Part II

Absolute Stability and $\mathbf{H}_\infty$ Control: Generalized Classical Problems

Lecture 5

Absolute Stability

The theory of *absolute stability* of nonlinear systems with sectoral restrictions dates back to A. I. Lurie's work (see [1], some historical reviews can be found in [2, 3]). The Lurie problem was reduced to the feasibility of a specific system of Linear Matrix Inequalities (LMI's) in [4] and later publications. Later, in a monograph [5], the advantages of the LMI approach to many issues in applied mathematics were thoroughly addressed, resulting in the widespread creation of a particular computer software for studying LMI's. The *Lurie theory* was expanded to include numerous nonlinearities in [6]-[8].

We'll go through two main notions proposed by V. A. Yakubovich in the 1960s and 1970s. The first is the *S-procedure* [2, 9, 10] (see also Lecture 1), which is particularly effective when dealing with several nonlinearities. The second is *integral-quadratic constraint* (IQC), which notion was first developed by V. A. Yakubovich in the context of the research of pulse-width modulated systems (a type of sampled-data system) [11]. Unfortunately, the last paper was never translated, and for a non-Russian speaking, it is almost unknown. It's worth noting that the influence of V. A. Yakubovich's early research on contemporary IQC theory was acknowledged in the review section of a well-known publication [12]. For the stability study of many kinds of nonlinear control systems, sectoral constraints and IQCs proven to be quite useful (see, e.g., [2]).

5.1 Linear systems with nonlinear feedbacks

Consider the dynamic system given by the following ODE:

$$\boxed{\left.\begin{array}{c} \dot{x}(t) = Ax(t) + Bu(t), \ t \geq t_0 = 0, \ x(0) = x_0, \\ \\ y(t) = Cx(t) \\ \\ A \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}, \ u \in \mathbb{R}^l, \ y \in \mathbb{R}^m, \end{array}\right\}} \tag{5.1}$$

where the control u is given by the nonlinear continuous mapping

$$\boxed{u = -\varphi(y),} \tag{5.2}$$

which corresponds to a *non-linear negative feedback*.

It can be interpreted (see Fig.5.1) as a linear system given by the transfer function

$$H(s) = C(sI - A)^{-1} B \tag{5.3}$$

with a nonlinear feedback $u = -\varphi(y)$.

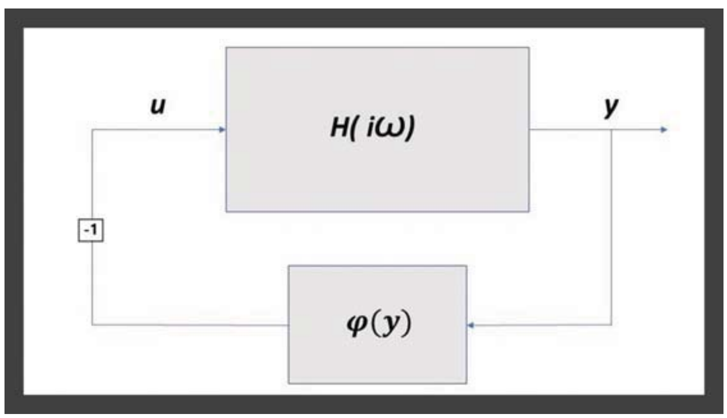

Figure 5.1: Linear systems with a nonlinear feedback

5.2 Generalized sector condition

5.2.1 Multidimensional case

Below we will consider the case when $l = m$.

Definition 5.1

1. *Denote by $\mathcal{F}_{\Gamma,k}$ the class of **nonlinear negative feedbacks** $\varphi(y)$ (5.2) which for all $y \in \mathbb{R}^m$, a given matrix*

$$0 < \Gamma = \Gamma^{\intercal} \in \mathbb{R}^{m\times m}$$

and a positive scalar $k > 0$ satisfies the inequality

$$\boxed{\varphi^{\intercal}(y)\,\Gamma\left(\varphi(y) - ky\right) \leq 0,} \tag{5.4}$$

*named below the **generalized sector condition**.*

2. *By $\mathcal{F}^{+}_{\Gamma,k}$ we will denote the class of nonlinear negative feedbacks $\varphi(y)$ (5.2) which for all $y \in \mathbb{R}^m$, $y \neq 0$, a given matrix*

$$0 < \Gamma = \Gamma^{\intercal} \in \mathbb{R}^{m\times m}$$

and a positive scalar $k > 0$ the inequality (5.4) is strictly fulfilled, namely,

$$\boxed{\varphi^{\intercal}(y)\,\Gamma\left(\varphi(y) - ky\right) < 0.} \tag{5.5}$$

5.2.2 Scalar output-control case

In the scalar case (when $m = l = 1$) the classes $\mathcal{F}_{\Gamma,k}$ and $\mathcal{F}^{+}_{\Gamma,k}$ contain all continuos functions $\varphi(y)$ (nonlinear feedbacks) satisfying the, so-called, **sector-condition (SC)** (see Fig.5.3) and **strict sector-condition (SSC)** (see Fig.5.3), namely,

$$\left.\begin{array}{ll} \text{SC:} & 0 \leq \dfrac{\varphi(y)}{y} \leq k \text{ for } y \neq 0,\ \varphi(0) = 0, \\ \\ \text{SSC:} & 0 < \dfrac{\varphi(y)}{y} < k \text{ for } y \neq 0,\ \varphi(0) = 0. \end{array}\right\} \tag{5.6}$$

Indeed, for $m = l = 1$ we have $\Gamma = \gamma > 0$ and deal with the classes $\mathcal{F}_{\gamma,k}$. In this case the inequalty (5.4) becomes

$$\gamma\varphi(y)\left(\varphi(y) - ky\right) \leq 0,$$

which is equivalent to

$$0 \leq |\varphi(y)|^2 = \varphi^2(y) \leq k\varphi(y)\,y = k\,|\varphi(y)\,y| = k\,|\varphi(y)|\,|y| \tag{5.7}$$

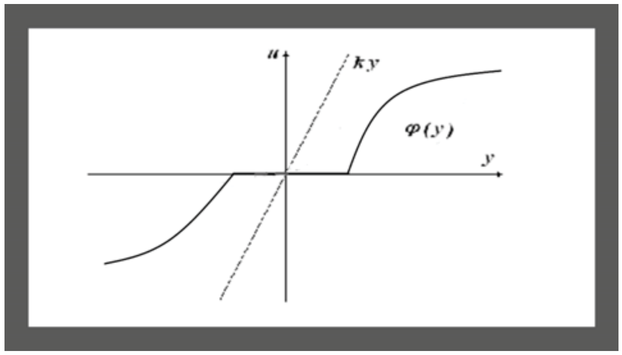

Figure 5.2: Sector condition

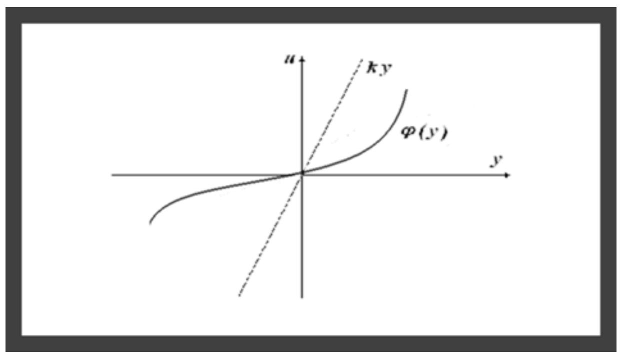

Figure 5.3: Strict sector-condition

and

$$\operatorname{sign}[\varphi(y)] = \operatorname{sign}(y) \;\; \text{for } y \neq 0 \tag{5.8}$$

or, there is also admitted that

$$\varphi(y) = 0 \text{ for some } y \neq 0.$$

For $y = 0$ from the inequality (5.7) it follows that $\varphi(0) = 0$. If $y \neq 0$ and $|\varphi(y)| \neq 0$ we have

$$|\varphi(y)| \leq k|y|,$$

implying in view of (5.8)

$$\frac{\varphi(y)}{y} = \frac{|\varphi(y)|}{|y|} \leq k$$

that proves (5.6) for SC-case (5.2). If we deal with SSC-case then the condition (5.5) leads to

$$|\varphi(y)|^2 < k|\varphi(y)||y|$$

and for $y \neq 0$ we obligatory have that $\varphi(y) \neq 0$ (see Fig.5.3).

Remark 5.1 *The generalized sector condition (5.4) can be expressed also as*

$$\boxed{u^{\intercal}\Gamma(u+ky) \leq 0.} \tag{5.9}$$

5.3 $\mathcal{F}_{\Gamma,k}$ - Lurie's systems and the problem of absolute stability

Definition 5.2 *The systems (5.1)-(5.2) with the negative nonlinear feedback*

$$u = -\varphi(y) \in \mathcal{F}_{\Gamma,k}$$

are called ***the*** $\mathcal{F}_{\Gamma,k}$ *-* ***Lurie's systems****.*

Problem 5.1 *Here we will interested in finding the conditions guaranteeing that the solution $x(t) \equiv 0$ be* ***absolutely*** *(in the class $\mathcal{F}_{\Gamma,k}$ of nonlinear feedbacks)* ***asymptotically stable****, that is, the origin be asymptotically stable for any $\mathcal{F}_{\Gamma,k}$ - Lurie's system.*

5.4 Conjectures of Aizerman and Kalman

Proposition 5.1 (Conjecture of M.A.Aizerman, 1949) *Let the scalar system (5.1) with*

$$n = m = l = 1$$

be globally ($\forall x(0) : \|x(0)\| < \infty$) stable for any

$$\varphi(y) = \alpha y, \ \alpha \in [0, k]$$

It seems to be true that this system remains globally stable for any feedback $\varphi(y)$ satisfying (5.6).

Proposition 5.2 (Conjecture of R.Kalman, 1957) *Let the scalar system (5.1) with*

$$n = m = l = 1$$

be globally stable for any

$$\varphi(y) = \alpha y, \ \ \alpha \in [0, k]$$

It seems to be true that this system remains globally stable for any feedback $\varphi(y)$ satisfying

$$\varphi(y) \ \text{ is differentiable, } \varphi(0) = 0,$$

$$0 \leq \varphi'(y) \leq k$$

Claim 5.1 ([13]) *Both conjectures of M.A.Aizerman and R.Kalman are not valid.*

To prove this claim it is sufficient to construct al least one counterexample. One of such counterexamples is as follows.

Counterexample . Let

$$H(s) = \frac{s^2}{\left[(s+0.5)^2 + (0.9)^2\right]\left[(s+0.5)^2 + (1.1)^2\right]}.$$

The close-loop system is stable (for example, by the Routh-Hurwitz criterion [14]) for any $u = -ky$ with

$$k \in [-0, 7124, \infty).$$

But for

$$\varphi(y) = \begin{cases} y^3 & \text{for} \quad |y| \leq \sqrt{|k|} \\ ky & \text{for} \quad |y| > \sqrt{|k|} \end{cases}$$

in this system the auto-oscillations arise, and, hence there is no asymptotic stability.

These conjectures were proposed before the keystone result of V.M. Popov who found the exact conditions of absolute stability of the linear system (5.1) with any scalar feedback satisfying (5.6).

Lecture 6

Analysis of Absolute Global Stability in Time-Domain

6.1 Scalar feedback

6.1.1 Lyapunov function of the Lurie-Postnikov type

To guarantee the absolute global stability of the system (5.1), (5.6), according to the Barbashin-Krasovskii Theorem (see, for example, Theorem 20.7 in [14]), it is sufficient the existence of the Lyapunov function $V(x)$ such that

a) $V(0) = 0$ and

$$V(x) > 0 \text{ for any } x \neq 0,$$

b) radially unboundedness:

$$V(x) \to \infty \text{ whereas } \|x\| \to \infty,$$

c) for any $t \geq t_0$

$$\frac{d}{dt} V(\bar{x}(t, x_0, t_0)) < 0,$$

where $\bar{x}(t, x_0, t_0)$ is the solution of (5.1) with the initial conditions $x(t_0) = x_0$.

In [15] there was suggested to find the function $V(x)$ as a quadratic form plus the integral of the nonlinear feedback in the scalar feedback case (when

$y = c^\intercal x$, $m = l = 1$ and $B = b \in \mathbb{R}^n$), that is,

$$\boxed{V(x) = x^\intercal P x - q \int_{y=0}^{y=c^\intercal x} \varphi(y)\, dy,} \tag{6.1}$$

where $P = P^\intercal$ is a real positive definite matrix and q is real number (may be positive, negative or zero). Calculating the time derivative of (6.1) on the trajectories of (5.1) we obtain

$$\left.\begin{array}{c} \frac{d}{dt} V(\bar{x}(t, x_0, t_0)) = 2x^\intercal(t) P (Ax(t) + bu(t)) + qu(t)\dot{y}(t), \\ \text{where} \\ \dot{y}(t) = c^\intercal (Ax(t) + bu(t)). \end{array}\right\} \tag{6.2}$$

The right-hand side is a quadratic form of variables x and u, namely,

$$\frac{d}{dt} V(t, \bar{x}(t, x_0, t_0)) = Q_0(x(t), u(t)) \tag{6.3}$$

where

$$\left.\begin{array}{c} Q_0(x, u) := 2x^\intercal P (Ax + bu) + quc^\intercal (Ax + bu) = \\ \begin{pmatrix} x \\ u \end{pmatrix}^\intercal Q_0 \begin{pmatrix} x \\ u \end{pmatrix} \\ \\ Q_0 := \begin{bmatrix} PA + A^\intercal P & Pb + \frac{q}{2} A^\intercal c \\ b^\intercal P + \frac{q}{2} c^\intercal A & qc^\intercal b \end{bmatrix}. \end{array}\right\} \tag{6.4}$$

Therefore, to fulfill the condition c), given above, one must fulfill the condition

$$Q_0(x, u) < 0 \text{ for all real } x \in \mathbb{R}^n \text{ and all } u \in \mathbb{R}, \;\; \|x\|^2 + u^2 > 0,$$

or equivalently,

$$\tilde{Q}_0(x, u) := -Q_0(x, u) > 0$$

for all real $x \in \mathbb{R}^n$ and all $u \in \mathbb{R}$ (simultaneously non equal to zero), satisfying the inequality (5.9) with $\Gamma = \gamma$:

$$u^\intercal \Gamma (u + ky) = \gamma u (u + ky) \leq 0, \;\; \gamma > 0. \tag{6.5}$$

implying

$$Q_1(x, u) := u^2 + k(x^\intercal c) u \leq 0 \tag{6.6}$$

Notice that $Q_1(x,u)$ is also a quadratic form of x and u, that is,

$$Q_1(x,u) = \begin{pmatrix} x \\ u \end{pmatrix}^{\mathsf{T}} Q_1 \begin{pmatrix} x \\ u \end{pmatrix}$$

with

$$Q_1 := \begin{bmatrix} 0 & k\dfrac{c}{2} \\ k\dfrac{c^{\mathsf{T}}}{2} & 1 \end{bmatrix}$$

and, in fact defines the additional constraint for these variables. In the equivalent format the constraint $Q_1(x,u) \leq 0$ may be expressed as

$$\tilde{Q}_1(x,u) := -Q_1(x,u) =$$

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & -k\dfrac{c}{2} \\ -k\dfrac{c^{\mathsf{T}}}{2} & -1 \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = -k\left(x^{\mathsf{T}}c\right)u - u^2 \geq 0,$$

which defines the subspace in the variables (x,u) which we are interested in.

6.1.2 Absolute stability in the time-domain

Based on the S-procedure (see Theorem 1.4) we may conclude the following.

Theorem 6.1 (Sufficient condition of absolute stability in time-domain) *The absolute global stability of the systems (5.1)-(5.6) with a nonlinear feedback $u = -\varphi(y)$ from the class $\mathcal{F}_{\gamma,k}$ with $m = l = 1$ takes place if there exist the positive definite matrix P, constants q and $\tau \geq 0$ such that*

$$\boxed{\begin{gathered}\tilde{Q}_\tau := \tilde{Q}_0 - \tau\tilde{Q}_1 = \\ \begin{bmatrix} -PA - A^{\mathsf{T}}P & -Pb - qA^{\mathsf{T}}\dfrac{c}{2} + \tau k\dfrac{c}{2} \\ -b^{\mathsf{T}}P - q\dfrac{c^{\mathsf{T}}}{2}A + \tau k\dfrac{c^{\mathsf{T}}}{2} & -qc^{\mathsf{T}}b + \tau \end{bmatrix} > 0\end{gathered}} \tag{6.7}$$

Proof. It directly follows from Theorem 1.4. ■

6.2 Vector feedback

To analyze the conditions of absolute stability for the $\mathcal{F}_{\Gamma,k}$ - Lurie's systems let us introduce the Lyapunov function of the generalized Lurie-Postnikov type:

$$\boxed{V(x) = x^{\intercal} P x - q \int\limits_{y=0}^{y=Cx} \varphi^{\intercal}(y)\, \Gamma dy,} \tag{6.8}$$

for which we have

$$\frac{d}{dt} V(\bar{x}(t, x_0, t_0)) = 2x^{\intercal}(t) P (Ax(t) + Bu(t)) + qu^{\intercal}(t) \Gamma \dot{y}(t),$$

where

$$\dot{y}(t) = (Ax(t) + Bu(t)).$$

The right-hand side of this equation can be represented as

$$\frac{d}{dt} V(\bar{x}(t, x_0, t_0)) = \begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} Q_0 \begin{pmatrix} x \\ u \end{pmatrix},$$

where

$$\boxed{Q_0 := \begin{bmatrix} PA + A^{\intercal}P & PB + \frac{q}{2} A^{\intercal} C^{\intercal} \Gamma \\ B^{\intercal}P + \frac{q}{2}\Gamma C A & \frac{q}{2}(\Gamma C B + B^{\intercal} C^{\intercal} \Gamma) \end{bmatrix}.} \tag{6.9}$$

So, we now are interesting in the condition c) of the Barbashin-Krasovskii Theorem

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} Q_0 \begin{pmatrix} x \\ u \end{pmatrix} < 0$$

or equivalently

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} \tilde{Q}_0 \begin{pmatrix} x \\ u \end{pmatrix} = - \begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} Q_0 \begin{pmatrix} x \\ u \end{pmatrix} > 0,$$

$$\tilde{Q}_0 = \begin{bmatrix} -PA - A^{\intercal}P & -PB - \frac{q}{2} A^{\intercal} C^{\intercal} \Gamma \\ -B^{\intercal}P - \frac{q}{2}\Gamma C A & -\frac{q}{2}(\Gamma C B + B^{\intercal} C^{\intercal} \Gamma) \end{bmatrix}$$

for all paires (x, u) $(\|x\|^2 + \|u\|^2 > 0)$, satisfying the generalized sector condition (5.9)

$$u^{\intercal}\Gamma (u + kCx) \leq 0,$$

which can be represented also as the quadratic form

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\intercal} Q_1 \begin{pmatrix} x \\ u \end{pmatrix} \leq 0 \tag{6.10}$$

where

$$Q_1 = \begin{bmatrix} 0 & \frac{k}{2}C^\intercal\Gamma \\ \frac{k}{2}\Gamma C & \Gamma \end{bmatrix}$$

The inequality (6.10) can be represented equivalently as

$$\begin{pmatrix} x \\ u \end{pmatrix}^\intercal \tilde{Q}_1 \begin{pmatrix} x \\ u \end{pmatrix} \geq 0 \tag{6.11}$$

with

$$\tilde{Q}_1 = \begin{bmatrix} 0 & -\frac{k}{2}C^\intercal\Gamma \\ -\frac{k}{2}\Gamma C & -\Gamma \end{bmatrix},$$

and we interested in the subspace of variables (x, u) satisfying the inequality (6.11), namely, when

$$\begin{pmatrix} x \\ u \end{pmatrix}^\intercal \tilde{Q}_1 \begin{pmatrix} x \\ u \end{pmatrix} = -kx^\intercal C^\intercal \Gamma u - u^\intercal \Gamma u \geq 0.$$

Based on the S-procedure (Theorem 1.4) we may conclude the following.

Theorem 6.2 (Sufficient condition of absolute stability in time-domain) *The absolute global stability of the systems (5.1)-(5.6) with a nonlinear feedback $u = -\varphi(y)$ from the class $\mathcal{F}_{\Gamma,k}$ takes place if there exist the positive definite matrix P, constants q and $\tau \geq 0$ such that*

$$\boxed{\begin{array}{c} \tilde{Q}_\tau := \tilde{Q}_0 - \tau\tilde{Q}_1 = \\ \begin{bmatrix} -PA - A^\intercal P & -PB - \frac{q}{2}A^\intercal C^\intercal \Gamma + \tau\frac{k}{2}C^\intercal\Gamma \\ -B^\intercal P - \frac{q}{2}\Gamma CA + \tau\frac{k}{2}\Gamma C & -\frac{q}{2}\left(\Gamma CB + B^\intercal C^\intercal \Gamma\right) + \tau\Gamma \end{bmatrix} > 0. \end{array}} \tag{6.12}$$

Proof. As in scalar contrrol case the proof directly follows from Theorem 1.4. ■

6.3 Exercise

Exercise 6.1 *Check if the linear system*

$$\left.\begin{array}{c}
\dot{x}(t)=\begin{bmatrix}-1 & 1 & 0\\ 0 & -2 & 1\\ -1 & 0 & -0.5\end{bmatrix}x(t)+\begin{bmatrix}1 & 1\\ 0 & 1\\ 1 & 0\end{bmatrix}u\left(t\right), \\[2ex]
t\geq t_0=0,\ x\left(0\right)=x_0, \\[2ex]
y(t)=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}x(t), \\[2ex]
n=3,\ m=l=2
\end{array}\right\}$$

is absolutely asymptotically stable in the class $\mathcal{F}_{\Gamma,k}$ *of nonlinear feedbacks with*

$$\Gamma=0.3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\quad \text{and } k=2.$$