Lecture 12

Observer-Based Feedback Design

12.1 State observer and the extended dynamic model

A state observer or state estimator is a system in control theory that estimates the internal state of a given real system based on observations of the actual system's input and output. It's usually computer-based, and it's the foundation for a lot of practical applications. Many control theory tasks, such as stabilizing a system through state feedback, need knowledge of the system state. The physical condition of a system cannot be known by direct observation in most circumstances. Instead, the system outputs are used to examine indirect consequences of the internal state.

Let us consider here the system (11.1)-(11.6) with the linear feedback designed as

$$\boxed{u\left(t\right)=K\hat{x}\left(t\right), \quad K\in\mathbb{R}^{m\times n},} \tag{12.1}$$

where we use the, so called, observer state $\hat{x}\in\mathbb{R}^{n}$, referred below to as the *state estimate*, which is generated by the *classical Luenberger observer* [16] having the structure

$$\boxed{\dot{\hat{x}}\left(t\right)=A\hat{x}\left(t\right)+Bu\left(t\right)+F(y\left(t\right)-C\hat{x}\left(t\right)),\ \hat{x}\left(0\right)=\hat{x}_{0},\ F\in\mathbb{R}^{n\times k}.} \tag{12.2}$$

Below we will show that the robust stabilization of the quasi-linear system (11.1) may be realized by the application of the feedback (12.1)-(12.2) using the Attractive Ellipsoid Method.

Problem 12.1 *The main problem of this section is to design the observer-based linear feedback control providing the boundedness of any trajectory of* *the system (11.1), (12.1), (12.2) within an attractive ellipsoid of the "minimal size", or, in other words, to find the "best" gain matrices K and F.*

Define the state estimation error $e(t)$ as

$$e(t) := x(t) - \hat{x}(t). \tag{12.3}$$

Then, in view of (11.3), (12.1) and (12.2), its time derivative satisfies

$$\dot{e}(t) = (A - FC)e(t) + \xi_x(t, x(t)) - F\xi_y(t, x(t)).$$

Introduce the extended vector

$$z := \begin{pmatrix} \hat{x} \\ e \end{pmatrix} \in \mathbb{R}^{2n}. \tag{12.4}$$

Obviously, it is governed by the following ordinary differential equation (ODE)

$$\dot{z}(t) = \hat{A}z(t) + \hat{F}w(t, x) \tag{12.5}$$

where

$$\left.\begin{aligned} \hat{A} &:= \begin{bmatrix} A + BK & FC \\ 0 & A - FC \end{bmatrix}, \\ \hat{F} &:= \begin{bmatrix} 0_{n\times n} & F \\ I_{k\times n} & -F \end{bmatrix}, \quad w(t, x) := \begin{pmatrix} \xi_x(t, x) \\ \xi_y(t, x) \end{pmatrix}. \end{aligned}\right\} \tag{12.6}$$

12.2 Stabilizing feedback gains K and F

Our aim here is to find the control gain matrix K and the observer gain matrix F providing a stabilization (boundedness) of the state dynamics $x(t)$ as well as state estimation $\hat{x}(t)$ of the system (12.5) with the corresponding attractive ellipsoid $\mathcal{E}_0(P_{attr})$ in the z-space.

The following theorem gives the solution of this problem.

Theorem 12.1 *If positive definite matrices $X_1, X_2, \bar{Q}, H, H_C, H_Y \in \mathbb{R}^{n\times n}$, matrices $Y_1 \in \mathbb{R}^{m\times n}$, $Y_2 \in \mathbb{R}^{n\times k}$ and positive scalars α and ε satisfy the* *system of LMI's*

$$\tilde{W}_T = \begin{bmatrix} \bar{W}_T^{(1,1)} & \begin{bmatrix} 0_{n\times n} & 0_{n\times k} \\ X_2 & -Y_2 \end{bmatrix} \\ \begin{bmatrix} 0_{n\times n} & 0_{n\times k} \\ X_2 & -Y_2 \end{bmatrix}^{\intercal} & \begin{bmatrix} -\varepsilon I_{n\times n} & -Y_2 \\ -Y_2 & -\varepsilon I_{k\times k} + H_y \end{bmatrix} \end{bmatrix} < 0\,,$$

where

$$\tilde{W}_T^{(1,1)} = \begin{bmatrix} \begin{array}{c}(A_\alpha X_1 + BY_1) + \\ (A_\alpha X_1 + BY_1)^{\intercal} + \\ \varepsilon\bar{Q} + 2H\end{array} & \varepsilon X_1 \left(Q_x + Q_y\right) \\ \varepsilon \left(Q_x + Q_y\right)^{\intercal} X_1^{\intercal} & \begin{array}{c} X_2 A_\alpha - Y_2 C + \\ (X_2 A_\alpha - Y_2 C)^{\intercal} + \\ \varepsilon\left(Q_x + Q_y\right) + H_C \end{array} \end{bmatrix} \tag{12.7}$$

with $A_\alpha := A + \frac{\alpha}{2} I_{n\times n}$ *and*

$$\left.\begin{array}{c} \begin{bmatrix} \bar{Q} & X_1^{\intercal} \\ X_1 & (Q_x + Q_y)^{-1} \end{bmatrix} > 0, \begin{bmatrix} H & I_{n\times n} \\ I_{n\times n} & X_2 \end{bmatrix} > 0, \\ \begin{bmatrix} H_C & Y_2 C \\ C^{\intercal} Y_2^{\intercal} & X_2 \end{bmatrix} > 0, \; \begin{bmatrix} H_Y & Y_2 \\ Y_2^{\intercal} & X_2 \end{bmatrix} > 0, \end{array}\right\} \tag{12.8}$$

then the set

$$\mathcal{E}_0(P_{attr}) = \{z \in \mathbb{R}^{2n} : z^{\intercal} P_{attr} z \le 1\}$$

with the matrix

$$P_{attr} = \frac{\alpha}{\varepsilon\left(c_{0,x} + c_{0,y}\right)} P, \;\; P = \begin{bmatrix} X_1^{-1} & 0 \\ 0 & X_2 \end{bmatrix} \tag{12.9}$$

is ***the attractive ellipsoid*** *of the system (12.5), (11.5), (11.6), (12.1) with the feedback control gain matrix*

$$\boxed{K = Y_1 X_1^{-1}} \tag{12.10}$$

and the observer gain matrix

$$\boxed{F = X_2^{-1} Y_2\,.} \tag{12.11}$$

Proof. Define the Lyapunov function as

$$V(z) := z^T P z,\ P = \begin{bmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{bmatrix} := \mathrm{diag}(P_1, P_2),$$

where $P > 0$ is the matrix of an attractive ellipsoid has to be minimized. Then

$$\dot{V} = 2z^T P \dot{z} = 2z^T P \left[\hat{A}z + \hat{F}w\right] =$$

$$\begin{pmatrix} z \\ w \end{pmatrix}^T \begin{bmatrix} \hat{A}^T P + P\hat{A} & P\hat{F} \\ \hat{F}^T P & 0 \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix} =$$

$$\begin{pmatrix} z \\ w \end{pmatrix}^T \begin{bmatrix} \hat{A}^T P + P\hat{A} + \alpha P & P\hat{F} \\ \hat{F}^T P & -\varepsilon I_{(n+k)\times(n+k)} \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix} - \alpha V + \varepsilon \|w\|^2 \leq$$

$$\begin{pmatrix} z \\ w \end{pmatrix}^T \begin{bmatrix} \hat{A}^T P + P\hat{A} + \alpha P & P\hat{F} \\ \hat{F}^T P & -\varepsilon I_{(n+k)\times(n+k)} \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix} - \alpha V +$$

$$\varepsilon\left(c_{0,x} + c_{0,y}\right) + \varepsilon x^{\mathsf{T}} \left(Q_x + Q_y\right) x$$

with $\alpha > 0$, $\varepsilon > 0$. Taking into account that

$$x = e + \hat{x} = \begin{bmatrix} I_{n\times n} & I_{n\times n} \end{bmatrix} \begin{pmatrix} \hat{x} \\ e \end{pmatrix} = \begin{bmatrix} I_{n\times n} & I_{n\times n} \end{bmatrix} z\,,$$

we get

$$\dot{V} \leq \begin{pmatrix} z \\ w \end{pmatrix}^T W\left(P, K, F, \alpha, \varepsilon\right) \begin{pmatrix} z \\ w \end{pmatrix} - \alpha V + \varepsilon\left(c_{0,x} + c_{0,y}\right) \qquad (12.12)$$

where

$$W\left(P, K, F, \alpha, \varepsilon\right) := \begin{bmatrix} \hat{A}^T P + P\hat{A} + \alpha P + \varepsilon Q_z & P\hat{F} \\ \hat{F}^T P & -\varepsilon I_{(n+k)\times(n+k)} \end{bmatrix},$$

$$Q_z := \begin{bmatrix} I_{n\times n} \\ I_{n\times n} \end{bmatrix} \left(Q_x + Q_y\right) \begin{bmatrix} I_{n\times n} & I_{n\times n} \end{bmatrix} = \begin{bmatrix} Q_x + Q_y & Q_x + Q_y \\ Q_x + Q_y & Q_x + Q_y \end{bmatrix}$$

In the open format the matrix $W(P,K,F,\alpha,\varepsilon)$ is as follows

$$W(P,K,F,\alpha,\varepsilon) =$$

$$\begin{bmatrix} \begin{bmatrix} \begin{matrix} P_1(A_\alpha+BK)+ \\ [P_1(A_\alpha+BK)]^\intercal \end{matrix} & P_1FC \\ (P_1FC)^\intercal & \begin{matrix} P_2(A_\alpha-FC)+ \\ [P_2(A_\alpha-FC)]^\intercal \end{matrix} \end{bmatrix} + \varepsilon Q_z & \begin{bmatrix} 0_{n\times n} & P_1F \\ P_2 & -P_2F \end{bmatrix} \\ \begin{bmatrix} 0_{n\times n} & P_1F \\ P_2 & -P_2F \end{bmatrix}^\intercal & -\varepsilon I_{(n+k)\times(n+k)} \end{bmatrix}$$

We will have $W<0$ if an only if $W_T := TWT^\intercal < 0$ for some non-singular matrix T. Taking

$$T = \begin{bmatrix} \underbrace{\begin{bmatrix} P_1^{-1} & 0_{n\times n} \\ 0_{n\times n} & I_{n\times n} \end{bmatrix}}_{T_{11}} & 0_{2n\times(n+k)} \\ 0_{(n+k)\times 2n} & I_{(n+k)\times(n+k)} \end{bmatrix}$$

we get

$$W_T =$$

$$\begin{bmatrix} T_{11} & 0 \\ 0 & I_{(n+k)\times(n+k)} \end{bmatrix} \begin{bmatrix} W^{(1,1)} & W^{(1,2)} \\ W^{(2,1)} & W^{(2,2)} \end{bmatrix} \begin{bmatrix} T_{11} & 0 \\ 0 & I_{(n+k)\times(n+k)} \end{bmatrix} =$$

$$\begin{bmatrix} T_{11}W^{(1,1)}T_{11}^\intercal & T_{11}W^{(1,2)} \\ W^{(2,1)}T_{11} & W^{(2,2)} \end{bmatrix} =$$

$$\begin{bmatrix} \begin{bmatrix} \begin{matrix} (A_\alpha+BK)P_1^{-1}+ \\ (A_\alpha+BK)^\intercal P_1^{-1} \end{matrix} & FC \\ (FC)^\intercal & \begin{matrix} P_2(A_\alpha-FC)+ \\ [P_2(A_\alpha-FC)]^\intercal \end{matrix} \end{bmatrix} + \varepsilon T_{11}Q_zT_{11}^\intercal & \begin{bmatrix} 0_{n\times n} & F \\ P_2 & -P_2F \end{bmatrix} \\ \begin{bmatrix} 0_{n\times n} & F \\ P_2 & -P_2F \end{bmatrix}^\intercal & -\varepsilon I_{(n+k)\times(n+k)} \end{bmatrix}.$$

Notice that

$$T_{11}Q_zT_{11}^\intercal = \begin{bmatrix} P_1^{-1}(Q_x+Q_y)\left(P_1^{-1}\right)^\intercal & P_1^{-1}(Q_x+Q_y) \\ (Q_x+Q_y)P_1^{-1} & Q_x+Q_y \end{bmatrix} <$$

$$\begin{bmatrix} \bar{Q} & P_1^{-1}(Q_x+Q_y) \\ (Q_x+Q_y)P_1^{-1} & Q_x+Q_y \end{bmatrix},$$

with the upper estimate

$$P_1^{-1}(Q_x+Q_y)\,P_1^{-1} < \bar{Q},$$

if and only if by the Schur's complement

$$\begin{bmatrix} \bar{Q} & \left(P_1^{-1}\right)^{\intercal} \\ P_1^{-1} & (Q_x+Q_y)^{-1} \end{bmatrix} > 0 \tag{12.13}$$

Using these facts, we obtain

$$W_T < \bar{W}_T,$$

where

$$\bar{W}_T := \begin{bmatrix} \begin{bmatrix} \begin{matrix}(A_\alpha{+}BK)\,P_1^{-1}{+}\\(A_\alpha{+}BK)^{\intercal}\,P_1^{-1}\end{matrix} & FC \\ (FC)^{\intercal} & \begin{matrix}P_2\,(A_\alpha{-}FC)+\\ \left[P_2\,(A_\alpha{-}FC)\right]^{\intercal}\end{matrix} \end{bmatrix} + \varepsilon \begin{bmatrix} \bar{Q} & P_1^{-1}(Q_x+Q_y) \\ (Q_x+Q_y)\,P_1^{-1} & Q_x+Q_y \end{bmatrix} & \begin{bmatrix} 0_{n\times n} & F \\ P_2 & -P_2F \end{bmatrix} \\ \begin{bmatrix} 0_{n\times n} & F \\ P_2 & -P_2F \end{bmatrix}^{\intercal} & -\varepsilon I_{(n+k)\times(n+k)} \end{bmatrix}.$$

In new variables

$$X_1 := P_1^{-1},\ Y_1 = KP_1^{-1},\ X_2 := P_2,\ Y_2 = P_2F \tag{12.14}$$

the matrix $\bar{W}_T$ looks as

$$\bar{W}_T = \begin{bmatrix} \bar{W}_T^{(1,1)} & \begin{bmatrix} 0_{n\times n} & X_2^{-1}Y_2 \\ X_2 & -Y_2 \end{bmatrix} \\ \begin{bmatrix} 0_{n\times n} & X_2^{-1}Y_2 \\ X_2 & -Y_2 \end{bmatrix}^{\intercal} & -\varepsilon I_{(n+k)\times(n+k)} \end{bmatrix} < 0\,,$$

where

$$\bar{W}_T^{(1,1)} = \begin{bmatrix} \begin{matrix}(A_\alpha X_1{+}BY_1)+\\(A_\alpha X_1{+}BY_1)^{\intercal}+\varepsilon\bar{Q}\end{matrix} & X_2^{-1}Y_2C+\varepsilon X_1\left(Q_x{+}Q_y\right) \\ \left(X_2^{-1}Y_2C\right)^{\intercal}+\varepsilon\left(Q_x{+}Q_y\right)^{\intercal}X_1^{\intercal} & \begin{matrix}X_2A_\alpha{-}Y_2C+\\(X_2A_\alpha{-}Y_2C)^{\intercal}\\+\varepsilon\left(Q_x{+}Q_y\right)\end{matrix} \end{bmatrix}$$

Bilinear blocks $X_2^{-1}Y_2C$ and $X_2^{-1}Y_2$ in the quadratic form $\begin{pmatrix} z \\ w \end{pmatrix}^T \bar{W}_T \begin{pmatrix} z \\ w \end{pmatrix}$ correspons to the terms

$$\hat{x}^\intercal X_2^{-1}Y_2Ce \text{ and } z^\intercal X_2^{-1}Y_2w \tag{12.15}$$

Applying Λ-inequality (see [5])

$$2x^\intercal y \leq x^\intercal \Lambda x + y^\intercal \Lambda^{-1} y, \tag{12.16}$$

valid for all $x, y \in \mathbb{R}^n$ and any positive definite matrix $\Lambda \in \mathbb{R}^{n\times n}$ ($\Lambda = \Lambda^\intercal > 0$), to both term in (12.15) we get

$$2\hat{x}^\intercal X_2^{-1}Y_2Ce = 2\left(X_2^{-1}\hat{x}\right)^\intercal (Y_2Ce) \leq \hat{x}^\intercal X_2^{-1}\Lambda X_2^{-1}\hat{x} + e^\intercal C^\intercal Y_2^\intercal \Lambda^{-1} Y_2Ce \overset{\Lambda=X_2}{=}$$

$$\hat{x}^\intercal \left(X_2^{-1}\right)\hat{x} + e^\intercal \left(C^\intercal Y_2^\intercal X_2^{-1}Y_2\right)Ce\,,$$

$$2z^\intercal \begin{bmatrix} 0_{n\times n} & X_2^{-1}Y_2 \\ X_2 & -Y_2 \end{bmatrix} w = 2\begin{pmatrix} \hat{x} \\ e \end{pmatrix}^\intercal \begin{bmatrix} 0_{n\times n} & X_2^{-1}Y_2 \\ X_2 & -Y_2 \end{bmatrix} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} =$$

$$2\begin{pmatrix} \hat{x} \\ e \end{pmatrix}^\intercal \begin{bmatrix} X_2^{-1}Y_2\xi_y \\ -Y_2\xi_y \end{bmatrix} = 2\hat{x}^\intercal X_2^{-1}Y_2\xi_y + 2e^\intercal(-Y_2)\xi_y,$$

$$2\hat{x}^\intercal X_2^{-1}Y_2\xi_y = 2\left(X_2^{-1}\hat{x}\right)^\intercal Y_2\xi_y \leq \hat{x}^\intercal X_2^{-1}\Lambda X_2^{-1}\hat{x} + \xi_y{}^\intercal Y_2^\intercal \Lambda^{-1}Y_2\xi_y \overset{\Lambda=X_2}{=}$$

$$\hat{x}^\intercal X_2^{-1}\hat{x} + \xi_y{}^\intercal Y_2^\intercal X_2^{-1}Y_2\xi_y.$$

Using upper estimates

$$X_2^{-1} < H \in \mathbb{R}^{n\times n},\ C^\intercal Y_2^\intercal X_2^{-1}Y_2C < H_C \in \mathbb{R}^{n\times n},\ Y_2^\intercal X_2^{-1}Y_2 < H_Y \in \mathbb{R}^{k\times k}$$

and the equivalent Schur's complements

$$\begin{bmatrix} H & I_{n\times n} \\ I_{n\times n} & X_2 \end{bmatrix} > 0,\ \begin{bmatrix} H_C & Y_2C \\ C^\intercal Y_2^\intercal & X_2 \end{bmatrix} > 0,\ \begin{bmatrix} H_Y & Y_2 \\ Y_2^\intercal & X_2 \end{bmatrix} > 0$$

we derive

$$\hat{x}^\intercal X_2^{-1}Y_2Ce < \hat{x}^\intercal H\hat{x} + e^\intercal H_c e\,,$$

$$z^\intercal X_2^{-1}Y_2w < \hat{x}^\intercal X_2^{-1}Y_2\xi_y + e^\intercal(-Y_2)\xi_y < \hat{x}^T H\hat{x} + \xi_y{}^\intercal H_y\xi_y + e^\intercal(-Y_2)\xi_y.$$

This permits estimate $\bar{W}_T < \tilde{W}_T$ with $\tilde{W}_T$ as in (12.7) and (12.8), which are linear with respect to the new variables X_1, Y_1, X_2, Y_2, H and H_c. If $\bar{W}_T < 0$, from (12.12) it follows

$$\dot{V} \leq -\alpha V + \varepsilon \left(c_{0,x} + c_{0,y}\right),$$

implying

$$\limsup_{t \to \infty} V\left(z\left(t\right)\right) \leq \frac{\varepsilon}{\alpha}\left(c_{0,x} + c_{0,y}\right),$$

which leads to (12.9). Theorem is proven. ■

12.3 Minimization of attractive ellipsoid

To minimize the attractive ellipsoid $\mathcal{E}_0(P_{attr})$ of the system (12.5) we need to resolve the following optimization problem

$$\left.\begin{array}{c} \operatorname{tr}\left\{P_{attr}^{-1}\right\} = \dfrac{\varepsilon}{\alpha}\left(c_{0,x} + c_{0,y}\right)\left[\operatorname{tr}\left\{X_1^{-1}\right\} + \operatorname{tr}(X_2)\right] \to \\ \\ \inf\limits_{X_1>0,\ X_2>0,\ \bar{Q}>0,\ \alpha>0,\ \varepsilon>0} \end{array}\right\} \quad (12.17)$$

subject to (12.7)-(12.8). Using the estimation from above

$$X_1^{-1} < H,$$

which is equivalent by the Schur's complement to the LMI (as in (**??**))

$$\begin{bmatrix} H & I_{n\times n} \\ I_{n\times n} & X_1 \end{bmatrix} > 0, \quad H > 0, \quad (12.18)$$

we are able to represent the problem (12.17) as follows:

$$\boxed{\left.\begin{array}{c} \operatorname{tr}\left\{P_{attr}^{-1}\right\} = \dfrac{\varepsilon}{\alpha}\left(c_{0,x} + c_{0,y}\right)\left[\operatorname{tr}\left\{H\right\} + \operatorname{tr}(X_2)\right] \to \\ \\ \inf\limits_{X_1>0,\ X_2>0,\ Y_1,\ Y_2,\ \bar{Q}>0,\ H>0,\ H_C>0,\ H_Y>0,\ \alpha>0,\ \varepsilon>0} \\ \\ \text{subject to the LMI's constraints (12.7) and(12.8).} \end{array}\right\}} \quad (12.19)$$

As before, this matrix optimization problem (12.19) with LMI's constraints can be resolved using MATLAB toolboxes SeDuMi and Yalmip. If the solutions of the optimization problem (12.19) are X_1^*, X_2^*, Y_1^* and Y_2^*, then the optimal control and observere matrix gaines are given by

$$\boxed{K^* = Y_1^*\left(X_1^*\right)^{-1}} \quad (12.20)$$

and the observer gain matrix

$$\boxed{F = (X_2^*)^{-1} Y_2^* .} \tag{12.21}$$

12.4 Adaptive version of AEM

Consider here again the quasi-Lipschitz affine control system with the quasi-Lipschitz state-output mapping, given by

$$\boxed{\left.\begin{array}{c} \dot{x}(t) = Ax(t) + Bu(t) + \xi_x(t,x), \\ x(0) = x_0 \in \mathbb{R}^n, \\ y(t) = Cx(t) + \xi_y(t,x), \end{array}\right\}} \tag{12.22}$$

and controlled by the feedback (12.1)

$$\boxed{u(t) = K_t \hat{x}(t), \quad K_t \in \mathbb{R}^{m \times n},} \tag{12.23}$$

where we use the observer state $\hat{x} \in \mathbb{R}^n$, generated by the Luenberger observer (12.2) [16]

$$\boxed{\left.\begin{array}{c} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + F_t[y(t) - C\hat{x}(t)], \\ \\ \hat{x}(0) = \hat{x}_0, F_t \in \mathbb{R}^{n \times k}. \end{array}\right\}} \tag{12.24}$$

Here, as before, we assume that

$$\left.\begin{array}{c} \xi_x^{\intercal}(t,x)\xi_x(t,x) \leq c_{0,x} + x^T Q_x x, \ 0 < Q_x \in \mathbb{R}^{n \times n}, \\ \\ \xi_y^{\intercal}(t,x)\xi_y(t,x) \leq c_{0,y}, \ Q_y = 0 \text{ in } (11.6). \end{array}\right\} \tag{12.25}$$

The optimal control matrices K^* and F^*, minimizing the attractive ellipsoid, are given by (12.20) and (12.21) with the according positive scalars α^* and ε^*. Suppose that we know these optimal parameters. The "size" of the optimal Attractive Ellipsoid may be made significantly smaller, if the gain parameters can be done variable in time with a special "*adaptation mechanism*" using available on-line data of the controlled process. Next theorem describes this adaptation mechanism.

Theorem 12.2 *If for the system (12.22)-(12.25) the gain matrices are adapted by the following adaptation (learning) algorithms*

$$\left.\begin{aligned} -\frac{d}{dt}K_t &= \frac{\alpha^*}{2}\left(K_t - K^*\right) + 2k_1 B^\intercal P_1^* \hat{x}\hat{x}^\intercal, \\ -\frac{d}{dt}F_t &= k_2\left[2P_1^*\hat{x} + P_2^*\left(F_t - F^*\right)\left(y - C\hat{x}\right)\right]\left(y - C\hat{x}\right)^\intercal \\ &+ \left(\frac{\alpha^*}{2}I_{n\times n} + \frac{k_2}{\varepsilon^*}P_1^*\hat{x}\hat{x}^\intercal P_1^* + k_2 c_{0,y} P_2^*\right)\left(F_t - F^*\right), \end{aligned}\right\} \quad (12.26)$$

with P_1^*, P_2^*, K^*, F^*, α^* and ε^*, found as the solution of the matrix optimization problem (12.19), where intead of the condition $\tilde{W}_T < 0$ the matrix inequality

$$\tilde{W}_{adapt,T} := \begin{bmatrix} \tilde{W}^{(1,1)}_{adapt,T} & \tilde{W}^{(1,2)}_{adapt,T} \\ \tilde{W}^{(2,1)}_{adapt,T} & \tilde{W}^{(2,2)}_{adapt,T} \end{bmatrix} < 0,$$

$$\tilde{W}^{(1,1)}_{adapt,T} = \tilde{W}^{(1,1)}_T + \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 2P_2^* \end{bmatrix},$$

$$\tilde{W}^{(1,2)}_{adapt,T} = \tilde{W}^{(1,2)}_T,\ \tilde{W}^{(2,1)}_{adapt,T} = \tilde{W}^{(2,1)}_T,\ \tilde{W}^{(2,2)}_{adapt,T} = \tilde{W}^{(2,2)}_T,$$

is applied, then the ellipsoid $\mathcal{E}_0\left(P^*_{adapt,attr}\right)$ is attractive in z-space with

$$P^*_{adapt,attr} = \frac{\alpha^*}{\varepsilon^*\left(c_{0,x} + 2c_{0,y}\right)}P,\ P = \begin{bmatrix} \left(X_1^*\right)^{-1} & 0 \\ 0 & X_2^* \end{bmatrix}$$

(X_1^* and X_2^* are one of solutions of (12.19)), providing the property ($z_t = (x_t^\intercal, e_t^\intercal)^\intercal$)

$$\boxed{\begin{aligned} &\limsup_{t\to\infty}\left(z_t^T P^*_{attr} z_t + \frac{1}{2k_1}\operatorname{tr}\left\{\left(K_t - K^*\right)^\intercal\left(K_t - K^*\right)\right\} + \right. \\ &\left.\frac{1}{2k_2}\operatorname{tr}\left\{\left(F_t - F^*\right)^\intercal\left(F_t - F^*\right)\right\}\right) \le \frac{\varepsilon^*}{\alpha^*}\left(c_{0,x} + 2c_{0,y}\right) \end{aligned}}$$

or, equivalently

$$\boxed{\begin{array}{c}\limsup_{t\to\infty} z_t^T P^*_{adapt,attr} z_t \leq 1- \\ \liminf_{t\to\infty} \frac{\alpha^*}{\varepsilon^*\left(c_{0,x}+2c_{0,y}\right)} \left(\frac{1}{2k_1} \mathrm{tr}\left\{ \left(K_t - K^*\right)^{\intercal} \left(K_t - K^*\right)\right\} + \right. \\ \left. \frac{1}{2k_2} \mathrm{tr}\left\{ \left(F_t - F^*\right)^{\intercal} \left(F_t - F^*\right)\right\} \right) \end{array}} \tag{12.27}$$

Proof. In view of (12.5), for the extended vector $z := \begin{pmatrix} \hat{x}^{\intercal} & e^{\intercal} \end{pmatrix}^{\intercal} \in \mathbb{R}^{2n}$ with the component $e(t) := x(t) - \hat{x}(t)$ we have

$$\dot{z}(t) = \hat{A}_t z(t) + \hat{F}_t w(t,x)$$

where

$$\hat{A}_t := \begin{bmatrix} A + BK_t & F_t C \\ 0 & A - F_t C \end{bmatrix},$$
$$\hat{F}_t := \begin{bmatrix} 0_{n\times n} & F_t \\ I_{k\times n} & -F_t \end{bmatrix}, \quad w(t,x) := \begin{pmatrix} \xi_x(t,x) \\ \xi_y(t,x) \end{pmatrix}.$$

For the energetic function $V_{adapt}(z)$, defined as

$$V_{adapt}(z) := z^T P^* z + \frac{1}{2k_1}\mathrm{tr}\left\{(\Delta K_t)^{\intercal} \Delta K_t\right\} + \frac{1}{2k_2}\mathrm{tr}\left\{(\Delta F_t)^{\intercal} \Delta F_t\right\},$$
$$P^* = \begin{bmatrix} P_1^* & 0_{n\times n} \\ 0_{n\times n} & P_2^* \end{bmatrix} := \mathrm{diag}(P_1^*, P_2^*),$$
$$\Delta K_t = K_t - K^*, \Delta F_t = F_t - F^*, \;\; k_1 > 0, \;\; k_2 > 0,$$

where $P^* > 0$ is found in (12.14), and taking into account that now the gain matrices are permitted to be time varying, namely,

$$K = K_t, \;\; K = F_t,$$

we get

$$\dot{V}(P^*) = 2z^T P^* \dot{z} + \frac{1}{k_1}\mathrm{tr}\left\{ \left(\tfrac{d}{dt}\Delta K_t\right)^{\intercal} \Delta K_t \right\} + \frac{1}{k_2}\mathrm{tr}\left\{ \left(\tfrac{d}{dt}\Delta F_t\right)^{\intercal} \Delta F_t \right\} =$$

$$\begin{pmatrix} z \\ w \end{pmatrix}^{\intercal} \begin{bmatrix} \hat{A}_t^{\intercal} P^* + P^* \hat{A}_t + \alpha^* P^* & P^* \hat{F}_t \\ \hat{F}_t^{\intercal} P^* & -\varepsilon^* I_{(n+k)\times(n+k)} \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix} - \alpha^* V(P^*)$$

$$+ \tfrac{\alpha}{2k_1}\mathrm{tr}\left\{(\Delta K_t)^{\intercal} \Delta K_t\right\} + \tfrac{\alpha}{2k_2}\mathrm{tr}\left\{(\Delta F_t)^{\intercal} \Delta F_t\right\} + \varepsilon^* \|w\|^2$$

Adding and subtracting the matrices K^* and F^* we obtain

$$\dot{V}(P^*) = \begin{pmatrix} z \\ w \end{pmatrix}^T \begin{bmatrix} W_{1,1} & W_{1,2} \\ W_{1,2}^\intercal & -\varepsilon^* I_{(n+k)\times(n+k)} \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix} +$$

$$\tfrac{1}{k_1}\mathrm{tr}\left\{\left(\tfrac{d}{dt}\Delta K_t\right)^\intercal \Delta K_t\right\} + \tfrac{1}{k_2}\mathrm{tr}\left\{\left(\tfrac{d}{dt}\Delta F_t\right)^\intercal \Delta F_t\right\}$$

$$-\alpha^* V^* + \frac{\alpha}{2k_1}\mathrm{tr}\left\{(\Delta K_t)^\intercal \Delta K_t\right\} + \frac{\alpha}{2k_2}\mathrm{tr}\left\{(\Delta F_t)^\intercal \Delta F_t\right\} + \varepsilon^* \|w\|^2 =$$

$$\begin{pmatrix} z \\ w \end{pmatrix}^T \tilde{W}_T \begin{pmatrix} z \\ w \end{pmatrix} - \alpha^* V^* + \frac{\alpha}{2k_1}\mathrm{tr}\left\{(\Delta K_t)^\intercal \Delta K_t\right\} + \frac{\alpha}{2k_2}\mathrm{tr}\left\{(\Delta F_t)^\intercal \Delta F_t\right\}$$

$$\tfrac{1}{k_1}\mathrm{tr}\left\{\left(\tfrac{d}{dt}\Delta K_t\right)^\intercal \Delta K_t\right\} + \tfrac{1}{k_2}\mathrm{tr}\left\{\left(\tfrac{d}{dt}\Delta F_t\right)^\intercal \Delta F_t\right\}$$

$$+\varepsilon^*\left(c_{0,x}^2 + c_{0,y}^2\right) + z^\intercal \begin{bmatrix} P_1^* B\Delta K_t + (P_1^* B\Delta K_t)^\intercal & P_1^*\Delta F_t C \\ (P_1^*\Delta F_t C)^\intercal & P_2^*\Delta F_t C + (P_2^*\Delta F_t C)^\intercal \end{bmatrix} z$$

$$+2\begin{pmatrix} \hat{x} \\ e \end{pmatrix}^\intercal \begin{bmatrix} 0_{n\times n} & P_1^*\Delta F_t \\ 0_{n\times n} & -P_2^*\Delta F_t \end{bmatrix} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} \tag{12.28}$$

where

$$W_{1,1} = \begin{bmatrix} \begin{matrix} P_1^*(A_\alpha + BK^*) + \\ [P_1^*(A_\alpha + BK^*)]^\intercal + \\ P_1^* B\Delta K_t + (P_1^* B\Delta K_t)^\intercal \end{matrix} & \begin{matrix} P_1^* F^* C + \\ P_1^*\Delta F_t C \end{matrix} \\ \begin{matrix} (P_1^* F^* C)^\intercal + \\ (P_1^*\Delta F_t C)^\intercal \end{matrix} & \begin{matrix} P_2^*(A_\alpha - F^* C) + \\ [P_2^*(A_\alpha - F^* C)]^\intercal \\ + P_2^*\Delta F_t C \\ + (P_2^*\Delta F_t C)^\intercal \end{matrix} \end{bmatrix}$$

$$W_{1,2} = \begin{bmatrix} P_1^* & P_1^* F^* + P_1^*\Delta F_t \\ P_2^* & -P_2^* F^* - P_2^*\Delta F_t \end{bmatrix}$$

and $\tilde{W}_T$ is defined in (12.7). Since

$$Ce = y - C\hat{x}$$

we get

$$z^{\intercal}\left[\begin{array}{cc} P_1^* B\Delta K_t + (P_1^* B\Delta K_t)^{\intercal} & P_1^*\Delta F_t C \\ (P_1^*\Delta F_t C)^{\intercal} & \begin{array}{c} P_2^*\Delta F_t C + \\ (P_2^*\Delta F_t C)^{\intercal} \end{array} \end{array}\right] z =$$

$$2\hat{x}^{\intercal} P_1^* B\Delta K_t \hat{x} + 2\hat{x}^{\intercal} P_1^* \Delta F_t (y - C\hat{x}) + 2e^{\intercal} P_2^* \Delta F_t (y - C\hat{x})$$

$$\overset{\text{by the }\Lambda\text{-inequality}}{\leq} \quad \operatorname{tr}\left\{2\hat{x}\hat{x}^{\intercal} P_1^* B\Delta K_t + 2(y - C\hat{x})\hat{x}^{\intercal} P_1^* \Delta F_t\right\} +$$

$$e^{\intercal} P_2^* \Lambda P_2^* e + (y - C\hat{x})^{\intercal} (\Delta F_t)^{\intercal} \Lambda^{-1} \Delta F_t (y - C\hat{x}) \overset{\Lambda = \left(P_2^*\right)^{-1}}{=} e^{\intercal} P_2^* e +$$

$$\operatorname{tr}\left\{(2B^{\intercal} P_1^* \hat{x}\hat{x}^{\intercal})^{\intercal} \Delta K_t + \left([2P_1^*\hat{x} + P_2^*\Delta F_t (y - C\hat{x})](y - C\hat{x})^{\intercal}\right)^{\intercal} \Delta F_t\right\}, \tag{12.29}$$

and (again by the Λ-inequality (12.16))

$$2\begin{pmatrix} \hat{x} \\ e \end{pmatrix}^{\intercal} \begin{bmatrix} 0_{n\times n} & P_1^*\Delta F_t \\ 0_{n\times n} & -P_2^*\Delta F_t \end{bmatrix} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = 2\begin{pmatrix} \hat{x} \\ e \end{pmatrix}^{\intercal} \begin{pmatrix} P_1^*\Delta F_t \xi_y \\ -P_2^*\Delta F_t \xi_y \end{pmatrix}$$

$$= 2\hat{x}^{\intercal} P_1^* \Delta F_t \xi_y - 2e^{\intercal} P_2^* \Delta F_t \xi_y = 2\left([\Delta F_t]^{\intercal} P_1^* \hat{x}\right)^{\intercal} \xi_y - 2(P_2^* e)^{\intercal} \left(\Delta F_t \xi_y\right)$$

$$\underset{\Lambda = \frac{1}{\varepsilon^*} I_{k\times k},\, \Lambda_1 = \left(P_2^*\right)^{-1}}{\overset{\text{by the }\Lambda\text{-inequality}}{\leq}} \hat{x}^{\intercal} P_1^* [\Delta F_t] \Lambda [\Delta F_t]^{\intercal} P_1^* \hat{x} + \xi_y^{\intercal} \Lambda^{-1} \xi_y$$

$$+ e^{\intercal} P_2^* \Lambda_1 P_2^* e + \xi_y^{\intercal} [\Delta F_t]^{\intercal} \Lambda_1^{-1} \Delta F_t \xi_y = \varepsilon^* \left\|\xi_y\right\|^2 + e^{\intercal} P_2^* e +$$

$$\operatorname{tr}\left\{\left(\tfrac{1}{\varepsilon^*} P_1^* \hat{x}\hat{x}^{\intercal} P_1^* \Delta F_t\right)^{\intercal} \Delta F_t\right\} + \operatorname{tr}\left\{P_2^* \Delta F_t \xi_y \xi_y^{\intercal} [\Delta F_t]^{\intercal}\right\}.$$

Using the upper estimates (12.25)

$$\xi_y^{\intercal}(t, x)\xi_y(t, x) \leq c_{0,y} + x^T Q_y x, \;\; 0 < Q_y \in \mathbb{R}^{n\times n},$$

and

$$\operatorname{tr}\left\{P_2^* \Delta F_t \xi_y \xi_y^{\intercal} [\Delta F_t]^{\intercal}\right\} \leq \left\|\xi_y\right\|^2 \operatorname{tr}\left\{P_2^* \Delta F_t [\Delta F_t]^{\intercal}\right\}$$

$$\leq \operatorname{tr}\left\{(c_{0,y} P_2^* \Delta F_t)^{\intercal} \Delta F_t\right\},$$

we obtain

$$\left.\begin{array}{c} 2\begin{pmatrix} \hat{x} \\ e \end{pmatrix}^{\intercal} \begin{bmatrix} 0_{n\times n} & P_1^* \Delta F_t \\ 0_{n\times n} & -P_2^* \Delta F_t \end{bmatrix} \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} \\ \le \varepsilon^* c_{0,y} + e^{\intercal} P_2^* e + \\ \operatorname{tr}\left\{ \left(\frac{1}{\varepsilon^*} P_1^* \hat{x}\hat{x}^{\intercal} P_1^* \Delta F_t + c_{0,y} P_2^* \Delta F_t \right)^{\intercal} \Delta F_t \right\} \end{array}\right\} \qquad (12.30)$$

Applying (12.29)-(12.30) to (12.28), we get

$$\dot{V}(P^*) \le \begin{pmatrix} z \\ w \end{pmatrix}^T \tilde{W}_T \begin{pmatrix} z \\ w \end{pmatrix} - \alpha^* V^* + \varepsilon^* \left(c_{0,x} + 2c_{0,y} \right) + e^{\intercal} \left(2P_2^* \right) e +$$

$$\operatorname{tr}\left\{ \underbrace{\left(2B^{\intercal} P_1^* \hat{x}\hat{x}^{\intercal} + \frac{\alpha}{2k_1} \Delta K_t + \frac{1}{k_1} \frac{d}{dt} \Delta K_t \right)^{\intercal}}_{S_{1,t}} \Delta K_t \right\} +$$

$$\operatorname{tr}\left\{ \underbrace{\left(\begin{array}{c} \left[2P_1^* \hat{x} + P_2^* \Delta F_t (y - C\hat{x}) \right] (y - C\hat{x})^{\intercal} + \\ \frac{1}{k_2} \frac{d}{dt} \Delta F_t + \left(\frac{\alpha}{2k_2} I_{n\times n} + \frac{1}{\varepsilon^*} P_1^* \hat{x}\hat{x}^{\intercal} P_1^* + c_{0,y} P_2^* \right) \Delta F_t \end{array} \right)^{\intercal}}_{S_{2,t}} \Delta F_t \right\}$$

The adaptation algorithm (12.26) corresponds to the conditions

$$S_{1,t} = 0 \text{ and } S_{2,t} = 0.$$

Including then the term $e^{\intercal}\left(2P_2^*\right)e$ into $\tilde{W}_T^{(1,1)}$ and defining

$$\tilde{W}_{adapt,T} := \begin{bmatrix} \tilde{W}_{adapt,T}^{(1,1)} & \tilde{W}_{adapt,T}^{(1,2)} \\ \tilde{W}_{adapt,T}^{(2,1)} & \tilde{W}_{adapt,T}^{(2,2)} \end{bmatrix}$$

$$\tilde{W}_T = \begin{bmatrix} \begin{array}{c} (A_\alpha X_1 + BY_1) + \\ (A_\alpha X_1 + BY_1)^{\intercal} + \\ \varepsilon \bar{Q} + 2H \end{array} & \varepsilon X_1 \left(Q_x + Q_y \right) \\ \varepsilon \left(Q_x + Q_y \right)^{\intercal} X_1^{\intercal} & \begin{array}{c} X_2 A_\alpha - Y_2 C + \\ (X_2 A_\alpha - Y_2 C)^{\intercal} + \\ \varepsilon \left(Q_x + Q_y \right) + H_C \end{array} \end{bmatrix}$$

with

$$\tilde{W}_{adapt,T}^{(1,1)} = \tilde{W}_T^{(1,1)} + \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & 2P_2^* \end{bmatrix}$$

and

$$\tilde{W}^{(1,2)}_{adapt,T} = \tilde{W}^{(1,2)}_{T},\ \tilde{W}^{(2,1)}_{adapt,T} = \tilde{W}^{(2,1)}_{T},\ \tilde{W}^{(2,2)}_{adapt,T} = \tilde{W}^{(2,2)}_{T},$$

with $\tilde{W}^{(1,1)}_{adapt,T} < 0$, we finally obtain

$$\dot{V}\left(P^{*}\right) \leq -\alpha^{*}V^{*} + \varepsilon^{*}\left(c_{0,x} + 2c_{0,y}\right),$$

which proves the theorem. ■

12.5 Exercise

Exercise 12.1 *Consider the model (as in Lecture 7) given by the strongly non-linear differential equations*

$$\left.\begin{array}{c} \dot{x}_1 = -x_1 + x_2 + 0.1x_1\text{sign}(x_2), \\ \dot{x}_2 = -x_1 + 0.2\text{sign}\left(x_1\right) + u, \\ y = x_1 + x_2 + \xi_y(t), \\ x_1, x_2 \in \mathbb{R},\ x_0 = (1,1)^{\mathsf{T}}, \end{array}\right\} \tag{12.31}$$

where

$$\left|\xi_y(t)\right|^2 \leq c_{0,y} = 0.01.$$

Design the control

a) using observer state estimates $\hat{x}$*;*

b) using the same controller as in a), but applying the gains adaptation procedure;

c) compare (graphically) both non-adaptive and adaptive gains ellipsoids.