Lecture 18

Sliding Mode Observers

18.1 General observer for nonlinear systems

Consider a dynamic model given in the quasi-linear format:

$$\left.\begin{array}{c}\dot{x}(t) = Ax(t) + Bu + \zeta\left(x(t), t\right), \; x(0) = x_0 \text{ is given,} \\ \\ y(t) = Cx(t), \\ x(t), \zeta\left(x(t), t\right) \in \mathbb{R}^n, \; u \in \mathbb{R}^k, \; y(t) \in \mathbb{R}^m \\ \\ \left\|\zeta\left(x, t\right)\right\|^2 \leq d_0 + d_1 \left\|x\right\|^2,\end{array}\right\} \tag{18.1}$$

and the *full-order observer with the Luenberger's Sliding-Mode structure* given by

$$\left.\begin{array}{c}\dfrac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu + L\sigma(t) + L_s \mathrm{SIGN}\left(\sigma(t)\right), \\ \hat{x}(0) \text{ is given,} \\ \sigma(t) := y(t) - C\hat{x}(t) = -Ce(t), \; L, L_s \in \mathbb{R}^{n\times m}, \\ \\ \mathrm{SIGN}\left(\sigma\right) := \left(\mathrm{sign}\left(\sigma_1\right), ..., \mathrm{sign}\left(\sigma_m\right)\right)^{\intercal}, \\ \\ e_t := \hat{x}_t - x_t \text{ is the observation error.}\end{array}\right\} \tag{18.2}$$

We also assume that the system (18.1) is BIBO-stable, i.e.,

$$\left\|x(t)\right\|^2 \leq X_+ < \infty$$

implying

$$\left\|\zeta\left(x(t), t\right)\right\|^2 \leq d_0 + d_1 \left\|x(t)\right\|^2 \leq d_0 + d_1 X_+ := d$$

Theorem 18.1 *If in the observer (18.2)*

$$L_s := \frac{\mu}{2} P^{-1} C^{\intercal} \tag{18.3}$$

and the gain matrix L fulfills the matrix inequality

$$\left.\begin{array}{c} W(P, L \mid \alpha, \varepsilon) := \\ \\ \begin{bmatrix} P(A - LC) + (A - LC)^{\intercal} P + \alpha P & -P \\ -P & -\varepsilon I_{n \times n} \end{bmatrix} < 0 \end{array}\right\} \tag{18.4}$$

for some matrix $P = P^{\intercal} > 0$ and some positive scalars α and ε, then we may guarantee that

1) *the state estimation error e_t converges to a bounded zone, namely,*

$$\|e(t)\|^2 \leq \frac{\varepsilon d}{\alpha \lambda_{\min}(P)} + O\left(e^{-\alpha t}\right) \tag{18.5}$$

2)

$$\int_0^{\infty} \|\sigma_\tau\| \, d\tau < \infty$$

which means that

$$\|\sigma(t)\| \to 0$$

excepting the time moments (spikes) of zero-measure.

Proof. Define the Lyapunov function as $V(e(t)) = e^{\intercal}(t) P e(t)$. Then its derivative on trajectories of the system (18.1) is

$$\left.\begin{array}{c} \dot{V}(e(t)) = 2e^{\intercal}(t) P \dot{e}(t) = \\ \\ 2e^{\intercal}(t) P \left[(A - LC) e(t) + L_s \mathrm{SIGN}(\sigma(t)) - \zeta(x(t), t)\right] \\ = 2e^{\intercal}(t) P (A - LC) e(t) + \\ 2e^{\intercal}(t) P L_s \mathrm{SIGN}(\sigma(t)) - 2e^{\intercal}(t) P \zeta(x(t), t) \end{array}\right\} \tag{18.6}$$

Select $L_s$as in (18.3). Then, because of the relation

$$2e^{\intercal}(t) P L_s = \mu e^{\intercal}(t) C^{\intercal} = \mu (Ce(t))^{\intercal} = -\mu \sigma^{\intercal}(t),$$

it follows

$$2e^{\intercal}(t) P L_s \mathrm{SIGN}(\sigma(t)) = -\mu \sigma^{\intercal}(t) \mathrm{SIGN}(\sigma(t)) = -\mu \sum_{i=1}^{m} |\sigma_i(t)|.$$

Using the estimate

$$\sigma^{\intercal}(t)\mathrm{SIGN}\left(\sigma(t)\right)=\sum_{i=1}^{m}\left|\sigma_i(t)\right|\geq\left\|\sigma(t)\right\|$$

we get

$$2e^{\intercal}(t)PL_s\mathrm{SIGN}\left(\sigma(t)\right)\leq-\mu\left\|\sigma(t)\right\|.$$

In view of that the right-hand side of the last identity (18.6) may be estimated as

$$\left.\begin{array}{c}
\dot{V}(e(t))\leq 2e^{\intercal}(t)P\left(A-LC\right)e(t)- \\[2ex]
2e^{\intercal}(t)P\zeta\left(x(t),t\right)-\mu\left\|\sigma(t)\right\|= \\[2ex]
\left(\begin{array}{c} e(t) \\ \zeta\left(x(t),t\right)\end{array}\right)^{\intercal}\left[\begin{array}{cc} P\left(A-LC\right)+ & -P \\ \left(A-LC\right)^{\intercal}P & \\ -P & 0\end{array}\right]\left(\begin{array}{c} e(t) \\ \zeta\left(x(t),t\right)\end{array}\right) \\[2ex]
-\mu\left\|\sigma(t)\right\|\leq\left(\begin{array}{c} e(t) \\ \zeta\left(x(t),t\right)\end{array}\right)^{\intercal}W\left(P,L\mid\alpha,\varepsilon\right)\left(\begin{array}{c} e(t) \\ \zeta\left(x(t),t\right)\end{array}\right) \\[2ex]
-\mu\left\|\sigma(t)\right\|-\alpha V(e(t))+\varepsilon\left\|\zeta\left(x(t),t\right)\right\|\leq \\[2ex]
\left(\begin{array}{c} e(t) \\ \zeta\left(x(t),t\right)\end{array}\right)^{\intercal}W\left(P,L\mid\alpha,\varepsilon\right)\left(\begin{array}{c} e(t) \\ \zeta\left(x(t),t\right)\end{array}\right) \\[2ex]
-\mu\left\|\sigma(e(t)\right\|-\alpha V(e(t))+\varepsilon d
\end{array}\right\}\quad(18.7)$$

Supposing $W\left(P,L\mid\alpha,\varepsilon\right)<0$ from (18.7) we get

$$\dot{V}(e_t)\leq-\mu\left\|\sigma(e(t)\right\|-\alpha V(e(t))+\varepsilon d.$$

For the new variable $\tilde{V}(t):=V(e(t))-\dfrac{\varepsilon d}{\alpha}$ it follows the inequalities

$$\frac{d}{dt}\tilde{V}(t)\leq-\mu\left\|\sigma(t)\right\|-\alpha\tilde{V}(t)\leq-\mu\left\|\sigma(t)\right\|\leq 0,$$

$$\frac{d}{dt}\tilde{V}(t)\leq-\mu\left\|\sigma(t)\right\|-\alpha\tilde{V}(t)\leq-\alpha\tilde{V}(t)\leq 0,$$

$$\mu\left\|\sigma(t)\right\|\leq-\frac{d}{dt}\tilde{V}(t),$$

implying

$$\mu \int_0^t \|\sigma(\tau)\| \, d\tau \leq -\left(\tilde{V}(t) - \tilde{V}(0)\right) =$$

$$V\left(e(0)\right) - V\left(e(t)\right) \leq V\left(e(0)\right) < \infty,$$

which for $t \to \infty$ lead to

$$\lambda_{\min}(P) \|e(t)\| \leq V(e(t)) \leq$$

$$V(e(0))e^{-\alpha t} + \frac{\varepsilon d}{\alpha}\left(1 - e^{-\alpha t}\right) = \frac{\varepsilon d}{\alpha} + O\left(e^{-\alpha t}\right)$$

and

$$\int_0^\infty \|\sigma(\tau)\| \, d\tau < \infty.$$

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18.2 SM observations for the class of mechanical models

18.2.1 Model of the system

Consider the class of mechanical systems given by the following dynamics

$$\left.\begin{array}{c} \dfrac{d^2}{dt^2}\bar{x}(t) = f\left(\bar{x}(t), \dfrac{d}{dt}\bar{x}(t), t\right) + \zeta\left(\bar{x}(t), t\right), \\ \\ \bar{x}(0) \text{ and } \dfrac{d}{dt}\bar{x}(0) \text{ are given,} \end{array}\right\} \tag{18.8}$$

where $\bar{x}(t) \in \mathbb{R}^n$ the systems states at time $t \geq 0$. Denoting $x_1 := \bar{x}(t) \in \mathbb{R}^n$ (and omitting the time-argument for simplicity) we can represent this system in the following extended form

$$\left.\begin{array}{c} \dot{x}_1 = x_2, \\ \\ \dot{x}_2 = f(x_1, x_2, t) + \zeta(x_1, t). \end{array}\right\} \tag{18.9}$$

Here $x_1 \in \mathbb{R}^n$ is the state vector, $x_2 \in \mathbb{R}^n$ is the velocities vector and $\zeta(x_1, t)$ is the uncertain term influencing the dynamics of the system (18.8).

18.2.2 Main assumptions

The following assumptions will be in force:

- the state variable x_1 is measurable only, that is,

$$y = x_1$$

- the velocity vector x_2 is bounded:

$$\|x_2\| \leq x_2^+ < \infty$$

- the uncertain term $\zeta(x_1, t)$ as well as the own dynamics $f(x_1, x_2, t)$ **may be uknown** but both are bounded as

$$\left.\begin{array}{c} \|f(x_1, x_2, t)\|^2 \leq c_0 + c_1 \|x\|^2, \\ \\ \|\zeta(x_1, t)\|^2 \leq d_0 + d_1 \|x\|^2. \end{array}\right\} \tag{18.10}$$

18.2.3 Observer structure

Select the observer structure as follows:

$$\frac{d}{dt}\hat{x}_1 = v$$

where $\hat{x}_1$ is the estimate of the state vector x_1 and v is the correction term to be designed. Notice that $\hat{x}_1$ is the auxiliary variable, since, in fact, the state estimation is not required because of the x_1 availability. Let

$$v = -\rho \mathrm{SIGN}(e_1), \ \rho > 0, \tag{18.11}$$

where

$$\mathrm{SIGN}(e_1) := (\mathrm{sign}(e_{1,1}), ...\mathrm{sign}(e_{1,n}))^{\intercal}$$

and

$$e_1 := \hat{x}_1 - x_1 \tag{18.12}$$

is the error of the state estimate.

18.2.4 Equivalent control concept application

For

$$V(e_1) := \frac{1}{2}\|e_1\|^2$$

we have

$$\dot{V}(e_1) = e_1^{\intercal}\dot{e}_1 = e_1^{\intercal}(-\rho\mathrm{SIGN}(e_1) - x_2) =$$

$$-\rho e_1^{\intercal}\mathrm{SIGN}(e_1) - e_1^{\intercal}x_2 \le$$

$$-\rho\sum_{i=1}^{n}|e_{1,i}| + \|e_1\|x_2^{+} \le -\|e_1\|\left(\rho - x_2^{+}\right) =$$

$$-\sqrt{2}\left(\rho - x_2^{+}\right)\sqrt{V(e_1)},$$

which for $\rho > x_2^{+}$ implies

$$e_1 = 0 \text{ after } t_{reach} = \frac{\|e_0\|}{\left(\rho - x_2^{+}\right)}$$

To maintain $e_1 = 0$ for all $t \ge t_{reach}$ we need to fulfill the condition

$$\dot{e}_1 = v - x_2 = 0$$

defining the equivalent control $v = v_{eqiv}$ as

$$v_{eqiv} = x_2 \tag{18.13}$$

As it was mentioned above, v_{eqiv} **is unrealizable** (x_2 is not available), but can be approximated by the vector $\hat{v}_{eqiv}$ generated by the the following law-pas filter

$$\mu\frac{d}{dt}\hat{v}_{eqiv} + \frac{d}{dt}\hat{v}_{eqiv} = v = -\rho\mathrm{SIGN}(e_1),\ \mu \le 0.1$$

So, we have

$$\hat{v}_{eqiv} \simeq v_{eqiv},$$

and according to (18.13) we may define the velocity estimation $\hat{x}_2$ as

$$\hat{x}_2 = \hat{v}_{eqiv}\,. \tag{18.14}$$

18.3 Exercises

18.3.1 Exercises

Exercise 18.1 *For the system*

$$\left.\begin{array}{c} \dfrac{d^2}{dt^2}\bar{x}(t) = a_1\dfrac{\bar{x}(t)}{1+|\bar{x}(t)|} + a_2 \arctan\left(\dfrac{d}{dt}\bar{x}(t)\right) + d_0\sin(\omega t), \\ \text{with} \\ \bar{x}(0) = 1,\ \dfrac{d}{dt}\bar{x}(0) = 0,\ a_1 = -0.5,\ a_2 = 0.1,\ d_0 = 0.01,\ \omega = 10. \\ y(t) = \bar{x}(t) \end{array}\right\}$$

design the observer for $\dfrac{d}{dt}\bar{x}(t)$

a) using general Sliding mode approach (18.2);

b) using Equivalent Control Concept.

Demonstrate for both cases a) and b) the graphics for

$$x_1(t), \hat{x}_1(t), e_1(t) \text{ and } x_2(t), \hat{x}_2(t), e_2(t).$$

Exercise 18.2 *For the same system as in Exercise (17.1), supposing that only $x(t)$ is available, construct the **observers** of $\dot{x}(t)$ using **the standard observer structure** and **observer based on the equivalent control method**. As a controller use a sliding mode controller with $u(\hat{x})$.*