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The admissible control problem from the moment problem point of view

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ABSTRACT

The admissible control problem occupies an important place in control theory. This problem involves the study of the set of *all* bounded controls which solve a given control problem, including the time optimal control. In this work we show with the help of the Fundamental Matrix Inequality of Potapov that the solution of the admissible control problem for the canonical linear case can be given in terms of the solution of a classical Hausdorff moment problem on a finite interval $[0, \theta]$.

1. Introduction and preliminaries

As is customary, we let $\mathbb{R}^n$ and $\mathbb{C}$ denote the n-dimensional Euclidean space ($\mathbb{R}$ is the set of real numbers) and the complex numbers, respectively. We will use $\mathcal{C}_{0,L}$ to denote the set of all measurable functions on $[0, \theta]$ such that $0 \le f(\tau) \le L$ for all $\tau \in [0, \theta]$. The symbol $\mathcal{M}[0, \theta]$ stands for the set of all nonnegative measures on $[0, \theta]$. The complex conjugate of a number z and of a function w is denoted by $\bar{z}$ and w^*, respectively.

1.1. Statement of the problem

Let $x \in \mathbb{R}^n$. Let A be an $n \times n$ matrix and b a given constant vector in $\mathbb{R}^n$. Consider the following completely controllable system

$$\dot{x} = Ax + b\tilde{u}. \tag{1}$$

For a given initial condition x_0 and $\theta \ge \theta_{\min}$, find the set $U_{x_0,\theta}$ of *all* controls $\tilde{u} = \tilde{u}_{x_0,\theta}(t)$ with $|\tilde{u}| \le 1$ such that the trajectory of the closed system $\dot{x} = Ax + b\tilde{u}_{x_0,\theta}(t)$ starting at x_0 terminates at the origin at time θ, i.e. $x(\theta) = 0$. Here $\theta_{\min}$ denotes the minimal possible time of this transfer, or the optimal time. Such a control problem is said to be *admissible*.

The time optimal control problem (TOC) is the problem of determining the optimal time $\theta_{\min}$ and, also, the corresponding control $u^{opt}_{x_0}(t)$ (it turns out to be unique and is called the optimal control); see [1].

Note that the case without restrictions on the control is relatively simple. Indeed, the control $\widehat{u}(t)$ of steering to null from the initial position x_0 should satisfy the equation $-x_0 = \int_0^\theta \mathrm{e}^{-A\tau} b\widehat{u}(\tau)\mathrm{d}\tau$. Let $\check{u}(t)$ be a control transferring x_0 to 0 at time θ, for example $\check{u}(t) = -b^*\mathrm{e}^{-A^*t}N^{-1}(\theta)x_0$, where $N(\theta) = \int_0^\theta \mathrm{e}^{-At}bb^*\mathrm{e}^{-A^*t}\mathrm{d}t$. Then the set of all nonrestricted controls includes those $\widehat{u}(t)$ that admit the representation $\widehat{u}(t) = u_0(t) + \check{u}(t)$, where $\int_0^\theta \mathrm{e}^{-At}bu_0(t)\mathrm{d}t = 0$.

In the case of bounded controls, the AC problem becomes complicated considerably and the description of the set $U_{x_0,\theta}$ becomes a difficult task.

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In the present work, we obtain the solution of the AC problem for the canonical system case, i.e.,

$$A := \{\delta_{jk+1}\}_{j,k=1}^{n}, \qquad b := (1, 0, \dots, 0)^{\mathrm{T}}. \tag{2}$$

Our solution method is based on some deep results of the moment problem.

Apparently Krasovskii [2] was the first to propose the use of moment problem methods for solving optimal control problems (OCP). He reduced linear OCP to moment problems by interpreting the cost function as a norm which in fact was an application of the Krein L-moment problem [3]. An analytical solution of the TOC problem was given by Korobov and Sklyar [4,5] based on the treatment of an equivalent Markov power moment problem, the so-called Markov moment min-problem [6].

Certainly, the search for TOC is one of the central goals in optimal control theory. However, in many situations, it is not so important to find a control on the minimal possible interval, but the desired control has to satisfy some additional requirements, for example, to be smooth. In these cases it is natural to enlarge the time of steering and to consider smooth controls only. This leads to the problem of describing all admissible controls realizing transfers from x_0 to 0 at a given time θ (AC problem).

To solve the AC problem, we propose in the present work a new step in the application of moment theory to control problems. We use the analysis of a classical Hausdorff moment problem with the help of Fundamental Matrix Inequality (FMI) of Potapov. We show that any solution of the Hausdorff moment problem (HMP) generates a solution of the AC problem and, conversely, any solution of the AC problem is generated by a solution of the HMP. What is more important, all solutions of the AC problem are described by means of the FMI of Potapov (Theorem 2.1).

We shall now introduce some notions about moment problems on $[0, \theta]$ which are necessary in the present work.

1.2. L-Markov and Hausdorff moment problems

The L-Markov moment problem (MMP) for an interval $[0, \theta]$ is stated as follows: Let a sequence of real numbers $(c_j)_{j=0}^{k}$ be given. Find the set of functions f belonging to $\mathcal{C}_{0,L}$ such that $c_j = \int_0^\theta \tau^j f(\tau)\mathrm{d}\tau,\ j \in \{0, \dots, k\}$ holds.

The Hausdorff moment problem (HMP) for an interval $[0, \theta]$ is stated as follows: Let a finite sequence of real numbers $(s_j)_{j=0}^{k}$ be given. Find the set of measures σ belonging to $\mathcal{M}[0, \theta]$ such that $s_j = \int_0^\theta \tau^j \mathrm{d}\sigma(\tau),\ j \in \{0, \dots, k\}$ holds. We use $\mathcal{M}([0, \theta], (s_j)_{j=0}^{k})$ to denote the set of solutions of the HMP.

1.3. Certain classes of holomorphic functions

We need the integral representation of two special classes of holomorphic functions $\mathcal{R}[0, \theta]$ and $\mathcal{S}[0, \theta]$ defined in [7]. The following result is concerned with the integral representation of functions of these two classes.

Theorem 1.1 ([7, *Theorem A6, Theorem A7*]). *The following statements hold:*

$$w \in \mathcal{R}[0, \theta] \quad \text{iff } w(z) = \int_0^\theta (\tau - z)^{-1} \mathrm{d}\sigma(\tau), \tag{3}$$

$$w \in \mathcal{S}[0, \theta] \quad \text{iff } w(z) = (\theta - z) \int_0^\theta (\tau - z)^{-1} \mathrm{d}\sigma(\tau) \tag{4}$$

where σ belongs to $\mathcal{M}[0, \theta]$.

The holomorphic function $s(z) = \int_0^\theta (t - z)^{-1} \mathrm{d}\sigma(t)$, defined in $\mathbb{C} \setminus [0, \theta]$, is called the Stieltjes transform of σ, where $\sigma \in \mathcal{M}[0, \theta]$. By (3) s belongs to $\mathcal{R}[0, \theta]$.

Let us recall the Stieltjes inverse formula. Given s in $\mathcal{R}[0, \theta]$ one gets a corresponding measure σ satisfying Eq. (3) by

$$\sigma(t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_0^t \operatorname{Im} s(x + \mathrm{i}\epsilon)\mathrm{d}x, \quad t \in [0, \theta]. \tag{5}$$

We can assume σ to be normalized, that is, $\sigma(t) = \frac{\sigma(t+0) - \sigma(t-0)}{2},\ \sigma(0) = 0$.

With these notations the HMP can be reformulated: Describe the set $\mathcal{R}([0, \theta], (s_j)_{j=0}^{k})$ of the Stieltjes transforms of all nonnegative measures which belong to $\mathcal{M}([0, \theta], (s_j)_{j=0}^{k})$.

1.4. Relation between the L-Markov moment and the Hausdorff moment problem

By Theorem 1.1 (see also [7]) there is a bijective relation between the set $\mathcal{C}_{0,L}$ and measures $\sigma \in \mathcal{M}[0, \theta]$ satisfying $\int_0^\theta \mathrm{d}\sigma(\tau) = 1$ given by

$$\int_0^\theta \frac{\mathrm{d}\sigma(\tau)}{\tau - z} = -\frac{1}{z} \exp\left(\frac{1}{L} \int_0^\theta \frac{f(\tau)\mathrm{d}\tau}{z - \tau} \right). \tag{6}$$

The formal asymptotic expansions of the left- and right-hand sides of (6) determine a unique and explicit relation between $(c_j)_{j=0}^{k-1}$ and $(s_j)_{j=0}^{k}$ (see [8]): $s_0 = 1, s_1 = c_0$,

$$s_j = \frac{1}{j!L^j} \begin{vmatrix} c_0 & -L & \cdots & 0 \\ 2c_1 & c_0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ (j-1)c_{j-2} & (j-2)c_{j-3} & \cdots & -(j-1)L \\ jc_{j-1} & (j-1)c_{j-2} & \cdots & c_0 \end{vmatrix}, \quad j \geq 2. \tag{7}$$

For the solvability properties of the MMP and HMP we recall the following result:

Proposition 1.1 ([7, Theorem 2.1]). *The L-Markov moment problem with entries $c_{j-1}(\theta, x_0), j \in \{1, \ldots, n\}$, is solvable iff the $[0, \theta]$-Hausdorff moment problem with entries $s_j(\theta, x_0),\ j \in \{0, \ldots, n\}$, is solvable.*

Using the bijective relation (7), the L-Markov moment problem can be solved in terms of the $[0, \theta]$-Hausdorff moment problem. The treatment of the last problem, we carry out with help of Potapov's FMI approach (cf. [9,10]). Let us remark that, in [9,10], an explicit solution of the nondegenerate matrix version of the Hausdorff moment problem was given.

1.5. Potapov's Fundamental Matrix Inequality

V.P. Potapov developed a powerful approach to matricial interpolation problems which we now use in its scalar version. This approach is based on a generalization of a classical lemma by H.A. Schwarz, and a modification of this result which goes back to G. Pick. Potapov converted the original problem in an equivalent matricial inequality, FMI. In the case, where the so-called information block of this inequality is nondegenerate (see the matrices H_1 and H_2 of Definitions 1.1 and 1.2), he created an ingenious factorization method which allows the determination of the solution set of the matrix inequality and, consequently, also of the HMP.

Note that, in the construction of the solution, there is a remarkable difference between the cases of an even and odd number of data (see Theorem 1.3). Taking this into account, we first introduce the matrices which appear in the FMI in the even case (scalar version).

Definition 1.1. Let $n = 2k + 1$. Using the moments $s_0, s_1, \ldots, s_{2k+1}$, we construct the following matrices:

$$H_1 := \{s_{i+j+1}\}_{i,j=0}^{k}, \qquad H_2 := \{\theta s_{i+j} - s_{i+j+1}\}_{i,j=0}^{k}, \qquad T := \{\delta_{ij+1}\}_{i,j=1}^{k}, \qquad v := (1, 0, \ldots, 0)^{\mathrm{T}}$$
$$u := (-s_0, -s_1, \ldots, -s_k)^{\mathrm{T}}, \qquad R_{\mathrm{T}}(z) := (I - zT)^{-1}.$$
$$u_1 := u, \qquad u_2 = -u + \theta T u,$$

Further, we introduce two auxiliary holomorphic functions

$$\tilde{s}_1(z) := zs(z), \qquad \tilde{s}_2(z) := (\theta - z)s(z), \quad z \in \mathbb{C} \setminus [0, \theta], \tag{8}$$

where $s(z)$ is the Stieltjes transform of a $\sigma \in \mathcal{M}[0, \theta]$.

In a similar way, we introduce the matrices that appear in Potapov's FMI in the odd case.

Definition 1.2. Let $n = 2k$. With T defined in Definition 1.1, set $T_1 := T$. Using the moments $s_0, s_1, \ldots, s_{2k}$, we construct the following matrices:

$$H_1 := \{s_{i+j}\}_{i,j=0}^{k}, \qquad H_2 := \{\theta s_{i+j+1} - s_{i+j+2}\}_{i,j=0}^{k-1}, \qquad R_{T_r}(z) := (I - zT_r)^{-1}, \quad r = 1, 2,$$
$$u_1 := (0, -s_0, \ldots, -s_{k-1})^{\mathrm{T}}, \qquad v_1 = (1, 0, \ldots, 0)^{\mathrm{T}}, \qquad \tilde{u}_1 := (-s_0, -s_1, \ldots, -s_{k-1})^{\mathrm{T}},$$
$$T_2 := \{\delta_{ij+1}\}_{i,j=1}^{k-1}, \qquad v_2 := (1, 0, \ldots, 0)^{\mathrm{T}}, \qquad \tilde{u}_3 := (-s_1, -s_2, \ldots, -s_k)^{\mathrm{T}},$$
$$u_2 := \theta\tilde{u}_1 - \tilde{u}_3.$$

Here $u_1, v_1 \in \mathbb{R}^{n+1}$ and $u_2, v_2 \in \mathbb{R}^n$. I represents the unitary matrix of the respective dimension.

Furthermore, we introduce two auxiliary holomorphic functions:

$$\tilde{s}_1(z) := s(z), \qquad \tilde{s}_2(z) := (\theta - z)zs(z) - s_0 z, \quad z \in \mathbb{C} \setminus [0, \theta], \tag{9}$$

where $s(z)$ is the Stieltjes transform of a $\sigma \in \mathcal{M}[0, \theta]$.

We now define the system of Potapov's FMI for the even and odd cases [9,10]. If $n = 2k + 1$, we set in the sequel $T_1 = T_2 = T$ and $v_1 = v_2 = v$.

Definition 1.3. A function s is called a solution of the associated system of Potapov's Fundamental Matrix Inequality (FMI), if s satisfies the following conditions: (i) s is holomorphic in $\mathbb{C} \setminus [0, \theta]$. (ii) For $r \in \{1, 2\}$, the inequality

$$\left[\begin{array}{c|c} H_r & R_{T_r}(z)[v_r\tilde{s}_r(z) - u_r] \\ \hline (R_{T_r}(z)[v_r\tilde{s}_r(z) - u_r])^* & \{\tilde{s}_r(z) - \tilde{s}_r^*(z)\}/\{z - \bar{z}\} \end{array}\right] \geq 0 \tag{10}$$

holds.

It turns out that the treatment of the matrix moment problem is equivalent to finding all solutions of the corresponding system of FMI (see [9,10]):

Theorem 1.2. *The function $s(z)$ is a Stieltjes transform of a $\sigma \in \mathcal{M}([0, \theta], (s_j)_{j=0}^k)$ iff $s(z)$ is a solution of the system of Potapov's Fundamental Matrix Inequalities (10).*

This theorem holds for both the even and odd cases of data. In this way, the problem of finding the Stieltjes transform of σ reduces to the problem of finding the holomorphic function $s(z)$ of Definition 1.3.

In the case when H_1 and H_2 are positive definite, also called completely indeterminate case, following the Potapov scheme, we introduce a polynomial 2×2 matrix function (see [9,10]), the so-called *resolvent matrix* of the HMP. Let v_r, T_r, H_r and u_r, $r = 1, 2$, be defined as in Definitions 1.1 and 1.2. In the even case (left column) and odd case (right column), we define

$$\begin{aligned} U_{11}^1(z) &:= 1 - zu_2^* R_{T^*}(z) H_2^{-1} v, \\ U_{12}^1(z) &:= u_1^* R_{T^*}(z) H_1^{-1} u_1, \\ U_{21}^1(z) &:= -(\theta - z) z v^* R_{T^*}(z) H_2^{-1} v, \\ U_{22}^1(z) &:= 1 + z v^* R_{T^*}(z) H_1^{-1} u_1. \end{aligned} \tag{11}$$

$$\begin{aligned} U_{11}^2(z) &:= 1 - zu_1^* R_{T_1^*}(z) H_1^{-1} v_1, \\ U_{12}^2(z) &:= M - zu_1^* R_{T_1^*}(z) H_1^{-1} v_1 M + zu_1^* R_{T_1^*}(z) H_1^{-1} u_1, \\ U_{21}^2(z) &:= -z v_1^* R_{T_1^*}(z) H_1^{-1} v_1, \\ U_{22}^2(z) &:= 1 - z v_1^* R_{T_1^*}(z) H_1^{-1} v_1 M + z v_1^* R_{T_1^*}(z) H_1^{-1} u_1, \end{aligned} \tag{12}$$

where $M = (1 + \theta[u_1^* H_1^{-1} v_1 - u_2 H_2^{-1} v_2])(\theta v_1^* H_1^{-1} v_1)^{-1}$.

The next theorem (see [9, Theorem 6.12] and [10, Theorem 6.14]) describes the solution set of the HMP in terms of classes of functions (3) and (4).

Theorem 1.3. *Let the polynomials U_{ij}^ℓ, $i, j = 1, 2$, $\ell = 1, 2$, be defined by (11) and (12). The fractional linear transformation*

$$s := \frac{U_{11}^\ell w + U_{12}^\ell}{U_{21}^\ell w + U_{22}^\ell} \tag{13}$$

yields the following bijections:

(a) *in the even case $\ell = 1$, between the parameter set $w \in \mathcal{R}[0, \theta] \cup \{\infty\}$ and the Stieltjes transform set $s \in \mathcal{R}([0, \theta], (s_j)_{j=0}^{2k+1})$;*
(b) *in the odd case $\ell = 2$, between the parameter set $w \in \mathcal{S}[0, \theta] \cup \{\infty\}$ and the Stieltjes transform set $s \in \mathcal{R}([0, \theta], (s_j)_{j=0}^{2k})$.*

2. Solution of the AC problem

In this section, we give the solution of the AC problem in the case of smooth controls. The optimal case $\theta = \theta_{\min}$ will be treated in a follow-up paper. The following result holds:

Theorem 2.1. *Let $\theta > \theta_{\min}$. The set $U_{x_0,\theta}$ of admissible controls of the system (1) is given by*

$$\tilde{u}(t) = -\frac{2}{\pi} \lim_{\varepsilon \to +0} \arg(-(t + \varepsilon i) s(t + \varepsilon i)) - 1, \quad t \in [0, \theta], \tag{14}$$

where s is the function associated to the solution of the HMP, with moment depending on x_0 and θ, corresponding to a functional parameter w in the completely indeterminate case.

Proof. Because of the complete controllability of (1), there exists θ such that $x(\theta) = 0$. The system (1) with initial condition $x(0) = x_0$ can be written as $x(t) = \mathrm{e}^{At}(x_0 + \int_0^t \mathrm{e}^{-A\tau} b\tilde{u}(\tau)\mathrm{d}\tau)$. Then the $x(\theta) = 0$ is equivalent to $-x_0 = \int_0^\theta \mathrm{e}^{-A\tau} b\tilde{u}(\tau)\mathrm{d}\tau$. Using the fact that A and b are canonical, the last relation can be written in the form $-x_0^j = \frac{(-1)^{j-1}}{(j-1)!} \int_0^\theta \tau^{j-1}\tilde{u}(\tau)\mathrm{d}\tau$, $j \in \{1, \ldots, n\}$. Introducing the notation $f := (\tilde{u} + 1)/2$, we get

$$\frac{\theta^j + (-1)^j j! x_0^j}{2j} = \int_0^\theta \tau^{j-1} f(\tau)\mathrm{d}\tau, \quad j \in \{1, \ldots, n\}. \tag{15}$$

Denoting the left-hand side of (15) by $c_{j-1}(\theta, x_0), j \in \{1, \dots, n\}$, the AC problem reduces to a Markov moment problem, i.e., to the problem of finding a set of functions f with $0 \le f(\tau) \le 1$ for $\tau \in [0, \theta]$ such that relation (15) holds. Now, using the relation (7) for $L = 1$, we obtain the data moments of the classical $[0, \theta]$-Hausdorff moment problem, which we symbolize by $s_j(\theta, x_0), j \in \{0, \dots, n\}$.

Using the sequence $(s_j(\theta, x_0))_{j=0}^{n}$, we construct Hankel matrices H_1 and H_2 for an even and odd number of data, and vectors $u_r, v_r, r = \{1, 2\}$, as described in Definitions 1.1 and 1.2. We assume that H_1 and H_2 are positive definite this implies $\det H_r \neq 0,\ r = \{1, 2\}$.

Using (11) and (12) and Theorem 1.3, we get the associated function $s(z)$, which has an integral representation equal to the left-hand side of (6), as a function of class $\mathcal{R}[0, \theta]$. Next, we rewrite the relation (6) in the form $\ln(-zs(z)) = \int_0^\theta \frac{\mathrm{d}\left(\int_0^t f(\tau)\mathrm{d}\tau\right)}{z-t}$, where ln denotes the principal value of complex logarithm. Applying in the last equation the Stieltjes inverse formula (5) to $\int_0^t f(\tau)\mathrm{d}\tau$ which is a nondecreasing function on $[0, \theta]$, we obtain (14). This completes the proof. $\square$

Example 2.1. Consider the system $\dot{x}_1 = \tilde{u},\ \dot{x}_2 = x_1,\ |\tilde{u}| \le 1$, with initial position $x_1^0 = 0, x_2^0 = 1$. For $\theta = 3$, the matrices H_1 and H_2 are positive definite. In this case the solution of the equivalent HMP is given by (13), where $U_{11} = 1 - \frac{12}{13}z$, $U_{12} = \frac{23}{15} - \frac{4}{5}z$, $U_{21} = \frac{1}{13}z(-31 + 12z)$, $U_{22} = 1 - \frac{41}{15}z - \frac{4}{5}z^2$. Let the parameter w have the form $w = (\theta - z)\int_0^\theta \frac{\mathrm{d}t}{t-z}$. Then the control is given by $\tilde{u}(t) = 2f(t) - 1$, where

$$f(t) := \begin{cases} \dfrac{1}{\pi}\left(\arctan \dfrac{g_1(t)}{h_1(t)} - \dfrac{1 + (-1)^k}{2}\pi\right), & t_k \le t < t_{k+1},\ k = 0, \dots, 3, \\ \dfrac{1}{\pi}\left(\arctan \dfrac{g_1(t)}{h_1(t)} - \pi\right), & t_4 \le t \le 3, \end{cases}$$

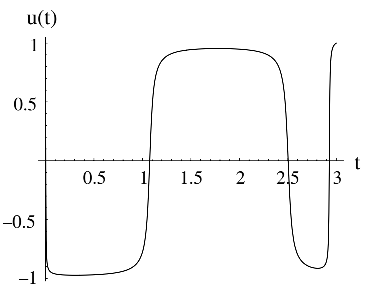
$$t_0 = 0, \quad t_1 = 0.019733, \quad t_2 = 1.115218, \quad t_3 = 2.526024, \quad t_4 = 2.9091237,$$

$$g_1(t) = \pi t(-3 + t),$$

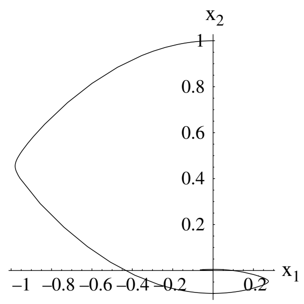
$$\begin{aligned} h_1(t) &= \frac{1}{152100}(3 - t)t(-900\pi^2 t(t - 3)(12t - 31)(12t - 13)) \\ &\quad - \left(130 - 312t + 15t(-31 + 12t)\ln\left[\frac{(-3 + t)^2}{t^2}\right]\right)\left(598 - 312t + 15(39 - 49t + 12t^2)\ln\left[\frac{(-3 + t)^2}{t^2}\right]\right). \end{aligned}$$

We have the following graphs:

Graph of the control



Phase trajectory beginning at $x_0 = (0, 1)^T$



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