Lecture 14

Robust Stabilization of Time-Delay Systems

We will look at the class of uncertain time delay affine-controlled systems where a delay is accepted in state variables as well as the control action separately. It will be demonstrated that the Attractive Ellipsoid Method application allows for the creation of a feedback that enables any state trajectory of the controlled system to be converged to an ellipsoid, whose "size" is determined by the parameters of the applied feedback. Finally, we provide a numerical approach for calculating these parameters that yields the "smallest" zone-convergence for all conceivable controlled trajectories. An overview of stability conditions in terms of the Lyapunov matrix for time-delay systems may be found in [37].

14.1 Affine systems with a delay in state variables

14.1.1 System description and problem formulation

Let us consider the time delay control system of the form

$$\dot{x}(t) = f\left(x(t), x(t-\tau), t\right) + Bu(t), \tag{14.1}$$

or, in quasi-linear format,

$$\boxed{\dot{x}(t) = Ax(t) + A_1 x(t-\tau) + Bu(t) + \xi\left(x(t), x(t-\tau), t\right)} \tag{14.2}$$

with the initial conditions

$$x(s) = \varphi\left(s\right), s \in [-\tau, 0] \tag{14.3}$$

where

$x \in \mathbb{R}^n$ is the vector of the system state,

$A, A_1 \in \mathbb{R}^{n\times n}$ are the system matrices,

$u \in \mathbb{R}^m$ is the vector of control inputs,

$B \in \mathbb{R}^{n\times m}$ is the matrix of the control gains,

the pair (A, B) is controllable,

$\tau > 0$ is the constant *state* time delay assumed to be known,

the vector-valued function $\xi(x(t), x(t-\tau), t)$, describing the unknown part of the model, is defined as

$$\xi(x(t), x(t-\tau), t) := f(x(t), x(t-\tau), t) - Ax(t) + A_1 x(t-\tau),$$

which assumed to be bounded as

$$\boxed{\begin{array}{c} \|\xi(x(t), x(t-\tau), t)\|^2 \le \\ c_0 + x^\intercal(t) Q_x x(t) + x^\intercal(t-\tau) Q_\tau x(t-\tau) \end{array}} \tag{14.4}$$

with the positive definite symmetric matrices $Q_x, Q_h \in \mathbb{R}^{n\times n}$ (supposed to be given),

the matrix B has a full $\text{rank}(B) = m \le n$.

Problem 14.1 *We need to design the control action $u \in \mathbb{R}^m$ as a linear feedback*

$$\boxed{\left.\begin{array}{c} u = Kx(t) + K_\tau x(t-\tau), \\ K, K_\tau \in \mathbb{R}^{m\times n}, \end{array}\right\}} \tag{14.5}$$

which stabilizes all possible trajectories $\{x(t)\}_{t\ge-\tau}$ in some bounded region, containing origin in the space $\mathbb{R}^n$, and to make this region as small as possible.

14.1.2 Lyapunov-Krasovskii's functional and stability analysis

Let us defined the Lyapunov-Krasovskii's functional as follows:

$$\left.\boxed{\begin{array}{c} V\left(x\left(t\right),t\right) := x^{\intercal}(t)Px(t) + \int\limits_{s=t-\tau}^{t} e^{h(s-t)} x^{\intercal}(s) P_1 x\left(s\right) ds, \\ \\ h>0,\ P,P_1 \in \mathbb{R}^{n\times n} \text{ are positive definite.} \end{array}}\right\} \tag{14.6}$$

and calculate its derivative over the trajectories of the system (14.2). We get

$$\begin{gathered} \dot{V}\left(x\left(t\right),t\right) = 2x^{\intercal}\left(t\right)P\dot{x}\left(t\right) + x^{\intercal}(t)P_1x\left(t\right) \\ -e^{-h\tau}x^{\intercal}(t-\tau)P_1x\left(t-\tau\right) - h\int\limits_{s=t-\tau}^{t} e^{h(s-t)}x^{\intercal}(s)P_1x\left(s\right)ds \\ = 2x^{\intercal}(t)P\left(Ax(t) + A_1x(t-\tau)\right. \\ \left.+B\left[Kx(t)+K_{\tau}x(t-\tau)\right] + \xi\left(x(t),x(t-\tau),t\right)\right) \\ + x^{\intercal}(t)P_1x(t) - e^{-h\tau}x^{\intercal}(t-\tau)P_1x\left(t-\tau\right) \\ -h\int\limits_{s=t-\tau}^{t} e^{h(s-t)}x^{\intercal}(s)P_1x\left(s\right)ds + 2x^{\intercal}(t)P\xi\left(x(t),x(t-\tau),t\right). \end{gathered}$$

This expression in new variables

$$z\left(t\right) = \begin{pmatrix} x(t) \\ x(t-\tau) \\ \xi\left(x(t),x(t-\tau),t\right) \end{pmatrix}$$

can be rewritten as

$$\left.\begin{array}{c} z^{\intercal}\left(t\right)\begin{bmatrix} \begin{array}{c} P\left(A+BK\right)+ \\ \left(A+BK\right)^{\intercal}P+P_1 \end{array} & P\left(A_1+BK_{\tau}\right) & P \\ \\ \left(A_1+BK_{\tau}\right)^{\intercal}P & -e^{-h\tau}P_1 & 0_{n\times n} \\ P & 0_{n\times n} & -\varepsilon I_{n\times n} \end{bmatrix} z\left(t\right) \\ \\ -h\int\limits_{s=t-\tau}^{t} e^{h(s-t)}x^{\intercal}(s)P_1x\left(s\right)ds + \varepsilon\left\|\xi\left(x(t),x(t-\tau),t\right)\right\|^2. \end{array}\right\} \tag{14.7}$$

Using the upper bound property (14.4)

$$\|\xi(x(t),x(t-\tau),t)\|^2 \le c_0 + x^\intercal(t)Q_x x(t) + x^\intercal(t-\tau)Q_\tau x(t-\tau)$$

we may represent (14.7) as differential inequality

$$\left.\begin{array}{c} \dot{V} \le \varepsilon c_0 - h \int\limits_{s=t-\tau}^{t} e^{h(s-t)} x^\intercal(s) P_1 x(s)\, ds + \\ z^\intercal \begin{bmatrix} \begin{array}{c} P(A+BK)+ \\ (A+BK)^\intercal P \\ +P_1+\varepsilon Q_x + \alpha P \end{array} & P(A_1+BK_\tau) \\ (A_1+BK_\tau)^\intercal P & -e^{-h\tau}P_1+\varepsilon Q_\tau \end{bmatrix} z. \end{array}\right\} \tag{14.8}$$

Adding and subtracting the term αV in the right-hand side of (14.8) we get

$$\boxed{\left.\begin{array}{c} \dot{V} \le -\alpha V + \varepsilon c_0 - (h-\alpha) \int\limits_{s=t-\tau}^{t} e^{h(s-t)} x^\intercal(s) P_1 x(s)\, ds \\ + z^\intercal(t) W_{\alpha,\varepsilon}(P,P_1,K,K_\tau) z(t) \end{array}\right\}} \tag{14.9}$$

where

$$\boxed{\left.\begin{array}{c} W_{\alpha,\varepsilon,h}(P,P_1,K,K_\tau) = \\ \begin{bmatrix} \begin{array}{c} P(A+BK)+ \\ (A+BK)^\intercal P \\ +P_1+\varepsilon Q_x + \alpha P \end{array} & P(A_1+BK_\tau) \\ (A_1+BK_\tau)^\intercal P & -e^{-h\tau}P_1+\varepsilon Q_\tau \end{bmatrix} \end{array}\right\}} \tag{14.10}$$

Now we are ready to formulate the following theorem.

Theorem 14.1 *If for the given matrices $P > 0, P_1 > 0, K, K_\tau$ and posive constants α, ε, h we have*

$$\boxed{W_{\alpha,\varepsilon,h}(P,P_1,K,K_\tau) < 0, h \ge \alpha,} \tag{14.11}$$

then the following stabilization pfoperty holds:

$$\boxed{\limsup_{t\to\infty} V(x(t),t) \le \frac{\varepsilon c_0}{\alpha}.} \tag{14.12}$$

Proof. It follows directly from (14.9), if the conditions (14.11) are fulfilled. ■

Corollary 14.1 *The attractive ellipsoid*

$$\mathcal{E}_0(P_{attr}) := \{x \in \mathbb{R}^{2n} : x^\intercal P_{attr} x < 1\}$$

for such system corresponds to the attractivity property

$$\boxed{\left.\begin{array}{c} \limsup\limits_{t\to\infty} x^\intercal(t) P_{attr} x(t) \le 1, \\ with \\ P_{attr} = \dfrac{\alpha}{\varepsilon c_0} P. \end{array}\right\}} \tag{14.13}$$

Proof. It follows from the inequality

$$x^\intercal(t)\, Px(t) \le V(x(t), t)$$

and (14.12). ■

14.1.3 Optimal feedback parameters

Following the same optimization scheme as in the previous lectures, we may formulate the problem of feedback parameters optimization in the following way:

$$\left.\begin{array}{c} \operatorname{tr}\left\{\dfrac{\varepsilon}{\alpha} P^{-1}\right\} \to \inf\limits_{P>0,\, P_1>0,\, K,\, K_\tau,\, \alpha>0,\, \varepsilon>0,\, h>0} \\ \text{subject to the constraints (14.11)} \end{array}\right\} \tag{14.14}$$

As usual, let us transform matrix $W_{\alpha,\varepsilon,h}(P, P_1, K, K_\tau)$ into

$$W_T = T^\intercal W_{\alpha,\varepsilon,h}(P, P_1, K, K_\tau)\, T$$

with $T = \begin{bmatrix} P^{-1} & 0_{n\times n} \\ 0_{n\times n} & P_1^{-1} \end{bmatrix}$, which leads to

$$\left.\begin{array}{c} W_T = \begin{bmatrix} P^{-1} W_{T,11} P^{-1} & P^{-1} W_{T,12} P_1^{-1} \\ P_1^{-1} W_{T,21} P^{-1} & P_1^{-1} W_{T,22} P_1^{-1} \end{bmatrix} = \\ \begin{bmatrix} \begin{array}{c} (A+BK)\, P^{-1} + \\ P^{-1}(A+BK)^\intercal + \\ P^{-1} P_1 P^{-1} + \\ \varepsilon P^{-1} Q_x P^{-1} + \alpha P^{-1} \end{array} & (A_1 + BK_\tau)\, P_1^{-1} \\ P_1^{-1}(A_1 + BK_\tau)^\intercal & -e^{-h\tau} P_1^{-1} + \varepsilon P_1^{-1} Q_\tau P_1^{-1} \end{bmatrix} \end{array}\right\} \tag{14.15}$$

The terms $P^{-1}P_1P^{-1}$, $P^{-1}Q_xP^{-1}$ and $P_1^{-1}Q_\tau P_1^{-1}$ in the diagonal blocks may be estimated as

$$P^{-1}P_1P^{-1} < H_1,\ P^{-1}Q_xP^{-1} < H_1,\ P_1^{-1}Q_\tau P_1^{-1} < H_3,$$

which by the Schur's complement equivalently expressed as the following matrix inequalities:

$$\begin{bmatrix} H_1 & P^{-1} \\ P^{-1} & P_1^{-1} \end{bmatrix} > 0, \begin{bmatrix} H_2 & P^{-1} \\ P^{-1} & Q_x^{-1} \end{bmatrix} > 0, \begin{bmatrix} H_3 & P^{-1} \\ P^{-1} & Q_\tau^{-1} \end{bmatrix} > 0 \qquad (14.16)$$

In new variables

$$X = P^{-1},\ Y = KP^{-1},\ X_1 = P_1^{-1},\ Y_1 = K_\tau P_1^{-1} \qquad (14.17)$$

the matrix inequalities $W_T < 0$ and (14.16) looks as LMI's:

$$\boxed{\bar{W}_T = \begin{bmatrix} \begin{matrix} AX + BY + \\ X^\intercal A^\intercal + Y^\intercal B^\intercal + \\ H_1 + \varepsilon H_2 + \alpha X \end{matrix} & A_1X_1 + BY_1 \\ X_1^\intercal A_1^\intercal + Y_1^\intercal B^\intercal & -e^{-h\tau}X_1 + \varepsilon H_3 \end{bmatrix} < 0} \qquad (14.18)$$

and

$$\boxed{\begin{bmatrix} H_1 & X \\ X & X_1 \end{bmatrix} > 0, \begin{bmatrix} H_2 & X \\ X & Q_x^{-1} \end{bmatrix} > 0, \begin{bmatrix} H_3 & X \\ X & Q_\tau^{-1} \end{bmatrix} > 0} \qquad (14.19)$$

So, the optimization problem (14.14) in the new variables (14.17) may be formulated as a matrix optimization problem with LMI's constraints:

$$\boxed{\left.\begin{matrix} \operatorname{tr}\left\{\dfrac{\varepsilon}{\alpha}X\right\} \to \inf\limits_{X>0,\, X_1>0,\, Y,\, Y_1,\, H_1>0,\, H_2>0,\, H_2>0,\, 0<\alpha\le h,\, \varepsilon>0} \\ \text{subject to the constraints (14.18)} \\ \text{and (14.19)} \end{matrix}\right\}} \qquad (14.20)$$

If X^*, Y^*, X_1^* and Y^* are the solutions of the optimization problem (14.20), then the optimal feedback parameters of the feedback stabilizer are

$$\boxed{K^* = Y^*\left(X^*\right)^{-1},\ K_\tau^* = Y_1^*\left(X_1^*\right)^{-1}.} \qquad (14.21)$$

14.2 Affine systems with a delay in control actions

14.2.1 System description

Here we will consider the following time-delay system

$$\dot{x}(t) = f\left(x(t), t\right) + Bu(t-\tau) + \eta\left(x(t), t\right), \tag{14.22}$$

with external perturbations $\eta\left(x(t), t\right)$, or, in quasi-linear format,

$$\boxed{\left.\begin{array}{c} \dot{x}(t) = Ax(t) + Bu(t-\tau) + \xi\left(x(t), t\right), \\ \\ A \in R^{n\times n},\ B \in R^{n\times m}, \\ \\ \xi\left(x(t), t\right) := f\left(x(t), t\right) - Ax(t) + \eta\left(x(t), t\right) \end{array}\right\}} \tag{14.23}$$

with the initial conditions

$$x(s) = \varphi\left(s\right), s \in [-\tau, 0].$$

Let us suppose that

$$\boxed{\left\|\xi\left(x(t), t\right)\right\|^2 \leq c_0 + x^{\intercal}(t)Q_x x(t),} \tag{14.24}$$

and additionally that the matrix A is Hurwitz (stable), the pair $\{A, B\}$ is controllable and the matrix B has a full rank m, i.e.,

$$\boxed{\operatorname{rank}(B) = m \leq n, B^{\intercal}B > 0.}$$

14.2.2 Prediction approach and unavoidable stabilization error

To stabilize the time delay control system (14.23) the *prediction approach* (see [18], [19], [20], and [21]) is used. The typical prediction equation for the system (14.23) has the form

$$\boxed{y(t) = e^{A\tau}x(t) + \int_{-\tau}^{0} e^{-sA}Bu(t+s)ds.} \tag{14.25}$$

Obviously, knowing the control function $u(t)$ on the time interval $[t-\tau, t)$ is required to calculate the *prediction variable* $y(t)$. This information is expected to be acceptable and usable for control design. It is simple to verify that the prediction variable y(t) obeys the following delay-free equation:

$$\dot{y}(t) = Ay(t) + Bu(t) + e^{A\tau}\xi\left(t\right). \tag{14.26}$$

According to *the predictor method* the stabilization of the original system (14.23) can be ensured by designing the stabilizing controller for the prediction system (14.26).

Lemma 14.1 *For the processes (14.23) and (14.25) the following relations holds:*

$$\boxed{x(t+\tau) = y(t) + \int_0^\tau e^{(\tau-s)A}\xi\,(t+s)\,ds.} \tag{14.27}$$

Proof. Using the formula for the general solution of the system (14.23) we obtain

$$x(t+\tau) = e^{A\tau}x(t) + \int_t^{t+\tau} e^{(t+\tau-s)A}Bu(s-\tau)ds+$$

$$\int_t^{t+\tau} e^{(t+\tau-s)A}\xi\,(s)\,ds = e^{A\tau}x(t)+$$

$$\int_{-\tau}^{0} e^{-sA}Bu(t+s)ds + \int_0^\tau e^{(\tau-s)A}\xi\,(t+s)\,ds,$$

from which the equality (14.27) follows. ■

This lemma describes dependence of the original system state $x(t)$ on the predictor variable $y(t)$ and the uncertain term $\xi\,(x(t),t)$. The integral term in the right-hand side (14.27) obviously does not depend on control inputs and the predictor variables, but it is linear functional of $\xi\,(x(t),t)$. So, it defines an *unavoidable stabilization error* of the system (14.23),

According to this lemma the original system state $x(t)$ is dependent on the predictor variable $y(t)$ and the uncertain term $\xi\,(x(t),t)$. The integral term on the right-hand side evidently is independent of the predictor variables and control inputs, but it is a linear function of $\xi\,(x(t),t)$. As a result, it specifies an *unavoidable system stabilization error* (14.23)

$$\boxed{w_\tau\,(t) := \int_0^\tau e^{(\tau-s)A}\xi\,(t+s)\,ds,} \tag{14.28}$$

produced by the prediction technique, namely,

$$\boxed{x(t+\tau) = y(t) + w_\tau\,(t)\,.}$$

Therefore, minimization of the attractive set for the original system (14.23) can be provided by the design of the appropriate controller for the prediction system 14.26.

Remark 14.1 *Unfortunately, under uncertainty presence (when* $\xi\,(x(t),t) \neq 0$*) the property* $y(t) \to 0$ *does not imply* $x(t) \to 0$*. Therefore, the time delay control system (14.23) may* ***only be practically stabilized*** *in some predetermined zone (attractive set).*

So, now *our aim* is to develop the predictor-based control design scheme, which minimizes (in some sense) the attractive set of the system (14.23). For this purpose we will use the *Attactive Ellipsoid Method* (AEM) as it is described in the previous lectures.

Assume that for the system (14.26) the standard proportional feedback controller

$$\boxed{u(t) = Ky(t)} \tag{14.29}$$

is applied.

14.2.3 Attractive ellipsoid

Introduce the "*energetic function*"

$$\boxed{\left.\begin{array}{c} V_\tau(x\,(t+\tau)\,, y(t)) = x^\intercal\,(t+\tau)\,P_x x\,(t+\tau) + \\ \\ y^\intercal(t) P_y y(t) + \beta \int\limits_{s=t}^{t+\tau} e^{h(s-t)} x^\intercal(s) Q_x x\,(s)\,ds, \\ \\ 0 < P_x \in \mathbb{R}^{n\times n},\ 0 < P_y \in \mathbb{R}^{n\times n},\ \beta > 0,\ h > 0. \end{array}\right\}} \tag{14.30}$$

Theorem 14.2 *If for the system (14.23) there exist positive definite matrices P_x, P_y and positive constants α, ε, β, h such that*

$$\boxed{\begin{array}{c} 0 > W_\tau = \\ \left[\begin{array}{cccc} \begin{array}{c} P_x A + A^\intercal P_x + \\ \alpha I_{n\times n} + \\ \left(1+\beta e^{h\tau}\right) Q_x \end{array} & P_x B K & P_x & 0_{n\times n} \\ (P_x B K)^\intercal & \begin{array}{c} P_y\,(A+BK) + \\ (A+BK)^\intercal\,P_y \\ +\alpha I_{n\times n} \end{array} & 0_{n\times n} & P_y e^{A\tau} \\ P_x & 0_{n\times n} & -\varepsilon I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & e^{A^\intercal \tau} P_y & 0_{n\times n} & -\varepsilon I_{n\times n} \end{array}\right] \end{array}} \tag{14.31}$$

and

$$\boxed{\varepsilon \le \beta, \alpha \le h,} \tag{14.32}$$

then the property

$$\boxed{\limsup_{t\to\infty} V_\tau(x\,(t+\tau)\,, y(t)) \le 2c_0 \frac{\varepsilon}{\alpha}} \tag{14.33}$$

is guaranteed.

Proof. For $V_\tau(x\left(t+\tau\right), y(t)$ we have

$$\left.\begin{aligned} V_\tau(x\left(t+\tau\right), y(t)) = x^\intercal\left(t+\tau\right) P_x x\left(t+\tau\right) + \\ y^\intercal(t) P_y y(t) + \beta \int\limits_{s=t}^{t+\tau} e^{h(s-t)} x^\intercal(s) Q_x x\left(s\right) ds = \\ \begin{pmatrix} x\left(t+\tau\right) \\ y(t) \end{pmatrix}^\intercal \begin{bmatrix} P_x & 0_{n\times n} \\ 0_{n\times n} & P_y \end{bmatrix} \begin{pmatrix} x\left(t+\tau\right) \\ y(t) \end{pmatrix} + \\ \beta \int\limits_{s=t}^{t+\tau} e^{h(s-t)} x^\intercal(s) Q_x x\left(s\right) ds, \end{aligned}\right\} \tag{14.34}$$

and calculate its derivative:

$$\dot{V}_\tau(x\left(t+\tau\right), y(t)) =$$

$$2 \begin{pmatrix} x\left(t+\tau\right) \\ y(t) \end{pmatrix}^\intercal \begin{bmatrix} P_x & 0_{n\times n} \\ 0_{n\times n} & P_y \end{bmatrix} \begin{pmatrix} \dot{x}\left(t+\tau\right) \\ \dot{y}(t) \end{pmatrix} -$$

$$h\beta \int\limits_{s=t}^{t+\tau} e^{h(s-t)} x^\intercal(s) Q_x x\left(s\right) ds + \beta e^{h\tau} x^\intercal(t+\tau) Q_x x\left(t+\tau\right) -$$

$$-\beta x^\intercal(t) Q_x x\left(t\right) =$$

$$2 \begin{pmatrix} x\left(t+\tau\right) \\ y(t) \end{pmatrix}^\intercal \begin{pmatrix} P_x A x(t+\tau) + P_x B K y(t) + P_x \xi\left(t+\tau\right) \\ P_y\left(A+BK\right) y(t) + P_y e^{A\tau} \xi\left(t\right) \end{pmatrix} +$$

$$-h\beta \int\limits_{s=t}^{t+\tau} e^{h(s-t)} x^\intercal(s) Q_x x\left(s\right) ds + \beta e^{h\tau} x^\intercal(t+\tau) Q_x x\left(t+\tau\right) -$$

$$-\beta x^\intercal(t) Q_x x\left(t\right),$$

which can be represented as

$$\dot{V}_{\tau}(x(t+\tau), y(t)) =$$
$$z(t)^{\intercal} \begin{bmatrix} \begin{matrix} P_x A + A^{\intercal} P_x \\ +\beta e^{h\tau} Q_x \end{matrix} & P_x BK & P_x & 0_{n\times n} \\ (P_x BK)^{\intercal} & \begin{matrix} P_y (A+BK) + \\ (A+BK)^{\intercal} P_y \end{matrix} & 0_{n\times n} & P_y e^{A\tau} \\ P_x & 0_{n\times n} & -\varepsilon I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & e^{A^{\intercal}\tau} P_y & 0_{n\times n} & -\varepsilon I_{n\times n} \end{bmatrix} z(t)$$
$$+\varepsilon \left(\|\xi(t+\tau)\|^2 + \|\xi(t)\|^2 \right) - \beta x^{\intercal}(t) Q_x x(t)$$
$$-h\beta \int\limits_{s=t}^{t+\tau} e^{h(s-t)} x^{\intercal}(s) Q_x x(s)\, ds,$$

where

$$z(t) = \begin{pmatrix} x(t+\tau) \\ y(t) \\ \xi(t+\tau) \\ \xi(t) \end{pmatrix}$$

Using the propertry (14.24), the right-hand side of the last differential equation can be estimated as

$$\left. \begin{array}{c} \dot{V}_{\tau}(x(t+\tau), y(t)) \leq z(t)^{\intercal} W_{\tau} z(t) \\ -\alpha V_{\tau}(x(t+\tau), y(t)) + 2\varepsilon c_0 + (\varepsilon - \beta) x^{\intercal}(t) Q_x x(t) \\ -(h-\alpha)\beta \int\limits_{s=t}^{t+\tau} e^{h(s-t)} x^{\intercal}(s) Q_x x(s)\, ds \end{array} \right\} \tag{14.35}$$

If $W_{\tau} < 0$, $\varepsilon \leq \beta$ and $\alpha \leq h$, then from (14.35) we get

$$\dot{V}_{\tau}(x(t+\tau), y(t)) \leq -\alpha V_{\tau}(x(t+\tau), y(t)) + 2\varepsilon c_0$$

implying (14.33). ■

Corollary 14.2 *In view of the inequality*

$$x^{\intercal}(t+\tau) P_x x(t+\tau) \leq V_{\tau}(x(t+\tau), y(t))$$

we may conclude that

$$\limsup_{t\to\infty} x^{\intercal}(t) P_x x(t) \leq 2c_0 \frac{\varepsilon}{\alpha},$$

and, as the result,

$$\boxed{\left.\begin{array}{c}\limsup\limits_{t\to\infty} x^{\intercal}(t)\,P_{attr}x(t)\le 1,\\ \\ P_{attr}=\dfrac{\alpha}{2c_0\varepsilon}P_x\end{array}\right\}}\qquad(14.36)$$

14.2.4 Minimal attractive ellipsoid for the original system

Following the standard technique, used in the previous lectures, instead of the matrix nonlinear constraint $W_\tau < 0$ we may consider the equivalent constraint

$$W_{\tau,T} := T^{\intercal}W_\tau T < 0$$

with

$$T=\begin{bmatrix} P_x^{-1} & 0_{n\times n} & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & P_y^{-1} & 0_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & I_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & I_{n\times n}\end{bmatrix}=\operatorname{diag}\left[P_x^{-1},P_y^{-1},I_{n\times n},I_{n\times n}\right].$$

We get

$$0>W_{\tau,T}=$$

$$\begin{bmatrix} \begin{array}{c}AP_x^{-1}+P_x^{-1}A^{\intercal}+\\ P_x^{-1}\left[\alpha I_{n\times n}+\right.\\ \left.\left(1+\beta e^{h\tau}\right)Q_x\right]P_x^{-1}\end{array} & BKP_y^{-1} & I_{n\times n} & 0_{n\times n}\\ \left(BKP_y^{-1}\right)^{\intercal} & \begin{array}{c}(A+BK)\,P_y^{-1}+\\ \left[(A+BK)\,P_y^{-1}\right]^{\intercal}\\ +\alpha\left(P_y^{-1}\right)^2\end{array} & 0_{n\times n} & e^{A\tau}\\ I_{n\times n} & 0_{n\times n} & -\varepsilon I_{n\times n} & 0_{n\times n}\\ 0_{n\times n} & e^{A^{\intercal}\tau} & 0_{n\times n} & -\varepsilon I_{n\times n}\end{bmatrix}.$$

In new variables

$$X=P_x^{-1},\ Y=P_y^{-1},\ Z=KP_y^{-1},$$

and using the upper estimates for quadratic elements in diagonal blocks

$$P_x^{-1}\left[\alpha I_{n\times n}+\ \left(1+\beta e^{h\tau}\right)Q_x\right]P_x^{-1}\ <Q_1,$$

$$\left(P_y^{-1}\right)^2<Q_2,$$

which by the Schur's complement can be represented as LMI's

$$\begin{bmatrix} Q_1 & P_x^{-1} \\ P_x^{-1} & \left[\alpha I_{n\times n} + \left(1+\beta e^{h\tau}\right) Q_x\right]^{-1} \end{bmatrix} > 0 \tag{14.37}$$

and

$$\begin{bmatrix} Q_2 & P_y^{-1} \\ P_y^{-1} & I_{n\times n} \end{bmatrix} > 0, \tag{14.38}$$

we are able to conclude that $W_{\tau,T} < \bar{W}_{\tau,T}$ where

$$\bar{W}_{\tau,T} = \begin{bmatrix} AX + X^{\intercal} P_x^{-1} + Q_1 & BZ & I_{n\times n} & 0_{n\times n} \\ (BZ)^{\intercal} & \begin{array}{c} AY + BZ + \\ {[AY+BZ]}^{\intercal} + \alpha Q_2 \end{array} & 0_{n\times n} & e^{A\tau} \\ I_{n\times n} & 0_{n\times n} & -\varepsilon I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & e^{A^{\intercal}\tau} & 0_{n\times n} & -\varepsilon I_{n\times n} \end{bmatrix} \tag{14.39}$$

We are ready to formulate the following result.

Theorem 14.3 *The optimal feedback matrix K^*, minimazing the attractive ellipsoid $\mathcal{E}_0\left(P_{attr}\right)$, is equal to*

$$K^* = Z^*\left(Y^*\right)^{-1}, \tag{14.40}$$

where Z^* and Y^* are the solution of the following matrix optimization problem

$$\boxed{\begin{array}{c} \operatorname{tr}\left\{P_{attr}^{-1}\right\} = 2c_0 \operatorname{tr}\left\{\dfrac{\varepsilon}{\alpha} X\right\} \to \inf\limits_{X>0, Y>0, Z, Q_1>0, Q_2>0, \varepsilon>0, \alpha>0} \\ \text{subject to LMI's constraints} \\ \bar{W}_{\tau,T} < 0, \\ \begin{bmatrix} Q_1 & X \\ X & \left[\alpha I_{n\times n} + \left(1+\beta e^{h\tau}\right) Q_x\right]^{-1} \end{bmatrix} > 0, \\ \begin{bmatrix} Q_2 & Y \\ Y & I_{n\times n} \end{bmatrix} > 0, \end{array}} \tag{14.41}$$

Proof. It follows from the estimate $W_{\tau,T} < \bar{W}_{\tau,T} < 0$, the representation (14.36) for P_{attr} and the LMI's (14.37) and (14.38) given in new variable X, Y. ■

14.2.5 Example and Exercise

Example 14.1 *For the dynamic system*

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-\tau) + \xi\,(t)\,, \\ u(t) &= 0 \text{ for } t \in [-\tau, 0] \end{aligned}\right\} \tag{14.42}$$

with

$$\left.\begin{aligned} A &= \begin{pmatrix} -1 & 2 & 0.4 \\ -1.5 & -0.7 & 2 \\ 0.5 & -0.6 & -1 \end{pmatrix},\ B = \begin{pmatrix} 0.3 \\ 0 \\ 1.2 \end{pmatrix},\ \tau = 0.5, \\ \xi\,(t) &= \begin{pmatrix} 0.0028\cos(0.6t) - 0.0879\sin(0.6t) \\ 0.0499\cos(0.6t) + 0.0049\sin(0.6t) \end{pmatrix} \end{aligned}\right\} \tag{14.43}$$

design a feedback controller using the ***predictive approach****.*

Solution 14.1 *It is easy to show that the upper bound (14.24)*

$$\|\xi\,(x(t), t)\|^2 \leq c_0 + x^{\mathsf{T}}(t) Q_x x(t)$$

is valid for

$$c_0 = 0.0103,\ Q_x = 0_{n\times n}.$$

The obtained numerical solution of the constrained optimization problem (14.41) is

$$X^* = \begin{bmatrix} 0.6280 & -0.2592 & -0.0494 \\ -0.2592 & 0.9754 & -0.2716 \\ -0.0494 & -0.2716 & 0.2775 \end{bmatrix},$$

$$Y^* = \begin{bmatrix} 0.5233 & -0.0260 & -0.0761 \\ -0.0260 & 0.5073 & -0.1453 \\ -0.0761 & -0.1453 & 0.1892 \end{bmatrix},$$

$$Z^* = \begin{bmatrix} 0.1072 & 0.0861 & -01798 \end{bmatrix}$$

$$\alpha^* = 0.7,\ \beta^* = 7,\ \varepsilon^* = 3.1,\ h = 2,$$

which leads to

$$\boxed{K^* = \begin{bmatrix} 0.055 & -0.712 & -0.1952 \end{bmatrix}}$$

and

$$\boxed{P_{attr} = \begin{bmatrix} 1.2772 & 0 & 0 \\ 0 & 1.2772 & 0 \\ 0 & 0 & 1.2772 \end{bmatrix}}.$$

The states behavior $\begin{pmatrix} x_1(t) & x_2(t) & x_3(t) \end{pmatrix}$ withe initial point

$$\begin{pmatrix} x_1(0) = 0, & x_2(0) = -1, & x_3(0) = 3 \end{pmatrix}$$

in the single and 3D-format are presented in figures 14.1 and 14.2.

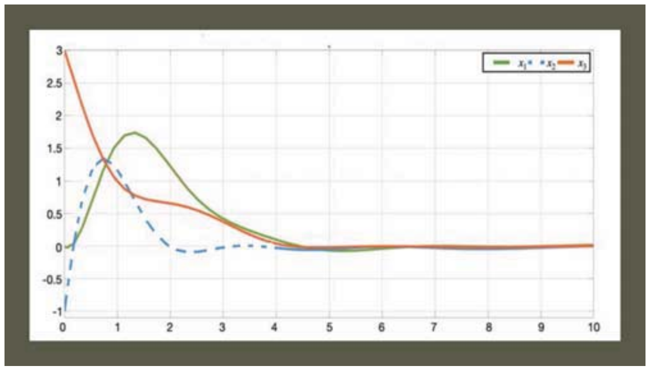

Figure 14.1: Trajectories of the controlled system.

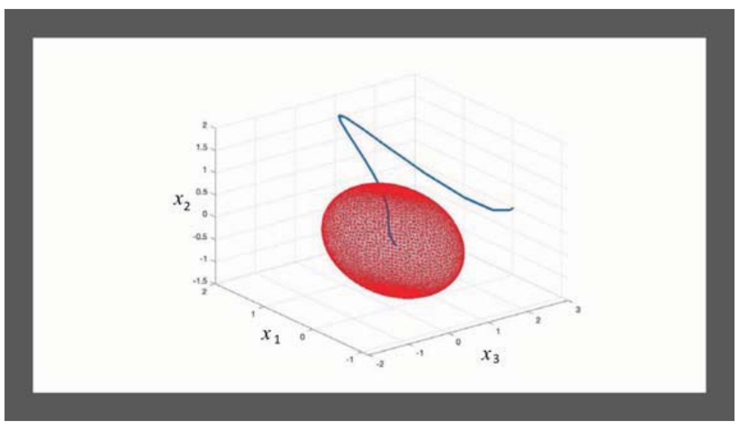

Figure 14.2: Convergence into the attractive ellipsoid.

Exercise 14.1 *For the dynamic system*

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-\tau) + \xi\,(t)\,, \\ u(t) &= 0 \text{ for } t \in [-\tau, 0] \end{aligned}\right\} \tag{14.44}$$

with

$$\left.\begin{aligned} A &= \begin{pmatrix} -1 & 2 \\ -1.5 & -0.7 \end{pmatrix},\ B = \begin{pmatrix} 3 \\ 1 \end{pmatrix},\ \tau = 0.25, \\ \xi\,(t) &= \begin{pmatrix} 0.0028\cos(0.6t) \\ 0.0049\sin(1.1t) \end{pmatrix} \end{aligned}\right\} \tag{14.45}$$

design a feedback controller using the ***predictive approach****.*