Lecture 8

H_∞ Control

8.1 The problem of perturbations attenuation in linear continuos-time systems

Consider the following linear plant

$$\boxed{\left.\begin{array}{c}\dot{x} = Ax + B_x u + \bar{D}_x \xi_x, \\ x(0) = x_0, \\ \\ y = C_y x + B_y u + \bar{D}_y \xi_y, \\ \\ z = C_z x + B_z u + \bar{D}_z \xi_z,\end{array}\right\}} \tag{8.1}$$

where

- $x \in \mathbb{R}^n$ is the state vector,
- $y \in \mathbb{R}^m$ is measurable output,
- $z \in \mathbb{R}^l$ is the controllable output,
- $u \in \mathbb{R}^k$ is the control action,
- $\xi_x \in \mathbb{R}^{\rho_x}$, $\xi_y \in \mathbb{R}^{\rho_y}$, $\xi_z \in \mathbb{R}^{\rho_z}$ are unmeasurable perturbations with bounded energy, that is,

$$\xi_x, \xi_y, \xi_z \in L_2[0,\infty) := \left\{ v(t), t \geq 0 : \int_{t=0}^{\infty} \|v(t)\|^2 \, dt < \infty \right\}. \tag{8.2}$$

Remark 8.1 *The property (8.2) means that all these perturbations are decreasing in time, but may have some bounded spikes on integrable intervals.*

Introduce the extended vector of perturbations

$$\xi := \begin{pmatrix} \xi_x \\ \xi_y \\ \xi_z \end{pmatrix} \in \mathbb{R}^\rho, \ \rho = \rho_x + \rho_y + \rho_z.$$

In view of this definition it is easy to check that the following relations take place:

$$\bar{D}_x \xi_x = \bar{D}_x H_x \xi = D_x \xi, \ H_x := \begin{pmatrix} I_{\rho_x \times \rho_x} & 0 & 0 \end{pmatrix}, \ D_x = \bar{D}_x H_x,$$

$$\bar{D}_y \xi_y = \bar{D}_y H_y \xi = D_y \xi, \ H_y := \begin{pmatrix} 0 & I_{\rho_y \times \rho_y} & 0 \end{pmatrix}, \ D_y = \bar{D}_y H_y$$

$$\bar{D}_z \xi_z = \bar{D}_z H_z \xi = D_z \xi, \ H_z := \begin{pmatrix} 0 & 0 & I_{\rho_z \times \rho_z} \end{pmatrix}, \ D_z = \bar{D}_z H_z,$$

and therefore the linear plant (8.1) can be rewritten as

$$\boxed{\left.\begin{array}{c} \dot{x} = Ax + B_x u + D_x \xi, \\ \\ y = C_y x + B_y u + D_y \xi, \\ \\ z = C_z x + B_z u + D_z \xi, \end{array}\right\}} \tag{8.3}$$

where the perturbation ξ has a bounded energy, i.e.,

$$\xi \in L_2 [0, \infty) .$$

Consider the following control problem.

Problem 8.1 (Attenuation of unmeasurable perturbations) *Design a dynamic feedback control $u(t)$ in the form*

$$\boxed{\left.\begin{array}{c} u(t) = C_r x_r(t) + D_r y(t), \\ \\ \dot{x}_r(t) = A_r x_r(t) + B_r y(t), \\ \\ x_r(0) = x_0^r, \end{array}\right\}} \tag{8.4}$$

which guarantees the given γ - ***attenuation level****, namely, providing the* *fulfilling of the inequality*

$$\boxed{\left.\begin{array}{c} \sup\limits_{\xi\in L_2[0,\infty)} \dfrac{\|z\|_{L_2}}{\|\xi\|_{L_2}} < \gamma \\ \\ \|v\|_{L_2} := \sqrt{\int\limits_{t=0}^{\infty} \|v(t)\|^2 \, dt} \end{array}\right\}} \tag{8.5}$$

ensuring that the worst "output - noise" ratio does not exceed the prespecified attenuation level γ.

8.2 H_∞ interpretation

8.2.1 Transfer functions

Let us apply the Laplace transformation to the plant and control equations (8.3) and (8.4). We get

$$\left.\begin{array}{r} sX = AX + B_x U + D_x \Xi, \\ \\ Y = C_y X + B_y U + D_y \Xi, \\ \\ Z = C_z X + B_z U + D_z \Xi, \end{array}\right\} \tag{8.6}$$

and for the control U, linearly depending on the output $Y$and and some auxilary dynamic signal X_r,

$$\left.\begin{array}{r} U = C_r X_r + D_r Y, \\ \\ sX_r = A_r X_r + B_r Y, \end{array}\right\} \tag{8.7}$$

it follows

$$U = \left[C_r (sI - A_r)^{-1} B_r + D_r\right] Y,$$

and

$$\left.\begin{array}{c} X = (sI - A)^{-1} B_x U + (sI - A)^{-1} D_x \Xi = \\ \\ (sI - A)^{-1} B_x \left[C_r (sI - A_r)^{-1} B_r + D_r\right] Y + (sI - A)^{-1} D_x \Xi, \end{array}\right\} \tag{8.8}$$

$$\left.\begin{aligned} Y &= \left[C_y (sI - A)^{-1} B_x + B_y\right] U + \left[C_y (sI - A)^{-1} D_x + D_y\right] \Xi \\ &= \left[C_y (sI - A)^{-1} B_x + B_y\right] \left[C_r (sI - A_r)^{-1} B_r + D_r\right] Y \\ &\qquad + \left[C_y (sI - A)^{-1} D_x + D_y\right] \Xi \end{aligned}\right\} \tag{8.9}$$

and

$$Z = \left[C_z (sI - A)^{-1} B_x + B_z\right] U + \left[C_z (sI - A)^{-1} D_x + D_z\right] \Xi. \tag{8.10}$$

From these representations we obtain

$$\left.\begin{gathered} Y = H_{\xi y} \Xi, \\ H_{\xi y} := \left(I - \left[C_y (sI - A)^{-1} B_x + B_y\right] \left[C_r (sI - A_r)^{-1} B_r + D_r\right]\right)^{-1} \times \\ \left[C_y (sI - A)^{-1} D_x + D_y\right], \end{gathered}\right\}$$

$$\left.\begin{gathered} U = H_{\xi u} \Xi, \\ H_{\xi u} := \left[C_r (sI - A_r)^{-1} B_r + D_r\right] H_{\xi y} = \\ \left(\left[C_r (sI - A_r)^{-1} B_r + D_r\right]^{-1} - \left[C_y (sI - A)^{-1} B_x + B_y\right]\right)^{-1} \times \\ \left[C_y (sI - A)^{-1} D_x + D_y\right] \end{gathered}\right\}$$

and

$$\boxed{\left.\begin{gathered} Z = H_{\xi z} \Xi \\ H_{\xi z} := \left[C_z (sI - A)^{-1} B_x + B_z\right] H_{\xi u} + \\ \left[C_z (sI - A)^{-1} D_x + D_z\right]. \end{gathered}\right\}} \tag{8.11}$$

8.2.2 Laplace transformation and $\mathbb{H}_\infty$ norm

Recall now one of the most important results of the Laplace transformation theory.

Theorem 8.1 (Plancherel, around 1800.) *If $f(t) \in L_2[0,\infty)$ and its* ***Laplace transformation*** *is $F(p) \in \mathbb{H}_2$, where $\mathbb{H}_2$ is the* ***Hardy space*** *defined by*

$$\mathbb{H}_2 := \left\{ F(p) = \mathcal{L}\{f\} := \int\limits_{t=0}^{\infty} f(t)\, e^{-pt} dt \mid \int\limits_{\omega=-\infty}^{\infty} F(-j\omega)\, F(j\omega)\, d\omega < \infty \right\},$$

then the following identity (known as the ***Parseval's identity*** *) holds:*

$$\boxed{\left.\begin{array}{c} \|f\|_{L_2} := \left(\int\limits_{t=0}^{\infty} |f(t)|^2 dt \right)^{1/2} = \|F\|_{\mathbb{H}_2} \\ \text{where} \\ \|F\|_{\mathbb{H}_2} := \left(\frac{1}{2\pi} \int\limits_{\omega=-\infty}^{\infty} F(-j\omega)\, F(j\omega)\, d\omega \right)^{1/2}. \end{array}\right\}} \tag{8.12}$$

Corollary 8.1 *In the vector case when $f(t) \in \mathbb{R}^n$ the* ***Parseval's identity*** *looks as follows:*

$$\boxed{\left.\begin{array}{c} \|f\|_{L_2} := \left(\int\limits_{t=0}^{\infty} \|f(t)\|^2 dt \right)^{1/2} = \\ \\ \|F\|_{\mathbb{H}_2} := \left(\frac{1}{2\pi} \int\limits_{\omega=-\infty}^{\infty} F^{\intercal}(-j\omega)\, F(j\omega)\, d\omega \right)^{1/2} \end{array}\right\}} \tag{8.13}$$

By the Parseval's identity (8.13) and in view of (8.11) the inequality (8.5) can be represented as

$$\boxed{\sup_{\xi \in L_2[0,\infty)} \frac{\|z\|_{L_2}}{\|\xi\|_{L_2}} = \sup_{\Xi \in \mathbb{H}_2} \frac{\|Z\|_{\mathbb{H}_2}}{\|\Xi\|_{\mathbb{H}_2}} = \sup_{\Xi \in \mathbb{H}_2} \frac{\|H_{\xi z} \Xi\|_{\mathbb{H}_2}}{\|\Xi\|_{\mathbb{H}_2}} < \gamma.} \tag{8.14}$$

Definition 8.1 *The norm $\|H_{\xi z}\|_{\mathbb{H}_\infty}$ of transfer matrix function $H_{\xi z}$ in the Hardy space $\mathbb{H}_\infty$ is defined as*

$$\boxed{\|H_{\xi z}\|_{\mathbb{H}_\infty} := \sup_{\Xi \in \mathbb{H}_2} \frac{\|H_{\xi z} \Xi\|_{\mathbb{H}_2}}{\|\Xi\|_{\mathbb{H}_2}}.}$$

That's why, the inequality (8.14) is equivalent to the following one:

$$\boxed{\|H_{\xi z}\|_{\mathbb{H}_\infty} < \gamma.} \tag{8.15}$$

The $\mathbb{H}_\infty$ norm of complex-valued matrix may be calculated based on the following lemma.

Lemma 8.1

$$\|H_{\xi z}\|_{\mathbb{H}_\infty} = \sup_{\omega \in (-\infty,\infty)} \|H_{\xi z}(j\omega)\| = \sup_{\omega \in (-\infty,\infty)} \max_i \sigma_i (H_{\xi z}(j\omega))$$

where

$$\|H_{\xi z}(j\omega)\| = \max_i \sigma_i (H_{\xi z}(j\omega))$$

and

$$\sigma_i(H) := \lambda_i^{1/2} \left(H_{\xi z} H_{\xi z}^{\sim} \right) = \lambda_i^{1/2} \left(H_{\xi z}(j\omega) H_{\xi z}^{\intercal}(-j\omega) \right),$$

$$H_{\xi z}^{\sim} := H_{\xi z}^{\intercal}(-j\omega)$$

($\sigma_i(H)$ is the singular value of the matrix $H_{\xi z}(j\omega)$).

Since

$$H_{\xi z}^{\intercal}(-j\omega) H_{\xi z}(j\omega) \le \max_i \sigma_i^2 (H_{\xi z}(j\omega)) = \|H_{\xi z}(j\omega)\|^2 I$$

$$\le \sup_{\omega \in (-\infty,\infty)} \|H_{\xi z}(j\omega)\|^2 I = \|H_{\xi z}\|_{\mathbb{H}_\infty}^2 I,$$

the properties (8.14)-(8.15) are equivalent to the following matrix inequality in the frequence domain:

$$\boxed{H_{\xi z}^{\intercal}(-j\omega) H_{\xi z}(j\omega) < \gamma^2 I} \tag{8.16}$$

valid for all $\omega \in (-\infty, \infty)$.

8.2.3 Problem formulation in the Hardy space $\mathbb{H}_\infty$

Now the original control problem (8.5) can be formulated in the frequency space $\mathbb{H}_\infty$.

Problem 8.2 *Design a feedback control in the form (8.4) which guarantees the given γ -* ***attenuation level*** *providing the fulfilling of the frequency matrix inequality (8.16) for all $\omega \in (-\infty, \infty)$.*

8.3 The Kalman-Yakubovich-Popov lemma

8.3.1 KYP - lemma

Consider the *Linear Time - Invariant* (LTI) system

$$\left.\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \\ A \in R^{n\times n},\ B &\in R^{n\times m},\ C \in R^{l\times n}, \end{aligned}\right\} \tag{8.17}$$

where for any fixed $t \geq 0$ and some fixed initial value $x(0) = x_0$ the vectors $x = x(t)$, $u = u(t)$ and $y = y(t)$ are referred to as the state, control (external input) and output respectively. Applying the Fourier transformation to (8.17) we obtain the model of the system in the frequency domain:

$$\left.\begin{aligned} i\omega X &= AX + BU,\ i^2 = -1, \\ Y &= CX. \end{aligned}\right\} \tag{8.18}$$

The transfer function $H_{uy}(i\omega)$ from the input u to the output y is

$$\boxed{H_{uy}(i\omega) = C\left(i\omega I_{n\times n} - A\right)^{-1} B.} \tag{8.19}$$

The following assumptions will be in force hereafter:

A1) Here we suppose that the matrix A *has no eigenvalues at the imaginary axis.*

A2) The pair (A, B) is *stabilizable*, i.e. there exists a matrix $K \in R^{m\times n}$ such that the matrix $(A + BK)$ is Hurwitz (stable).

Lemma 8.2 (The KYP frequency lemma, 1973) *Let the assumptions **A1** and **A2** are met. To guarantee the existance of a real symmetric matrix $P = P^{\mathsf{T}}$ satisfying the the inequality*

$$\boxed{2\,\mathrm{Re}\, X^* P(AX + BU) - \mathcal{L}(X, U) < 0} \tag{8.20}$$

for all $\|X\| + \|U\| > 0$ *and for a given hermitian quadratic form*

$$\boxed{\left.\begin{array}{c} \mathcal{L}(X,U) := \begin{pmatrix} X \\ U \end{pmatrix}^* \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{pmatrix} X \\ U \end{pmatrix} = \\ \\ X^*\mathcal{L}_{11}X + 2\operatorname{Re} X^*\mathcal{L}_{12}U + U^*\mathcal{L}_{11}U, \\ \\ \mathcal{L}_{11} = \mathcal{L}_{11}^\intercal,\ \mathcal{L}_{22} = \mathcal{L}_{22}^\intercal,\ \mathcal{L}_{21} = \mathcal{L}_{12}^\intercal \text{ - real matrices} \end{array}\right\}} \tag{8.21}$$

it is neccessary and sufficient the fulfilling of the following ***frequency condition***

$$\boxed{\mathcal{L}([j\omega I - A]^{-1} BU, U) > 0} \tag{8.22}$$

for all $U \neq 0$ *and all* $\omega \in (-\infty, \infty)$.

Proof.

a) *Necessity.* Suppose that (8.20) holds. Then

$$2\operatorname{Re} X^*P(AX + BU) < \mathcal{L}(X,U)$$

for all $\|X\| + \|U\| > 0$. Then for $X = (i\omega I_{n\times n} - A)^{-1} B$ in view of the relation (8.18) we have

$$2\operatorname{Re} i\omega X^*PX = 0 < \mathcal{L}((i\omega I_{n\times n} - A)^{-1} B, U)$$

and the necessity trivially follows. Notice also that from the last inequality it follows that

$$\mathcal{L}_{22} = \mathcal{L}_{22}^\intercal > 0.$$

b) *Sufficiency.* First, let us show that the Hermitian form $\mathcal{L}(X,U)$ can be represented as

$$\mathcal{L}(X,U) = 2\operatorname{Re} X^*P(AX + BU) + (U - H^\intercal X)^* \mathcal{L}_{22} (U - H^\intercal X) \tag{8.23}$$

with real matrix H. To do that it sufficient to open both quadratic form in the left and right hand sides and to equal the corresponding parameters:

$$X^*\mathcal{L}_{11}X + 2\operatorname{Re} X^*\mathcal{L}_{12}U + U^*\mathcal{L}_{11}U =$$

$$2\operatorname{Re} X^*PAX + 2\operatorname{Re} X^*PBU + U^*\mathcal{L}_{22}U -$$

$$X^*H\mathcal{L}_{22}U - U^*\mathcal{L}_{22}H^\intercal X + X^*H\mathcal{L}_{22}H^\intercal X =$$

$$X^*\left(PA + A^\intercal P + H\mathcal{L}_{22}H^\intercal\right)X + X^*\left(PB - H\mathcal{L}_{22}\right)U +$$

$$U^*\left(B^\intercal P - \mathcal{L}_{22}H^\intercal\right)X + U^*\mathcal{L}_{22}U$$

and

$$\boxed{\left.\begin{array}{c} PA + A^{\intercal}P + H\mathcal{L}_{22}H^{\intercal} = \mathcal{L}_{11}, \\ \\ PB - H\mathcal{L}_{22} = \mathcal{L}_{12}, \\ \\ \mathcal{L}_{11} = \mathcal{L}_{22}. \end{array}\right\}} \tag{8.24}$$

The algebraic relations (8.24) is referred to as the *Lurie's equations*. Since $\mathcal{L}_{22} > 0$, from the second equation in (8.24) we get

$$H = (PB - \mathcal{L}_{12})\,\mathcal{L}_{22}^{-1}. \tag{8.25}$$

Substitution this representation of matrix H in the first equation in (8.24) we finally get

$$PA_0 + A_0^{\intercal}P + PRP + Q = 0 \tag{8.26}$$

where

$$A_0 := A - B\mathcal{L}_{22}^{-1}\mathcal{L}_{12}^{\intercal},$$

$$R := B\mathcal{L}_{22}^{-1}B^{\intercal},$$

$$Q := \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{12}^{\intercal} - \mathcal{L}_{11}.$$

But, according to Theorem 10.4 in [14] the symmetric solution P of the algebraic Riccati matrix equation (8.26) exists in view of the assumptions A1 and A2. Indeed, the assumptions there require the stabilizability of the pare (A_0, B), and our assumption A2 deals with the stabilizability of the pare (A, B). But this is sufficient, since the matrix $K_0 = K - \mathcal{L}_{22}^{-1}\mathcal{L}_{12}^{\intercal}$ provides the stability (the Hurwitz property) of the matrix

$$A_0 - BK_0 = A - B\mathcal{L}_{22}^{-1}\mathcal{L}_{12}^{\intercal} - B\left(K - \mathcal{L}_{22}^{-1}\mathcal{L}_{12}^{\intercal}\right) = A - BK$$

if $(A - BK)$ is Hurwitz one. Hence, by (8.25) there exists the matrix H satisfying (8.23). So, if (8.22) holds, then from (8.23) (since $\mathcal{L}_{22} = \mathcal{L}_{22}^{\intercal} > 0$) we have

$$\mathcal{L}(X, U) - 2\,\mathrm{Re}\,X^* P(AX + BU) = (U - H^{\intercal}X)^*\,\mathcal{L}_{22}\,(U - H^{\intercal}X) > 0.$$

Suffiiency is proven. ■

Corollary 8.2 *By the Schur's complement lemma the inequality (8.20) is equivalent to the following LMI*

$$\boxed{\begin{bmatrix} PA + A^{\intercal}P - \mathcal{L}_{11} & PB - \mathcal{L}_{12} \\ B^{\intercal}P - \mathcal{L}_{21} & -\mathcal{L}_{22} \end{bmatrix} < 0,} \tag{8.27}$$

and the inequality (8.22) looks as

$$\boxed{\begin{bmatrix} [-j\omega I - A]^{-1} B \\ I \end{bmatrix}^{\intercal} \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} [j\omega I - A]^{-1} B \\ I \end{bmatrix} > 0.} \tag{8.28}$$

8.4 LMI representation of the perturbations attenuation problem

Consider the simplified version of the system (8.3) with $B_y = 0$, $D_y = 0$ and $B_z = 0$:

$$\boxed{\left.\begin{aligned} \dot{x} &= Ax + B_x u + D_x \xi, \\ y &= C_y x, \\ z &= C_z x + D_z \xi. \end{aligned}\right\}} \tag{8.29}$$

Together with the dynamic controller (8.4) for the extended vector $\tilde{x} = \begin{pmatrix} x \\ x_r \end{pmatrix}$ we have

$$\left.\begin{aligned} \frac{d}{dt}\tilde{x} &= A_{cl}\tilde{x} + D_{cl}\xi, \\ z &= C_{cl}\tilde{x} + D_z\xi, \end{aligned}\right\} \tag{8.30}$$

where

$$A_{cl} = A_0 + B_0 \Theta C_0,$$

$$A_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \; B_0 = \begin{bmatrix} B & 0 \\ 0 & I_{n\times n} \end{bmatrix},$$

$$C_0 = \begin{bmatrix} C & 0 \\ 0 & I_{n\times n} \end{bmatrix}, \; D_{cl} = \begin{bmatrix} D_x \\ 0 \end{bmatrix},$$

$$C_{cl} = C_z \begin{bmatrix} I_{n\times n} & 0_{n\times n} \end{bmatrix}$$

and

$$\Theta = \begin{bmatrix} D_r & C_r \\ B_r & A_r \end{bmatrix}.$$

The Laplace transformation of (8.30) is

$$s\tilde{X} = A_{cl}\tilde{X} + D_{cl}\Xi,$$

$$Z = C_{cl}\tilde{X} + D_z\Xi,$$

which leads to the following expression of the transfer matrix $H_{\xi z}$:

$$\boxed{H_{\xi\tilde{x}}(i\omega) = C_{cl}\left[sI_{2n\times 2n} - A_{cl}\right]^{-1} D_{cl} + D_z.} \tag{8.31}$$

Let us select the hermitian form $\mathcal{L}(Z,\Xi)$ as

$$\begin{aligned}\mathcal{L}(Z,\Xi) &:= \gamma^2\Xi^*\Xi - Z^*Z = \gamma^2\Xi^*\Xi - \left(C_{cl}\tilde{X} + D_z\Xi\right)^*\left(C_{cl}\tilde{X} + D_z\Xi\right)\\ &= \begin{pmatrix}\tilde{X}\\ \Xi\end{pmatrix}^* \begin{bmatrix} -C_{cl}^{\intercal}C_{cl} & -C_{cl}^{\intercal}D_z\\ -D_z^{\intercal}C_{cl} & \gamma^2 I - D_z^{\intercal}D_z\end{bmatrix}\begin{pmatrix}\tilde{X}\\ \Xi\end{pmatrix}\\ &= \begin{pmatrix}\tilde{X}\\ \Xi\end{pmatrix}^* \begin{bmatrix}\mathcal{L}_{11} & \mathcal{L}_{12}\\ \mathcal{L}_{21} & \mathcal{L}_{22}\end{bmatrix}\begin{pmatrix}\tilde{X}\\ \Xi\end{pmatrix}\end{aligned} \tag{8.32}$$

with

$$\boxed{\left.\begin{array}{c}\mathcal{L}_{11} = -C_{cl}^{\intercal}C_{cl},\ \mathcal{L}_{12} = -C_{cl}^{\intercal}D_z,\\[2ex] \mathcal{L}_{21} = -D_z^{\intercal}C_{cl},\ \ \mathcal{L}_{22} = \gamma^2 I - D_z^{\intercal}D_z.\end{array}\right\}} \tag{8.33}$$

Show now that the inequality (8.28) coincides with (8.16) where hermitian form $\mathcal{L}(Z,\Xi)$ as in (8.32).

Theorem 8.2 *If*
1) the matrix $[i\omega I_{2n\times 2n} - A_{cl}]$ is non-singular,
2) the pare (A_{cl}, D_{cl}) is stabilizable,
then for all $\omega \in (-\infty,\infty)$

$$\boxed{\mathcal{L}(\left[i\omega I_{2n\times 2n} - A_{cl}\right]^{-1} D_{cl}, I) = \gamma^2 I - H_{\xi\tilde{x}}^{\intercal}(-i\omega)\, H_{\xi\tilde{x}}(i\omega) > 0.} \tag{8.34}$$

Proof. Defining

$$S := \left[i\omega I_{2n\times 2n} - A_{cl}\right]^{-1} D_{cl},$$

we get

$$0 < \mathcal{L}([i\omega I_{2n\times 2n} - A_{cl}]^{-1} D_{cl}, I) =$$

$$\begin{bmatrix} S \\ I \end{bmatrix}^* \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} S \\ I \end{bmatrix} =$$

$$\begin{bmatrix} S \\ I \end{bmatrix}^* \begin{bmatrix} -C_{cl}^{\intercal} C_{cl} & -C_{cl}^{\intercal} D_z \\ -D_z^{\intercal} C_{cl} & \gamma^2 I - D_z^{\intercal} D_z \end{bmatrix} \begin{bmatrix} S \\ I \end{bmatrix}^* =$$

$$\begin{bmatrix} S \\ I \end{bmatrix}^* \begin{bmatrix} -C_{cl}^{\intercal} C_{cl} [i\omega I_{2n\times 2n} - A_{cl}]^{-1} D_{cl} - C_{cl}^{\intercal} D_z \\ -D_z^{\intercal} C_{cl} [i\omega I_{2n\times 2n} - A_{cl}]^{-1} D_{cl} + \gamma^2 I - D_z^{\intercal} D_z \end{bmatrix}$$

$$= \begin{bmatrix} S \\ I \end{bmatrix}^* \begin{bmatrix} -C_{cl}^{\intercal} H_{\xi\tilde{x}}(i\omega) \\ -D_z^{\intercal} H_{\xi\tilde{x}}(i\omega) + \gamma^2 I \end{bmatrix}$$

$$= -\left[[-i\omega I_{2n\times 2n} - A_{cl}]^{-1} D_{cl}\right]^{\intercal} C_{cl}^{\intercal} H_{\xi\tilde{x}}(i\omega) - D_z^{\intercal} H_{\xi\tilde{x}}(i\omega) + \gamma^2 I$$

$$= -H_{\xi\tilde{x}}^{\intercal}(-i\omega) H_{\xi\tilde{x}}(i\omega) + \gamma^2 I.$$

Theorem is proven. ■

By the Corrolary 8.2, the condition (8.34)

$$\mathcal{L}([j\omega I - A_{cl}]^{-1} D_{cl} U, U) > 0$$

guarantees the desired tolerance level γ (8.16)

$$H_{\xi z}^{\intercal}(-j\omega) H_{\xi z}(j\omega) < \gamma^2 I,$$

and is equivalent to the following LMI

$$\boxed{\begin{bmatrix} P A_{cl}(\Theta) + A_{cl}^{\intercal}(\Theta) P + C_{cl}^{\intercal} C_{cl} & P D_{cl} + C_{cl}^{\intercal} D_z \\ D_{cl}^{\intercal} P + D_z^{\intercal} C_{cl} & D_z^{\intercal} D_z - \gamma^2 I \end{bmatrix} < 0.} \tag{8.35}$$

This means that any parameter Θ, participating in

$$A_{cl}(\Theta) = A_0 + B_0 \Theta C_0$$

and satisfying LMI (8.35) for some symmetric matrix P and a scalar γ, solves the perturbations attenuation problem (8.16) with the tolerance level γ.

Remark 8.2 *The explicit relationship between sum of squares (SOS) decompositions of univariate polynomial matrices and the Kalman-Yakubovich-Popov (KYP) lemma can be found in [17]. There an efficient algorithm for explicitly finding an SOS decomposition of such matrices, inspired by the Hamiltonian-type methods for the solution of Riccati equations, is presented.*

8.5 Exercise

Exercise 8.1 *Design the dynamic feedback controller*

$$\Theta = \begin{bmatrix} D_r & C_r \\ B_r & A_r \end{bmatrix},$$

providing the attenuation tolerance level $\gamma = 0.5$, *fulfilling the estimate*

$$H_{\xi\tilde{x}}^{\intercal}(-i\omega)\, H_{\xi\tilde{x}}(i\omega) < \gamma^2 I,$$

for the system

$$\left.\begin{array}{c} \dot{x} = \begin{bmatrix} 1 & -0.1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0.1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xi, \\ \\ y = \begin{bmatrix} 1 & 1 \end{bmatrix} x, \;\; z = x + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \xi, \\ \\ x \in \mathbb{R}^2, \;\; u \in \mathbb{R}^1, \;\; y \in \mathbb{R}^1, \;\; \xi = \begin{bmatrix} 0.1e^{-0.01t}\sin(10t) \\ -0.1e^{-0.01t}\cos(2t) \end{bmatrix}. \end{array}\right\}$$

Hint. *To find* $P > 0$ *and* Θ *for which the LMI (8.35)*

$$\begin{bmatrix} PA_{cl}(\Theta) + A_{cl}^{\intercal}(\Theta)P + C_{cl}^{\intercal}C_{cl} & PD_{cl} + C_{cl}^{\intercal}D_z \\ D_{cl}^{\intercal}P + D_z^{\intercal}C_{cl} & D_z^{\intercal}D_z - \gamma^2 I \end{bmatrix} < 0$$

with $\gamma = 0.5$ *is fuldilled.*

References to Part II

1. A. I. Luré, Some Non-linear Problems in the Theory of Automatic Control, H. M. Stationery Office, London, 1957.

2. V. A. Yakubovich, G. A. Leonov, and A. Kh. Gelig, Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities, vol. 14 of Series on Stability, Vibration and Control of Systems. Series A: Textbooks, Monographs and Treatises, World Scientific Publishing Co., Inc., River Edge, NJ, 2004.

3. M. R. Liberzon, "Essays on the absolute stability theory," Automation and Remote Control, vol. 67, no. 10, pp. 1610–1644, 2006.

4. V. A. Yakubovich, "The solution of certain matrix inequalities in automatic control theory," Soviet Mathematics - Doklady, vol. 3, no. 2, pp. 620–623, 1962.

5. S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, Pa, USA, 1994.

6. V. M. Popov, "Hyperstability and optimality of automatic systems with several control functions," Revue roumain des sci. techn., sér. électrotechn. et énergét, vol. 9, no. 4, pp. 629–690, 1964.

7. V. A. Yakubovich, "Frequency domain conditions for absolute stability of control systems with several nonlinear or linear nonstationary blocks," Automation and Remote Control, no. 6, pp. 857–880, 1967.

8. V. M. Popov, Hyperstability of Control Systems, Springer, New York, NY, USA, 1973.

9. V. A. Yakubovich, "The S-procedure in nonlinear control theory," Vestnik Leningrad University. Mathematics, vol. 4, pp. 62–77, 1971.

10. A. L. Fradkov and V. A. Yakubovich, "The S-procedure and duality relations in nonconvex problems of quadratic programming," Vestnik Leningrad University. Mathematics, vol. 6, pp. 101–109, 1979.

11. V. A. Yakubovich, "On impulsive control systems with a pulse width modulation," Doklady Akademii Nauk SSSR, vol. 180, pp. 283–285, 1968 (in Russian).

12. A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," IEEE Transactions on Automatic Control, vol. 42, no. 6, pp. 819–830, 1997.

13. V. Pliss, Nonlocal Problems of Oscillation Theory, Nauka, Moscow-Leningrad (in Russian), 1964.

14. Poznyak A. (2008). Advanced mathematical tools for automatic control engineers: Deterministic techniques. V.1, Amsterdam: Elsevier.

15. Lurie A. and Postnikov V. , "To the theory of control systems", Problems of Mathematics and Mechanics (PMM), 8(3), 1944.

16. Gelig, A., Leonov, G. and Yakubovich, V. (1978), Stability of Nonlinear Systems with Nonunique Steady State, Nauka, Moscow, Russia (in Russian).

17. Aylward, Erin M., Sleiman M. Itani, and Pablo A. Parrilo (2007), Explicit SOS decompositions of univariate polynomial matrices and the Kalman-Yakubovich-Popov lemma, 2007 46th IEEE Conference on Decision and Control, IEEE.