Part III

Attractive Ellipsoid Method

- In the 3-rd part of this course, which includes the lectures 9-15, the methodology, providing a successful designing of feedbacks for tracking or stabilization of nonlinear systems in *presence of a sufficiently general type of uncertainties or disturbances,* is presented. Nonlinear uncertain systems, considered here, are governed by a vector Ordinary Differential Equation (ODE) with the, so-called, quasi-Lipschitz right-hand sides admitting a wide class of external and internal uncertainties (including discontinuous nonlinearities such as relay and hysteresis elements, time-delay blocks and so on).

- The class of stabilizing feedbacks are assumed to be of a *linear format* with gain parameters given by the corresponding BMI's (Bilinear Matrix Inequalities) or LMI's (Linear matrix Inequalities). The sufficient conditions guarantying the boundedness of all possible trajectories of controlled systems are presented.

- If the corresponding matrix inequilities are fulfilled, then one may guarantee that *all possible trajectories* of the considered class of controllable systems *are bounded*.

- Since any bounded dynamics may be imposed inside of some ellipsoid, then the *"best parameters" of the used feedback* are suggested to be associated with the minimal size of this ellipsoid.

- Unfortunately, this finite-dimensional optimization problem with matrix constrains can not be resolve analytically. Therefore, the associated *numerical procedure* is suggested for designing robust and adaptive-robust feedbacks for a wide class (quasi-Lipshitz) nonlinear uncertain systems.

- The considered control design includes

- state and output feedbacks,
- the full-order dynamic feedback,
- sample-data and quantized output feedbacks.

All these subclasses of possible feedback controllers are treated by the unuique methodology based on *Attractive Ellipsoid Method* (AEM). Several numerical and experimental illustrative examples are considered.

Lecture 9

Control Under Complete and Incomplete information

Sections below discuss the "attractive ellipsoid approach," a recently discovered robust-control design methodology for a broad class of continuous-time dynamic systems. Here we examine nonlinear affine control systems in the presence of uncertainty and offer a constructive and easily implementable control strategy that ensures certain stability features, as well as a coherent introduction to the suggested control design and associated issues. In the framework of the above-mentioned systems, linear-style feedback control synthesis is explored. It is addressed the creation of high-performance robust-feedback controllers that function in the lack of complete knowledge. The focus is on understanding and applying the theory to real-world problems, and on proving theorems in a methodical manner.

In this part we follow [1].

9.1 Complete Information Case: Classical Optimal Control

Optimal control is the rapidly expanding field developed during last half-century to analyze optimal behavior of a constrained process that evolves in time according to prescribed laws. Its applications now embrace a variety of new disciplines such as economics, production planning and etc. The main supposition of the *Classical Optimal Control Theory* (OCT) is that the mathematical technique, especially designed for analysis and synthesis of an optimal control of dynamic models, is based on the assumption that a designer (or an analyst) possesses *the complete information* on the considered model as well on an environment where this controlled model has to evolve.

There exist two principal approaches in solving *optimal control problems* in the presence of complete information on considered dynamic models:

- the first one is *Maximum Principle* (MP) of L.Pontryagin [2],

- and the second one is *Dynamic Programming Method* (DPM) of R. Bellman [3].

9.1.1 System description

Formally, the description of the optimal control problem in its classical form is as follows.

- *the controlled plant dynamics* is given by the system of Ordinary Differential Equations (ODE)

$$\left.\dot{x}(t) = f(x(t), u(t), t), \quad \text{a.e. } t \in [0, T], \; x(0) = x_0\right\}, \tag{9.1}$$

  where $x = (x^1, ..., x^n)^T \in \mathbb{R}^n$ is its state vector, $u = (u^1, ..., u^r)^T \in \mathbb{R}^r$ is the control that may run over a given control region $U \subset \mathbb{R}^r$;

- *the cost functional* is defined as

$$J(u(\cdot)) := h_0(x(T)) + \int_{t=0}^{T} h_1(x(t), u(t), t)\, dt, \tag{9.2}$$

  containing the integral term as well as terminal one, and the time process or *horizon* T is supposed to be fixed or non fixed and may be finite or infinite;

- *the terminal set* $\mathcal{M} \subseteq \mathbb{R}^n$ given by the inequalities

$$\mathcal{M} = \{x \in \mathbb{R}^n : g_l(x) \le 0 \quad (l = 1, ..., L)\}; \tag{9.3}$$

- the function (9.2) is said to be given in *Bolza form*. If in (9.2)

$$h_0(x) = 0$$

  we obtain the cost functional in the *Lagrange form*, that is,

$$J(u(\cdot)) = \int_{t=0}^{T} h_1(x(t), u(t), t)\, dt. \tag{9.4}$$

If in (9.2)

$$h_1(x, u, t) = 0,$$

we obtain the cost functional in the *Mayer form*, that is,

$$J(u(\cdot)) = h_0(x(T)). \tag{9.5}$$

9.1.2 Feasible and admissible control

A function $u(t)$, $t_0 \le t \le T$, is said to be an *feasible control* if it is measurable and $u(t) \in U$ for all $t \in [0, T]$. Denote the set of all feasible controls by

$$\mathcal{U}[0, T] := \{u(\cdot) : [0, T] \to U \mid u(t) \quad \text{is measurable}\}. \tag{9.6}$$

The control $u(t)$, $t_0 \le t \le T$ is also said to be *admissible* or, *realizing the terminal condition (9.3),* if the corresponding trajectory $x(t)$ satisfies the terminal condition, that is, satisfies the inclusion $x(T) \in \mathcal{M}$. Denote the set of all admissible controls by

$$\mathcal{U}_{admis}[0, T] := \{u(\cdot) : u(\cdot) \in \mathcal{U}[0, T], \; x(T) \in \mathcal{M}\}. \tag{9.7}$$

In view of the theorem on the existence of the solutions to ODE (see [4] or [5]), it follows that under the assumptions **(A1)-(A2)** for any $u(t) \in \mathcal{U}[0, T]$ the equation (9.1) admits a unique solution $x(\cdot) := x(\cdot, u(\cdot))$ and the functional (9.2) is well defined.

9.1.3 Problem setting in the general Bolza form

Based on the definitions given above, the classical optimal control problem (OCP) can be formulated as follows.

Problem 9.1 (OCP in Bolza form)

$$\textit{Minimize (9.2) over } \mathcal{U}_{admis}[0, T]. \tag{9.8}$$

Problem 9.2 (OCP with a fixed terminal term) *If in the problem (9.8)*

$$\left.\begin{array}{c} \mathcal{M} = \{x_f \in \mathbb{R}^n\} = \\ \{x \in \mathbb{R}^n : g_1(x) = x - x_f \le 0, \; g_2(x) = -(x - x_f) \le 0\} \\ \textit{or, equivalently, } x = x_f \,, \end{array}\right\} \tag{9.9}$$

*then it is called the **optimal control problem** with a fixed terminal term x_f.*

Any control $u^*(\cdot) \in \mathcal{U}_{admis}[0,T]$, satisfying

$$J(u^*(\cdot)) = \min_{u(\cdot)\in\mathcal{U}_{admis}[0,T]} J(u(\cdot)), \tag{9.10}$$

is called an *optimal control*, the corresponding state trajectory $x^*(\cdot) := x^*(\cdot, u^*(\cdot))$ and $(x^*(\cdot), u^*(\cdot))$ are called an *optimal state trajectory* and an *optimal pair*.

9.1.4 Specific features of the classical optimal control

The main specific features of both (MP and DPM) approaches are

1) the cost functional, defining the quality of the applied control, is given in the Bolza form (9.2) containing terminal term as well as the integral term, characterizing the losses of a designer during all time of the control process;

2) the function f in (9.1) is **a priory known** an may be used in the control designing process, in other words, the right-hand side of the dynamic equation (9.1) **does not contain any uncertainty or disturbances** which are unavailable during the control process;

3) the **state vector** $x(t)$ is assumed to be **available for control designing** on all time interval $[0,T]$.

The solution of the classical control problem can be found, for example, in [6]. **If one of this three features does not hold, then the classical Optimal Control approach is not applicable.**

9.2 Incomplete Information Case

When we don't have all of the information we need to manage a dynamic model, the key challenge is to build an acceptable control that is "near to optimal or desirable" (having a modest sensitivity to any unknown (unpredictable) component from a given range of options).To put it another way, the intended control should be robust (resilient) in the face of unknown factors. In the presence of any type of uncertainty (parametric, unmodeled dynamics, external perturbations, and so on), the main methodology for obtaining a solution suitable for a class of given models is to formulate a corresponding *tracking control problem*, where we are interested in the "best approaching" (a zone stabilization) or in the practical stability of a desired trajectory. For the past two decades, the *robust stabilization problem* has been a hot area of research for many types of nonlinear systems (see for example [7], [8], [9] and[10]).

9.3 Robust tracking problem formulation

Formally, the robust tracking problem can be described as follows.

- *The controlled plant dynamics,* which is affine (linear) in control , is given by

$$\left.\begin{array}{c}\dot{\bar{x}}(t)=\bar{f}(\bar{x}(t))+B\bar{u}(t)+\bar{\xi}_x(t), \quad \text{a.e. } t\in[0,T],\\ \bar{x}(0)=\bar{x}_0,\\ y(t)=\bar{h}(\bar{x}(t))+\xi_y(t),\end{array}\right\} \tag{9.11}$$

  where

  $\bar{x}=\left(\bar{x}^1,...,\bar{x}^n\right)^T\in\mathbb{R}^n$ is its state vector which may be unavailable during the plant dynamics,

  $\bar{u}=\left(\bar{u}^1,...,\bar{u}^r\right)^T\in\mathbb{R}^r$ is the control to be designed,

  $y=\left(y^1,...,y^m\right)^T\in\mathbb{R}^m$ is the measurable output of the system available for a designer at any time $t\geq 0$,

  the functions $\bar{\xi}_x(t)$ and $\xi_y(t)$ represent external perturbations which are not measurable (unavailable) for a designer.

- The *desired dynamics* $x^*(t)$ is governed by the following *reference model*

$$\dot{x}^*(t)=\varphi(x^*(t),t), \tag{9.12}$$

  where $x^*=\left(x^{*1},...,x^{*n}\right)^T\in\mathbb{R}^n$, and $x^*(t)$ is supposed to be measurable (available) at any time $t\geq 0$. The matrix $B\in\mathbb{R}^{n\times r}$ characterizing the actuator properties is also assumed to be known.

- The tracking error $x(t)$ is defined as

$$x(t)=\bar{x}(t)-x^*(t). \tag{9.13}$$

So, the ODE (Ordinary Differential Equation), describing the dynamics of the tracking error, is

$$\left.\begin{array}{c} \dot{x}(t) = f(x(t),t) + B\bar{u}(t) - \varphi(x^*(t),t) + \bar{\xi}_x(t) \\ \text{a.e. } t \in [0,T], \\ y(t) = h(x(t),t) + \xi_y(t), \end{array}\right\} \quad (9.14)$$

where

$$\left.\begin{array}{c} f(x(t),t) := f(x(t) + x^*(t)), \ x(0) = \bar{x}_0 + x^*(0), \\ \\ h(x(t),t) := \bar{h}(x(t) + x^*(t)). \end{array}\right\} \quad (9.15)$$

- The control action $\bar{u}$ usually constitutes of two terms:

$$\bar{u}(t) := u(t) + u_{comp}(t), \quad (9.16)$$

where the *compensating control* $u_{comp}(t)$ is selected in such a way that the effect of the dynamics $\varphi(x^*(t),t)$ of the desired trajectory would be compensated or minimized, namely,

$$\boxed{\begin{aligned} u_{comp}(t) &= \arg\min_{u_{comp}} \|Bu_{comp} - \varphi(x^*(t),t)\|^2 \\ &= B^+\varphi(x^*(t),t), \end{aligned}} \quad (9.17)$$

where

$$B^+ := (B^\intercal B)^{-1} B^\intercal,$$

if we assume that

$$B^\intercal B > 0.$$

If so, the model (9.14) may be represented as

$$\left.\begin{array}{c} \dot{x}(t) = f(x(t),t) + Bu(t) + \xi_x(t), \ \text{a.e. } t \in [0,T], \\ y(t) = h(x(t),t) + \xi_y(t). \end{array}\right\} \quad (9.18)$$

where

$$\xi_x(t) := \bar{\xi}_x(t) + \left(BB^+ - I\right)\varphi(x^*(t),t). \quad (9.19)$$

For the tracking error dynamics (9.18) the following assumptions usually are supposed:

(B1) The dynamic plant (9.18) is *controllable* and *observable* (see, for example, [11]).

(B2) The functions f and h may be unknown but belong to the *given classes* $\mathcal{C}_f$ and $\mathcal{C}_h$ of nonlinear functions, respectively. In this lecture both classes consist of the *quasi-Lipschitz functions* whose exact definition is given below.

(B3) The unmeasured functions $\xi_x(t)$ and $\xi_y(t)$ are *bounded,* but admitting the existence of the solution to the ODE (9.18).

(B4) The control $u(t)$ is designed as a feedback (static or dynamic) in a given structure containing the set of parameters $\mathcal{P}$, that is,

$$u(t) = u\left(y(\tau)\mid_{0\leq\tau\leq t}, t, \mathcal{P}\right), \tag{9.20}$$

so that $u(t)$ depends on all measurable data $y(\tau)\mid_{\tau\in[0,t]}$ on the time interval $[0, t]$.

9.4 What is effectiveness of a designed control in incomplete information case?

If we have the non-zero terms $\xi_x(t)$ (9.19) and $\xi_y(t)$ (9.18), which are unmeasurable during the control process, obviously, the application of the classical Optimal Control Approach (as it is describe above) is impossible. The situations looks much more difficult if the functions $f \in \mathcal{C}_f$ and $h \in \mathcal{C}_h$, describing the dynamic process, are unknown a priory. In this case the following questions seems to be important:

"*How can one describe the control issue in the unknown model situation, and how can we assess the efficacy of a given control strategy based on particular performances?*"

Several approaches may be considered in this situation:

- One of them suggests to formulate the corresponding control problem as the, so-called, *Min-Max Optimal Control,* where the *maximum* is taken over all existing uncertainties and *minimum* is realized within an admissible control set (see, for example, H^{∞} approach [12] and Robust Maximum Principle [6]). Such Min-Max consideration works successfully if the set of uncertain terms has a sufficiently simple structure (external perturbations are quadratically integrable, parametric uncertainties are from finite sets or belong to a measurable compact of a simple nature).

- Here we will discuss another approach referred below to as the *Attractive Ellipsoid Method (AEM)* [1] which turns out to be workable for significantly wide spectrum of uncertainties participating in a model description.

9.5 Ellipsoid Based Feedback Control Design

The main features of AEM are as follows:

- Since in the uncertain case the optimization of the cost functional (such as (9.2)) can not be realized exactly because of uncertain factor participation, the control problem is formulated as *a tracking problem*, which equivalently is reduced to the minimization of the vector-trajectory $x(t)$ (9.18) by an adequate selection of control strategies $u(t)$.

- The set of considered control strategies is suggested to belong to a *parametrized class of nonlinear (may be, nonstationary) feedbacks* (9.20)

  $$u(t) = u(y(\tau)|_{0\leq\tau\leq t}, t, \mathcal{P}),$$

  whose parameters $\mathcal{P}$ selected in such a way that all possible trajectories $x(t)$ of the closed controlled systems remain bounded around the origin (which approach is our "*ideal aim*").

- Taking into account that any set of bounded trajectories may be imposed within a convex bounded set and, particularly, within an ellipsoid, the AEM suggests to select the feed-back parameters $\mathcal{P} = \mathcal{P}^*$ providing a minimal "size" of the ellipsoid, which contains all possible bound trajectories of any dynamic system from the considered class of dynamics containing uncertain elements. In this case we talk about the, so-called, *zone-convergence* or on the "*practical stability*" (with a prescribed convex convergence zone) if the size of the convergence sone is of a predetermined value. That's why *the effectiveness of such robust control strategies is associated with the "size" of the corresponding attractive ellipsoid set.*

- During the control process these "optimal" parameters may be adjusted on-line (*learning* or *adaptive* version of AEM) making the attractive ellipsoid of a smaller size.

9.6 Example and Exercise

Example 9.1 *For the dynamic controlled system*

$$\frac{d}{dt}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} \bar{x}_1 + \operatorname{arctg}(\bar{x}_2) \\ -\bar{x}_1 - \sin\bar{x}_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{u}(t) + \bar{\xi}_x(t) \tag{9.21}$$

design the compensating control

$$u_{comp}(t) = arg \min_{u_{comp}} \|Bu_{comp} - \varphi(x^*(t), t)\|^2 = B^+\varphi(x^*(t), t),$$
$$\dot{x}^*(t) = \varphi(x^*(t), t)$$

in the tracking problem for the desired trajectory $x^*(t) \in \mathbb{R}^2$ *satisfying*

$$\ddot{x}^*(t) + \omega^2 x^*(t) = 0,\ \omega = 2,$$
$$x^*(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \dot{x}^*(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9.22}$$

Solution 9.1 *Let us represent (9.22) in the component-wise ordinari differential equations (ODE's):*

$$\ddot{x}_i^*(t) + \omega^2 x_i^*(t) = 0,\ i = 1, 2$$

which have the solution

$$x_i^*(t) = A_i \sin\omega t + B_i \cos\omega t,$$
$$\dot{x}_i^*(t) = A_i\omega \cos\omega t - B_i\omega \sin\omega t,$$
$$\ddot{x}_i^*(t) = -A_i\omega^2 \sin\omega t - B_i\omega^2 \cos\omega t.$$

Using the intial conditions in (9.22) we get

$$x_1^*(0) = A_1 \sin\omega 0 + B_1 \cos\omega 0 = B_1 = 1,$$
$$\dot{x}_1^*(0) = A_1\omega \cos\omega 0 - B_1\omega \sin\omega 0 = A_0\omega = 0$$
$$A_1 = 0,\ B_1 = 1$$

$$x_2^*(0) = A_2 \sin\omega 0 + B_2 \cos\omega 0 = B_2 = 1,$$
$$\dot{x}_2^*(0) = A_2\omega \cos\omega 0 - B_2\omega \sin\omega 0 = A_2\omega = 1$$
$$A_2 = \omega^{-1} = 0.5,\ B_2 = 1$$

and

$$\frac{d}{dt}\begin{pmatrix} x_1^*(t) \\ x_2^*(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0.5 \end{pmatrix} \omega \cos \omega t - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \omega \sin \omega t = \varphi(x^*(t), t)$$

We also have

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; B^\intercal B = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 > 0,$$
$$B^+ := (B^\intercal B)^{-1} B^\intercal = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Hence, by the formula (9.17) it follows

$$\boxed{\begin{gathered} u_{comp}(t) = B^+ \varphi(x^*(t), t) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1(x^*(t), t) \\ \varphi_2(x^*(t), t) \end{pmatrix} \\ = \varphi_1(x^*(t), t) = -\omega \sin \omega t = -2 \sin 2t. \end{gathered}}$$

Exercise 9.1 *For the same plant (9.21) design the compensating control* $u_{comp}(t)$ *in the tracking problem for the desired trajectory* $x^*(t) \in \mathbb{R}^2$ *satisfying*

$$\ddot{x}^*(t) + \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} x^*(t) = 0$$
$$x^*(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \dot{x}^*(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$