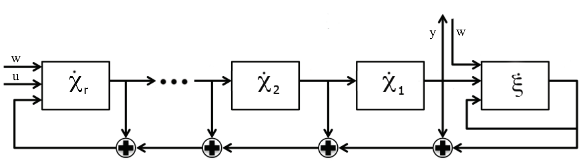

Fig. 1. Block representation of system (3) with Assumptions 2.

Proposition 1: If system (1) is controllable and observable, then the pair (A_0, B_{01}) of (4) will be controllable and the pair (A_0, C_2) of (4) will be observable, where $C_2 := A_{n1}$.

Once controllable and observable pairs are found, a controllable and observable reduced order system (5) can be derived from (3), defining a virtual output

$$y_v := \ddot{y} - A_{n3}\dot{y} - A_{n2}y - u,$$

and a virtual control [15] $u_v := \chi_1$.

$$\begin{array}{rcl} \dot{\xi} & = & A_0\xi + B_{01}u_v + B_{11}w_1 \\ y_v & = & C_2\xi + D_{21}w_2. \end{array} \tag{5}$$

To calculate the H_∞ controller one has to define a penalty variable $J = C_1\xi + D_{12}u_v$ that assigns weights to the state ξ and the control u_v and whose parameters must satisfy $D_{12}^T \left[\begin{array}{cc} C_1 & D_{12} \end{array} \right] = \left[\begin{array}{cc} 0 & \alpha I \end{array} \right]$ for an $\alpha \neq 0$ [9] . Theorem 1 states the rest of the conditions that have to be satisfied to calculate the controller, and shows its form.

Theorem 1: [9] If the pair (A_0, B_{11}) is stabilizable, the equality $\left[\begin{array}{c} B_{11} \\ D_{21} \end{array} \right] D_{21}^T = \left[\begin{array}{c} 0 \\ I \end{array} \right]$ holds and there exist matrices X_∞ and Y_∞ such that for a positive γ,

$$\begin{array}{c} A_0^T X_\infty + X_\infty A_0 + X_\infty(\gamma^{-2}B_{11}B_{11}^T - B_{01}B_{01}^T)X_\infty = -C_1^T C_1 \\ A_0 Y_\infty + Y_\infty A_0^T + Y_\infty(\gamma^{-2}C_1^T C_1 - C_2^T C_2)Y_\infty = -B_{11}B_{11}^T, \end{array} \tag{6}$$

then the state for a controller

$$u_v := F_\infty h, \tag{7}$$

for (5), such that $||T_{Jw}||_\infty < \gamma$, is:

$$\dot{h} = \hat{A}_\infty h - Z_\infty L_\infty y_v, \tag{8}$$

where $\hat{A}_\infty$, F_∞, L_∞ and Z_∞ are constant gains calculated with the parameters of the system and the solutions of (6). $||T_{Jw}||_\infty$ is the H_∞ norm of the transfer function that maps $w_1 \mapsto J$. The nomenclature used here is the usual H_∞ nomenclature, the details of this procedure, the calculations of the controller parameters and the proof of this theorem can be found in [9].

V. SECOND ORDER SLIDING SURFACE AND CONTROL

In this section a dynamic sliding surface is designed such that when reached, the virtual control u_v defined in (5) takes exactly the values of the H_∞ dynamic controller (7). Then, to enforce the sliding mode, discontinuous control law will be defined.

Recall that the virtual output was defined as $y_v := \ddot{y} - A_{n3}\dot{y} - A_{n2}y - u$ which is necessary to construct the virtual control. This virtual output is only available obtaining the first and second derivatives of the output y through some differentiator. The following proposition allows the construction of u_v using only the first derivative of y. The reduction on the order of differentiation will decrease the cost and complexity of the solution as well as the risk of introducing undesired dynamics provoked by a higher order differentiator.

Proposition 2: Define the auxiliary variable

$$\psi := Z_\infty L_\infty(\beta_1 - \beta_3) + \beta_5,$$

and let the dynamic system β be

$$\begin{array}{rcl} \dot{\beta}_1 & := & u \\ \dot{\beta}_2 & := & \beta_3 = \dot{y} \\ \dot{\beta}_3 & := & \beta_4 = \ddot{y} \\ \dot{\beta}_4 & := & y^{(3)} \\ \dot{\beta}_5 & := & \hat{A}_\infty\beta_5 + (Z_\infty L_\infty a_{33} - \hat{A}_\infty Z_\infty L_\infty)\beta_3 + \\ & & +Z_\infty L_\infty a_{32}\beta_2 + \hat{A}_\infty Z_\infty L_\infty \beta_1. \end{array} \tag{9}$$

Then the virtual control (7) is equivalent to

$$u_v = F_\infty \psi.$$

The first derivative of the output y, necessary to construct ψ, can be robustly obtained, in finite time, by a SOSM differentiator [11] that has the form:

$$\begin{array}{rcl}\dot{\zeta}_0 & = & v_0 \;=\; -2\,\theta^{\frac{1}{3}}|\zeta_0 - y|^{\frac{2}{3}}\operatorname{sign}(\zeta_0 - y) + \zeta_1 \\ \dot{\zeta}_1 & = & v_1 \;=\; -1.5\,\theta^{\frac{1}{2}}|\zeta_2 - v_0|^{\frac{1}{2}}\operatorname{sign}(\zeta_1 - v_0) + \zeta_2 \\ \dot{\zeta}_2 & = & -1.1\,\theta\operatorname{sign}(\zeta_2 - v_1),\end{array} \qquad (10)$$

where $\zeta_2 := \hat{\chi}_2$ and θ is an upper bound for $|y^{(3)}|$ which is assumed to be known.

Remark In the presence of noise in the output y, the output of the differentiator deteriorates, however, it offers the best possible differentiation with respect to noise [11]. For a deeper analysis of second order sliding modes differentiator error of noisy signals, see [3].

Theorem 2: A relative deegree $r_s = 2$ sliding surface $S = \{(\beta_1, \beta_2, \beta_3, \beta_5)\,|\,s(\beta_1, \beta_2, \beta_3, \beta_5) = 0\}$ for (3), that guarantees the minimization $||T_{Jw}||_\infty < \gamma$ is

$$s = \beta_2 - F_\infty(Z_\infty L_\infty(\beta_1 - \beta_3) + \beta_5). \qquad (11)$$

It is quite clear that when the sliding surface is reached the output y, and thus the state χ_1, becomes the central H_∞ controller output, guaranteeing the exact H_∞ bound $||T_{Jw}|| < \gamma$ attenuating the effect of w_1.

The first and second derivatives of (11) are:

$$\dot{s} = \alpha_1 x_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4 w_2 + \alpha 5\beta_5 + \alpha_6\beta_1$$

$$\ddot{s} = \delta_1 x_1 + \delta_2\beta_2 + \delta_3\beta_3 + \delta_4\beta_1 + \delta_5 u + \delta_6 w_1 + \delta_7 w_2 + \delta_8\dot{w}_2 + \delta_9\beta_5,$$

where α_{1-6} and δ_{1-9} are constants calculated with the parameters of system (4) and those of the H_∞ controller.

Assuming $||w_1|| < \bar{w}_1$, $||w_2|| < \bar{w}_2$, $||\dot{w}_2|| < \bar{\bar{w}}_2$ and $||x_1(0)|| < \bar{x}_1$ for some known $\bar{w}_1$, $\bar{w}_2$, $\bar{\bar{w}}_2$ and $\bar{x}_1$ a SOSM control law that will take the trajectories of (3) to s in finite time, can be the twisting controller [12], [1], [10] [14]:

$$u = \eta - k_1\operatorname{sign}(s) - k_2\operatorname{sign}(\dot{s}), \qquad (12)$$

where $\eta := -(\delta_2\beta_2 + \delta_3\beta_3)$, $k_1 > k_2 + \bar{w}_2$ and $k_2 > \bar{w}_2$.

VI. NUMERICAL EXAMPLE

Consider the system

$$\begin{array}{rcl}\dot{x} & = & Ax + Bu + Dw \\ y & = & Cx,\end{array} \qquad (13)$$

where $A = \begin{bmatrix} -3 & 2 & 0 \\ 0 & 0 & 1 \\ 2 & -3 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

The pair (A, B) is controllable and the pair (A, C) is observable. System (13) has already a regular form, it is of order $n = 3$ and the output's relative degree is clearly $r = 2$. Variable $w \in R^2$ represents an unmatched disturbance and a matched one.

The parameters of the penalty variable are:

$$C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\; D_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

assigning equal weights to the virtual control and the state.

The controllable and observable reduced order system is:

$$\begin{array}{rcl}\dot{x}_1 & = & -3x_1 + 2u_v + w_1 \\ J & = & C_1 x_1 + D_{12} u_v \\ y_v & = & 2x_1 + w_2.\end{array} \qquad (14)$$

An H_∞ controller for (14) is:

$$u_v = F_\infty\psi = -1.11\,\psi$$
$$\dot{\psi} = \hat{A}_\infty\psi - Z_\infty L_\infty y_v = 6.702\,\psi + 0.6325\,y_v$$

that satisfies $||T_{Jw}||_\infty < 0.2809$.

The sliding variable, defined in (11) has the form $s = \beta_2 - F_\infty(Z_\infty L_\infty(\beta_1 - \beta_3) + \beta_5)$ where the dynamic system β is constructed as in (9) and values $\hat{A}_\infty$, $Z_\infty L_\infty$ and F_∞ are the ones shown above.

With initial conditions $x_1(0) = 0.5$, $x_2(0) = 1$, $x_3(0) = 0.5$, perturbations $w_1 = 0.2 + 0.5\sin(5t)$ and $w_2 = 1 + .04\sin(2t)$ and gains $k_1 = 25$ and $k_2 = 10$ for the SOSM controller (12), the following results are obtained:

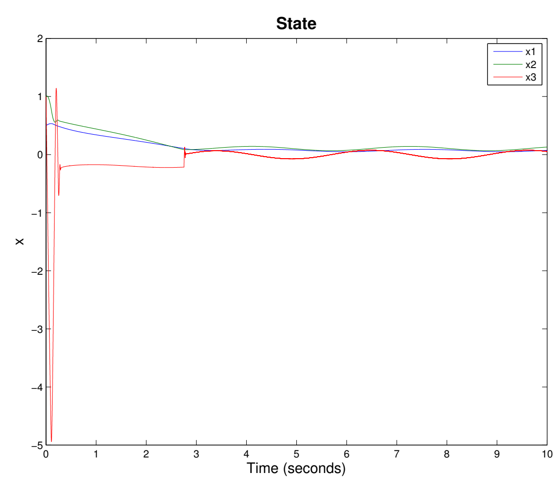

Fig. 2. States x_1, x_2 and x_3

Figure 2 shows that the complete state converges to a neighborhood of the origin, figure 3 shows how the sliding surface s and its derivative $\dot{s}$ converge in finite time and

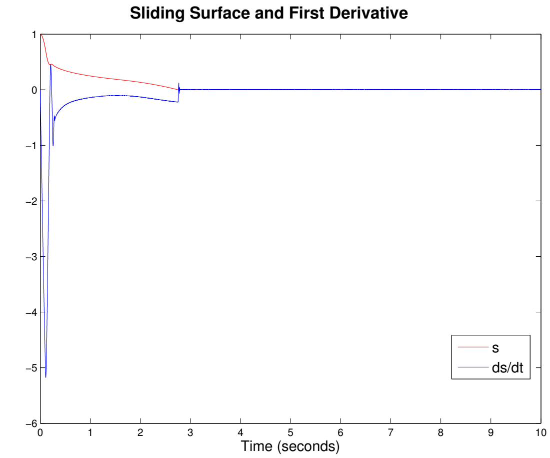

Fig. 3. Sliding surface and first derivative

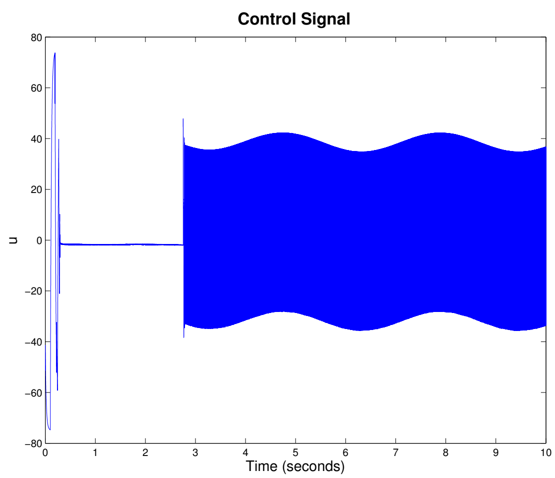

Fig. 4. Control Signal

Figure 4 shows the SOSM control action.

A zoom of the states shown in Figure 2 can be found in the upper image of Figure 5. Here it is more evident the attenuation of the disturbances. For comparison purposes an H_∞ controller was designed for the complete order system (13). The results of the states behavior can be seen on the lower image of Figure 5, where one can appreciate how a better attenuation was achieved by the combination with sliding modes.

VII. CONCLUSIONS

A generalized way of transforming a linear uncertain system with only part of the state available in the output, regardless of its relative degree, into a regular form was presented, as well as a procedure of obtaining a reduced order system that maintains controllability and observability properties for any relative degree and dimensions. A reduced order H_∞ controller was found that satisfies $||T_{Jw_1}||_\infty < \gamma$

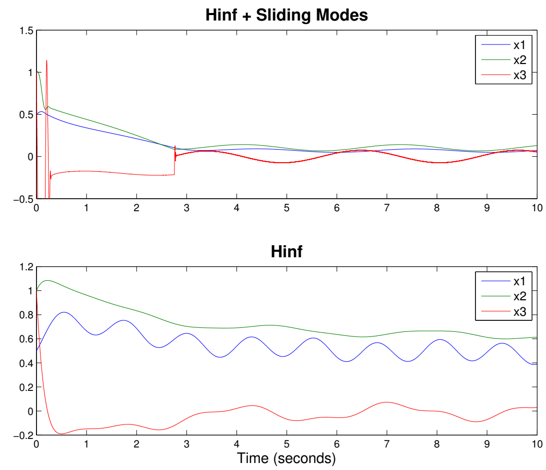

Fig. 5. Sliding modes with H_∞ and H_∞ only

being w_1 an unmatched disturbance. The minimization was guaranteed to be achieved in finite time through a relative degree $r_s = 2$ sliding surface and the sliding modes were enforced by a SOSM control law resulting in a closed loop that stabilizes the original, full order closed loop. A full measure of the state is not needed for the implementation of this strategy and it also overcomes the main restrictions of relative degree one and matchedness of the disturbances that output feedback sliding modes control strategies found in the literature impose over the system.

REFERENCES

[1] L. Aguilar, Y. Orlov, and J. C. Cadiou. Switched chattering control vs. backlash/friction phenomena in electrical servo-motors. *International Journal of Control*, 76(9-10):959–967, 2003.

[2] M. T. Angulo, L. Fridman, and A. Levant. Output-feedback finite-time stabilization of disturbed lti systems. *Automatica*, 48(4):606 – 611, 2012.

[3] M. T. Angulo, J.A. Moreno, and L. Fridman. The differentiation error of noisy signals using the generalized super-twisting differentiator. In *Decision and Control (CDC), 2012 IEEE 51st Annual Conference on*, pages 7383–7388, 2012.

[4] S.K. Bag, S.K. Spurgeon, and C. Edwards. Output feedback sliding mode design for linear uncertain systems. *Control Theory and Applications, IEE Proceedings* -, 144(3):209–216, 1997.

[5] G. Bartolini, A. Pisano, and E. Usai. A survey of applications of second order sliding mode control to mechanical systems. *International Journal of Control*, 76:875–892, 2003.

[6] F. Castaños and L. Fridman. Dynamic switching surfaces for output sliding mode control: An h_∞ approach. *Automatica*, 47(9):1957 – 1961, 2011.

[7] H.H. Choi. Sliding mode output feedback control design. *Transactions on Industrial Electronics*, 55(11), 2008.

[8] J. Davila. Exact tracking using backstepping control design and high-order sliding modes. *IEEE Transactions on Automatic Control*, page 452–457, 2013.

[9] J.C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis. State-space solutions to standard h2 and h infin; control problems. *IEEE Transactions on Automatic Control*, 34(8):831–847, 1989.

[10] A. Levant. Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, 58(6):1247–1263, 1993.

[11] A. Levant. Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control*, 76(9-10):924–941, 2003.

[12] A. Levant. Principles of 2-sliding mode design. *Automatica*, 43(4):576–586, 2007.
[13] A. Pisano, S. Scodina, and E. Usai. Load swing suppresion in the 3-d overhead crane via second-order sliding-modes. *Proc. 11th Int. Workshop Variable Struct. Syst*, page 452–457, 2010.
[14] A. Polyakov and A. Poznyak. Lyapunov function design for finite-time convergence analysis: ”twistingçontroller for second-order sliding mode realization. *Automatica*, 45(2):444–448, 2009.
[15] V. I. Utkin. *Sliding Modes in Electro Mechanical Systems*. Taylor and Francis, 1999.