Lecture 2

Linear Matrix Inequalities

This lecture covers the definitions of Linear Matrix Inequalities (LMIs) with respect to a vector and a matrix, as well as their equivalence. The viability (feasibility) of LMI is explored, and all possible solutions are parametrized. The equivalent representation of various nonlinear matrix inequalities (such as matrix norm constraint, nonlinear trace norm constraint, Lyapunov inequality, algebraic Riccati-matrix Lurie's inequality) as LMIs is demonstrated. Here only the elements of the LMI theory, required for understanding of the following materials, are presented. More profound information on this theory can be found in [10] and [11].

2.1 Matrix inequality with respect to a vector and a matrix

Definition 2.1 (Matrix inequality with respect to a vector) *The matrix inequality*

$$\boxed{F(x) := F_0 + x_1 F_1 + ... + x_n F_n > 0} \tag{2.1}$$

is said to be a ***Linear (Affine) Matrix Inequality (LMI)*** *if*

- $$x = (x_1, ..., x_n) \in \mathbb{R}^n$$

  and

- F_0, F_1 ... F_n *are real symmetric* $n \times n$ *matrices, that is,*

$$F_i = F_i^{\mathsf{T}} \in \mathbb{R}^{n \times n}.$$

Definition 2.2 *Denote by $\mathcal{S}^n$ the set of all real symmetric $n \times n$ matrices, namely,*

$$\mathcal{S}^n = \left\{ M \mid M = M^{\intercal} \in \mathbb{R}^{n \times n} \right\} .$$

Notice that the space $\mathcal{S}^n$ is isomorphic to the standard Euclidian space $\mathbb{R}^m$ with

$$m = n(n+1)/2 .$$

Definition 2.3 (Matrix inequality with respect to a matrix) *A **linear matrix inequality** (LMI) with respect to a matrix argument X has the following form*

$$\boxed{0 < F(X) := S + GXH + H^{\intercal} X G^{\intercal}} \tag{2.2}$$

*where the matrices $X, S \in \mathbb{R}^{n \times n}$ are **symmetric** and $G, H \in \mathbb{R}^{n \times n}$ such that the function $F(X)$ is an **affine transformation** (mapping) from $\mathbb{R}^{n \times n}$ to $\mathbb{R}^{n \times n}$, that is,*

$$F : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n} .$$

This inequality means that $F(X)$ is positive definite, i.e.,

$$u^{\intercal} F(X) u > 0$$

for all nonzero $u \in \mathbb{R}^n$.

Definition 2.4 *A **non strict LMI** has the form*

$$F(X) \geq 0 . \tag{2.3}$$

Both inequalities (2.2) and (2.3) are closely related since the last one is equivalent the following inequality

$$\tilde{F}(X) := F(X) + Q \geq Q > 0,$$

where $Q \in \mathbb{R}^{n \times n}$ is any positive definite matrix. So, $\tilde{F}(X)$ has the same form as (2.2), but with $\tilde{S} = S + Q$. In view of that **without loss of generality we will consider below only strict LMI** (2.2).

Multiple LMI's

$$F^{(1)}(X) > 0, ..., F^{(p)}(X) > 0 \tag{2.4}$$

can be expressed as a single LMI using the block-diagonal representation

$$\operatorname{diag}\left(F^{(1)}(X), ..., F^{(p)}(X)\right) > 0\,. \tag{2.5}$$

Therefore we will make no distinction between a set of LMI's (2.4) and a single LMI (2.2).

Remark 2.1 (on the equivalence of vector and matrix LMIs) *Let* $\{E_1, ..., E_m\}$ *be a basis in* $\mathcal{S}^n$*, that is, any matrix* $X \in \mathcal{S}^n$ *can be represented as*

$$X = \sum_{i=1}^{m} x_i E_i$$

and, hence, the LMI (2.2) can be represented as

$$0 < F(X) := S + G\left(\sum_{i=1}^{m} x_i E_i\right) H + H^{\intercal}\left(\sum_{i=1}^{m} x_i E_i^{\intercal}\right) G^{\intercal}$$

$$= S + \sum_{i=1}^{m} x_i \left(G E_i H + H^{\intercal} E_i^{\intercal} G^{\intercal}\right) = S + \sum_{i=1}^{m} x_i F_i\,.$$

This means that the matrix LMI (2.2) consides with the vector LMI (2.1) if take

$$F_i := G E_i H + H^{\intercal} E_i^{\intercal} G^{\intercal},\ i = 1, ..., m$$

and

$$F_0 = S.$$

For example, if $n = 2$ *we may take the basis as* $(m = 3)$

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ E_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and, as the result, we get

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} = x_1 E_1 + x_2 E_2 + x_3 E_3 =$$

$$x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

2.2 LMI´s feasibility

Here we consider the neccesary conditions for the feasibility of the LMI of the special form (which is the most common in Modern Control Theory):

$$\boxed{\Psi + P^{\intercal} X^{\intercal} Q + Q^{\intercal} X P < 0,} \tag{2.6}$$

where

$\Psi \in \mathcal{S}^n$ is a given symmetric $(n \times n)$-matrix,

$P \in \mathbb{R}^{l \times n}$, $Q \in \mathbb{R}^{k \times n}$ are given matrices of the orders $(l \times n)$ and $(k \times n)$, respectively,

$X \in \mathbb{R}^{k \times l}$ is a unknown matrix of the order $(k \times l)$.

Here we are interested when the LMI (2.6) is feasible (has a solution) wih respect to matrix X (the original version see in [5]).

Theorem 2.1 (on the feasibility of LMI's)

1) If

$$\text{rank}P = n \textit{ and } \text{rank}Q := r_Q < n\,, \tag{2.7}$$

then the LMI (2.6)

$$\Psi + P^{\intercal} X^{\intercal} Q + Q^{\intercal} X P < 0$$

has a solution with respect to X if and only if

$$W_Q^{\intercal} \Psi W_Q < 0 \tag{2.8}$$

where the columns of the matrix W_Q constitute the basis of the kernel (right-nullspace)

$$\mathcal{N}(Q) = \ker Q := \{x \in \mathbb{R}^n \mid Qx = 0\} \tag{2.9}$$

of the matrix Q, that is, W_Q satisfies the condition

$$QW_Q = 0\,. \tag{2.10}$$

2) If

$$\text{rank}P := r_P < n \textit{ and } \text{rank}Q := r_Q < n\,, \tag{2.11}$$

then the LMI (2.6)

$$\Psi + P^{\intercal} X^{\intercal} Q + Q^{\intercal} X P < 0$$

has a solution with respect to X if and only if

$$W_P^\intercal \Psi W_P < 0 \text{ and } W_Q^\intercal \Psi W_Q < 0 \tag{2.12}$$

where the columns of the matrix W_P constitute the basis of the kernal (left-nullspace)

$$\mathcal{N}(P) = \ker P := \{x \in \mathbb{R}^n \mid Px = 0\}$$

of the matrix P, that is, W_P, analogousely to (2.10), satisfies the condition

$$PW_P = 0\,. \tag{2.13}$$

Proof of this theorem see in Appendix to this chapter.

The following lemma represents the conditions of feasibility of LMI (2.6) equivalent to the conditions of Theorem 2.1.

Lemma 2.1 (On the feasibility of the basic LMI)

- *For the case (2.7) the LMI (2.6) has a solution with respect to X if and only if there exists a parameter $\mu > 0$ such that*

$$\Psi - \mu Q^\intercal Q < 0\,. \tag{2.14}$$

- *For the case (2.11) the LMI (2.6) has a solution with respect to X if and only if there exists a parameter $\mu > 0$ such that*

$$\Psi - \mu P^\intercal P < 0 \text{ and } \Psi - \mu Q^\intercal Q < 0\,. \tag{2.15}$$

Proof. The results of this statement directly follows from Finsler's lemma 1.2.

1) In the case (2.7), by Theorem 1.2, the feasibility criterion is given by

$$W_Q^\intercal \Psi W_Q < 0,\ QW_Q = 0$$

By Corolary 1.5 the conditions above are equivalent to the following statement:

$$\Psi - \mu Q^\intercal Q < 0$$

2) In the case (2.11) by Theorem 1.2 the LMI (2.6) is feasible if and only if

$$W_P^\intercal \Psi W_P < 0,\ QW_P = 0 \text{ and } W_Q^\intercal \Psi W_Q < 0,\ QW_Q = 0$$

which by Corollary 1.5 is equivalent to (2.15). ■

2.3 Parametrization of all solutions

Here we will discuss the possibility to parametrize all solutions X of the LMI (2.6) following [5]. To do that let us represent P and Q in the form (QR-decomposition in terms of Matlab description)

$$P = P_L P_R, \; Q = Q_L Q_R \tag{2.16}$$

where all matrices P_L, P_R, Q_L and Q_R are the matrices of a full rank. For example, one can take as P_L (or Q_L) any r_P (or r_Q) linear independent columns of the matrix P (or Q). Then any other j-column of P may be represented as a linear combination of the columns of P_L so that the j-column of P be such linear combination with the coefficients forming this j-column. By the fact that P_L, P_R, Q_L and Q_R has a full ranks, it follows that

$$P_L P_L^{\intercal} > 0, \; P_R P_R^{\intercal} > 0, \; Q_L Q_L^{\intercal} > 0, \; Q_R Q_R^{\intercal} > 0. \tag{2.17}$$

Theorem 2.2 (on the parametrization of all solutions) *Let the LMI (2.6) be feasible. Then there exist a large enough constant $\mu > 0$ and matrices $Z \in \mathbb{R}^{k \times l}$ and L (of the corresponding dimensions), satisfying the condition*

$$LL^{\intercal} < I_{r_Q \times r_Q} = \operatorname{diag}\left\{ \underbrace{1, ..., 1}_{r_Q} \right\},$$

such that any solution $X \in \mathbb{R}^{k \times l}$ of the LMI (2.6) may be presented as

$$\boxed{X = \left(Q_L^{\intercal}\right)^{+} K P_L^{+} + Z - \left(Q_L^{\intercal}\right)^{+} Q_L^{\intercal} Z P_L P_L^{+}} \tag{2.18}$$

where

$$\left.\begin{array}{c} K = \mu \left(S^{1/2} L - Q_R \Phi P_R^{\intercal}\right) \left(P_R \Phi P_R^{\intercal}\right)^{-1/2} \\ S = \mu^{-1} I_{r_Q \times r_Q} - Q_R \Phi \left[\Phi^{-1} - P_R^{\intercal} \left(P_R \Phi P_R^{\intercal}\right)^{-1} P_R\right] \Phi Q_R^{\intercal} \\ \Phi^{-1} = \mu Q_R^{\intercal} Q_R - \Psi > 0, \end{array}\right\} \tag{2.19}$$

and the operator $[\cdot]^{+}$ corresponds to Moore-Penrose's pseudo-invers operation, that is,

$$P_L^{+} = P_L^{\intercal} \left(P_L P_L^{\intercal}\right)^{-1}, \; Q_L^{+} = \left(Q_L Q_L^{\intercal}\right)^{-1} Q_L.$$

Proof of this theorem see also in Appendix to this chapter.

2.4 Nonlinear matrix inequalities equivalent to LMI

In this subection we follow [6].

2.4.1 Matrix norm constraint

The **matrix norm constraint**

$$\boxed{\|Z(X)\| < 1} \tag{2.20}$$

where $Z(X) \in \mathbb{R}^{n\times q}$ depends affinely on X, that is,

$$Z(X) = Z_0 + Z_1 X Z_2 + Z_3 X^{\mathsf{T}} Z_4\,,$$

or, equivalently,

$$I_{n\times n} - Z(X)Z^{\mathsf{T}}(X) > 0$$

is represented as (following to the Schur's complement 1.1)

$$\boxed{\begin{bmatrix} I_{n\times n} & Z(X) \\ Z^{\mathsf{T}}(X) & I_{n\times n} \end{bmatrix} > 0} \tag{2.21}$$

2.4.2 Nonlinear weighted norm constraint

The **nonlinear weighted norm constraint**

$$\boxed{c^{\mathsf{T}}(X)P^{-1}(X)\,c(X) < 1} \tag{2.22}$$

(where $c(X) \in \mathbb{R}^n$, $0 < P(X) \in \mathbb{R}^{n\times n}$ depend affinely on X) is expressed by the Schur's complement (Theorem 1.1) as the following LMI

$$\boxed{\begin{bmatrix} P(X) & c(X) \\ c^{\mathsf{T}}(X) & 1 \end{bmatrix} > 0.} \tag{2.23}$$

2.4.3 Nonlinear trace norm constraint

The **nonlinear trace norm constraint**

$$\boxed{\mathrm{Tr}\big(S^{\mathsf{T}}(X)P^{-1}(X)\,S(X)\big) < 1} \tag{2.24}$$

(where $S(X) \in \mathbb{R}^{n\times q}$, $0 < P(X) \in \mathbb{R}^{n\times n}$ depend affinely on X) is handled by introducing a new (slack) variable $Q = Q^{\mathsf{T}} \in \mathbb{R}^{p\times p}$, satisfying

$$S^{\mathsf{T}}(X)P^{-1}(X)\,S(X) < Q,\ \mathrm{Tr}\,(Q) < 1,$$

and by the Schur's complement (Theorem 1.1) the following LMI in X:

$$\boxed{\operatorname{Tr}(Q) < 1, \begin{bmatrix} Q & S^{\intercal}(X) \\ S(X) & P(X) \end{bmatrix} > 0.} \tag{2.25}$$

2.4.4 Lyapunov's inequality

The **Lyapunov's inequality**

$$\boxed{XA + A^{\intercal}X < 0} \tag{2.26}$$

where $A \in \mathbb{R}^{n\times n}$ is a stable matrix and already has the LMI formate.

2.4.5 Algebraic Riccati - Lurie's matrix inequality

The **algebraic Riccati - Lurie's matrix inequality**

$$\boxed{XA + A^{\intercal}X + XBR^{-1}B^{\intercal}X + Q < 0} \tag{2.27}$$

where $A, B, Q = Q^{\intercal}, R = R^{\intercal} > 0$ are given matrices of appropriate sizes and $X = X^{\intercal}$ is variable, is a quadratic matrix inequality in X. By the Schur's complement (Theorem 1.1) it may be represented as the following LMI:

$$\boxed{\begin{bmatrix} -XA - A^{\intercal}X - Q & XB \\ B^{\intercal}X & R \end{bmatrix} > 0.} \tag{2.28}$$

2.5 Appendix

2.5.1 Some simple properties of Linear Matrix Equations

In this course two simple lemmas [5], given below, will be use throughout.

Lemma 2.2 *If the matrix equation*

$$AX = C \tag{2.29}$$

(with $A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{m\times q}$) is resolvable with respect to unknown matrix $X \in \mathbb{R}^{n\times q}$, then among its solutions there exists a solution $\mathring{X}$ of a minimal rank such that

$$\operatorname{rank}\mathring{X} = \operatorname{rank}C := r_c, \tag{2.30}$$

which can be presented as

$$\mathring{X} = VC, \tag{2.31}$$

where $V \in \mathbb{R}^{n\times m}$ is some matrix.

Proof. Without any the loss of generality, one may suggest that the first r_c columns of the matrix C are linearly independent and other ones are linear combinations of these first ones. This exactly means that

$$C = \left[C_1 \vdots C_2\right], \; C_2 = C_1 D$$

for some matrix $D \in R^{(m-r_c)\times q}$. Represent X in the form

$$X = \left[X_1 \vdots X_2\right], \; X_1 \in \mathbb{R}^{n\times r_c}, X_2 \in \mathbb{R}^{(n-r_c)\times q},$$

where X_1 is a solution of (2.29), i.e.,

$$AX_1 = C_1,$$

so that the columns of X_1 are linearly independent. Define

$$\mathring{X}_2 = X_1 D$$

satisfying

$$A\mathring{X}_2 = C_2.$$

Then we can say that thematrix $\mathring{X} = \left[X_1 \vdots \mathring{X}_2\right]$ can be taken as a solution of (2.29) with a minimal rank. Since we have $A\mathring{X} = C$, then the matrix C is a linear combination of the rows of the matrix $\mathring{X}$. And inverse, by (2.30), it follows that the rows of $\mathring{X}$ is a linear combination of the rows of the matrix C, that can be expressed as (2.31). ■

Lemma 2.3 *The matrix equation*

$$AXB = C \tag{2.32}$$

is feasible (resolvable) with respect to the matrix X if and only if the following two matrix equations

$$AY = C, \; ZB = C \tag{2.33}$$

are feasible with respect to the unknown matrices Y and Z.

Proof.

a) *Necessity.* If X is a solution of (2.32) then obviously that

$$Y = XB \text{ and } Z = AX$$

satisfy (2.33).

b) *Sufficiency.* Let Y and Z be solutions of (2.33). Then, by lemma 2.2 the first equation in (2.33) has a solution $\mathring{Y}$ of a minimal rank r_c such that it may be presented as $\mathring{Y} = VC$. Hence,

$$C = A\mathring{Y} = AVC = AVZB$$

and, as the result, the matrix $X := VZ$ be a solution of (2.32). ■

2.5.2 Proofs of the main theorems on LMI's

Proof of Theorem 2.1.

1) Consider first the case (2.7)

Necessity. Suppose that (2.6) fulfilled. Multiplying (2.6) by $W_Q^\intercal$ from left and by W_Q we obtain (2.8).

Sufficiency. Suppose that (2.8) fulfilled. Let us represent the space R^n as

$$\mathcal{R}(Q) \oplus \mathcal{N}(Q),$$

where

$$\mathcal{R}(Q) = \mathrm{Im}A := \{y \in \mathbb{R}^k : y = Qx,\ x \in \mathbb{R}^n\}$$

is the image (or range) of the matrix Q and $N(Q)$ is its kernal (2.9). Select the corresponding basis in R^n such that the matrix will have the following presentation

$$Q = \left[Q_1 \vdots 0_{k\times(n-r_Q)}\right],$$

where $Q_1 \in R^{k\times r_Q}$ has the full rank. In this basis the matrices P and Ψ have the following structure

$$P = \left[P_1 \vdots P_2\right],\ \Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^\intercal & \Psi_{22} \end{bmatrix},$$

where

$$P_1 \in \mathbb{R}^{k\times r_Q},\ P_2 \in \mathbb{R}^{k\times(n-r_Q)},\ \Psi_{11} \in \mathbb{R}^{k\times k},\ \Psi_{22} \in \mathbb{R}^{r_Q\times r_Q}$$

Remember that the matrix W_Q has the maximal rank r_Q (since it constitutes a basis) and satisfies (2.10). That's why W_Q may be taken as

$$W_Q = \begin{bmatrix} 0_{k\times r_Q} \\ I_{r_Q\times r_Q} \end{bmatrix} \text{ or } W_Q^\intercal = \begin{bmatrix} 0_{r_Q\times k} & I_{r_Q\times r_Q} \end{bmatrix} \tag{2.34}$$

Then the condiion (2.8) may be rewritten as

$$\Psi_{22} < 0$$

since

$$W_Q^\intercal \Psi W_Q =$$

$$\begin{bmatrix} 0_{r_Q\times k} & I_{r_Q\times r_Q} \end{bmatrix} \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^\intercal & \Psi_{22} \end{bmatrix} \begin{bmatrix} 0_{k\times r_Q} \\ I_{r_Q\times r_Q} \end{bmatrix}$$

$$= \begin{bmatrix} 0_{r_Q\times k} & I_{r_Q\times r_Q} \end{bmatrix} \begin{bmatrix} \Psi_{12} \\ \Psi_{22} \end{bmatrix} = \Psi_{22}.$$

Moreover, the main inequality becomes

$$\begin{bmatrix} \Psi_{11} + Q_1^\intercal X P_1 + P_1^\intercal X Q_1 & \Psi_{12} + Q_1^\intercal X P_2 \\ \Psi_{12}^\intercal + P_2^\intercal X Q_1 & \Psi_{22} \end{bmatrix} < 0$$

According to Lemma 2.3, for a given matrix $K = \begin{bmatrix} K_1 \vdots K_2 \end{bmatrix}$ the matrix equation

$$Q_1^\intercal X P_1 = K \tag{2.35}$$

has a solution with respect to X if and only if the following two matrix equations

$$Q_1^\intercal Y = K \text{ and } ZP = ZP = Z\begin{bmatrix} P_1 \vdots P_2 \end{bmatrix} = K = \begin{bmatrix} K_1 \vdots K_2 \end{bmatrix}$$

are feasible. Since $(r_Q \times k)$ - matrix $Q_1^\intercal$ has the rank $r_Q \leq k$ and $(l \times n)$ - matrix $P = \begin{bmatrix} P_1 \vdots P_2 \end{bmatrix}$ has rank$P = n$ both equations above are resolvable with respect to the matrices Y and Z, then always there exist X fulfilling (2.35), since all three blocks may be done any of you wish selecting X.

2) Now consider the case (2.11).

Necessity. Suppose that (2.6) fulfilled. Multiplying, first, (2.6) by $W_P^\intercal$ from left and by W_P from right, and then, analogously, multiplying, (2.6) by $W_Q^\intercal$ from left and by W_Q from right, we obtain (2.12).

Sufficiency. Suppose now that both inequalities (2.12) are fulfilled. Let us represent R^n as the direct sum:

$$\mathbb{R}^n = (\mathcal{N}(P) \diagdown [\mathcal{N}(P) \cap \mathcal{N}(Q)]) \oplus [\mathcal{N}(P) \cap \mathcal{N}(Q)]$$

$$\oplus (\mathcal{N}(Q) \diagdown [\mathcal{N}(P) \cap \mathcal{N}(Q)]) \oplus \mathcal{M}$$

where is the compliment of $[\mathcal{N}(P) \cap \mathcal{N}(Q)]$ such that

$$\mathbb{R}^n = [\mathcal{N}(P) \cap \mathcal{N}(Q)] \oplus \mathcal{M}$$

Selecting the specific basis, we will have the following representation

$$P = \left[0 \vdots 0 \vdots P_1 \vdots P_2 \right], \; Q = \left[Q_1 \vdots 0 \vdots 0 \vdots Q_2 \right]$$

$$\Psi = [\Psi_{ij}]_{i,j=1,2,3,4}$$

Obviously, in such format the matrices W_P and W_Q has the following forms

$$W_P = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \; W_Q = \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix}$$

and the inequalities (2.12) become

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12}^{\intercal} & \Psi_{22} \end{bmatrix} < 0, \; \begin{bmatrix} \Psi_{22} & \Psi_{23} \\ \Psi_{23}^{\intercal} & \Psi_{33} \end{bmatrix} < 0 \tag{2.36}$$

So, now we need to check the feasibility of the following matrix inequality

$$\Psi + P^{\intercal} X^{\intercal} Q + Q^{\intercal} X P =$$

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} + L_{11} & \Psi_{14} + L_{12} \\ \Psi_{12}^{\intercal} & \Psi_{22} & \Psi_{23} & \Psi_{24} \\ \Psi_{13}^{\intercal} + L_{11}^{\intercal} & \Psi_{23}^{\intercal} & \Psi_{33} & \Psi_{34} + L_{21}^{\intercal} \\ \Psi_{14}^{\intercal} + L_{12}^{\intercal} & \Psi_{24}^{\intercal} & \Psi_{34}^{\intercal} + L_{21} & \Psi_{44} + L_{22} + L_{22}^{\intercal} \end{bmatrix} < 0 \tag{2.37}$$

where

$$L_{ij} = Q_i^{\intercal} X P_j, \; i, j = 1, 2$$

First, show that the matrix equation

$$\begin{bmatrix} Q_1^{\intercal} \\ Q_2^{\intercal} \end{bmatrix} X \begin{bmatrix} P_1 & P_2 \end{bmatrix} = K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

is resolvable with respect to X for any matrix K of the corresponding size. Indeed, according to Lemma 2.3, to fulfill this demands it is necessary and sufficient to prove the feasibility of the following two matrix equations

$$\begin{bmatrix} Q_1^{\intercal} \\ Q_2^{\intercal} \end{bmatrix} Y = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \; Z \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{2.38}$$

Notice that since $(k \times r_Q)$ - matrix $\left[Q_1 \vdots Q_2 \right]$ has the rank $r_Q \leq k$ and $(l \times r_P)$ - matrix $\left[P_1 \vdots P_2 \right]$ has the rank $r_P \leq l$, both these these equations are resolvable. Therefore, for any K_{ij} $(i, j = 1, 2)$ there exists the matrix X such that the relations (2.38) hold.

Define now

$$\Phi := \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} + L_{11} \\ \Psi_{12}^{\intercal} & \Psi_{22} & \Psi_{23} \\ \Psi_{13}^{\intercal} + L_{11}^{\intercal} & \Psi_{23}^{\intercal} & \Psi_{33} \end{bmatrix}$$

Then, by the Schur's complement (Theorem 1.1), the relation (20.22) is fulfilled if and only if

$$\Phi < 0$$

$$(\Psi_{44} + L_{22} + L_{22}^{\intercal}) - \begin{bmatrix} \Psi_{14} + L_{12} \\ \Psi_{24} \\ \Psi_{34} + L_{21}^{\intercal} \\ \Psi_{44} + L_{22} + L_{22}^{\intercal} \end{bmatrix}^{\intercal} \Phi^{-1} \begin{bmatrix} \Psi_{14} + L_{12} \\ \Psi_{24} \\ \Psi_{34} + L_{21}^{\intercal} \\ \Psi_{44} + L_{22} + L_{22}^{\intercal} \end{bmatrix} < 0$$

Fulfilling the second inequality one can provide by the corresponding selection of L_{22}. So, to finish the proof it is sufficient demonstrate that $\Phi < 0$. Taking in to account that by (2.36) $\Psi_{22} < 0$, consider the corresponding quadratic form

$$z^{\intercal} \Phi z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}^{\intercal} \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} + L_{11} \\ \Psi_{12}^{\intercal} & \Psi_{22} & \Psi_{23} \\ \Psi_{13}^{\intercal} + L_{11}^{\intercal} & \Psi_{23}^{\intercal} & \Psi_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} =$$

$$\left(z_2 + \Psi_{22}^{-1} \Psi_{12}^{\intercal} z_1 + \Psi_{22}^{-1} \Psi_{23} z_3 \right)^{\intercal} \Psi_{22} \left(z_2 + \Psi_{22}^{-1} \Psi_{12}^{\intercal} z_1 + \Psi_{22}^{-1} \Psi_{23} z_3 \right) +$$

$$\begin{pmatrix} z_1 \\ z_3 \end{pmatrix}^{\intercal} \begin{bmatrix} \Psi_{11} - \Psi_{12} \Psi_{22}^{-1} \Psi_{12}^{\intercal} & \Psi_{13} + L_{11} - \Psi_{12} \Psi_{22}^{-1} \Psi_{23} \\ \left[\Psi_{13} + L_{11} - \Psi_{12} \Psi_{22}^{-1} \Psi_{23} \right]^{\intercal} & \Psi_{33} - \Psi_{23}^{\intercal} \Psi_{22}^{-1} \Psi_{23} \end{bmatrix} \begin{pmatrix} z_1 \\ z_3 \end{pmatrix}.$$

The first term is negative since $\Psi_{22} < 0$. Taking L_{11}, for example, as

$$\Psi_{13} + L_{11} - \Psi_{12} \Psi_{22}^{-1} \Psi_{23} = 0$$

we obtain that the second term is negative too, since again by the Schur's complement (Lemma 1.1), applied to (2.36), it follows that

$$\Psi_{11} - \Psi_{12} \Psi_{22}^{-1} \Psi_{12}^{\intercal} < 0$$

and

$$\Psi_{33} - \Psi_{23}^{\intercal} \Psi_{22}^{-1} \Psi_{23} < 0$$

Theorem is proven. ■

Proof of Theorem 2.2.

a) Let X be a solution of (2.6). Then $K = Q_L^{\intercal} X P_L$ exists and satisfies

$$\Psi + P_R^{\intercal} K^{\intercal} Q_R + Q_R^{\intercal} K P_R < 0$$

and, hence, there exists large enough $\mu > 0$ such that

$$\Psi + P_R^{\intercal} K^{\intercal} Q_R + Q_R^{\intercal} K P_R + \mu^{-1} P_R^{\intercal} K^{\intercal} K P_R < 0.$$

The last inequality can be rewritten as

$$\begin{aligned} 0 &> \mu \left(\mu^{-1} K P_R + Q_R\right)^{\intercal} \left(\mu^{-1} K P_R + Q_R\right) + \Psi - \mu Q_R^{\intercal} Q_R \\ &= \mu \left(\mu^{-1} K P_R + Q_R\right)^{\intercal} \left(\mu^{-1} K P_R + Q_R\right) - \Phi^{-1}. \end{aligned} \tag{2.39}$$

So, for large enough μ we have

$$\Phi^{-1} = \mu Q_R^{\intercal} Q_R - \Psi > 0.$$

b) By Schur' complement Theorem 1.1, the property $\Phi > 0$ and (2.39) are equivalent to the inequality

$$\begin{bmatrix} \mu^{-1} I_{r_Q \times r_Q} & \mu^{-1} K P_R + Q_R \\ \left(\mu^{-1} K P_R + Q_R\right)^{\intercal} & \Phi^{-1} \end{bmatrix} > 0,$$

which in turn is equivalent to the following one

$$\mu^{-1} I_{r_Q \times r_Q} > \left(\mu^{-1} K P_R + Q_R\right) \Phi \left(\mu^{-1} K P_R + Q_R\right)^{\intercal}.$$

Transforming the last inequality to

$$\bar{L} \left(P_R \Phi P_R^{\intercal}\right) \bar{L}^{\intercal} < S \tag{2.40}$$

with

$$\bar{L} = \mu^{-1} K + Q_R \Phi P_R^{\intercal} \left(P_R \Phi P_R^{\intercal}\right)^{-1}$$

and

$$S = \mu^{-1} I_{r_Q \times r_Q} - Q_R \Phi \left[\Phi^{-1} - P_R^{\intercal} \left(P_R \Phi P_R^{\intercal}\right)^{-1} P_R\right] \Phi Q_R^{\intercal}$$

for large enough μ, one can see that $S > 0$. So, the basic LMI (2.6) is equivalent to (2.40), which is equivalent to the condition $LL^\intercal < I_{r_Q \times r_Q}$:

$$LL^\intercal = S^{-1/2}\bar{L}\left(P_R \Phi P_R^\intercal\right)\bar{L}^\intercal S^{-1/2} < I_{r_Q \times r_Q}.$$

From the definition of $\bar{L}$ one can find K as in (2.19):

$$\bar{L} = S^{1/2}L\left(P_R \Phi P_R^\intercal\right)^{-1/2}$$

$$K = \mu\bar{L} - \mu Q_R \Phi P_R^\intercal \left(P_R \Phi P_R^\intercal\right)^{-1} =$$

$$\mu S^{1/2}L\left(P_R \Phi P_R^\intercal\right)^{-1/2} - \mu Q_R \Phi P_R^\intercal \left(P_R \Phi P_R^\intercal\right)^{-1}$$

But

$$K = Q_L^\intercal X P_L, \tag{2.41}$$

which implies (2.18) for any matrix $Z \in \mathbb{R}^{k \times l}$ of the same dimensions as X. The last step can be verified by the direct substitution of (2.18) in (2.41). ■

2.6 Examples

Example 2.1 *Check to see if the LMI*

$$\left.\begin{array}{c} \Psi + P^\intercal X^\intercal Q + Q^\intercal X P < 0 \\ \\ \Psi = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, P = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, Q = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \end{array}\right\} \tag{2.42}$$

has a solution. If the answer is yes, parametrize the entire solutions.

Solution. Notice that

$$\text{rank}P = n = 2 \text{ and } \text{rank}Q := r_Q = 1 < n = 2.$$

So, we may apply Theorem 2.1 (the version 1) and verify the condition (2.8)

$$W_Q^\intercal \Psi W_Q < 0.$$

In our case the nullspace basis is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, that is,

$$W_Q = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and hence

$$W_Q^{\intercal}\Psi W_Q = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1.$$

This means that the LMI (2.42) has at least one solution and the set of all possible solutions $X \in \mathbb{R}^{1\times 2}$ can be parametrized as in Theorem (2.2. In our case we may calculate (using QR - decomposition command in Matlab)

$$P = P_L P_R : P_L = P = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, P_R = I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$Q = Q_L Q_R = \begin{bmatrix} -1 & 1 \end{bmatrix} : Q_L = -1,\, Q_R = \begin{bmatrix} 1 & -1 \end{bmatrix},$$

so that

$$P_L^{+} = P_L^{\intercal}\left(P_L P_L^{\intercal}\right)^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}, P_L P_L^{+} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix},$$

$$Q_L^{+} = \left(Q_L Q_L^{\intercal}\right)^{-1} Q_L = -1,\, \left(Q_L^{\intercal}\right)^{+} Q_L^{\intercal} = 1.$$

Then, in view of (2.18),

$$\boxed{X = -\left(\mu^{1/2}L - \mu\begin{bmatrix} 1 & -1 \end{bmatrix}\Phi^{1/2}\right)\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} + Z\begin{bmatrix} -1 & 3 \\ 3 & -4 \end{bmatrix}},$$

where Z is any matrix from $\mathbb{R}^{1\times 2}$ and by (2.19)

$$\mu > 3,\ \Phi = \frac{1}{\mu - 2}\begin{bmatrix} \mu + 2 & \mu \\ \mu & \mu - 1 \end{bmatrix},\ L \in R^{1\times 2},\ \ \|L\| < 1$$

with

$$\Phi^{-1} = \mu Q_R^{\intercal} Q_R - \Psi = \mu\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} \mu - 1 & -\mu \\ -\mu & \mu + 2 \end{bmatrix} > 0$$
$$\text{iff (by Sylvester's criterion) } \mu > 3,$$

and (see (2.19))

$$S = \mu^{-1} I_{r_Q \times r_Q} - Q_R \Phi\left[\Phi^{-1} - P_R^{\intercal}\left(P_R \Phi P_R^{\intercal}\right)^{-1} P_R\right]\Phi Q_R^{\intercal} = \mu^{-1} I_{1\times 1} = \mu^{-1}.$$

Example 2.2 *To find at least one solution $X = X^{\intercal} \in R^{2\times 2}$ satisfying the matrix inequality*

$$\operatorname{Tr}\left(S^{\intercal}(X) P^{-1}(X)\, S(X)\right) < 1$$

with

$$P(X)=\begin{bmatrix}1&1\\1&4\end{bmatrix}X+X\begin{bmatrix}1&1\\1&4\end{bmatrix}-\begin{bmatrix}5&0\\0&5\end{bmatrix},$$
$$S(X)=X\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}.$$

Solution. By the Schur's complement the considered matrix inequality is equivalent to (2.25),

$$\begin{bmatrix}Q & S^{\intercal}(X)\\ S(X) & P(X)\end{bmatrix}>0,\ \operatorname{Tr}(Q)<1,$$

which in the open format is

$$\begin{bmatrix} Q & \begin{bmatrix}1 & -1\end{bmatrix}X+\begin{bmatrix}1 & 1\end{bmatrix}\\ X\begin{bmatrix}1\\-1\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix} & \begin{bmatrix}1&1\\1&4\end{bmatrix}X+X\begin{bmatrix}1&1\\1&4\end{bmatrix}-\begin{bmatrix}5&0\\0&5\end{bmatrix}\end{bmatrix}>0$$

with

$$\operatorname{Tr}(Q)<1,$$

Using the packages SEDUMI and YALMIP of Matlab we get

$$Q=0.5889,\ X=\begin{bmatrix}1.6667 & 0.8333\\0.8333 & 0.4167\end{bmatrix}.$$

2.7 Exercises

Exercise 2.1 *To find at least one $X\in R^{2\times 2}$ such that*

$$c^{\intercal}(X)P^{-1}(X)\,c(X)<1$$

for

$$c(X)=\begin{bmatrix}1\\-1\end{bmatrix}+X\begin{bmatrix}1\\1\end{bmatrix}$$

and

$$P=\begin{bmatrix}2&0\\0&2\end{bmatrix}+X\begin{bmatrix}2&1\\-1&1\end{bmatrix}+\begin{bmatrix}2&-1\\1&1\end{bmatrix}X$$

Exercise 2.2 *Check to see if the LMI*

$$\left.\begin{array}{c}\Psi+P^{\intercal}X^{\intercal}Q+Q^{\intercal}XP<0\\[2ex] \Psi=\begin{bmatrix}2&0\\0&-2\end{bmatrix},\ P=\begin{bmatrix}0&-1\\1&2\end{bmatrix},\ Q=\begin{bmatrix}-1&1\\0&0\end{bmatrix}\end{array}\right\}$$

has a solution. If the answer is yes, parametrize the entire solutions.