Lecture 10: DNN - Backstepping Control

Plan of presentation

- Historical comments
- Block Controllable Form (a general view)
- Backstepping method
- Backstepping method in mechanical systems
- Backstepping application to DNN-control of mechanical systems
- Comparison of DNN-SM and DNN-Backstepping-SM metods

Remark

- **A.G. Lukianov and V.I. Utkin** *in 1981 demonstrated that a principle of control system construction based on the sequential use of some of the state vector's coordinates as controls can be used.*
- *This allows the original synthesis problem to be divided into independent subtasks of smaller dimensions, including elementary ones, where the dimensions of the state and control vectors coincide.*
- *In relation to the fundamental issues of theory control, the capabilities of this approach are illustrated by* **Drakunov et all** *(1990) for the assignment of the spectrum of a closed-loop system, and for designing of filters (state estimators) invariant to perturbations.*

Block Controllable Form

Initially the block (cascade) control was developed for systems presented in the, so-called, *Block Controllable Form*

$$\left.\begin{array}{c}\dot{x}_i = f_i\left(x_1, ..., x_i, t\right) + B_i\left(x_1, ..., x_i, t\right) x_{i+1},\ 1 \le i \le r-1, \\ \\ \dot{x}_r = f_r\left(x_1, ..., x_r, t\right) + B_i\left(x_1, ..., x_i, t\right) x_{i+1}, \\ \\ \dot{x}_r = f_r\left(x_1, ..., x_r\right) + B_r\left(x_1, ..., x_r, t\right) u + D_r\left(x_1, ..., x_r, t\right) w(t), \\ \\ y = h(x) = x_1,\end{array}\right\} \quad (1)$$

where the vector x is decomposed as $x = \left(x_1^{\intercal}, ..., x_r^{\intercal}\right)^{\intercal}$, $x_i \in \mathbb{R}^{n_i}$ so that

$$n_1 \le n_2 \le \cdots \le n_r \le m, u \in \mathbb{R}^m, \sum_{i=1}^{r} n_i = n.$$

Backstepping method (a general view)

The *Backstepping method* has further been developed by **P.Kokotovich** (1992) for affine control systems of arbitrary relative degree k and then it has been applied for adaptive control of systems presented in the form of n-blocks **V.Krstich-P.Kokotovich** and **R.Lozano-B.Bragliato** 1992). The concept, extending the block-cascade approach, is treated as a technique for designing stabilizing controls for a special class of nonlinear dynamic systems presented in the form of n-blocks (or cascaded) subsystems (the strict-feedback form):

$$\left.\begin{array}{c} \dot{x}_i = x_{i+1} + f_i\left(x_1, ..., x_i\right), \ 1 \leq i \leq n-1, \\ \dot{x}_n = f_n\left(x_1, ..., x_n\right) + g\left(x_1, ..., x_n\right) u. \end{array}\right\} \tag{2}$$

These systems are composed of irreducible subsystems which can be stabilized using some other method. Because of this recursive structure, the designer can start the design process at the known-stable system and "back out" new controllers that progressively stabilize each outer subsystem. The process terminates when the final external control is reached. That's why this process is known as backstepping (see **H.K.Khalil** 2002). Some applications of this approach was done by **R.Suarez** (2000).

Backstepping method in mechanical systems

In mechanical systems, given by the model

$$\boxed{D\left(q\right)\ddot{q}+b(q,\dot{q},t)+c(q,t)=\tau,} \tag{3}$$

where $q\in R^{n}$, $D\left(q\right)$ is invertible and known, vectors $b(q,\dot{q},t)$ and $c(q,t)$ are unknown, τ is a torque control to be designed, the trajectory tracking problem

$$\boxed{\left\|q\left(t\right)-q^{*}\left(t\right)\right\|\underset{t\to\infty}{\rightarrow}0,} \tag{4}$$

($q^{*}\left(t\right)$ is a smooth desired tracking trajectory) can be represented as two-block subsystems:

$$\boxed{\left.\begin{array}{c}\dot{z}_{1}=z_{2},\\ \dot{z}_{2}=D^{-1}\left(q\right)\left[\tau-\zeta\right],\end{array}\right\}} \tag{5}$$

where $z_{1}=q$, $z_{2}=\dot{q}$ and $\zeta=b(q,\dot{q},t)+c(q,t)$ is the uncertain vector (which assumed to be restricted by a known upper bound function as $\left\|\zeta\right\|\leq\zeta^{+}\left(q,\dot{q},t\right)$).

The first step

In *the first step*, let us consider the coordinate z_2 as an auxiliary control u_1 (forgetting the second equation), that is,

$$\boxed{\dot{z}_1 = u_1,} \tag{6}$$

and select it in such a way that the difference $\Delta_1(t) := z_1(t) - q^*(t)$ will tend to zero, providing the smoothness property for u_1.
In variable $\Delta(t)$ we have

$$\boxed{\dot{\Delta}_1(t) = u_1(t) - \dot{q}^*(t),}$$

and, defining u_1 as

$$\boxed{u_1(t) = \dot{q}^*(t) - \alpha\Delta_1(t), \alpha > 0.} \tag{7}$$

we obtained $\dot{\Delta}_1(t) = -\alpha\Delta(t)$, guarantying the desired property

$$\boxed{\Delta_1(t) = \Delta_1(0)\exp(-\alpha t) \underset{t\to\infty}{\longrightarrow} 0.}$$

The second step

In the second step, let us design the original torque-control τ such that $\Delta_2(t) := z_2(t) - u_1(t)$ converges to zero in a finite time in the presence of uncertainty ζ. Indeed, for the Lyapunov function

$$\boxed{V(\Delta_2(t)) = \frac{1}{2}\|\Delta_2(t)\|^2 = \frac{1}{2}\|\dot{q}(t) - u_1(t)\|^2} \tag{8}$$

we derive

$$\boxed{\left.\begin{array}{c} \dot{V}(\Delta_2(t)) = \Delta_2^{\intercal}(t)\dot{\Delta}_2(t) = \\ \\ \Delta_2^{\intercal}(t)\left[D^{-1}(q)[\tau - \zeta] - \dot{u}_1(t)\right] = \\ \\ \Delta_2^{\intercal}(t)D^{-1}(q)\left[\tau - \left[\zeta + \ddot{q}^*(t) + \alpha^2(q - q^*)\right]\right]. \end{array}\right\}} \tag{9}$$

The backstepping control design (1)

Designing τ as

$$\left.\begin{array}{c}\tau=-k\left(t\right)D\left(q\right)\mathrm{SIGN}(\Delta_{2}\left(t\right))=\\-k\left(t\right)D\left(q\right)\mathrm{SIGN}(\dot{q}(t)-\dot{q}^{*}\left(t\right)+\alpha\left[q\left(t\right)-q^{*}\left(t\right)\right]),\end{array}\right\}$$

from (9), we get

$$\left.\begin{array}{c}\dot{V}(\Delta_{2}\left(t\right))=\Delta_{2}^{\intercal}(t)D^{-1}\left(q\right)\tau+\\\Delta_{2}^{\intercal}(t)D^{-1}\left(q\right)\left[-\zeta-\ddot{q}^{*}\left(t\right)-\alpha^{2}\left(q\left(t\right)-q^{*}\left(t\right)\right)\right]\leq\\-k\left(t\right)\Delta_{2}^{\intercal}(t)\mathrm{SIGN}(\Delta_{2}\left(t\right))+\\\left\|\Delta_{2}(t)\right\|\left\|D^{-1}\left(q\right)\right\|\left(\left\|\zeta\right\|+\left\|\ddot{q}^{*}\left(t\right)+\alpha^{2}\left(q\left(t\right)-q^{*}\left(t\right)\right)\right\|\right)\\\leq-k\left(t\right)\Delta_{2}^{\intercal}(t)\mathrm{SIGN}(\Delta_{2}\left(t\right))+\\\left\|\Delta_{2}(t)\right\|\left\|D^{-1}\left(q\right)\right\|\left(\zeta^{+}\left(q,\dot{q},t\right)+\right.\\\left.\left\|\ddot{q}^{*}\left(t\right)+\alpha^{2}\left(q\left(t\right)-q^{*}\left(t\right)\right)\right\|\right)\leq-\left\|\Delta_{2}(t)\right\|\left[k\left(t\right)-\right.\\\left\|D^{-1}\left(q\right)\right\|\left(\zeta^{+}\left(q,\dot{q},t\right)+\left\|\ddot{q}^{*}\left(t\right)+\alpha^{2}\left(q\left(t\right)-q^{*}\left(t\right)\right)\right\|\right),\end{array}\right\}$$

The backstepping control design (2)

And with

$$k\,(t) = \left\| D^{-1}\,(q) \right\| \zeta^{+}\,(q, \dot{q}, t) + \left\| \ddot{q}^{*}\,(t) + \alpha^{2}\,(q\,(t) - q^{*}\,(t)) \right\| + \rho,\ \rho > 0,$$

we obtain (as in the previous lectures) that $\Delta_2(t) = 0$ for all

$$t \geq t_f = \frac{\sqrt{2V\,(\Delta_2\,(0))}}{\rho} = \frac{\|\Delta_2(0)\|}{\rho},$$

which solves the problem.

Remark

Notice that in the case when $b(q, \dot{q}, t) = C\dot{q}$ *and if the Lyapunov function (8) is chosen as* $V(\Delta_2\,(t)) = \Delta_2^{\intercal}(t) D\,(q)\,\Delta_2(t)$ *(which is usual in the robot control), then the conservative term* $b(q, \dot{q}, t)$ *disappears in the Lyapunov function derivative, since the matrix* $\left[\dot{D}\,(q) - 2C\right]$ *occurs to be a skew-symmetric, that leads to a more simple control law, namely,* $\tau = -k\,(t)\,\mathrm{Sign}(\Delta_2\,(t))$ *without the matrix multiplier* $D\,(q)$.

Backstepping application to DNN-control of mechanical systems (1)

Consider again at the first step the DNN system

$$\left.\begin{array}{c} \dfrac{d}{dt}\hat{x}_{1,t} = \hat{x}_{2,t} \\ \dfrac{d}{dt}\hat{x}_{2,t} = f_{NN}\left(\hat{x}_t, t\right) + B_{NN}\left(\hat{x}_t, t\right) u_t, \\ \\ f_{NN}\left(\hat{x}_t, t\right) := A\hat{x}_t + L\left[y_t - C\hat{x}_t\right] + W_{0,t}\varphi\left(\hat{x}_t\right), \\ B_{NN,t} := B + W_{1,t}\psi\left(\hat{x}_t\right). \end{array}\right\} \qquad (10)$$

and, according to backstepping approach, take $\hat{x}_{2,t} = u_1(t)$, obtaining $\frac{d}{dt}\hat{x}_{1,t} = u_1(t)$.In variable $\Delta_{1,t} = \hat{x}_{1,t} - x_t^*$ we get $\dot{\Delta}_{1,t} = u_1(t) - \dot{x}_t^*$, and, defining u_1 as

$$\boxed{u_1(t) = \dot{x}_t^* - \alpha_1 \Delta_{1,t},\ \alpha_1 > 0.}$$

we obtained $\dot{\Delta}_1(t) = -\alpha\Delta(t)$, guarantying the desired property

$$\Delta_{1,t} = \Delta_{1,t=0}\exp\left(-\alpha_1 t\right) \to 0$$

Backstepping application to DNN-control of mechanical systems (2)

In the second step, let us design the original torque-control u such that $\Delta_{2,t} := \hat{x}_{2,t} - u_1(t)$ converges to zero in a finite time in the presence of uncertainty. To do that, we can select the Lyapunov function again as

$$V_t = \frac{1}{2} s_t^\intercal \mathcal{M}_t s_t, \, s_t = \Delta_{2,t} = \hat{x}_{2,t} - u_1(t) = \hat{x}_{2,t} - \dot{x}_t^* + \alpha_1 \Delta_{1,t},$$
$$\mathcal{M}_t = \mathcal{M}_t^\intercal := \left(B_{NN,t} B_{NN,t}^\intercal \right)^+ \geq 0,$$

obtaining

$$\dot{V}_t = s_t^\intercal \mathcal{M}_t \dot{s}_t + \frac{1}{2} s_t^\intercal \dot{\mathcal{M}}_t s_t =$$
$$s_t^\intercal \mathcal{M}_t \left(\frac{d}{dt} \hat{x}_{2,t} - \ddot{x}_t^* + \alpha_1 \dot{\Delta}_{1,t} \right) + \frac{1}{2} s_t^\intercal \dot{\mathcal{M}}_t s_t = s_t^\intercal \left(g_t + \mathcal{M}_t B_{NN,t} u_t \right),$$

where

$$g_t := \mathcal{M}_t \left[f_{NN}(\hat{x}_t, t) - \ddot{x}_t^* + \alpha_1 \dot{\Delta}_1(t) \right] + \frac{1}{2} \dot{\mathcal{M}}_t s_t \, .$$

Backstepping application to DNN-control of mechanical systems (3)

Take, as in the previouse lecture,

$$
\boxed{
\begin{array}{c}
u_t := -\alpha_t B_{NN,t}^{\intercal} \mathcal{M}_t \mathrm{SIGN}\left(\mathcal{M}_t s_t\right), \\
\\
\alpha_t = \left\| \left[f_{NN}\left(\hat{x}_t, t\right) - \ddot{x}_t^{*} \right] \right\| + \dfrac{s_t^{\intercal} \dot{\mathcal{M}}_t s_t}{2 \left\| \mathcal{M}_t s_t \right\|} + \varrho, \;\; \varrho > 0.
\end{array}
}
$$

we get

$$
\boxed{\dot{V}_t \leq -\varrho \left\| \mathcal{M}_t s_t \right\| < 0,}
$$

and, as the result, fulfilling the "**property of asymptotic convergence**"

$$
\boxed{V_t = \frac{1}{2} \Delta_t^{\intercal} \mathcal{M}_t \Delta_t \underset{k \to \infty}{\to} 0.}
$$

Comparison of DNN-SM and DNN-Backstepping-SM metods

The structure of the controller is the same SM structure:

$$\left.\begin{array}{c} u_t := -\alpha_t B_{NN,t}^{\intercal} \mathcal{M}_t \mathrm{SIGN}\left(\mathcal{M}_t s_t\right), \\ \\ \alpha_t = \left\| \left[f_{NN}\left(\hat{x}_t, t\right) - \ddot{x}_t^* \right] \right\| + \dfrac{s_t^{\intercal} \dot{\mathcal{M}}_t s_t}{2 \left\| \mathcal{M}_t s_t \right\|} + \varrho, \ \ \varrho > 0. \end{array}\right\}$$

But

- for DNN-SM controller

$$\boxed{s_t = \Delta_{1,t} = \hat{x}_{1,t} - x_t^*,} \tag{11}$$

- for DNN-Backstepping-SM controller

$$\boxed{s_t = \Delta_{2,t} = \hat{x}_{2,t} - \dot{x}_t^* + \alpha_1 \Delta_{1,t},\ \alpha_1 > 0.} \tag{12}$$