Lecture 17

Sliding mode control

17.1 Sliding mode surface as desired dynamics

Consider the special case where the function $f(t,x)$ is discontinuous on a smooth surface S given by the equation

$$\boxed{s(x) = 0,\ s : \mathbb{R}^n \to \mathbb{R},\ s(\cdot) \in C^1} \tag{17.1}$$

The surface separates its neighborhood (in $\mathbb{R}^n$) into domains $\mathcal{G}^+$ and $\mathcal{G}^-$. For $t = \text{const}$ and for the point x^* approaching the point $x \in S$ from the domains $\mathcal{G}^+$ and $\mathcal{G}^-$ let us suppose that the function $f(t,x^*)$ has the following limits:

$$\left.\begin{aligned} \lim_{(t,x^*)\in\mathcal{G}^-,\ x^*\to x} f(t,x^*) = f^-(t,x), \\ \lim_{(t,x^*)\in\mathcal{G}^+,\ x^*\to x} f(t,x^*) = f^+(t,x). \end{aligned}\right\} \tag{17.2}$$

Then by the Filippov's definition, $\mathcal{F}(t,x)$ is a linear segment joining the endpoints of the vectors $f^-(t,x)$ and $f^+(t,x)$. Two situations are possible:

- If for $t \in (t_1,t_2)$ this segment lies on one side of the plane $\mathcal{P}$ tangent to the surface S at the point x, the solutions for these t pass from one side of the surface S to the other one (see Fig.17.1 depicted at the point $x = 0$);

- If this segment intersects the plane $\mathcal{P}$, the intersection point is the endpoint of the vector $f^0(t,x)$ which defines the velocity of the motion

$$\boxed{\dot{x}_t = f^0(t,x_t)} \tag{17.3}$$

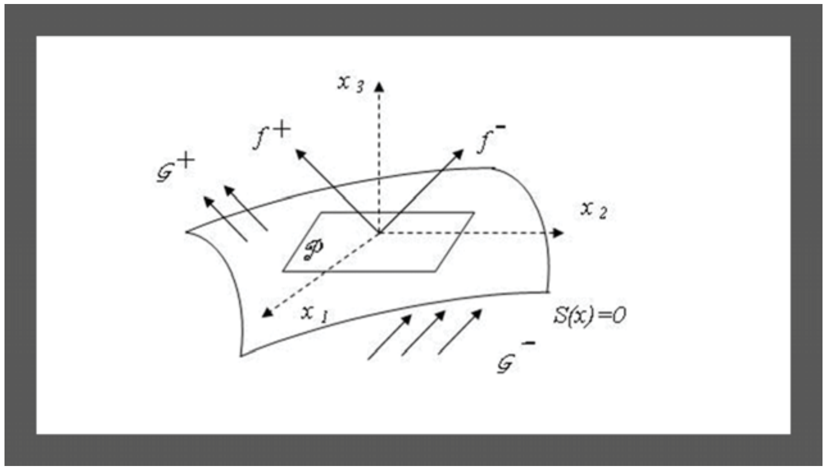

Figure 17.1: The sliding surface and the rate vector field at the point $x = 0$.

along the surface S in $\mathbb{R}^n$ (see Fig.17.2 depicted at the point $x = 0$). Such a solution, lying on S for all $t \in (t_1, t_2)$, is often called a **sliding motion** *(*or*,* ***mode****)*. Defining the projections of the vectors $f^-(t, x)$ and $f^+(t, x)$ to the surface S ($\nabla s(x) \neq 0$) as

$$p^-(t,x) := \left(\frac{\nabla s(x)}{\|\nabla s(x)\|}, f^-(t,x) \right),$$

$$p^+(t,x) := \left(\frac{\nabla s(x)}{\|\nabla s(x)\|}, f^+(t,x) \right)$$

one can find that when $p^-(t, x) < 0$ and $p^+(t, x) > 0$ we have that

$$f^0(t,x) = \alpha f^-(t,x) + (1-\alpha) f^+(t,x).$$

Here α can be easily found from the equation

$$\left(\nabla s(x), f^0(t,x)\right) = 0,$$

or, equivalently,

$$0 = \left(\nabla s(x), \alpha f^-(t,x) + (1-\alpha) f^+(t,x)\right)$$
$$= \alpha p^-(t,x) + (1-\alpha) p^+(t,x),$$

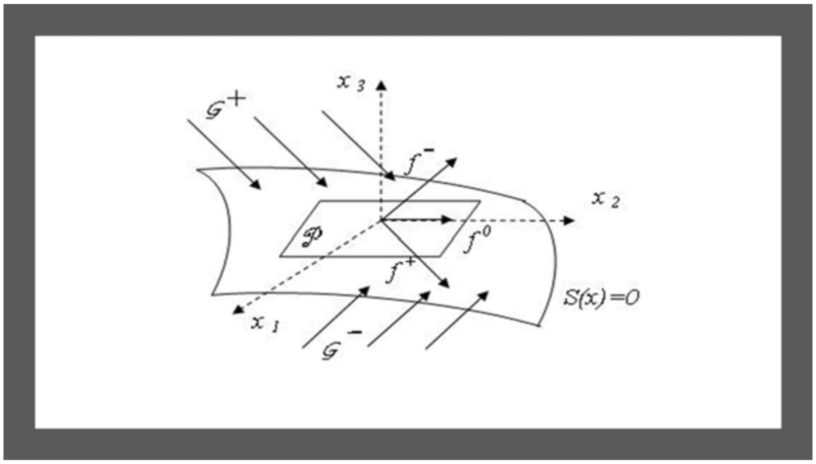

Figure 17.2: The velocity vector on a sliding surface.

which implies

$$\alpha = \frac{p^{+}(t,x)}{p^{+}(t,x) - p^{-}(t,x)}.$$

Finally, we obtain that

$$\boxed{\begin{array}{c} f^{0}(t,x) = \dfrac{p^{+}(t,x)}{p^{+}(t,x) - p^{-}(t,x)} f^{-}(t,x) + \\ \left(1 - \dfrac{p^{+}(t,x)}{p^{+}(t,x) - p^{-}(t,x)}\right) f^{+}(t,x) \end{array}} \tag{17.4}$$

Consider now in this subsection several examples demonstrating that a desired dynamic behavior of a controlled system may be expressed not only in the traditional manner, using some cost (or payoff) functionals as possible performance indices, but also representing a nominal (desired) dynamics in the form of a surface (or, manifold) in a space of coordinates.

17.1.1 First-order tracking system

Consider a first-order scalar system given by the following ODE:

$$\boxed{\dot{x}(t) = f(t, x(t)) + u(t)} \tag{17.5}$$

where u_t is a control action and $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is supposed to be bounded, that is,

$$|f(t, x(t))| \leq f^+ < \infty.$$

Assume that the desired dynamics (signal), which should be tracked, is given by a smooth function r_t ($|\dot{r}_t| \leq \rho$), such that the tracking error e_t is (see Fig.17.3)

$$e(t) := x(t) - r(t).$$

Select a desired surface s as follows

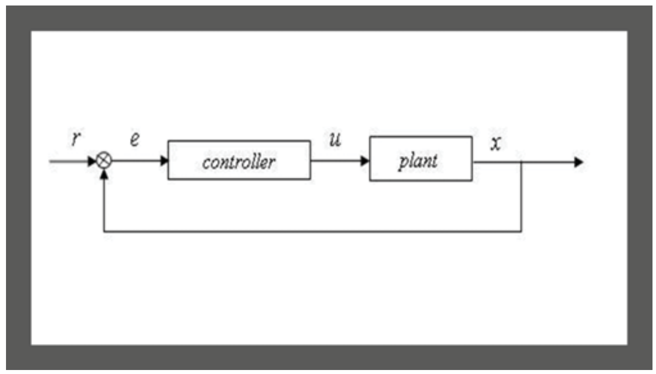

Figure 17.3: A tracking system.

$$\boxed{s(e) = e = 0} \tag{17.6}$$

that exactly corresponds to an "ideal tracking" process. Then, designing the control $u(t)$ as

$$\boxed{u(t) := -k\mathrm{sign}(e_t),}$$

we derive that

$$\dot{e}(t) = f(t, x(t)) - \dot{r}(t) - k\mathrm{sign}(e(t))$$

and for $V(e) = e^2/2$ one has

$$\dot{V}(e(t)) = e(t)\dot{e}(t) = e(t)[f(t, x(t)) - \dot{r}(t) - k\mathrm{sign}(e(t))] =$$

$$e(t)[f(t, x(t)) - \dot{r}(t)] - k|e(t)| \leq |e(t)|[f^+ + \rho] - k|e(t)| =$$

$$|e(t)|[f^+ + \rho - k] = -\sqrt{2}[k - f^+ - \rho]\sqrt{V(e(t))},$$

and, hence,

$$\sqrt{V\left(e(t)\right)} \leq \sqrt{V\left(e\left(0\right)\right)} - \frac{1}{\sqrt{2}}\left[k - f^{+} - \rho\right] t$$

So, taking $k > f^{+} + \rho$ implies the finite time convergence of $e(t)$ to the surface (17.6) with the reaching time

$$t_f = \frac{\sqrt{2V\left(e\left(0\right)\right)}}{k - f^{+} - \rho} = \frac{\left|e\left(0\right)\right|}{k - f^{+} - \rho}$$

(see Fig.17.4 and Fig.17.5).

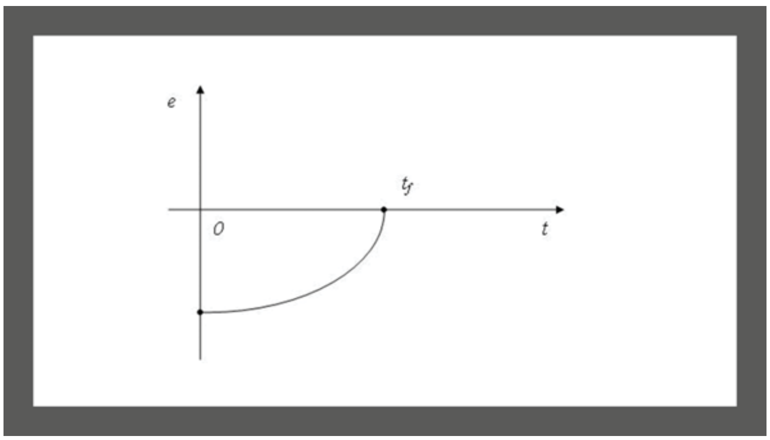

Figure 17.4: The finite time tracking to the surface $s = e = 0$.

17.1.2 Stabilization of a second order relay-system

Let us consider now a second order relay-system given by the following ODE

$$\left.\begin{array}{c} \ddot{x}(t) + a_2\dot{x}(t) + a_1 x(t) = u(t) + \xi(t), \\ \\ u(t) = -k\mathrm{sign}\left(s(t)\right) \text{ - the relay-control,} \\ \\ s = s\left(x(t), \dot{x}(t)\right) := \dot{x}(t) + cx(t),\ c > 0, \\ \\ \left|\xi_t\right| \leq \xi^{+} \text{ - a bounded unknown disturbance.} \end{array}\right\} \tag{17.7}$$

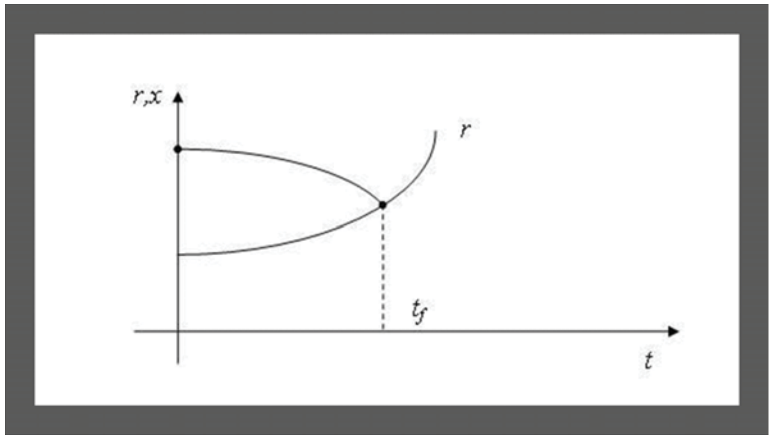

Figure 17.5: The finite time tracking the tragectories $r(t)$.

One may rewrite the dynamic ($x_1 := x$) as

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -a_1 x_1(t) - a_2 x_2(t) + u(t) + \xi(t), \\ u(t) &= -k \operatorname{sign}(x_2(t) + c x_1(t)). \end{aligned}\right\} \tag{17.8}$$

Select here the *sliding surface* s as

$$\boxed{s(x) = x_2 + c x_1,\ c > 0.}$$

So, the sliding motion, corresponding the dynamics

$$s := \dot{x}(t) + c x(t) = 0,$$

is given by (see Fig.17.6)

$$x_t = x_0 e^{-ct}.$$

Let us introduce the following Lyapunov function candidate:

$$\boxed{V(s) = s^2/2,}$$

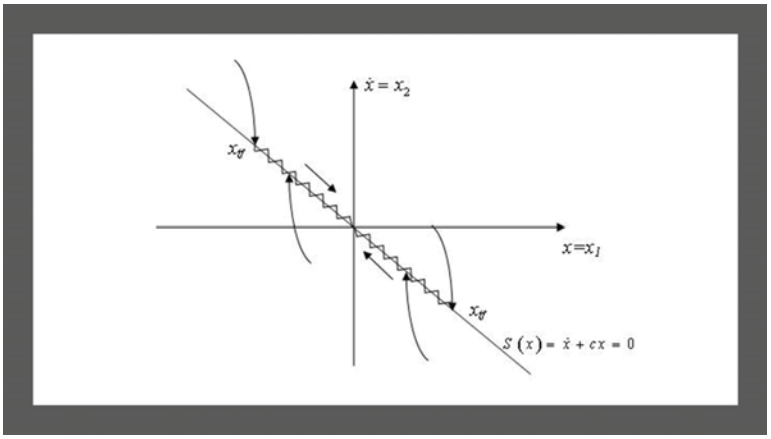

Figure 17.6: The sliding motion on the sliding surface $s(x) = x_2 + cx_1$.

for which we have:

$$\dot{V}(s) = s\dot{s} = s(x(t)) \left[\frac{\partial s(x(t))}{\partial x_1} \dot{x}_1(t) + \frac{\partial s(x_t)}{\partial x_2} \dot{x}_2(t) \right] =$$

$$s(x(t)) \left[cx_2(t) - a_1 x_1(t) - a_2 x_2(t) + u(t) + \xi(t) \right] \leq$$

$$|s(x(t))| \left[|a_1| |x_1(t)| + (c + |a_2|) |x_2(t)| + \xi^+ \right] - ks(x(t)) \operatorname{sign}(s(x(t)))$$

$$= -\left[k - |a_1| |x_1(t)| - (c + |a_2|) |x_2(t)| - \xi^+ \right] |s(x(t))| \leq 0,$$

if take

$$\boxed{k = |a_1| |x_1(t)| + (c + |a_2|) |x_2(t)| + \xi^+ + \rho, \rho > 0} \tag{17.9}$$

This implies

$$\dot{V}(s) \leq -\rho \sqrt{2V(s)}$$

and, hence, the reaching time t_f (see Fig.17.6) is

$$\boxed{t_f = \frac{\sqrt{2V(s(0))}}{\rho} = \frac{|\dot{x}(0) + cx(0)|}{\rho}.} \tag{17.10}$$

17.2 Equivalent control method

17.2.1 Equivalent control construction

Here a formal procedure will be described to obtain sliding equations along the intersection of sets of discontinuity for a nonlinear system given by

$$\boxed{\left.\begin{array}{c} \dot{x}(t) = f\left(t, x(t), u(t)\right), \\ \\ x(0) \text{ is given, } x(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}^r \end{array}\right\}} \tag{17.11}$$

and the manifold $\mathcal{M}$ (16.25) defined as

$$\boxed{S(x) = \left(S_1(x), ..., S_m(x)\right)^{\intercal} = 0,} \tag{17.12}$$

representing an intersection of m submanifolds $S_i(x)$ $(i = 1, ..., m)$.

Definition 17.1 *Hereinafter the control $u(t)$ will be referred to (according to [1]) as the* ***equivalent control*** *$u^{(eq)}(t)$ in the system (17.11) if it satisfies the equation*

$$\boxed{\left.\begin{array}{c} \dot{S}(x(t)) = G(x(t))\,\dot{x}(t) = G(x(t))\, f\left(t, x(t), u(t)\right) = 0 \\ \\ G(x(t)) \in \mathbb{R}^{m \times n}, \ G(x(t)) = \dfrac{\partial}{\partial x} S(x(t)) \end{array}\right\}} \tag{17.13}$$

It is quite obvious that, by virtue of the condition (17.13), a motion starting at $S(x(t_0)) = 0$ in time t_0 will proceed along the trajectories

$$\boxed{\dot{x}(t) = f\left(t, x(t), u^{(eq)}(t)\right)} \tag{17.14}$$

which lies on the manifold $S(x) = 0$.

Definition 17.2 *The above procedure is called the* ***equivalent control method*** *[1], [3], [5] and the equation (17.14), obtained as a result of applying this method, will be regarded as the* ***sliding mode equation*** *describing the motion on the manifold $S(x) = 0$.*

From the geometric viewpoint, the equivalent control method implies a replacement of the undefined discontinued control on the discontinuity boundary with a continuous control which directs the velocity vector in the system state space along the discontinuity surface intersection. In other words, it exactly realizes the velocity $f^0\left(t, x, u^{(eq)}(t)\right)$ (17.4) corresponding to the Filippov's definition of the differential inclusion in the point x.

Consider now the equivalent control procedure for an important particular case of a nonlinear system which is affine on u, the right-hand side of whose differential equation is a linear function of the control, that is,

$$\boxed{\dot{x}(t) = f\left(t, x(t)\right) + B\left(t, x(t)\right) u(t),} \tag{17.15}$$

where $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and $B : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^{n\times r}$ are all argument continuous vector and matrix, respectively, and $u(t) \in \mathbb{R}^r$ is a control action. The corresponding equivalent control should satisfies (17.13), namely,

$$\left.\begin{array}{c} \dot{S}\left(x(t)\right) = G\left(x(t)\right)\dot{x}(t) = G\left(x(t)\right) f\left(t, x(t), u(t)\right) = \\ \\ G\left(x(t)\right) f\left(t, x(t)\right) + G\left(x(t)\right) B\left(t, x(t)\right) u(t) = 0 \end{array}\right\} \tag{17.16}$$

Assuming that the matrix $G\left(x(t)\right) B\left(t, x(t)\right)$ is nonsingular for all x and t, one can find the equivalent control from (17.16) as

$$\boxed{u^{(eq)}(t) = -\left[G\left(x(t)\right) B\left(t, x(t)\right)\right]^{-1} G\left(x(t)\right) f\left(t, x(t)\right).} \tag{17.17}$$

Substitution this control into (17.15) yields the following ODE:

$$\boxed{\dot{x}(t) = \left[I_{n\times n} - \left[G\left(x(t)\right) B\left(t, x(t)\right)\right]^{-1} G\left(x(t)\right)\right] f\left(t, x(t)\right),} \tag{17.18}$$

which describes the sliding mode motion on the manifold $S\left(x\right) = 0$. Below the corresponding trajectories in (17.18) will be referred to as $x(t) = x^{(sl)}(t)$.

Remark 17.1 *If we deal with an uncertain dynamic model (17.11) with (17.15) when the function $f\left(t, x(t)\right)$ is not known a priory, then the equivalent control $u^{(eq)}(t)$* ***is not physically realizable****.*

Below we will show that $u^{(eq)}(t)$ may be successfully approximated (in some sense) by the output of the first order low-pass filter with the input equal to the corresponding sliding mode control.

17.2.2 Sliding mode motion

Let us try to stabilize the system (17.15) applying sliding mode approach. For the Lyapunov function

$$V\left(x\right) := \left\|S\left(x\right)\right\|^2/2,$$

considered on the trajectories of the controlled system (17.15), one has

$$\dot{V}(x(t)) = \left(S(x(t)), \dot{S}(x(t))\right) =$$

$$(S(x(t)), G(x(t)) f(t, x(t)) + G(x(t)) B(t, x(t)) u(t)) =$$

$$(S(x(t)), G(x(t)) f(t, x(t))) + (S(x(t)), G(x(t)) B(t, x(t)) u(t)) \leq$$

$$\|S(x(t))\| \|G(x(t)) f(t, x(t))\| + (S(x(t)), G(x(t)) B(t, x(t)) u(t)).$$

Taking $u(t)$ as a *sliding mode control*, i.e.,

$$\boxed{\left.\begin{array}{c} u(t) = u^{(sl)}(t), \\ u^{(sl)}(t) := -k(t)\left[G(x(t)) B(t, x(t))\right]^{-1} \mathrm{SIGN}(S(x(t))), \\ k_t > 0, \\ \mathrm{SIGN}(S(x(t))) := (\mathrm{sign}(S_1(x(t))), ..., \mathrm{sign}(S_m(x(t))))^{\intercal}, \end{array}\right\}} \tag{17.19}$$

we obtain

$$\dot{V}(x(t)) \leq \|S(x(t))\| \|G(x(t)) f(t, x(t))\| - k(t) \sum_{i=1}^{m} |S_i(x(t))|$$

that, in view of the inequality $\sum_{i=1}^{m} |S_i| \geq \|S\|$, implies

$$\dot{V}(x(t)) \leq -\|S(x(t))\| (k(t) - \|G(x(t)) f(t, x(t))\|)$$

Assuming that for all $t \geq 0$

$$\|f(t, x)\| \leq f_0 + f_1 \|x\| \tag{17.20}$$

we get the upper estimate

$$\dot{V}(x(t)) \leq -\|S(x(t))\| k(t) + \|S(x(t))\| \|G(x(t))\| \|f(t, x(t))\| \leq$$

$$-\|S(x(t))\| [k(t) - \|G(x(t))\| (f_0 + f_1 \|x(t)\|)]$$

The selection

$$\boxed{k(t) = \|G(x(t))\| (f_0 + f_1 \|x(t)\|) + \rho, \ \rho > 0} \tag{17.21}$$

gives

$$\dot{V}(x(t)) \leq -\rho \|S(x(t))\| = -\rho \sqrt{2V(x(t))}$$

that provides the reaching phase in time

$$\boxed{t_f = \frac{\sqrt{2V(x_0)}}{\rho} = \frac{\|S(x_0)\|}{\rho}} \tag{17.22}$$

Remark 17.2 *If the sliding motion on the manifold $S(x) = 0$ is stable, then there exists a constant $k^0 \in (0, \infty)$ such that*

$$\|G(x(t))\| (f_0 + f_1 \|x(t)\|) \leq k^0,$$

and hence, $k(t)$ (17.21) may be selected as a constant

$$\boxed{k(t) := k^0 + \rho.} \tag{17.23}$$

17.2.3 Low-pass filtering

As it follows from the presentation

$$u^{(sl)}(t) := -k(t) [G(x(t)) B(t, x(t))]^{-1} \operatorname{SIGN}(S(x(t))),$$

after the reaching phase, the dynamics of the controlled system around the sliding surface $S(x(t)) \simeq 0$ must have considerable "jumpings" (the **chattering effect**) because of the presence of the discontinuous function $\operatorname{SIGN}(S(x(t)))$ in the control action $u^{(sl)}(t)$.

To minimize the influence of the chattering effect arising after the reaching phase let us consider the property of the signal obtained as an output of a *low-pass filter* (the first order ODE) with the input equal to the sliding mode control, that is,

$$\boxed{\mu \dot{u}^{(av)}(t) + u^{(av)}(t) = u^{(sl)}(t), \;\; u^{(av)}(0) = 0, \, \mu > 0,} \tag{17.24}$$

where $u_t^{(sl)}$ is given by (17.19). The solution of (17.24) is

$$\boxed{u^{(av)}(t) = \frac{1}{\mu} \int_{s=0}^{t} e^{-(t-s)/\mu} u_s^{(sl)} ds} \tag{17.25}$$

The amplitude-frequency characteristic $A(\omega)$ of the filter is

$$\boxed{A(\omega) = \frac{1}{\sqrt{1 + (\mu\omega)^2}}, \, \omega \in [0, \infty)} \tag{17.26}$$

and its plot is depicted at Fig.17.7 for $\mu = 0.01$, where $y = A(\omega)$ and $x = \omega$.

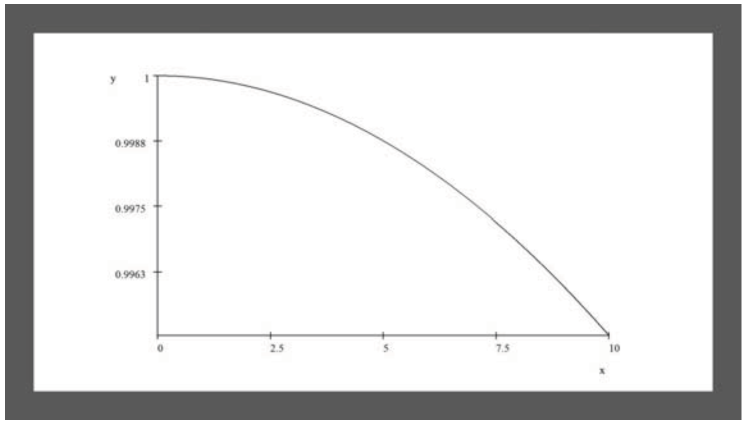

Figure 17.7: The amplitude-frequency characteristic of the filter.

17.2.4 The realizable approximation of the equivalent control

By (17.25) $u_t^{(av)}$ may be represented as

$$u^{(av)}(t) = \int\limits_{s=0}^{t} u^{(sl)}(s) d\left(e^{-(t-s)/\mu}\right)$$

Consider the dynamics $x_t^{(av)}$ of the system (17.11) controlled by $u_t^{(av)}$ (17.25) at two time intervals:

- during the *reaching phase*: $u(t) = u^{(av)}(t)$,
- and during the *sliding mode regime:* $u(t) = u^{(eq)}(t)$.

1. **Reaching phase** ($t \in [0, t_f]$). Here the integration by part implies

$$u^{(av)}(t) = \int\limits_{s=0}^{t} u^{(sl)}(s)\, d\left(e^{-(t-s)/\mu}\right) =$$

$$u^{(sl)}(t) - u^{(sl)}(0)\, e^{-t/\mu} - \int\limits_{s=0}^{t} \dot{u}^{(sl)}(s)\, e^{-(t-s)/\mu} ds.$$

Supposing that $u_t^{(sl)}$ (17.19) is bounded almost everywhere together with its derivative when we are in the reaching phase, i.e.,

$$\left\| \dot{u}^{(sl)}(t) \right\| \leq d,$$

the identity above leads to the following estimation:

$$\left\|u^{(av)}(t)-u^{(sl)}(t)\right\| \leq \left\|u^{(sl)}(0)\right\| e^{-t/\mu}+d\int_{s=0}^{t} e^{-(t-s)/\mu}ds =$$

$$\left\|u^{(sl)}(0)\right\| e^{-t/\mu}+\mu d\int_{s=0}^{t} e^{-(t-s)/\mu}d\left(s/\mu\right) =$$

$$\left\|u^{(sl)}(0)\right\| e^{-t/\mu}+\mu d\int_{\tilde{s}=0}^{t/\mu} e^{-(t/\mu-\tilde{s})}d\tilde{s} =$$

$$\left\|u^{(sl)}(0)\right\| e^{-t/\mu}+\mu d\left(1-e^{-t/\mu}\right)=\mu d+O\left(e^{-t/\mu}\right)$$

So, $u_t^{(av)}$ may be represented as

$$\boxed{u^{(av)}(t)=u^{(sl)}(t)+\xi(t),} \tag{17.27}$$

where $\xi(t)$ may be done as small as you wish taking μ tending to zero, since

$$\|\xi(t)\| \leq \mu d+O\left(e^{-t/\mu}\right).$$

As a result, the trajectories $x_t^{(sl)}$ and $x_t^{(av)}$ will be somewhat different. Indeed, we have

$$\dot{x}^{(sl)}(t)=f\left(t,x^{(sl)}(t)\right)-B\left(t,x^{(sl)}(t)\right)u^{(sl)}(t),$$

$$\dot{x}^{(av)}(t)=f\left(t,x^{(av)}(t)\right)-B\left(t,x^{(av)}(t)\right)u^{(av)}(t)$$

Defining

$$\tilde{B}(t)=B\left(t,x^{(av)}(t)\right),\tilde{G}(t)=G\left(t,x^{av}(t)\right),$$

$$\tilde{f}(t)=f\left(t,x^{(av)}(t)\right),$$

the last equation may be represented as

$$\dot{x}^{(sl)}(t)=f(t)-B(t)\,u^{(sl)}(t),$$

$$\dot{x}^{(av)}(t)=\tilde{f}(t)-\tilde{B}(t)\,u^{(av)}(t).$$

Hence by (17.27), the difference

$$\Delta(t):=x^{(sl)}(t)-x^{(av)}(t)$$

satisfies

$$\Delta(t) = \Delta(0) + \int_{s=0}^{t} \left[\left(f(s) - \tilde{f}(s) \right) - Bu^{(sl)}(s) + \tilde{B}u^{(av)}(s) \right] ds =$$
$$\Delta_0 + \int_{s=0}^{t} \left[\left(f(s) - \tilde{f}(s) \right) - Bu^{(sl)}(s) + \tilde{B}\left(u^{(sl)}(s) + \xi(s) \right) \right] ds$$

Taking into account that $\Delta_0 = 0$ (the system starts with the same initial conditions independently on an applied control) and that $f(t,x)$ satisfies (17.20) and $B(x)$ is Lipschitz (with the constant L_B) on x it follows

$$\|\Delta(t)\| \le \int_{s=0}^{t} \left[\left\| f(s) - \tilde{f}(s) \right\| + \left\| \left(\tilde{B} - B \right) u^{(sl)}(s) + \tilde{B}\xi(s) \right\| \right] ds$$
$$\le \int_{s=0}^{t} \left[f_1 \|\Delta(s)\| + L_B \|\Delta(s)\| \left\| u^{(sl)}(s) \right\| + \left\| \tilde{B} \right\| \|\xi(s)\| \right] ds \le$$
$$\int_{s=0}^{t} \left[\left(f_1 + L_B \left\| u^{(sl)}(s) \right\| \right) \|\Delta(s)\| + \left\| \tilde{B} \right\| \left(\mu d + O\left(e^{-s/\mu} \right) \right) \right] ds$$

Since for any $\varepsilon > 0$ after large enough time

$$O\left(e^{-s/\mu}\right) = \mu O\left(\frac{1}{\mu} e^{-s/\mu}\right) = \mu o(1) \le \mu\varepsilon$$

and

$$\left\| u_s^{(sl)}(s) \right\| \le u_+^{(sl)} < \infty, \quad \left\| \tilde{B} \right\| \le B_+ < \infty,$$

we finally get

$$\|\Delta(t)\| \le \int_{s=0}^{t} \left[\left(f_1 + L_B u_+^{(sl)} \right) \|\Delta(s)\| + B_+ \mu (d + \varepsilon) \right] ds$$
$$\le B_+ \mu (d + \varepsilon) t_f + \int_{s=0}^{t} \left(f_1 + L_B u_+^{(sl)} \right) \|\Delta(s)\| ds$$

Now let us apply the Bihari lemma (see Lemma 19.1 in [7]), which says that if $v(t)$ and $\xi(t)$ are nonnegative continuous functions on $[t_0, \infty)$ verifying

$$v(t) \le c + \int_{s=t_0}^{t} \xi(s) v(s) ds \qquad (17.28)$$

then for any $t \in [t_0, \infty)$ the following inequality holds:

$$v(t) \leq c \exp\left(\int_{s=t_0}^{t} \xi(s)\, ds\right). \tag{17.29}$$

This results remains true if $c = 0$. In our case

$$v(t) = \|\Delta(t)\|, \; c = B^{+}\mu(d+\varepsilon)\, t_f, \; \xi(s) = f_1 + L_B u_{+}^{(sl)}, \; t_0 = 0$$

for any $t \in [0, t_f)$. So,

$$\boxed{\|\Delta(t)\| \leq B_{+}\mu(d+\varepsilon)\, t_f \exp\left(\left(f_1 + L_B u_{+}^{(sl)}\right) t_f\right)} \tag{17.30}$$

Claim. *For any finite reaching time t_f and any small value $\delta > 0$ there exists a small enough μ such that $\|\Delta_t\|$ is less than δ*, that is

$$\boxed{\left\|x^{(sl)}(t) - x^{(av)}(t)\right\| \leq \delta.}$$

2. **Sliding mode phase** $(t > t_f)$. During the sliding mode phase we have

$$\left.\begin{array}{c} S\left(x^{(sl)}(t)\right) = 0 \\ \text{and} \\ \dot{S}\left(x^{(sl)}(t)\right) = G\ \left(f(t) - Bu^{(eq)}(t)\right) = 0 \end{array}\right\} \tag{17.31}$$

if $u_t = u_t^{(eq)}$ for all $t > t_f$. Applying $u_t = u_t^{(av)}$ we can not guarantee (17.31) already. Indeed,

$$S\left(x^{(av)}(t)\right) = S\left(x^{(av)}(t_f)\right) + \int_{s=t_f}^{t} \dot{S}\left(x^{(av)}(s)\right) ds,$$

and, by (17.30),

$$\left\|S\left(x^{(av)}(t_f)\right)\right\| = \left\|S\left(x^{(av)}(t_f)\right) - \underbrace{S\left(x_{t_f}^{(sl)}\right)}_{0}\right\| \leq$$

$$\left\|G\left(x^{(sl)}(t_f)\right)\Delta(t_f)\right\| \leq O(\mu).$$

Hence, in view of (17.31), $\left\|S\left(x^{(av)}(t)\right)\right\| = O(\mu)$.

Claim 17.1 *During the sliding-mode phase*

$$\boxed{\left\|S\left(x^{(av)}(t)\right)\right\| = O(\mu).} \tag{17.32}$$

17.3 Exercise

Exercise 17.1 *Consider the model*

$$\left.\begin{array}{c}\ddot{x}(t)+\alpha\dot{x}(t)+\beta x(t)=u+\xi(t),\\ \textit{where } x(0),\dot{x}(0) \textit{ are given,}\\ \\ \xi(t)=\xi_0\sin(\omega t) \textit{ is unmeasurable signal,}\\ \alpha \textit{ and } \beta \textit{ are unknown but positive.}\end{array}\right\}$$

Take some parameters $\alpha > 0$, $\beta > 0$, ξ_0, $\omega > 0$ and, supposing that $x(t)$ and $\dot{x}(t)$ are available, design a standard ***SM controller*** *providing convergence $x(t)$, $\dot{x}(t) \to 0$. To compare the equivalent control (its low-frequency realization) with external perturbation $\xi(t)$. Estimate $\xi(t)$.*