Part IV

SLIDING MODE CONTROL

Lecture 16

About Sliding Mode Control

Robust Control Theory
without Sliding Mode approach
is like a nightingale without songs
(a version of one of Russian proverb).

Sliding Mode Control (SMC) is a nonlinear control approach in control systems that changes the dynamics of a nonlinear system by applying a discontinuous control signal (or, more precisely, a set-valued control signal) that causes the system to "slide" along a cross-section of its usual behavior. The state-feedback control law is not a time-dependent function. Instead, depending on where it is in the state space, it can flip from one continuous structure to another. As a result, sliding mode control is a form of variable structure control. Trajectories are continually moving toward a neighboring zone with a different control structure, therefore the end trajectory will not reside fully inside one control structure. Instead, it will "slide" along the control structures' borders. The sliding mode [1]-[5] is the motion of the system as it slides along these limits, and the sliding (hyper) surface is the geometrical locus containing the boundaries. Any variable structure system, such as a system under SMC, may be seen as a particular instance of a hybrid dynamical system in the framework of contemporary control theory since it flows through a continuous state space but also moves through different discrete control modes.

16.1 Tracking as Stabilization

Condider two different control problems:

Problem 16.1 (Tracking problem) *Design the admissible control action $u(t)$ in such a way that the trajectories $x(t)$ of the considered dynamic model*

$$\left.\begin{array}{c} \dot{x}(t) = f(x(t), t) + g(x(t), t)\, u(t), \;\; x(0) = x_0, \\ \\ x(t) \in \mathbb{R}^n, \;\; u(t) \in \mathbb{R}^r \end{array}\right\} \tag{16.1}$$

follows the intended way to proceed

$$\left.\begin{array}{c} \dot{x}^*(t) = \varphi(x^*(t), t), \;\; x^*(0) = x_0^*, \\ \\ x^*(t) \in \mathbb{R}^n \end{array}\right\} \tag{16.2}$$

as close as possible. Note that the nonlinear term $f(x(t), t)$ may include an external perturbation as well.

Problem 16.2 (Stabilization problem) *Design the control action $u(t)$ which stabilizes the trajectories $x(t)$ of the system (16.1) in the origin, that is, providing the property*

$$x(t) \to 0 \text{ when } t \to \infty. \tag{16.3}$$

Introduce now the tracking error variable

$$\Delta(t) := x(t) - x^*(t),$$

which dynamics is governed by the following differential equation

$$\dot{\Delta}(t) = f(x(t), t) + g(x(t), t)\, u_t - \varphi(x^*(t), t) =$$

$$f(\Delta(t) + x^*(t), t) + g(\Delta(t) + x^*(t), t)\, u(t) - \varphi(x^*(t), t),$$

implying

$$\dot{\Delta}(t) = F(\Delta(t), t) + G(\Delta(t), t)\, u_t, \;\; \Delta(0) = \Delta_0 = x_0 - x_0^* \tag{16.4}$$

with

$$F(\Delta(t), t) := f(\Delta(t) + x^*(t), t) - \varphi(x^*(t), t),$$

$$G(\Delta(t), t) := g(\Delta(t) + x^*(t), t).$$

Summary 16.1 *It is easy to see that the* ***tracking problem*** *(16.1) of the system (16.1) - (16.2)* ***may be treated as the stabilization problem*** *(16.2) of the system (16.4).*

So, below without loss of generality we will consider only the stabilization problem (16.2) of the plant

$$\boxed{\left.\begin{array}{c} \dot{x}(t) = f(x(t), t) + g(x(t), t)\, u(t), \; x(0) = x_0, \\ \\ x(t) \in \mathbb{R}^n, \; u(t) \in \mathbb{R}^r, \; t \geq 0, \end{array}\right\}} \tag{16.5}$$

keeping in mind that the general tracking problem (16.1) may be converted in the first one if instead of the state variable $x(t)$ to work with the tracking error variable $\Delta(t)$.

16.2 Desired dynamics in the state space

The SMC approach's major characteristics are as follows:

- the *realization of the dynamics of the controlled plant in two steps*,

  - the first of which is to reach a desired dynamics in finite time even if the model of the considered plant is not completely known,

  - and the second of which is the adequate selection of the desired dynamics;

- the use of a *discontinuous feedback* as the main instrument providing the robustness property for the controlled system at the first step of the control process.

Introduce the auxilary variable $s := s(t, x) \in \mathbb{R}^m$, often known as the "*sliding mode variable*," which is independent of any plant features (16.5). Only the appropriate dynamics properties are defined, which corresponds to the condition when

$$s(t, x(t)) = 0 \text{ for all } t \geq t_0 \geq 0, \tag{16.6}$$

where t_0 denotes the start of the desired dynamcs. Let us consider some examples of sliding mode variables.

Example 16.1 (Simplest stability property) *For the plant*

$$\dot{x}(t) = f(x(t), t) + g(x(t), t) u(t)$$

the sliding mode variable may be selected as

$$s = x. \tag{16.7}$$

Example 16.2 (Models with 2-nd derivative) *For the plant*

$$\ddot{x}(t) = f(x(t), \dot{x}(t), t) + g(x(t), t) u(t),$$

represented as

$$\left.\begin{array}{c} \dot{x}_1(t) = x_2(t), \ \ x_1(t) = x(t), \\ \\ \dot{x}_2(t) = f(x_1(t), x_2(t), t) + g((x_1(t), x_2(t), t), t) u(t), \end{array}\right\}$$

the sliding mode variable may be selected as

$$s = x_2 + Cx_1 \tag{16.8}$$

with the matrix $C \in \mathbb{R}^{n \times n}$ providing the stability property $x(t) \to 0$ for the dynamic equation

$$\dot{x}(t) + Cx(t) = 0.$$

Example 16.3 (Models with m-th derivative) *For the plant*

$$\begin{array}{c} x^{(m)}(t) = f\left(x(t), \dot{x}(t), ..., x^{(m-1)}(t), t\right) + \\ g\left(x(t), \dot{x}(t), ..., x^{(m-1)}(t), t\right) u(t), \end{array}$$

represented as

$$\left.\begin{array}{c} \dot{x}_i(t) = x_{i+1}(t), \ \ x_1(t) = x(t), \ \ i = 1, ..., m-1, \\ \\ \dot{x}_m(t) = f\left(x(t), \dot{x}(t), ..., x^{(m-1)}(t), t\right) + \\ \\ g\left(x(t), \dot{x}(t), ..., x^{(m-1)}(t), t\right) u(t) \ , \end{array}\right\}$$

the sliding mode variable may be selected as

$$s = \sum_{i=1}^{m} c_i x_i \tag{16.9}$$

with the parameters c_i providing the stability property $x(t) \to 0$ for the dynamic equation

$$c_m x^{(m)}(t) + c_{m-1} x^{(m-1)}(t) + ... + c_1 x(t) = 0,$$

or, in other words, such that the polynomial

$$P(\lambda) = c_m \lambda^m + c_{m-1} \lambda^{m-1} + ... + c_1$$

would be Hurwitz (stable).

Example 16.4 (Integral sliding variable) *For the plant*

$$\dot{x}(t) = f(x(t), t) + g(x(t), t) u(t)$$

the sliding mode variable may be selected also as

$$\left. \begin{aligned} s = s(x(t), \zeta(t), t), \\ \zeta(t) := \int_{\tau=t_0}^{t} \varphi(\tau, x(\tau)) \, d\tau \end{aligned} \right\} \tag{16.10}$$

containing the integral $\zeta(t)$ which includes the prehistory of the desired dynamics step.

Remark 16.1 *In light of the above arguments, the value $\|s\|$ may be seen as the* ***distance between the intended and the current dynamics****.*

16.3 ODE with Discontinuous Right-Hand Side

In this lecture we will follow [1] and [6]. For the simplicity we also will use the following abbreviation:

- ODE meaning an *ordinary differential equation*,
- DRHS meaning the *discontinuous right-hand side*.

16.3.1 Why ODE with DRHS are important in Robust Control

We shall provide several compelling reasons for continuing our investigation of ODE with DRHS in section. Let's start with the simplest scalar example, which is the following basic scalar ODE with affine (linear) control, that is:

$$\dot{x}(t) = f(x(t)) + u(t), \ x_0 \text{ is given,} \tag{16.11}$$

where $x(t), u(t) \in \mathbb{R}$ are interpreted here as the *state* of the system (16.11) and, respectively, the *control action* applied to it at time $t \in [0, T]$. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a *Lipschitz function* satisfying the, so-called, Lipschitz condition, that is, for any $x, x' \in \mathbb{R}$

$$\left| f(x) - f(x') \right| \leq L \left| x - x' \right|, \ 0 \leq L < \infty. \tag{16.12}$$

Problem 16.3 *Let us try to stabilize this system at the point $x^* = 0$ using the **feedback control***

$$u := u(x) \tag{16.13}$$

and considering the following informative situations:

1) Case 1: *the Lipschitz constant L is exactly known and $f(0) = 0$ (see Fig.16.1);*

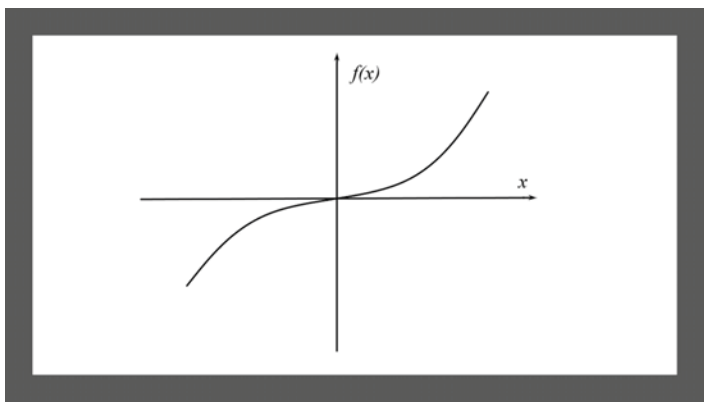

Figure 16.1: A function with the property $f(0) = 0$.

2) Case 2: *it is only known that the function $f(x)$ is bounded as (see Fig.16.2)*

$$|f(x)| \leq f_0 + f^+ |x|, \ f_0 < \infty, \ f^+ < \infty \tag{16.14}$$

(this inequality is assumed to be valid for any $x \in \mathbb{R}$ and admits that $f_0 \neq 0$).

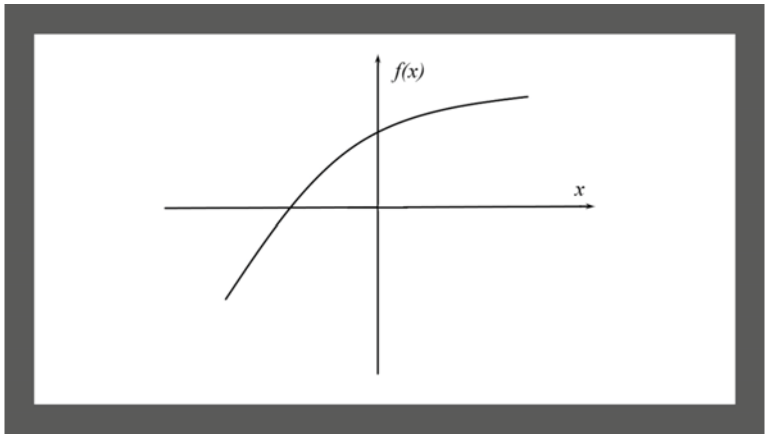

Figure 16.2: A function with $f_0 \neq 0$.

There are two possibilities to do that:

- to use any **continuous control**, namely, take $u : \mathbb{R} \longrightarrow \mathbb{R}$ as a continuous function of x, i.e., $u \in C$;

- to use **a discontinuous control** which will be defined below.

Case 1: $f(0) = 0$

Evidently that at the stationary point $x^* = 0$ any continuous control $u(t) := u(x(t))$ should satisfy the following identity

$$f(0) + u(0) = 0 \tag{16.15}$$

Taking into account that $f(0) = 0$, this may be fulfilled if we use, for example,

$$u(x) := -kx, \ k > 0 \tag{16.16}$$

impliying

$$\dot{x}(t) = f(x(t)) - kx(t)$$

and, as the result, for $V(x) = \dfrac{1}{2}x^2$

$$\dot{V}(x(t)) = x(t)\dot{x}(t) = x\left[f\left(x(t)\right) - kx(t)\right] =$$

$$-2kV(x(t)) + x(t)f\left(x(t)\right) \leq -2kV(x(t)) + |x(t)|\left|f\left(x(t)\right)\right| =$$

$$-2kV(x(t)) + |x(t)|\left|f\left(x(t)\right) - f(0)\right| \leq$$

$$-2kV(x(t)) + |x_t|\, L\, |x(t) - 0| = -2\left(k - L\right) V(x(t))$$

Taking k a little bit more than L, namely,

$$k = L + \rho,\ \ \rho > 0,$$

we get

$$V(x(t)) = V(x(0))e^{-\rho t} \underset{t\to\infty}{\rightarrow} 0.$$

Conclusion 16.1 *So, in this case 1 the continuous control (16.16), for example, linear control with a large enouth gain parameter solves the stabilization problem (16.3).*

Case 2: $\left|f\left(x\right)\right| \leq f_0 + f^{+}\left|x\right|,\ f_0 \neq 0.$

In this case we have

$$\left.\begin{array}{c} \dot{V}(x(t)) = x(t)\dot{x}(t) = x(t)\left[f\left(x(t)\right) + u\left(x(t)\right)\right] \leq \\ \\ |x(t)|\left|f\left(x(t)\right)\right| + x(t)u\left(x(t)\right) \leq \\ \\ \left(f_0 + f^{+}\left|x(t)\right|\right)|x(t)| + x(t)u\left(x(t)\right). \end{array}\right\} \tag{16.17}$$

a) Select $u\left(x\right)$ as before in (16.16):

$$u\left(x\right) := -kx,\ \ k > f^{+} > 0.$$

From (16.17) we get

$$\dot{V}(x(t)) \leq \left(f_0 + f^{+}\left|x(t)\right|\right)|x(t)| - kx^2(t) =$$

$$-2\left(k - f^{+}\right)V(x(t)) + f_0\sqrt{2V(x(t)} =$$

$$-2\left(k - f^{+}\right)\sqrt{V(x(t))}\left(\sqrt{V(x(t))} - \frac{f_0}{\sqrt{2}\left(k - f^{+}\right)}\right) < 0$$

if
$$\sqrt{V(x(t))} = \frac{|x(t)|}{\sqrt{2}} > \frac{f_0}{\sqrt{2}\,(k - f^+)}$$
or equivalently,
$$|x(t)| > \frac{f_0}{\sqrt{2}\,(k - f^+)}. \tag{16.18}$$
This means that when $f_0 \neq 0$ we have convergence property $\dot{V}(x(t)) < 0$ only outside of the sphere of the radius $\dfrac{f_0}{\sqrt{2}\,(k - f^+)}$ and inside of this sphere trajectories do not converge to the origin.

b) Select now
$$u(x) = -kx(t) - K\operatorname{sign}(x(t)),\ k > 0,\ K > 0, \tag{16.19}$$
where
$$\operatorname{sign}(x) := \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ \in [-1, 1] & \text{if } x = 0 \end{cases} \tag{16.20}$$
(see Fig.16.3).The use of (16.19) in (16.17) leads to the following dif-

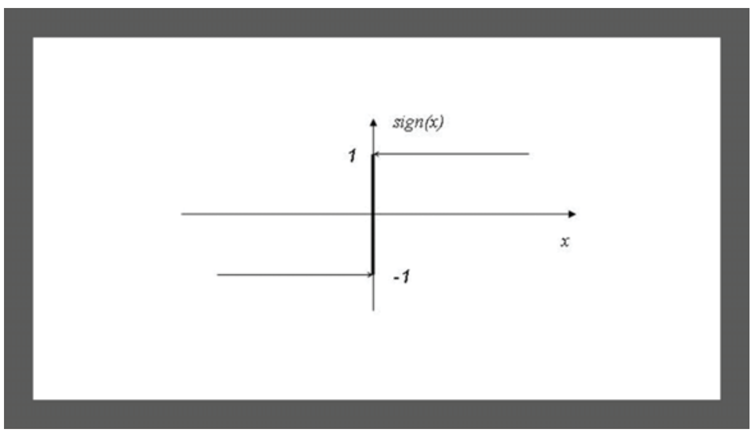

Figure 16.3: Signum function.

ferential inequality:
$$\dot{V}(x(t)) \leq (f_0 + f^+|x(t)|)\,|x(t)| + x(t)\,u(x(t)) =$$
$$f_0\,|x(t)| + f^+x^2(t) - kx^2(t) - K\,|x_t|$$

Selecting $k \geq f^{+}$ (the "*high-gain proportional control*") and taking

$$K = f_0 + \rho > 0,$$

we get

$$\dot{V}(x_t) \leq -\sqrt{2}\rho\sqrt{V(x_t)},$$

$$2\left(\sqrt{V(x_t)} - \sqrt{V(x_0)}\right) \leq -\sqrt{2}\rho t,$$

$$0 \leq \sqrt{2V(x_t)} = |x_t| \leq \sqrt{2V(x_0)} - \rho t = |x_0| - \rho t$$

This means that in the considered informative situation *the "high-gain proportional control" together with a **discontinuous controller** solves the stabilization problem in finite time*

$$t_{reach} = \frac{|x_0|}{\rho}. \tag{16.21}$$

Conclusion 16.2 *As it follows from the considerations above,* ***the discontinuous (in this case, sliding-mode) control*** *(16.19)* ***can stabilize the class of the dynamic systems*** *(16.11), (16.12), (16.14)* ***in finite time*** *(16.21) without the exact knowledge of its model. Besides,* ***the reaching phase may be done as small as you wish*** *by the simple selection of the gain parameter K in (16.21). In other words, the discontinuous control is* ***robust with respect to the presence of uncertainties in the description*** *of the model (16.11) which means that it is capable to stabilize a wide class of "black/grey-box" systems. Pure continuos control can not do that.*

Remark 16.2 *Evidently, that using such discontinuous control, the trajectories of the controlled system can not stay in the stationary point $x^* = 0$ since it arrives to it in finite time but with a nonzero rate, namely, with $\dot{x}_t$ such that*

$$\dot{x}(t) = \begin{cases} f(0) + k^0 & \text{if} \quad x(t) \longrightarrow +0 \\ f(0) - k^0 & \text{if} \quad x(t) \longrightarrow -0 \end{cases},$$

*that provokes the, so-called, "**chattering effect**" (see Fig.16.4). Simple engineering considerations show that some sort of smoothing (or, low-pass filtering) may be applied to keep dynamics close to the stationary point $x^* = 0$.*

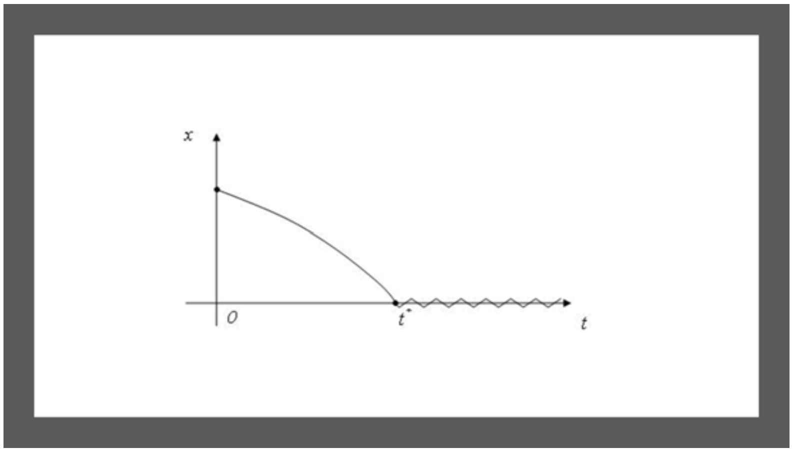

Figure 16.4: The chattering effect.

Remark 16.3 *Notice that when $x(t) = x^* = 0$ we only know that*

$$\boxed{\dot{x}(t) \in \left[f(0) - k^0,\ f(0) + k^0\right]} \tag{16.22}$$

This indicates that we are dealing with a ***differential inclusion*** *(rather than an equation) (16.22). As a result, we must define what a mathematically valid solution of a differential inclusion is, as well as what it is.*

All these questions, arising in the remarks above, will be considered below in details and be illustrated by the corresponding examples and figures.

16.3.2 ODE with DRHS and differential inclusions

General requirements to a solution

As it is well known, a solution of the differential equation

$$\dot{x}(t) = f(t, x(t)) \tag{16.23}$$

with a continuous right-hand side (RHS) is a function x_t which has a derivative and satisfies (16.23) everywhere (more exactly, almost everywhere) on a given interval time-interval. This definition is not, however, valid for DE with DRHS since in some points of discontinuity the derivative of $x(t)$ does not exists. That's why the consideration of DE with DRHS requires a generalization of the concept of a solution. Anyway, such generalized concept should necessarily meet the following requirements:

- For differential equations with a continuous right-hand side the definition of a solution must be equivalent to the usual (standard) one.

- For the equation $\dot{x}(t) = f(t)$ the solution should be the functions $x(t) = \int f(t)\,dt + c$ only.

- For any initial data $x(t_0) = x_{init}$ within a given region the solution $x(t)$ should exist (at least, locally) for any $t > t_0$ and admit the possibility to be continued up to the boundary of this region or up to infinity (when $(t, x) \to \infty$).

- The limit of a uniformly convergent sequences of solutions should be a solution too.

- Under the commonly used changes of variables a solution must be transformed into a solution.

The definition of a solution

Here we follow [5] and [6].

Definition 16.1 *A vector-valued function $f(t,x)$, defined by a mapping $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^p$, is said to be* ***piecewise continuous*** *in a finite domain $\mathcal{G} \subseteq \mathbb{R}^{n+1}$ if $\mathcal{G}$ consists of a finite numbers of a domains $\mathcal{G}_i$ $(i = 1, ..., l)$, i.e.,*

$$\mathcal{G} = \bigcup_{i=1}^{l} \mathcal{G}_i$$

such that in each of them the function $f(t,x)$ is continuous up to the boundary

$$\mathcal{M}_i := \bar{\mathcal{G}}_i \backslash \mathcal{G}_i \ \ (i = 1, ..., l) \tag{16.24}$$

of a measure zero.

The most frequent case is the one where the set

$$\mathcal{M} = \bigcup_{i=1}^{l} \mathcal{M}_i$$

of all discontinuity points consists of a finite number of hypersurfaces

$$0 = S_k(x) \in C^1, \ k = 1, ..., m,$$

where $S_k(x)$ is a smooth function.

Definition 16.2 *The set $\mathcal{M}$ defined as*

$$\boxed{\mathcal{M} = \{x \in \mathbb{R}^n \mid S(x) = (S_1(x), ..., S_m(x))^\intercal = 0\}} \tag{16.25}$$

is called a ***manifold*** *in $\mathbb{R}^n$. It is referred to as* ***a smooth manifold*** *if*

$$S_k(x) \in C^1, k = 1, ..., m.$$

Now we are ready to formulate the main definition of this section.

Definition 16.3 (A solution in the Filippov's sense [6].) *An* ***absolutely continuous*** *on $[t_0, t_f]$ function x_t (which can be represented as a Lebesque integral of another function) satisfying*

$$\boxed{\dot{x}(t) \in \mathcal{F}(t, x_t)} \tag{16.26}$$

almost everywhere on $[t_0, t_f]$, where the set $\mathcal{F}(t, x)$ is ***the smallest convex closed set containing all limit values of the vector-function*** *$f(t, x^*)$ for $(t, x^*) \notin \mathcal{M}$, $x^* \to x$, $t = \text{const}$, is referred to as a solution of the* ***differential inclusion*** *(16.26)* ***in the Filippov's sense***

Remark 16.4 *The set $\mathcal{F}(t, x)$*

1) *consists of one point $f(t, x)$ at points of continuity of the function $f(t, x)$;*

2) *is a segment (a convex polygon, or polyhedron), which in the case when $(t, x) \in \mathcal{M}_i$ (16.24) has the vertices*

$$f_i(t, x) := \lim_{(t,x^*) \in \mathcal{G}_i,\ x^* \to x} f(t, x^*). \tag{16.27}$$

All points $f_i(t, x)$ are contained in $\mathcal{F}(t, x)$, but it is not obligatory that all of them are vertices.

Example 16.5 *For the scalar differential inclusion*

$$\dot{x}(t) \in -\text{sign}(x(t))$$

the set $\mathcal{F}(t, x)$ is as follows (see Fig.16.5):

1. *$\mathcal{F}(t, x) = -1$ if $x > 0$;*

2. *$\mathcal{F}(t, x) = 1$ if $x < 0$;*

3. *$\mathcal{F}(t, x) = [-1, 1]$ if $x = 0$.*

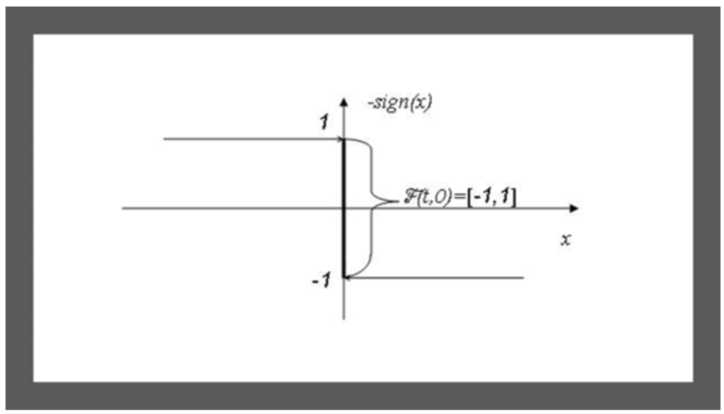

Figure 16.5: The RHS of the differential inclusion $\dot{x}(t) = -\mathrm{sign}\,(x(t))$.