Lecture 7

Analysis of Absolute Global Stability in Frequency-Domain

7.1 On equivalency of Hermitian and quadratic forms

Let $A \in \mathbb{C}^{n \times n}$ be an *Hermitian* matrix so that

$$A = A^* := \left(\bar{A}\right)^{\intercal},$$

$$A = U + iV, \; \bar{A} := U + iV$$

and z be a conlex vector:

$$z = u + iv \in \mathbb{C}^n, \; i^2 = -1.$$

Definition 7.1 *The function*

$$\boxed{f_A(z) := (z, Az) = z^* Az = (u - iv)^{\intercal} A (u + iv)} \tag{7.1}$$

with the Hermitian matrix A is called

- *the **Hermitian form**,*
- *converting to the **quadratic form**, if $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix $(A = A^{\intercal})$ and $z = u \in \mathbb{R}^n$.*

If $\mathcal{E} = \left\{ z^{(1)}, ..., z^{(n)} \right\}$ is a basic in $\mathbb{C}^n$ such that

$$z = \sum_{i=1}^{n} \alpha_i z^{(i)},$$

then $f_A(x)$ may be represented as

$$\left.\begin{array}{c} f_A(z) := (z, Az) = \left(\sum_{j=1}^{n} \alpha_j z^{(j)}, A \sum_{i=1}^{n} \alpha_i z^{(i)} \right) = \\ \left(\sum_{j=1}^{n} \alpha_j z^{(j)}, \sum_{i=1}^{n} \alpha_i A z^{(i)} \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} \alpha_i \bar{\alpha}_j, \end{array}\right\} \tag{7.2}$$

where

$$\gamma_{ij} = \left(z^{(j)}, Az^{(i)} \right).$$

Proposition 7.1 *If $A = A^{\intercal}$ is real and $z = u + iv$, $\left(i^2 = -1\right)$, then*

$$\boxed{f_A(z) := (z, Az) = f_A(u) + f_A(v).} \tag{7.3}$$

Proof. Indeed,

$$f_A(z) := (z, Az) = (u, Au) + (v, Av)$$

$$-i(v, Au) + i(u, Av) = (u, Au) + (v, Av).$$

■

Corollary 7.1 (on the extension)

- *Any real quadratic form $f_A(u)$ can be uniquely extended up to the corresponding Hermitian form $f_A(z)$, using the formula (7.3).*
- *If $f_A(u)$ has a special form*

$$\boxed{f_A(u) = \left|a^{\intercal}u\right|^2 + \left(b^{\intercal}u\right)\left(u^{\intercal}c\right),} \tag{7.4}$$

with complex vectors a, b and c, then

$$\boxed{f_A(z) = \left|a^{*}z\right|^2 + \operatorname{Re}\left(b^{*}z\right)\left(z^{*}c\right)} \tag{7.5}$$

Proof. Indeed,

$$f_A(u) = |a^\intercal u|^2 + (b^\intercal u)(u^\intercal c) =$$

$$\left(\sum_{i=1}^n a_i u_i\right)\left(\sum_{j=1}^n a_j u_j\right) + \left(\sum_{i=1}^n b_i u_i\right)\left(\sum_{j=1}^n c_j u_j\right) =$$

$$\sum_{i=1}^n \sum_{j=1}^n (a_i a_j + b_i c_j) u_i u_j.$$

Implying by (7.3) we get

$$\left.\begin{aligned} f_A(z) = f_A(u) + f_A(v) = \sum_{i=1}^n \sum_{j=1}^n (a_i a_j + b_i c_j) u_i u_j + \\ \sum_{i=1}^n \sum_{j=1}^n (a_i a_j + b_i c_j) v_i v_j = \sum_{i=1}^n \sum_{j=1}^n (a_i a_j + b_i c_j) \underbrace{(u_i u_j + v_i v_j)}_{\operatorname{Re} z_i z_j} \\ = \sum_{i=1}^n \sum_{j=1}^n a_i a_j \operatorname{Re}(z_i z_j) + \sum_{i=1}^n \sum_{j=1}^n b_i c_j \operatorname{Re}(z_i z_j). \end{aligned}\right\} \tag{7.6}$$

But

$$\left.\begin{aligned} |a^* z|^2 = \left(\sum_{i=1}^n a_i (u_i + i v_i)\right)\left(\sum_{j=1}^n a_j (u_j - i v_j)\right) = \\ \sum_{i=1}^n \sum_{j=1}^n a_i a_j (u_i + i v_i)(u_j - i v_j) = \\ \sum_{i=1}^n \sum_{j=1}^n a_i a_j (u_i u_j + v_i v_j) = \sum_{i=1}^n \sum_{j=1}^n a_i a_j \operatorname{Re}(z_i z_j), \end{aligned}\right\} \tag{7.7}$$

and

$$\left.\begin{array}{c} \operatorname{Re}\left[(b^* z)(z^* c)\right] = \operatorname{Re}\left[b^\intercal (u+iv)(u-iv)^\intercal c\right] \\ \operatorname{Re}\left[\left(\sum_{i=1}^{n} b_i (u_i + iv_i)\right)\left(\sum_{j=1}^{n} c_j (u_j - iv_j)\right)\right] = \\ \operatorname{Re}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} b_i c_j (u_i + iv_i)(u_j - iv_j)\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i c_j (u_i u_j + v_i v_j) \\ = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i c_j \operatorname{Re}(z_i z_j). \end{array}\right\} \quad (7.8)$$

Substitution (7.7) and (7.8) into (7.6) leaads to (7.5). ■

Corollary 7.2 *Evidently that, by (7.3),* $f_A(z) > 0$ *(*$f_A(z) \geq 0$*) for any* $z \in \mathbb{C}^n$ *if and only if* $f_A(u) > 0$ *(*$f_A(u) \geq 0$*) for all* $u \in \mathbb{R}^n$*.*

Proposition 7.2 *If*

$$z = \begin{pmatrix} z^{(1)} \\ z^{(2)} \end{pmatrix} \text{ and } A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix},$$

then

$$\boxed{\begin{array}{c} f_A(z) := (Az, z) = \\ \left(A_{11} z^{(1)}, z^{(1)}\right) + 2\operatorname{Re}\left(A_{12} z^{(2)}, z^{(1)}\right) + \left(A_{22} z^{(2)}, z^{(2)}\right). \end{array}} \quad (7.9)$$

7.2 Representation of the stability conditions in frequency domain

7.2.1 Scalar feedback

So, by (7.5) the condition (6.7), emposed to all x and u such that $\|x\|^2 + \|u\|^2 > 0$,

$$\tilde{Q}_\tau(x,u) = \tilde{Q}_0(x,u) - \tau \tilde{Q}_1(x,u) > 0$$

may be expanded to the following one:

$$\left.\begin{array}{c} \tilde{Q}_\tau(X,U) = \operatorname{Re}\left(\tilde{Q}_0(X,U) - \tau \tilde{Q}_1(X,U)\right) > 0 \\ \text{for all complex } X \in \mathbb{C}^n \text{ and all complex } U \in \mathbb{C}, \end{array}\right\} \quad (7.10)$$

where X and U are the Furier transformations of x and u processes.

Theorem 7.1 ([16]) *Suppose that*

- *the matrix A in (5.1) has no pure imaginary eigenvalues;*

- *the nonlinear feedback $\varphi(y)$ is from the class $\mathcal{F}_{\gamma,k}$ (5.6).*

To guarantee the existence of the function $V(x)$ of the form (6.1) for which

$$\frac{d}{dt}V(x(t,x_0,t_0)) < 0$$

in any points $x(t,x_0,t_0) \in \mathbb{R}^n$ (which is the solution of (5.1) with the initial condition x_0 at time t_0) and any $u = u(t) \in \mathbb{R}$, realized by a feedback from the class $\mathcal{F}_{\gamma,k}$, satisfying the constraints (5.6), it is ***necessary and sufficient*** *that for all $\omega \in [-\infty,\infty]$ the following "**frequency inequality**" (it is known as* ***the "Popov's inequality***) *would be fulfilled:*

$$\boxed{q\omega \operatorname{Im} H(i\omega) + k \operatorname{Re} H(i\omega) + 1 > 0,} \tag{7.11}$$

where the complex function $H(s)$ (5.3) is the transfer function of the linear subsystem and q is a real number.

Proof.

a) Sufficiency. Let now X, U and Y be the Furier transformations of x, u and y processes, connected as

$$i\omega X = AX + bU,$$

$$Y = c^{\mathsf{T}} X$$

where ω is a real number for which

$$\det[A - i\omega I] \neq 0.$$

By (7.10) with $\tau = 1$ (see Remark 1.5) we get

$$\tilde{Q}_\tau (X, U) := \tilde{Q}_0 (X, U) - \tau \tilde{Q}_1 (X, U) \overset{\tau=1}{\equiv}$$

$$X^* \left(-PA - A^\intercal P\right) X + X^* \left(-Pb - qA^\intercal \frac{c}{2} + k\frac{c}{2}\right) U +$$

$$U^* \left(-b^\intercal P - q\frac{c^\intercal}{2} A + k\frac{c^\intercal}{2}\right) X + U^* \left(-qc^\intercal b + 1\right) U =$$

$$-2 \operatorname{Re} X^* P (AX + bU) - q \operatorname{Re} \left[U^* c^\intercal (AX + bU)\right] +$$

$$\operatorname{Re} \left[U^* (U + kY)\right] > 0$$

for all $\|X\| + |U| \neq 0$. Also we have

$$\operatorname{Re} X^* P (AX + bU) = \operatorname{Re} i\omega (X^* P X) = 0$$

and

$$q \operatorname{Re} \left[U^* c^\intercal (AX + bU)\right] = q \operatorname{Re} \left[i\omega U^* c^\intercal X\right] = q \operatorname{Re} \left[i\omega U^* Y\right].$$

So, the relation $\tilde{Q}_\tau (X, U) > 0$ becomes

$$\tilde{Q}_\tau (X, U) = -2 \operatorname{Re} X^* P (AX + bU) - q \operatorname{Re} \left[U^* c^\intercal (AX + bU)\right] +$$

$$\operatorname{Re} \left[U^* (U - Y)\right] = -q \operatorname{Re} \left[i\omega U^* Y\right] + \operatorname{Re} \left[U^* (U - Y)\right] =$$

$$\operatorname{Re} \left[-qi\omega U^* Y + U^* \left(k^{-1} U - Y\right)\right] > 0$$

By (5.3), we also have that

$$Y = H (i\omega) U,$$

where $H (i\omega)$ is the complex value (non matrix). So, we obtain

$$\tilde{Q}_\tau (X, U) = \operatorname{Re} \left[-qi\omega U^* H (i\omega) U + U^* U + U^* \left[kH (i\omega)\right] U\right] > 0. \qquad (7.12)$$

Notice that $U^* U = |U|^2$,

$$\operatorname{Re} U^* H (i\omega) U = \operatorname{Re} U^* \left[\operatorname{Re} H (i\omega) + i \operatorname{Im} H (i\omega)\right] U =$$

$$\left[\operatorname{Re} H (i\omega) + \operatorname{Re} i \operatorname{Im} H (i\omega)\right] U^* U = \operatorname{Re} H (i\omega) |U|^2,$$

and

$$q\,\mathrm{Re}\left[i\omega U^{*} H\left(i\omega\right) U\right] = q\,\mathrm{Re}\left[i\omega U^{*}\left[\mathrm{Re}\, H\left(i\omega\right) + i\,\mathrm{Im}\, H\left(i\omega\right)\right] U\right] =$$

$$q\,\mathrm{Re}\left[i\omega\left[\mathrm{Re}\, H\left(i\omega\right) + i\,\mathrm{Im}\, H\left(i\omega\right)\right] U^{*}U\right] =$$

$$q\,\mathrm{Re}\left[\left[i\omega\,\mathrm{Re}\, H\left(i\omega\right) - \omega\,\mathrm{Im}\, H\left(i\omega\right)\right] U^{*}U\right] = q\,\mathrm{Re}\left[-\omega\,\mathrm{Im}\, H\left(i\omega\right)\right] U^{*}U,$$

which permits to represent (7.12) as

$$\left[q\omega\,\mathrm{Im}\, H\left(i\omega\right) + k\,\mathrm{Re}\, H\left(i\omega\right) + 1\right] \left|U\right|^{2} > 0$$

whereas $|U| \neq 0$, implying (7.11).

b) Necessity. It follows directly from the properties of S-procedure (1.4), if take into account that it gives necessary and sufficient conditions of the "equivalency" of the sets defined by the inequalities $\tilde{Q}_0\left(x,u\right) > 0$ under the constraint $\tilde{Q}_1\left(x,u\right) > 0$ and

$$\tilde{Q}_{\tau}\left(x,u\right) > 0\left(\left\|x\right\| + \left|u\right| \neq 0\right).$$

Theorem is proven. ■

Remark 7.1 *In the considered class $\mathcal{F}_{\gamma,k}$ of nonlinear feedbacks we still have not proved the validity of the properties a) and b) of the Barbashin-Krasovskii theorem given above. But these properties follows directly from the accepted assumptions since*

- *if $P = P^{\mathsf{T}} > 0$ and $q \geq 0$ the properties a) and b) obviousely are guranteed,*
- *if $P = P^{\mathsf{T}} > 0$ and $q < 0$ may be guaranteed for sufficiently small $|q|$.*

7.2.2 The Popov's line

Define new coordinates

$$x_H\left(\omega\right) := \mathrm{Re}\, H\left(i\omega\right),\ y_H\left(\omega\right) := \omega\,\mathrm{Im}\, H\left(i\omega\right).$$

The parametric curve $x_H\left(\omega\right)$, $y_H\left(\omega\right)$ $\left(\omega \in \left(-\infty,\infty\right)\right)$ is referred to as *the Popov's plot*, and the line

$$\boxed{k x_H + q y_H + 1 = 0} \tag{7.13}$$

as *the Popov's line.* The figure Fig.7.1 represents the disposition of Popov's line

$$\boxed{y_H = -q^{-1}\left(k x_H + 1\right), q \neq 0.}$$

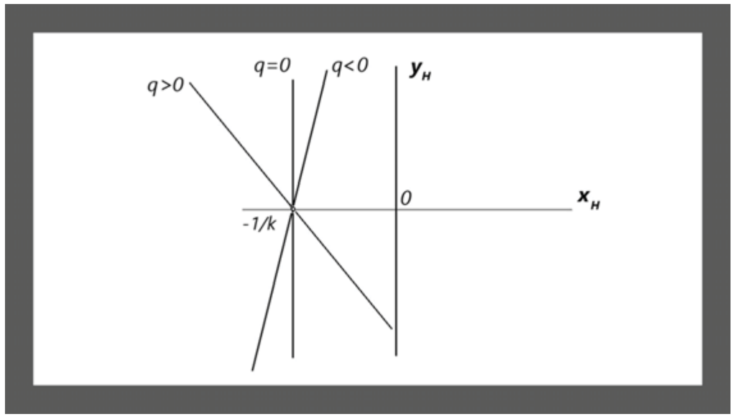

Figure 7.1: Popov's lines for $q = 0$, $q < 0$ and $q > 0$

Proposition 7.3 *According to the inequality (7.11), if the Popov's plot is only on one side of the Popov's line (7.13) enclosing the origin, the linear system (5.1) with the class $\mathcal{F}_{\gamma,k}$ of nonlinear feedbacks (5.2) is* ***globally stable****.*

Next figures (7.2),(7.3) and (7.4) illustrate how to test the frequency condition (7.11) using the Popov's line (7.13).

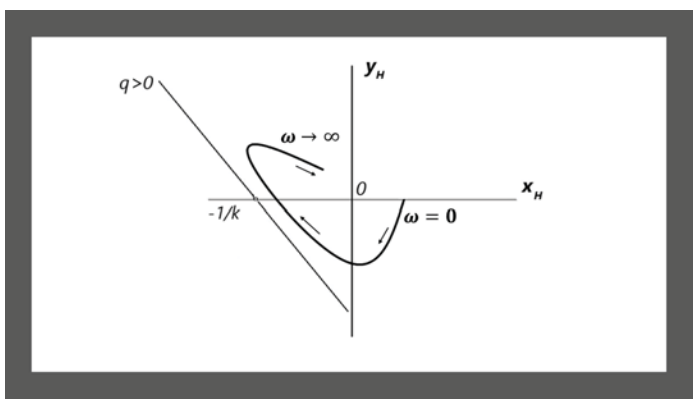

Figure 7.2: The class of the systems is absolutely stable: the Popov's plot admits the existance of a Popv's line with $q > 0$

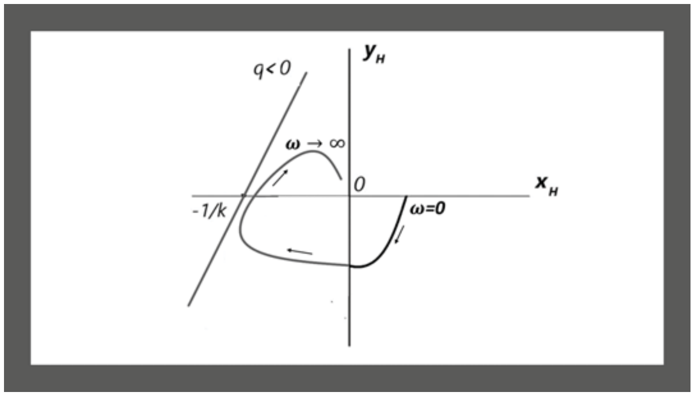

Figure 7.3: The class of the systems is absolutely stable: the Popov's plot admits the existance of a Popv's line with $q < 0$

7.2.3 Vector feedback

Following the same scheme as in the scalar feedback case, we have that the the linear system (5.1) - (5.6) with a nonlinear feedback $u = -\varphi(y)$ from the class $\mathcal{F}_{\Gamma,k}$ is absolutely globally stable if

$$\tilde{Q}_\tau(X,U) := \tilde{Q}_0(X,U) - \tau\tilde{Q}_1(X,U) \overset{\tau=1}{\equiv}$$

$$X^*(-PA - A^\intercal P)X + X^*\left(-PB - \frac{q}{2}A^\intercal C^\intercal\Gamma + \frac{k}{2}C^\intercal\Gamma\right)U +$$

$$U^*\left(-B^\intercal P - \frac{q}{2}\Gamma CA + \frac{k}{2}\Gamma C\right)X + U^*\left(-\frac{q}{2}\left(\Gamma CB + B^\intercal C^\intercal\Gamma\right) + \Gamma\right)U,$$

or equivalently,

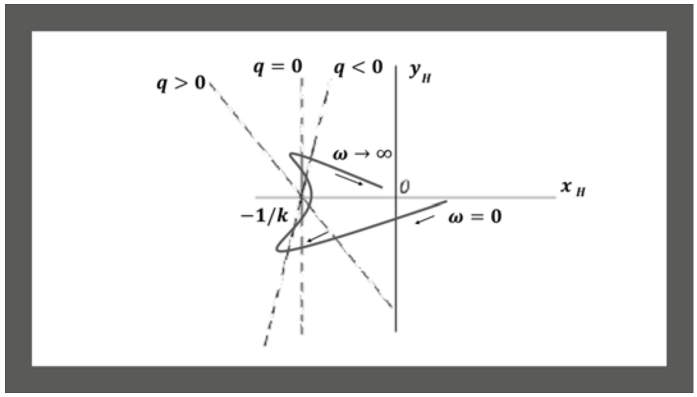

Figure 7.4: The class of the systems can not be considered as absolutely stable: there no exists a Popov's line such that the Popov's plot would be located on one side of it.

$$\tilde{Q}_\tau (X, U) = -2 \operatorname{Re} X^* P (AX + BU) -$$

$$\frac{q}{2} \operatorname{Re} [U^* \Gamma CAX + U^* \Gamma CBU + (U^* \Gamma CAX + U^* \Gamma CBU)^\intercal] +$$

$$\operatorname{Re} \left[U^* \left(\Gamma U + \frac{k}{2} (Y \Gamma + Y^\intercal \Gamma) \right) \right] = -2 \operatorname{Re} X^* P (AX + BU)$$

$$-\frac{q}{2} \operatorname{Re} [U^* \Gamma C (AX + BU) + (AX + BU)^* C^\intercal \Gamma U] +$$

$$\operatorname{Re} \left[U^* \left(\Gamma U + \frac{k}{2} (\Gamma Y + Y^* \Gamma) \right) \right] > 0.$$

Since

$$i\omega X = AX + BU,$$

$$Y = CX, \; Y = H (i\omega) U,$$

where ω is a real number for which $\det[A - i\omega I] \neq 0$, the last inequality becomes

$$\tilde{Q}_{\tau}(X,U) = -2\underbrace{\operatorname{Re}(i\omega X^{*}PX)}_{0} -$$

$$-\frac{q}{2}\operatorname{Re}\left[U^{*}\Gamma C(AX+BU) + (AX+BU)^{*}C^{\intercal}\Gamma U\right] +$$

$$\operatorname{Re}\left[U^{*}(\Gamma U + kH(i\omega)U)\right] =$$

$$-\frac{q}{2}\operatorname{Re}\left[i\omega(U^{*}\Gamma Y + Y^{*}\Gamma U)\right] +$$

$$\operatorname{Re}\left[U^{*}\Gamma U + \frac{k}{2}U^{*}(\Gamma H(i\omega) + H^{*}(i\omega)\Gamma)U\right] > 0$$

which, in fact, is

$$\tilde{Q}_{\tau}(X,U) = -\frac{q}{2}\operatorname{Re}\left[i\omega U^{*}(\Gamma H(i\omega) + H^{*}(i\omega)\Gamma)U\right] +$$

$$\operatorname{Re}\left[U^{*}\Gamma U + \frac{k}{2}U^{*}(\Gamma H(i\omega) + H^{*}(i\omega)\Gamma)U\right] =$$

$$\frac{q}{2}\left[U^{*}\omega\operatorname{Im}(\Gamma H(i\omega) + H^{*}(i\omega)\Gamma)U\right] +$$

$$\left[\frac{k}{2}U^{*}\operatorname{Re}(\Gamma H(i\omega) + H^{*}(i\omega)\Gamma)U\right] + U^{*}\Gamma U > 0,$$

implying

$$\tilde{Q}_{\tau}(X,U) = U^{*}\left[\frac{q}{2}\omega\operatorname{Im}(\Gamma H(i\omega) + H^{*}(i\omega)\Gamma) +\right.$$

$$\left.\frac{k}{2}\operatorname{Re}(\Gamma H(i\omega) + H^{*}(i\omega)\Gamma) + \Gamma\right]U =$$

$$U^{*}\mathcal{H}(\omega \mid q,k,\Gamma)U > 0 \text{ forall } U \quad (\|U\| > 0),$$

where the Hermitian matrix

$$\left.\begin{aligned} \mathcal{H}(\omega \mid q,k,\Gamma) &:= \frac{q}{2}\omega\operatorname{Im}(\Gamma H(i\omega) + H^{*}(i\omega)\Gamma) \\ &+ \frac{k}{2}\operatorname{Re}(\Gamma H(i\omega) + H^{*}(i\omega)\Gamma) + \Gamma \end{aligned}\right\} \tag{7.14}$$

is strictly positive, that is,

$$\mathcal{H}(\omega \mid q, k, \Gamma) > 0 \text{ for all } \omega \in (-\infty, \infty) \text{ such that } \det[A - i\omega I] \neq 0.$$

So, we are ready to formulate the following result.

Theorem 7.2 *Suppose that*

- *the matrix A in (5.1) has no pure imaginary eigenvalues;*
- *the nonlinear feedback $\varphi(y)$ is from the class $\mathcal{F}_{\Gamma,k}$ (5.6).*

To guarantee the existence of the function $V(x)$ of the form (6.8) for which

$$\frac{d}{dt} V(x(t, x_0, t_0)) < 0$$

in any points $x(t, x_0, t_0) \in \mathbb{R}^n$ and any $u = u(t) \in \mathbb{R}^m$ from the class $\mathcal{F}_{\Gamma,k}$, satisfying the constraints (5.9), it is ***necessary and sufficient*** *that for all $\omega \in [-\infty, \infty]$ the following "**generalized frequency inequality**" would be met for some real number q:*

$$\boxed{\begin{array}{c} \mathcal{H}(\omega \mid q, k, \Gamma) := \dfrac{q}{2}\omega \operatorname{Im}(\Gamma H(i\omega) + H^*(i\omega)\Gamma) \\ \\ +\dfrac{k}{2}\operatorname{Re}(\Gamma H(i\omega) + H^*(i\omega)\Gamma) + \Gamma > 0 \end{array}} \tag{7.15}$$

Remark 7.2 *Notice that for scalar case, when $m = 1$ and $\Gamma = \gamma > 0$, the condition (7.15) consides with (7.11).*