Lecture 11

Robust Linear Output Feedback Control

In this lecture, we'll look at three different forms of linear feedbacks that may be implemented using only the current output data:

- the static feedback proportional to the output measurable signal,
- the observer-based feedback proportional to the state estimation vector with its daptive version,
- and the full order linear dynamic controller.

We propose that for each form of considered linear feedback, a system of the corresponding LMI's be used to describe the set of all stabilizing gain-feedback matrices, ensuring the boundedness of all possible trajectories of any controlled plant from the examined class of uncertain systems. We also recommend picking one of the best feedback gain matrices from the stated class of stabilizing feedbacks to reduce the "size" of the attractive ellipsoid comprising all potential limited dynamic trajectories. For each form of studied feedback, the related numerical processes for creating the optimum feedback gain matrices are introduced and addressed. The usefulness of the recommended strategy is demonstrated by a number of cases.

11.1 System description and problem formulation

Consider here the quasi-Lipschitz affine control system with the quasi-Lipschitz state-output mapping given by

$$\boxed{\left.\begin{array}{c} \dot{x}\left(t\right)=f(x,t)+Bu\left(t\right)+\tilde{\xi}_{x}(t,x),\ x\left(0\right)=x_{0}, \\ \\ y\left(t\right)=h\left(x\left(t\right)\right)+\tilde{\xi}_{y}(t,x), \end{array}\right\}} \quad (11.1)$$

where

- $x(t) \in \mathbb{R}^n$ is the state vector at time $t \geq 0$,
- $u(t) \in \mathbb{R}^m$ is the control input,
- $B \in \mathbb{R}^{n \times m}$ is known matrux,
- $y(t) \in \mathbb{R}^k$ is the measurable system output,
- $\tilde{\xi}_x(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\tilde{\xi}_y(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^k$ are unknown bounded nonlinear functions
$$\left\| \tilde{\xi}_x(t,x) \right\| \leq \tilde{\xi}_x^+ < \infty, \ \left\| \tilde{\xi}_y(t,x) \right\| \leq \tilde{\xi}_y^+ < \infty, \tag{11.2}$$
treated below as external distrurbances,
- $f(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is unknown nonlinea mappining characterizing the dynamics of the system,
- $h(x(t)) : \mathbb{R}^n \to \mathbb{R}^n$ is unknown output mapping.

As in the previose lecture we may rewrite the given dynamics (11.1) in, the so-called, quasi-linear format, namely,

$$\boxed{\left.\begin{array}{c} \dot{x}(t) = Ax + Bu(t) + \xi_x(t,x), \ x(0) = x_0, \\ \\ y(t) = Cx(t) + \xi_y(t,x), \end{array}\right\}} \tag{11.3}$$

where
$$\left.\begin{array}{c} \xi_x(t,x) = \tilde{\xi}_x(t,x) + f(x,t) - Ax \\ \\ \xi_y(t,x) = \tilde{\xi}_y(t,x) + h(x) - Cx \end{array}\right\} \tag{11.4}$$
with $A \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{k \times n}$ as the system matrices supposed to be known.

The following assumptions will be in force throughout:

A1) the nonlinear function $f(x,t)$ is *quasi-Lipschitz*, namely, it belongs to the class of functions satisfying
$$f^{\mathsf{T}}(x,t) f(x,t) \leq c_f + x^T Q_f x,$$
where Q_f is known nonnegative (usually normalizing) matrix and $c_f \geq 0$ is a known constant, and $h(x)$ satisfies the inequality
$$h^{\mathsf{T}}(x) h(x) \leq x^T Q_h x, \ Q_h > 0;$$

A2) the nonlinear functions $\xi_x(t,x)$ and $\xi_y(t,x)$, characterizing uncertainties of the system, in the dynamics and output satisfy the similar quasi-Lipschitz constraints:

$$\left.\begin{array}{c} \xi_x^{\intercal}(t,x)\xi_x(t,x) \leq c_{0,x} + x^T Q_x x,\ 0 < Q_x \in \mathbb{R}^{n\times n}, \\ c_{0,x} = \underbrace{2\left(\tilde{\xi}_x^{+}\right)^2 + 4c_f}_{c_{0,x}} + 4x^T \underbrace{(Q_f + A^{\intercal}A)}_{Q_x} x \end{array}\right\} \tag{11.5}$$

$$\left.\begin{array}{c} \xi_y^{\intercal}(t,x)\xi_y(t,x) \leq c_{0,y} + x^T Q_y x,\ 0 < Q_y \in \mathbb{R}^{n\times n}, \\ c_{0,x} = \underbrace{2\left(\tilde{\xi}_y^{+}\right)^2}_{c_{0,y}} + 4x^T \underbrace{(Q_h + C^{\intercal}C)}_{Q_y} x \end{array}\right\} \tag{11.6}$$

where Q_x, Q_y are known nonnegative definite matrixes and $c_{0,x}$, $c_{0,y}$ are known nonnegative number;

A3) the matrices B and C have *full ranks*, i.e.,

$$B^T B > 0 \text{ and } CC^T > 0. \tag{11.7}$$

In this section we consider the *static linear output feedback* of the form

$$u = Ky\,, \tag{11.8}$$

where $K \in \mathbb{R}^{m\times k}$ is the control gains matrix to be design.

The closed-loop dynamics is given by the following ODE:

$$\boxed{\left.\begin{array}{c} \dot{x}\,(t) = Ax\,(t) + BK\left[Cx\,(t) + \xi_y(t,x)\right] + \xi_x(t,x) = \\ (A + BKC)\,x\,(t) + BK\xi_y(t,x) + \xi_x(t,x). \end{array}\right\}} \tag{11.9}$$

Problem 11.1 *We need to find the **output gain control matrix** K which provides the convergence of all possible trajectories into an attractive ellipsoid $\mathcal{E}_0(P_{attr})$ of a minimal possible size.*

11.2 Special orthonormal state space transformation

Introduce the following orthogonal matrices

$$\left.\begin{array}{c} G_B \in \mathbb{R}^{n\times n},\ G_B G_B^T = G_B^T G_B = I_{n\times n},\ G_B B = \begin{bmatrix} 0_{(n-m)\times m} \\ \tilde{B} \end{bmatrix}, \\ \\ \tilde{B} \in \mathbb{R}^{m\times m},\ \det(\tilde{B}) \neq 0\,. \end{array}\right\} \tag{11.10}$$

The corresponding matrices G_B and $\tilde{B}$ can be easily found in MATLAB using the function *null*. For given matrix M the function *null(M)* returns the matrix that columns represent the orthonormal basis of the null space of the matrix M. In this case we get

$$G_B = \begin{pmatrix} B^\perp \\ B' \end{pmatrix},\ \text{where } B^\perp = (\text{null}(B^\intercal))^\intercal \text{ and } B' = \left(\text{null}\left(B^\perp\right)\right)^\intercal. \tag{11.11}$$

Applying this transformation to the system (11.9) we obtain

$$\frac{d}{dt}\left(G_B x\left(t\right)\right) = \left(G_B A + G_B BKC\right) G_B^T \left(G_B x\left(t\right)\right) +$$

$$G_B BK\xi_y(t,x) + G_B\xi_x(t,x),$$

which in new variables

$$\tilde{x} := G_B x = \begin{pmatrix} \tilde{x}_1 \in R^{n-m} \\ \tilde{x}_2 \in R^m \end{pmatrix},\ \tilde{A} := G_B A G_B^T,\ \bar{\xi}_x := G_B \xi_x,\ Y := \tilde{B}K \tag{11.12}$$

is as follows

$$\boxed{\left.\begin{array}{c} \dfrac{d}{dt}\tilde{x}\left(t\right) = \left(\tilde{A} + \begin{bmatrix} 0_{(n-m)\times m} \\ Y \end{bmatrix}\right) C G_B^T \tilde{x}\left(t\right) + \\ \\ \begin{bmatrix} 0_{(n-m)\times m} \\ Y \end{bmatrix} \xi_y(t,x) + \bar{\xi}_x(t,x). \end{array}\right\}} \tag{11.13}$$

Now we are ready to describe the class of the feeadback matrices $K = \tilde{B}^{-1}Y$ providing the boundedness of all possible trajectories of the quasi-Lipschits class of nonlinear systems (11.1), closed by the linear feed back (11.8).

11.3 Attractive ellipsoid design

Using the same technique as above, let us introduce the Lyapunov function

$$V(\tilde{x}) = \tilde{x}^{\intercal} P \tilde{x}, \; P = \begin{bmatrix} P_1 & 0_{(n-m)\times m} \\ 0^{\intercal}_{(n-m)\times m} & P_2 \end{bmatrix} > 0\,, \tag{11.14}$$

which derivative on the trajectpries of (11.13) is as follows:

$$\dot{V}(\tilde{x}) = 2\tilde{x}^{\intercal} P \frac{d}{dt}\tilde{x} =$$

$$\tilde{x}^{\intercal} P \left[\left(\tilde{A} + \begin{bmatrix} 0_{(n-m)\times m} \\ Y \end{bmatrix} \right) C G_B^T \tilde{x} + \begin{bmatrix} 0_{(n-m)\times m} \\ Y \end{bmatrix} \xi_y + \tilde{\xi}_x \right] =$$

$$\begin{pmatrix} \tilde{x} \\ \tilde{\xi}_x \\ \xi_y \end{pmatrix}^{\intercal} \begin{bmatrix} P_\alpha & P & P \begin{bmatrix} 0_{(n-m)\times m} \\ Y \end{bmatrix} \\ P & -\varepsilon I_{n\times n} & 0_{n\times k} \\ \left(P \begin{bmatrix} 0_{(n-m)\times m} \\ Y \end{bmatrix} \right)^{\intercal} & 0_{k\times n} & -\varepsilon I_{k\times k} \end{bmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\xi}_x \\ \xi_y \end{pmatrix}$$

$$-\alpha V(\tilde{x}) + \varepsilon \left(\left\| \tilde{\xi}_x \right\|^2 + \left\| \xi_y \right\|^2 \right)$$

with

$$\begin{gathered} P_\alpha := P \left(\tilde{A} + \begin{bmatrix} 0_{(n-m)\times m} \\ Y \end{bmatrix} \right) C G_B^T + \\ \left[P \left(\tilde{A} + \begin{bmatrix} 0_{(n-m)\times m} \\ Y \end{bmatrix} \right) C G_B^T \right]^{\intercal} + \alpha P \end{gathered}$$

Using the upper estimates (11.5) and (11.6), we get

$$\left. \begin{gathered} \bar{\xi}_x^{\intercal}(t,x)\bar{\xi}_x(t,x) = \xi_x^{\intercal}(t,x)\xi_x(t,x) \le c_{0,x} + x^T Q_x x = c_{0,x} + \tilde{x}^T \tilde{Q}_x \tilde{x}, \\ \tilde{Q}_x := G_B Q_x G_B^{\intercal}, \\ \\ \xi_y^{\intercal}(t,x)\xi_y(t,x) \le c_{0,y} + x^T Q_y x = c_{0,y} + \tilde{x}^T \tilde{Q}_y \tilde{x}, \\ \tilde{Q}_y := G_B Q_y G_B^{\intercal}. \end{gathered} \right\} \tag{11.15}$$

Applying these estimates to the term in the expressin for $\dot{V}(\tilde{x})$ we obtain

$$\dot{V}(\tilde{x}) \le \begin{pmatrix} \tilde{x} \\ \tilde{\xi}_x \\ \xi_y \end{pmatrix}^{\intercal} W_{\alpha,\varepsilon}\left(P_1, P_2, \tilde{Y}\right) \begin{pmatrix} \tilde{x} \\ \tilde{\xi}_x \\ \xi_y \end{pmatrix} - \alpha V(\tilde{x}) + \varepsilon\left(c_{0,x} + c_{0,y}\right) \tag{11.16}$$

where

$$W_{\alpha,\varepsilon}\left(P_1, P_2, \tilde{Y}\right) = \begin{bmatrix} W_{\alpha,\varepsilon}^{(1,1)}\left(P_1, P_2, \tilde{Y}\right) & W_{\alpha,\varepsilon}^{(1,2)}\left(P_1, P_2, \tilde{Y}\right) & W_{\alpha,\varepsilon}^{(1,3)}\left(P_1, P_2, \tilde{Y}\right) \\ W_{\alpha,\varepsilon}^{(1,2)}\left(P_1, P_2, \tilde{Y}\right)^{\intercal} & -\varepsilon I_{n\times n} & 0_{n\times k} \\ W_{\alpha,\varepsilon}^{(1,3)}\left(P_1, P_2, \tilde{Y}\right)^{\intercal} & 0_{k\times n} & -\varepsilon I_{k\times k} \end{bmatrix} \tag{11.17}$$

with

$$\tilde{Y} := P_2 Y$$

and

$$\begin{aligned} W_{\alpha,\varepsilon}^{(1,1)}\left(P_1, P_2, \tilde{Y}\right) = {} & \begin{bmatrix} P_1 & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & P_2 \end{bmatrix} \tilde{A} C G_B^T + \begin{bmatrix} 0_{(n-m)\times(n-m)} & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & \tilde{Y} \end{bmatrix} C G_B^T \\ & + \left(\begin{bmatrix} P_1 & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & P_2 \end{bmatrix} \tilde{A} C G_B^T \right)^{\intercal} \\ & + \left(\begin{bmatrix} 0_{(n-m)\times(n-m)} & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & \tilde{Y} \end{bmatrix} C G_B^T \right)^{\intercal} \\ & + \alpha \begin{bmatrix} P_1 & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & P_2 \end{bmatrix} + \varepsilon \left(\tilde{Q}_x + \tilde{Q}_y \right), \end{aligned}$$

$$W_{\alpha,\varepsilon}^{(1,2)}\left(P_1, P_2, \tilde{Y}\right) = P = \begin{bmatrix} P_1 & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & P_2 \end{bmatrix},$$

$$W_{\alpha,\varepsilon}^{(1,3)}\left(P_1, P_2, \tilde{Y}\right) = \begin{bmatrix} 0_{(n-m)\times(n-m)} & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & \tilde{Y} \end{bmatrix}.$$

Notice that matrix $W_{\alpha,\varepsilon}\left(P_1, P_2, \tilde{Y}\right)$ is linear with respect to its matrix arguments.

Theorem 11.1 *If for the closed-loop system (11.9) there exist positive definite matrices $P_1 \in \mathbb{R}^{(n-m)\times(n-m)}$, $P_2 \in \mathbb{R}^{m\times m}$ and matrix $\tilde{Y} \in \mathbb{R}^{m\times m}$ such that the following LMI holds*

$$\boxed{W_{\alpha,\varepsilon}\left(P_1, P_2, \tilde{Y}\right) < 0,} \tag{11.18}$$

*then the ellipsoid $\mathcal{E}_0(P_{attr})$ is **attractive**, that is, contains asymptotically all possible trajectories $x(t)$ with*

$$\boxed{P_{attr} = \frac{\alpha}{\varepsilon\left(c_{0,x} + c_{0,y}\right)} G_B^{\intercal} \begin{bmatrix} P_1 & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & P_2 \end{bmatrix} G_B,} \tag{11.19}$$

meanwhile the corresponding output control gains matrix is

$$\boxed{K = \tilde{B}^{-1} P_2^{-1} \tilde{Y}\,.} \tag{11.20}$$

Proof. In view of (11.18) from (11.16) we have

$$\dot{V}(\tilde{x}) \le -\alpha V(\tilde{x}) + \varepsilon\left(c_{0,x} + c_{0,y}\right),$$

which leads to following inequality

$$V(\tilde{x}(t)) \le \tfrac{\varepsilon(c_{0,x}+c_{0,y})}{\alpha} + \left(V(x(0)) - \tfrac{\varepsilon(c_{0,x}+c_{0,y})}{\alpha}\right)e^{-\alpha t}$$

$$\underset{t\to\infty}{\to} \frac{\varepsilon\left(c_{0,x}+c_{0,y}\right)}{\alpha},$$

and hence,

$$\limsup_{t\to\infty} \tilde{x}(t)^{\intercal} P \tilde{x}(t) = \limsup_{t\to\infty} x(t)^{\intercal} G_B^{\intercal} P G_B x(t) \le \frac{\varepsilon\left(c_{0,x}+c_{0,y}\right)}{\alpha},$$

or equivalently,

$$\limsup_{t\to\infty} x(t)^{\intercal} \left[\frac{\alpha}{\varepsilon\left(c_{0,x}+c_{0,y}\right)} G_B^{\intercal} P G_B\right] x(t) \le 1.$$

Theorem is proven. ■

11.4 Optimal output feedback

Since the "size" of the attractive ellipsoid $\mathcal{E}_0(P_{attr})$ is smaller if the matrix P_{attr} is "more", the "optimal" feedback gain matrix will correspons to the solution of the following matrix optimization problem:

$$\mathrm{tr} P_{attr} = \frac{\alpha}{\varepsilon\left(c_{0,x}+c_{0,y}\right)} \mathrm{tr}\left\{ G_B^{\intercal} \begin{bmatrix} P_1 & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & P_2 \end{bmatrix} G_B \right\} =$$

$$\frac{\alpha}{\varepsilon\left(c_{0,x}+c_{0,y}\right)} \left(\mathrm{tr} P_1 + \mathrm{tr} P_2\right) \to \sup_{P_1>0,\, P_2>0,\, \tilde{Y},\, \alpha>0,\, \varepsilon>0}$$

subject to the constraint (11.18). The optimal parameters may be equivalently represented as the solution of matrix minimization problem

$$\boxed{\left.\begin{array}{c} \left(P_1^*, P_2^*, \tilde{Y}^*\right) = \underset{P_1>0,\, P_2>0,\, \tilde{Y},\, \alpha>0,\, \varepsilon>0}{\arg\inf} \left[\dfrac{\alpha}{\varepsilon}\left(\mathrm{tr} P_1^{-1} + \mathrm{tr} P_2^{-1}\right)\right] \\ \text{subject to the } LMI \text{ constraint (11.18)} \\ W_{\alpha,\varepsilon}\left(P_1, P_2, \tilde{Y}\right) < 0. \end{array}\right\}} \tag{11.21}$$

By the upper estimates

$$P_1^{-1} < H_1, \; P_2^{-1} < H_2$$

which equivalently, based on the Schur's complement, may be represented as

$$\boxed{\left.\begin{array}{l} \begin{bmatrix} H_1 & I_{(n-m)\times(n-m)} \\ I_{(n-m)\times(n-m)} & P_1 \end{bmatrix} > 0, \\ \begin{bmatrix} H_2 & I_{(n-m)\times(n-m)} \\ I_{(n-m)\times(n-m)} & P_2 \end{bmatrix} > 0, \end{array}\right\}} \tag{11.22}$$

the problem (11.21) can be expressed as

$$\boxed{\left.\begin{array}{c} \left(P_1^*, P_2^*, \tilde{Y}^*\right) = \\ \underset{P_1>0,\, P_2>0,\, H_1>0,\; H_2>0,\; \tilde{Y},\, \alpha>0,\, \varepsilon>0}{\arg\inf} \left[\frac{\alpha}{\varepsilon}\left(\mathrm{tr}H_1 + \mathrm{tr}H_2\right)\right] \\ \text{subject to the } LMI \text{ constraints} \\ \text{(11.18) and (11.22).} \end{array}\right\}} \tag{11.23}$$

The "optimal" feedback gain matrix K^* will be equal to

$$\boxed{K^* = \tilde{B}^{-1}\left(P_2^*\right)^{-1}\tilde{Y}^*} \tag{11.24}$$

with the corresponding matrix P_{attr}^* of the attractive ellipsoid $\mathcal{E}_0(P_{attr}^*)$ as

$$\boxed{P_{attr}^* = \frac{\alpha^*}{\varepsilon^*\left(c_{0,x} + c_{0,y}\right)} G_B^{\intercal} \begin{bmatrix} P_1^* & 0_{(n-m)\times m} \\ 0_{m\times(n-m)} & P_2^* \end{bmatrix} G_B.} \tag{11.25}$$

11.5 Example: Stabilization of a discontinuous system

Consider the model given by the strongly non-linear differential equations

$$\left.\begin{array}{c} \dot{x}_1 = -x_1 + x_2 + 0.1x_1\mathrm{sign}(x_2), \\ \dot{x}_2 = -x_1 + 0.2\mathrm{sign}\left(x_1\right) + u, \\ y = x_1 + x_2 + \xi_y(t), \\ x_1, x_2 \in \mathbb{R}, \; x_0 = (1, 1)^{\intercal}, \end{array}\right\} \tag{11.26}$$

where

$$|\xi_y(t)|^2 \le c_{0,y} = 0.01.$$

The given system can be represented in the quasi-linear form

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + \xi_x(t, x(t)) \\ y\,(t) &= Cx(t) + \xi_y(t) \end{aligned}\right\} \tag{11.27}$$

with the correspondingly defined system matrices

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

and the nonlinear function

$$\xi_x(t, x) = \begin{pmatrix} 0.1x_1\mathrm{sign}(x_2) \\ 0.2\mathrm{sign}(x_1) \end{pmatrix}$$

satisfying quasi-Lipschitz condition

$$\|\xi_x\|^2 \le c_{0,x} + x^T Q_x x$$

where

$$c_{0,x} = 0.04,\ Q_x = 0.01 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Solving the optimization problem (11.21) with the MATLAB solver SEDUMI (or YALMIP) and using (11.24) we obtained

$$P = \begin{pmatrix} 247.9034 & 0 \\ 0 & 75.3833 \end{pmatrix},\ K^* = -2.2886.$$

The figures 11.1 - 11.2 present the results of the numerical simulation of the closed-loop system (11.26) with static output feedback $u = K^* y$.

To simulated the deterministic noise of measurements the function $\xi_y(t)$ was selected as

$$\xi_y(t) = 0.1\cos(100t).$$

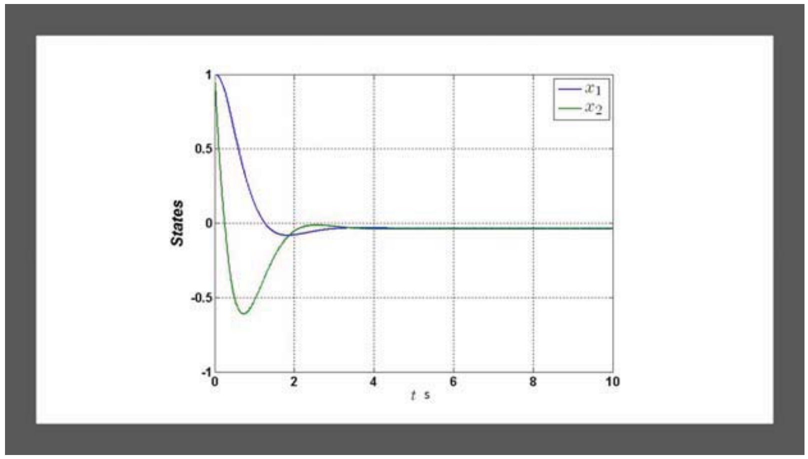

Figure 11.1: States dynamics.

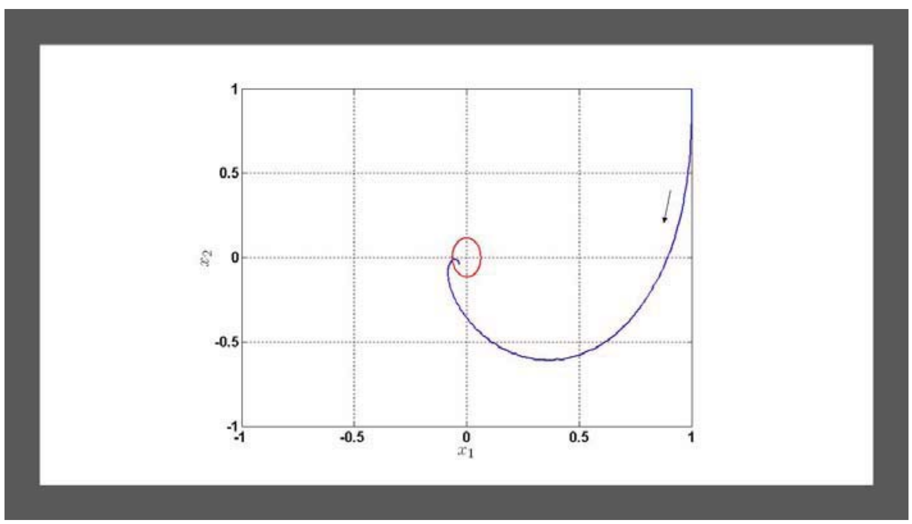

Figure 11.2: Attractive ellipsoid.