Lecture 6

Analysis of Absolute Global Stability in Time-Domain

6.1 Scalar feedback

6.1.1 Lyapunov function of the Lurie-Postnikov type

To guarantee the absolute global stability of the system (5.1), (5.6), according to the Barbashin-Krasovskii Theorem (see, for example, Theorem 20.7 in [14]), it is sufficient the existence of the Lyapunov function $V(x)$ such that

a) $V(0) = 0$ and
$$V(x) > 0 \text{ for any } x \neq 0,$$

b) radially unboundedness:
$$V(x) \to \infty \text{ whereas } \|x\| \to \infty,$$

c) for any $t \geq t_0$
$$\frac{d}{dt} V(\bar{x}(t, x_0, t_0)) < 0,$$
where $\bar{x}(t, x_0, t_0)$ is the solution of (5.1) with the initial conditions $x(t_0) = x_0$.

In [15] there was suggested to find the function $V(x)$ as a quadratic form plus the integral of the nonlinear feedback in the scalar feedback case (when

$y = c^{\intercal}x$, $m = l = 1$ and $B = b \in \mathbb{R}^n$), that is,

$$\boxed{V(x) = x^{\intercal}Px - q \int\limits_{y=0}^{y=c^{\intercal}x} \varphi(y)\, dy,} \tag{6.1}$$

where $P = P^{\intercal}$ is a real positive definite matrix and q is real number (may be positive, negative or zero). Calculating the time derivative of (6.1) on the trajectories of (5.1) we obtain

$$\left.\begin{array}{c} \dfrac{d}{dt}V(\bar{x}(t, x_0, t_0)) = 2x^{\intercal}(t)\, P(Ax(t) + bu(t)) + qu(t)\, \dot{y}(t)\,, \\ \text{where} \\ \dot{y}(t) = c^{\intercal}(Ax(t) + bu(t))\,. \end{array}\right\} \tag{6.2}$$

The right-hand side is a quadratic form of variables x and u, namely,

$$\frac{d}{dt}V(t, \bar{x}(t, x_0, t_0)) = Q_0(x(t), u(t)) \tag{6.3}$$

where

$$\left.\begin{array}{c} Q_0(x, u) := 2x^{\intercal}P(Ax + bu) + quc^{\intercal}(Ax + bu) = \\ \begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} Q_0 \begin{pmatrix} x \\ u \end{pmatrix} \\ \\ Q_0 := \begin{bmatrix} PA + A^{\intercal}P & Pb + \dfrac{q}{2}A^{\intercal}c \\ b^{\intercal}P + \dfrac{q}{2}c^{\intercal}A & qc^{\intercal}b \end{bmatrix}. \end{array}\right\} \tag{6.4}$$

Therefore, to fulfill the condition c), given above, one must fulfill the condition

$$Q_0(x, u) < 0 \text{ for all real } x \in \mathbb{R}^n \text{ and all } u \in \mathbb{R}, \;\; \|x\|^2 + u^2 > 0,$$

or equivalently,

$$\tilde{Q}_0(x, u) := -Q_0(x, u) > 0$$

for all real $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}$ (simultaneously non equal to zero), satisfying the inequality (5.9) with $\Gamma = \gamma$:

$$u^{\intercal}\Gamma(u + ky) = \gamma u(u + ky) \leq 0, \;\; \gamma > 0. \tag{6.5}$$

implying

$$Q_1(x, u) := u^2 + k(x^{\intercal}c)\, u \leq 0 \tag{6.6}$$

Notice that $Q_1(x,u)$ is also a quadratic form of x and u, that is,

$$Q_1(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^{\mathsf{T}} Q_1 \begin{pmatrix} x \\ u \end{pmatrix}$$

with

$$Q_1 := \begin{bmatrix} 0 & k\frac{c}{2} \\ k\frac{c^{\mathsf{T}}}{2} & 1 \end{bmatrix}$$

and, in fact defines the additional constraint for these variables. In the equivalent format the constraint $Q_1(x,u) \le 0$ may be expressed as

$$\tilde{Q}_1(x,u) := -Q_1(x,u) =$$

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & -k\frac{c}{2} \\ -k\frac{c^{\mathsf{T}}}{2} & -1 \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = -k\left(x^{\mathsf{T}} c\right) u - u^2 \ge 0,$$

which defines the subspace in the variables (x,u) which we are interested in.

6.1.2 Absolute stability in the time-domain

Based on the S-procedure (see Theorem 1.4) we may conclude the following.

Theorem 6.1 (Sufficient condition of absolute stability in time-domain) *The absolute global stability of the systems (5.1)-(5.6) with a nonlinear feedback $u = -\varphi(y)$ from the class $\mathcal{F}_{\gamma,k}$ with $m = l = 1$ takes place if there exist the positive definite matrix P, constants q and $\tau \ge 0$ such that*

$$\boxed{\begin{gathered} \tilde{Q}_\tau := \tilde{Q}_0 - \tau\tilde{Q}_1 = \\ \begin{bmatrix} -PA - A^{\mathsf{T}}P & -Pb - qA^{\mathsf{T}}\frac{c}{2} + \tau k\frac{c}{2} \\ -b^{\mathsf{T}}P - q\frac{c^{\mathsf{T}}}{2}A + \tau k\frac{c^{\mathsf{T}}}{2} & -qc^{\mathsf{T}}b + \tau \end{bmatrix} > 0 \end{gathered}} \tag{6.7}$$

Proof. It directly follows from Theorem 1.4. ■

6.2 Vector feedback

To analyze the conditions of absolute stability for the $\mathcal{F}_{\Gamma,k}$ - Lurie's systems let us introduce the Lyapunov function of the generalized Lurie-Postnikov type:

$$\boxed{V(x) = x^{\intercal} P x - q \int_{y=0}^{y=Cx} \varphi^{\intercal}(y)\, \Gamma dy,} \tag{6.8}$$

for which we have

$$\frac{d}{dt} V(\bar{x}(t, x_0, t_0)) = 2x^{\intercal}(t) P (Ax(t) + Bu(t)) + q u^{\intercal}(t) \Gamma \dot{y}(t),$$

where

$$\dot{y}(t) = (Ax(t) + Bu(t)).$$

The right-hand side of this equation can be represented as

$$\frac{d}{dt} V(\bar{x}(t, x_0, t_0)) = \begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} Q_0 \begin{pmatrix} x \\ u \end{pmatrix},$$

where

$$\boxed{Q_0 := \begin{bmatrix} PA + A^{\intercal} P & PB + \frac{q}{2} A^{\intercal} C^{\intercal} \Gamma \\ B^{\intercal} P + \frac{q}{2} \Gamma C A & \frac{q}{2} (\Gamma C B + B^{\intercal} C^{\intercal} \Gamma) \end{bmatrix}.} \tag{6.9}$$

So, we now are interesting in the condition c) of the Barbashin-Krasovskii Theorem

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} Q_0 \begin{pmatrix} x \\ u \end{pmatrix} < 0$$

or equivalently

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} \tilde{Q}_0 \begin{pmatrix} x \\ u \end{pmatrix} = - \begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} Q_0 \begin{pmatrix} x \\ u \end{pmatrix} > 0,$$

$$\tilde{Q}_0 = \begin{bmatrix} -PA - A^{\intercal} P & -PB - \frac{q}{2} A^{\intercal} C^{\intercal} \Gamma \\ -B^{\intercal} P - \frac{q}{2} \Gamma C A & -\frac{q}{2} (\Gamma C B + B^{\intercal} C^{\intercal} \Gamma) \end{bmatrix}$$

for all paires (x, u) $(\|x\|^2 + \|u\|^2 > 0)$, satisfying the generalized sector condition (5.9)

$$u^{\intercal} \Gamma (u + kCx) \leq 0,$$

which can be represented also as the quadratic form

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} Q_1 \begin{pmatrix} x \\ u \end{pmatrix} \leq 0 \tag{6.10}$$

where

$$Q_1 = \begin{bmatrix} 0 & \frac{k}{2}C^\intercal \Gamma \\ \frac{k}{2}\Gamma C & \Gamma \end{bmatrix}$$

The inequality (6.10) can be represented equivalently as

$$\begin{pmatrix} x \\ u \end{pmatrix}^\intercal \tilde{Q}_1 \begin{pmatrix} x \\ u \end{pmatrix} \geq 0 \tag{6.11}$$

with

$$\tilde{Q}_1 = \begin{bmatrix} 0 & -\frac{k}{2}C^\intercal \Gamma \\ -\frac{k}{2}\Gamma C & -\Gamma \end{bmatrix},$$

and we interested in the subspace of variables (x, u) satisfying the inequality (6.11), namely, when

$$\begin{pmatrix} x \\ u \end{pmatrix}^\intercal \tilde{Q}_1 \begin{pmatrix} x \\ u \end{pmatrix} = -kx^\intercal C^\intercal \Gamma u - u^\intercal \Gamma u \geq 0.$$

Based on the S-procedure (Theorem 1.4) we may conclude the following.

Theorem 6.2 (Sufficient condition of absolute stability in time-domain) *The absolute global stability of the systems (5.1)-(5.6) with a nonlinear feedback $u = -\varphi(y)$ from the class $\mathcal{F}_{\Gamma,k}$ takes place if there exist the positive definite matrix P, constants q and $\tau \geq 0$ such that*

$$\boxed{\begin{gathered} \tilde{Q}_\tau := \tilde{Q}_0 - \tau \tilde{Q}_1 = \\ \begin{bmatrix} -PA - A^\intercal P & -PB - \frac{q}{2}A^\intercal C^\intercal \Gamma + \tau \frac{k}{2}C^\intercal \Gamma \\ -B^\intercal P - \frac{q}{2}\Gamma C A + \tau \frac{k}{2}\Gamma C & -\frac{q}{2}\left(\Gamma C B + B^\intercal C^\intercal \Gamma\right) + \tau \Gamma \end{bmatrix} > 0. \end{gathered}} \tag{6.12}$$

Proof. As in scalar contrrol case the proof directly follows from Theorem 1.4. ■

6.3 Exercise

Exercise 6.1 *Check if the linear system*

$$\left.\begin{array}{c}
\dot{x}(t)=\begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ -1 & 0 & -0.5 \end{bmatrix} x(t)+\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u\left(t\right), \\[2ex]
t \geq t_0=0,\ x\left(0\right)=x_0, \\[2ex]
y(t)=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} x(t), \\[2ex]
n=3,\ m=l=2
\end{array}\right\}$$

is absolutely asymptotically stable in the class $\mathcal{F}_{\Gamma,k}$ *of nonlinear feedbacks with*

$$\Gamma=0.3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and } k=2.$$