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INVERSE MODEL APPROACH TO DISTURBANCE REJECTION AND DECOUPLING CONTROLLER DESIGN

Leonid Lyubchyk

National Technical University of Ukraine
“Kharkov Polytechnic Institute”

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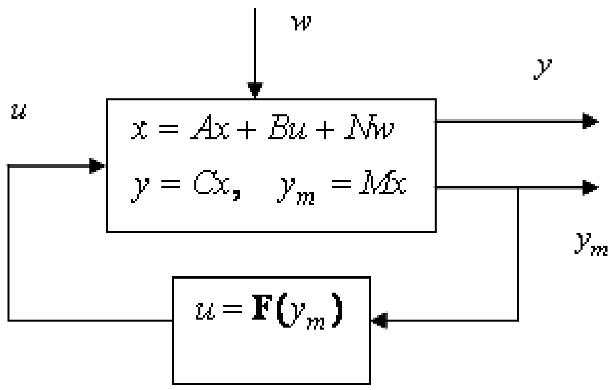
Disturbance rejection problem

Plant (controlled object) $\dot{x} = Ax + Bu + Nw$

$y = Cx, \quad y_m = Mx$



Disturbance $w \in \Omega_w = \{w \mid \|w\| \le c_w\}$

Control law $u = \mathbf{F}(y_m)$

Control process $\Omega_w \to \Omega_y$, performance index $v(\Omega_y)$

Find control $u \in \Omega_u$, so that $v(\Omega_y) \to \min, \quad u \in \Omega_u$

and closed-loop system $\dot{x} = Ax + BF(y_m) + Nw$ will be stable.

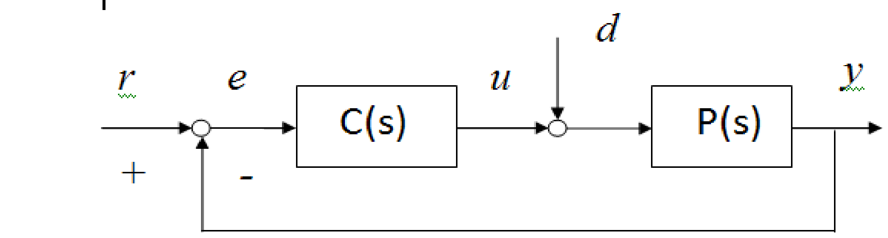
$v(\Omega_y) = 0$ absolute invariance

$v(\Omega_y) \le \varepsilon$ ε - invariance under the stability and robustness requirements

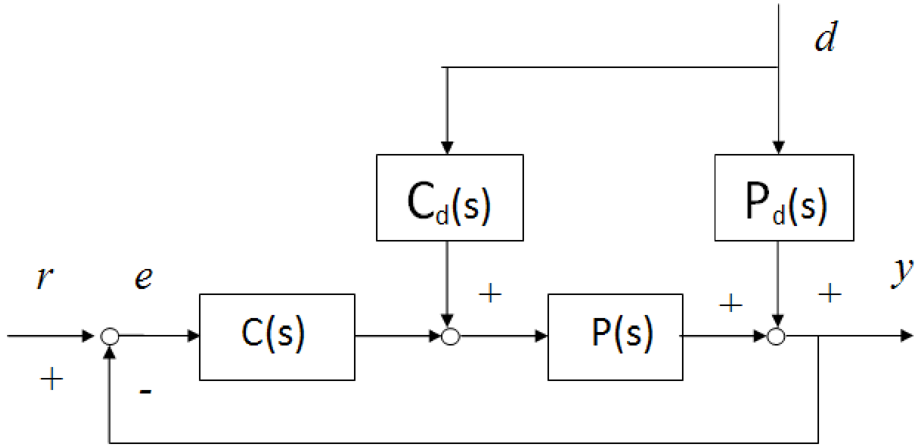
Attainable level of disturbance rejection

$$\overline{\lim}\, v(\Omega_y) = \bar{v}(\Omega_y) \le \gamma(\Omega_w, \Omega_\Sigma), \quad w \in \Omega_w, \quad \{A, B, C\} \in \Omega_\Sigma$$

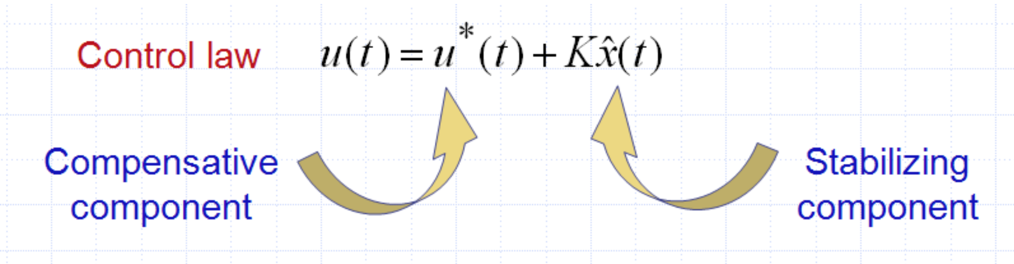
Control structures for disturbance rejection



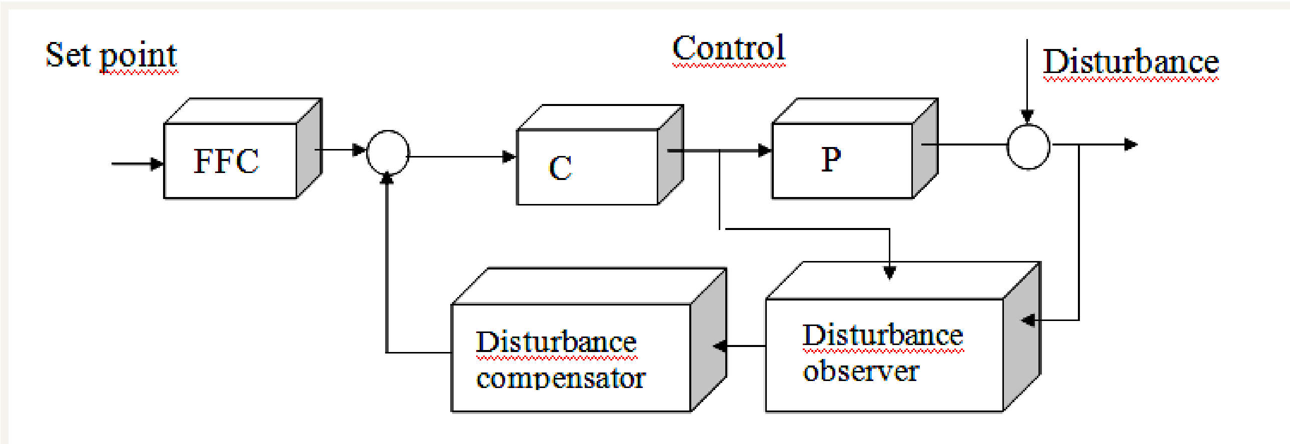
Feedback control
Disturbance attenuation

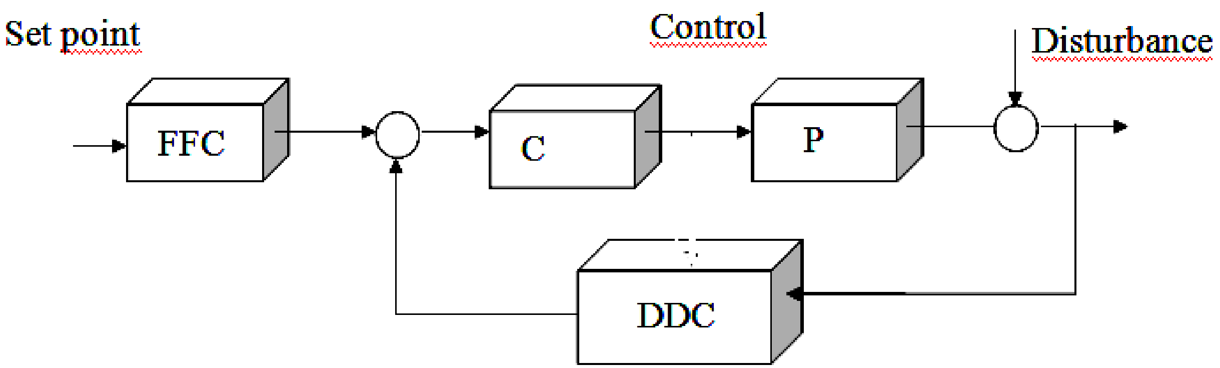


Feedback / feed-forward control
Disturbance compensation



Control system with disturbance observer structure





Output control problem

Consider a linear multivariable system described by the state-space model

$$\dot{x}(t) = Ax(t) + Bu(t) + Nf(x(t),t),$$
$$y_c(t) = Cx(t),\ y_m(t) = Mx(t), \qquad (1)$$

where $x(t) \in \boldsymbol{R}^n$ - state vector, $u(t) \in \boldsymbol{R}^m$ - control, $f(x(t),t) \in \boldsymbol{R}^q$ - unknown disturbance from certain class $f(x(t),t) \in \mathrm{N} = \{ f, \| f \| \le c_{f_1} \| x \| + c_{f_2} \}$,
$y_c(t) \in \boldsymbol{R}^r$, $y_m(t) \in \boldsymbol{R}^p$, - output controlled and measured variables respectively.

We will assume that $\operatorname{rank} B = m$, $\operatorname{rank} C = r$, $\operatorname{rank} N = q$, $\operatorname{rank} M = p$.

Matrices $S_{CB}(\alpha_1) = CA^{\alpha_1 - 1}B$, $S_{MN}(\alpha_2) = MA^{\alpha_2 - 1}N$ are known as Markov parameters of system (1). The integers α_1, α_2 are relative orders of control and disturbance transfer functions i.e. the minimal integers so that $S_{CB}(\alpha_1) \ne 0$, $S_{MN}(\alpha_2) \ne 0$.

Output control problem

Let the following assumptions take place:

$$\begin{aligned} &\text{(a)} \qquad \operatorname{rank} B = \operatorname{rank} S_{CB}(\alpha_1) = r, \\ &\text{(b)} \qquad \operatorname{rank} N \le \operatorname{rank} S_{MN}(\alpha_2) = p. \end{aligned} \tag{2}$$

Without loss of generality for simplicity reason we will assume that $\alpha_1 = \alpha_2 = 1$ and use the notation $S_{CB}(1) = S_{CB}, S_{MN}(1) = S_{MN}$.

The control problem is to find the control $u(t)$, depending from the measured variables, which ensure the reference signal $y^*(t)$ tracking, which formed by the given reference model $\dot{y}^*(t) = A^* y^*(t) + y_{ref}(t)$ and disturbance $f((x),t)$ decoupling for all disturbances from certain class N.

Formally the control goal is

$$\overline{\lim} \, \| e_c(t) \| \le \varepsilon^*, \quad t \to \infty,$$

where $e_c(t) = y^*(t) - y_c(t)$ - control error, ε^* - pre-established sufficiently small constant.

Inverse model based disturbance observer design

The first step of the DDC design procedure is the state and disturbance observer design using UIO approach. Let $z(t)=Rx(t)\in \boldsymbol{R}^{n-p}$ be an aggregated auxiliary variables, where R is the appropriate aggregate matrix such as $\operatorname{rank}\left(M^{\mathrm{T}} \mid R^{\mathrm{T}} \right) = n$.

Then the state vector estimation may be obtained as follows

$$\begin{pmatrix} y_m \\ z \end{pmatrix} = \begin{pmatrix} M \\ R \end{pmatrix} \cdot x, \quad \begin{pmatrix} M \\ R \end{pmatrix}^{-1} = (P \quad Q), \quad \hat{x}(t) = Py_m(t) + Q\overline{x}(t) \tag{3}$$

where matrices $P \in \boldsymbol{R}^{n\times p}, Q \in \boldsymbol{R}^{n\times n-p}$ are defined as

$$\begin{aligned} &MP = I_p, \; RQ = I_{n-p}, \; PM + QR = I_n, \\ &MQ = 0_{p,n-p}, \; RP = 0_{n-p,p}. \end{aligned} \tag{4}$$

Unknown input observer. Structural synthesis

The aggregated vector estimation $\overline{x}(t)$ is given by minimal-order UIO

$$\dot{\overline{x}}(t) = \overline{F}\overline{x}(t) + \overline{G}_1 y_m(t) + H\dot{y}_m(t) + \overline{G}_0 u(t). \tag{5}$$

The UIO parameters are determined from "invariance conditions"

$$\begin{gathered} R - \overline{H}M \quad A - \overline{F} \quad R - \overline{H}M \quad = \overline{G}M, \\ RN - \overline{H}MN = 0, \overline{G}_0 - RB = 0, \overline{G}_1 = \overline{G} - \overline{F}\overline{H}. \end{gathered} \tag{6}$$

If assumption (2b) takes place, a solution of (5) may be obtained as

$$\begin{gathered} \overline{F} = R\Pi_N AQ, \ \overline{G}_0 = RB, \ \overline{G}_1 = R\Pi_N AP, \\ \overline{H} = RNS_{MN}^{+}, \quad \Pi_N = I_n - BS_{MN}^{+}M, \end{gathered} \tag{7}$$

Inverse model-based disturbance observer

Taking the unknown disturbance estimation in the form

$$\hat{f}(t) = N^{+}\left[\dot{\hat{x}}(t) - A\hat{x}(t) - Bu(t)\right]. \qquad (8)$$

The minimal-order state and disturbance observer (SDO) equation:

$$\begin{aligned} \dot{\bar{x}}(t) &= R\Pi_N AQ\bar{x}(t) + R\Pi_N APy_m(t) + RNS_{MN}^{+}\dot{y}_m(t) + R\Pi_N Bu(t), \\ \hat{f}(t) &= \bar{C}_N(\dot{y}_m(t) - MAQ\bar{x}(t) - MAPy_m(t) - S_{MB}u(t)), \\ C_N &= S_{MN}^{+} + N^{+}P\Omega_N. \end{aligned} \qquad (9)$$

The estimation errors $e_x(t) = x(t) - \hat{x}(t),\ e_f(t) = f(x,t) - \hat{f}(t)$

$$\begin{aligned} \dot{\bar{e}}_x(t) &= \bar{F}\left[R\right]\bar{e}_x(t),\ e_x(t) = Q\bar{e}_x(t), \\ \bar{e}_f(t) &= -C_N MAQ\bar{e}_x(t). \end{aligned} \qquad (10)$$

Unknown input observer. Parametric synthesis

Concretely define the matrices $P \mid Q \ = \begin{pmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{pmatrix},$

with $P_1 = I_p, \ Q_1 = 0_{p,n-p}$, than $R = Q_2^{-1} \ -P_2 \mid I_{n-p}$ and P_1 , Q_2

are arbitrary matrices with $\det \ Q_2 \neq 0$.

For system representation

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \ M = \ I_p \quad 0_{n-p,p} \ , \ N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}_{n-p}^{p} \tag{11}$$

the observer dynamics matrix has the form:

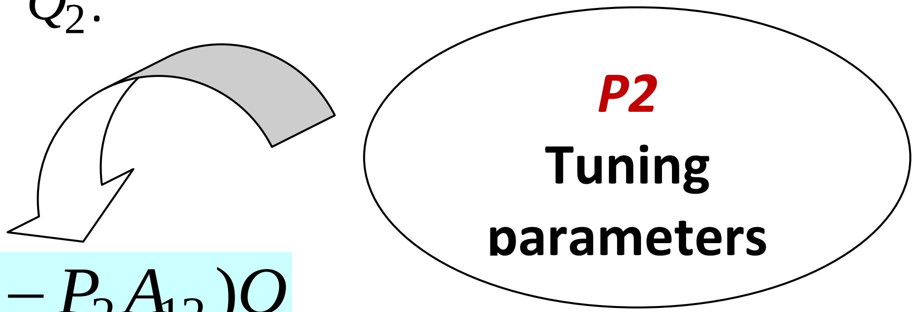
$$\bar{F} \ \ R \ = Q_2^{-1} \ \bar{A}_{22} - P_2 \bar{A}_{12} \ \ Q_2,$$

$$\bar{A}_{12} = \Omega_{N_{11}} A_{12}, \quad \tilde{A}_{22} = A_{22} - N_2 N_1^{+} A_{12} \tag{12}$$

$$\Omega_{N_1} = I_q - N_1 N_1^{+}.$$

Unknown input observer. Parametric synthesis

Thus the matrix Q_2 defines the similarity transformation and doesn't change the spectrum of $\bar{F}_1 \; R_1$, which completely determined by arbitrary matrix $P_2 \in \boldsymbol{R}^{n-p\times p}$. The last may be choused by pole placement method if pair $(\bar{A}_{22}, \bar{A}_{12})$ is observable. Such a condition is equivalent to the well-known UIO design solvability condition, namely observable of matrix pair (Π_N, M). The aggregate matrix R is determined up to an arbitrary nonsingular matrix Q_2.



$$\bar{F}(R) = Q(A_{22} - P_2 A_{12})Q$$

UIO pole placement

$$\lambda(F(P_2)) \in \sigma^*(\lambda_1^* \ldots \lambda_{n-p}^*)$$

Solvability conditions

$$(\bar{A}_{22}, \bar{A}_{12}) \text{ is observable}$$

Regularized disturbance observer design

The observability condition is violated in the case when $p = q$.

At that $\Omega_{N_1} = 0$ and $\bar{F}(R)$ doesn't depend from P_2.

In such singular case for the tuning properties guarantee it is possible to use the so-called „regularized“ UIO , which ensure the apprixmately invariance with respect the the unknown disturbance

$$\left\| RN - \bar{H}CN \right\|^2 + \nu \left\| \bar{H} \right\|^2 \to \min_{H} \qquad (13)$$

where $\nu > 0$ -regularization parameter.

Then

$$\bar{H}(\nu) = RNS^{\mathrm{T}}\left(\nu I_q + S_{MN}S_{MN}^{\mathrm{T}}\right)^{-1}, \Pi_N(\nu) = I_n - H(\nu)M \qquad (14)$$

Regularized SDO design problem solution

$$\bar{F}\ \nu\ = \tilde{A}_{22}\ \nu\ - P_2 \Omega_{N_1}\ \nu\ A_{12},\ \tilde{A}_{22}\ \nu\ = A_{22} - N_2 \Psi_{N_1}\ \nu\ A_{12},$$

$$\Psi_{N_1}\ \nu\ = N_1{}^{\mathrm{T}}\ \nu I_q + N_1 N_1{}^{\mathrm{T}}\ , \tag{15}$$

$$\Omega_{N_1}\ \nu\ = I_q - N_1 N_1{}^{\mathrm{T}}\ \nu I_q + N_1 N_1{}^{\mathrm{T}}\ {}^{-1} = \nu\ \nu I_q + N_1 N_1{}^{\mathrm{T}}\ {}^{-1}.$$

Estimation error equations for the regularized state and disturbance observer are the following:

$$\dot{\bar{e}}_x(t) = \bar{F}\ \nu\ \bar{e}_x(t) + \nu R N\ \nu I_q + S_{MN}{}^{\mathrm{T}} S_{MN}\ {}^{-1} f(x,t),$$

$$e_f(t) = -N^{+}\ P\Omega_N\ \nu\ + H_N\ \nu\ \ MAQ\bar{e}_x(t) + \tag{16}$$

$$+\nu N^{+}\ I_n - PM\ \ \nu I_q + S_{MN}{}^{\mathrm{T}} S_{MN}\ {}^{-1} f(x,t)$$

and for small value of ν may be done sufficiently small.

Disturbance compensator design

The disturbance compensative control is a function of reference signal and disturbance estimation in the form of TDF controller. In the usual case of "square plant" ($r = m$) under the assumption (2a)

$$u^*(t) = S_{CB}^{-1}(y_{ref}(t) + C_A\hat{x}(t) - S_{CN}\hat{f}(t)), \quad C_A = A^*C - CA. \tag{17}$$

If system structure non-singularity condition take place

$$\operatorname{rank} \bar{S} = m + q, \quad \bar{S} = \begin{pmatrix} I_m & S_{CB}^{-1}S_{CN} \\ S_{MN}^{+}S_{MB} & I_q \end{pmatrix} \tag{18}$$

then disturbance estimation may be eliminated from the controller equation and DDC has the form of *two-degree-of-freedom controller.*

Disturbance decoupling controller design

- $\det \bar{S}(C,B,N,M) \neq 0, \quad S_{CN} = 0$

DDC equations are:

$$\dot{\bar{x}}(t) = F^0\bar{x}(t) + R\Pi_N A^0(P\Omega_N + H_N)y_m(t) + \Pi_N H_B y_{ref}(t), \qquad (19)$$

$$u^*(t) = S_{CB}^{-1}(y_{ref}(t) + C_A Q\bar{x}(t)) + S_{CB}^{-1}C_A(P\Omega_N + H_N)y_m(t)),$$

$$F^0 = R\Pi_N A^0 Q, \quad A^0 = A + H_B C_A, \quad H_B = BS_{CB}^{-1}, \quad H_N = NS_{MN}^{+}.$$

- $\det \bar{S}(C,B,N,M) = 0$

The realizable controller may be obtained using the disturbance estimations dynamically transformed by the internal dynamic filter:

$$u^*(t) = S_{CB}^{-1}(y^*(t) + C_A\hat{x}(t) - S_{CN}\tilde{f}(t)) \qquad 0 < \varepsilon << 1, \quad 0 < \mu << 1$$

- small filter parameters (20)

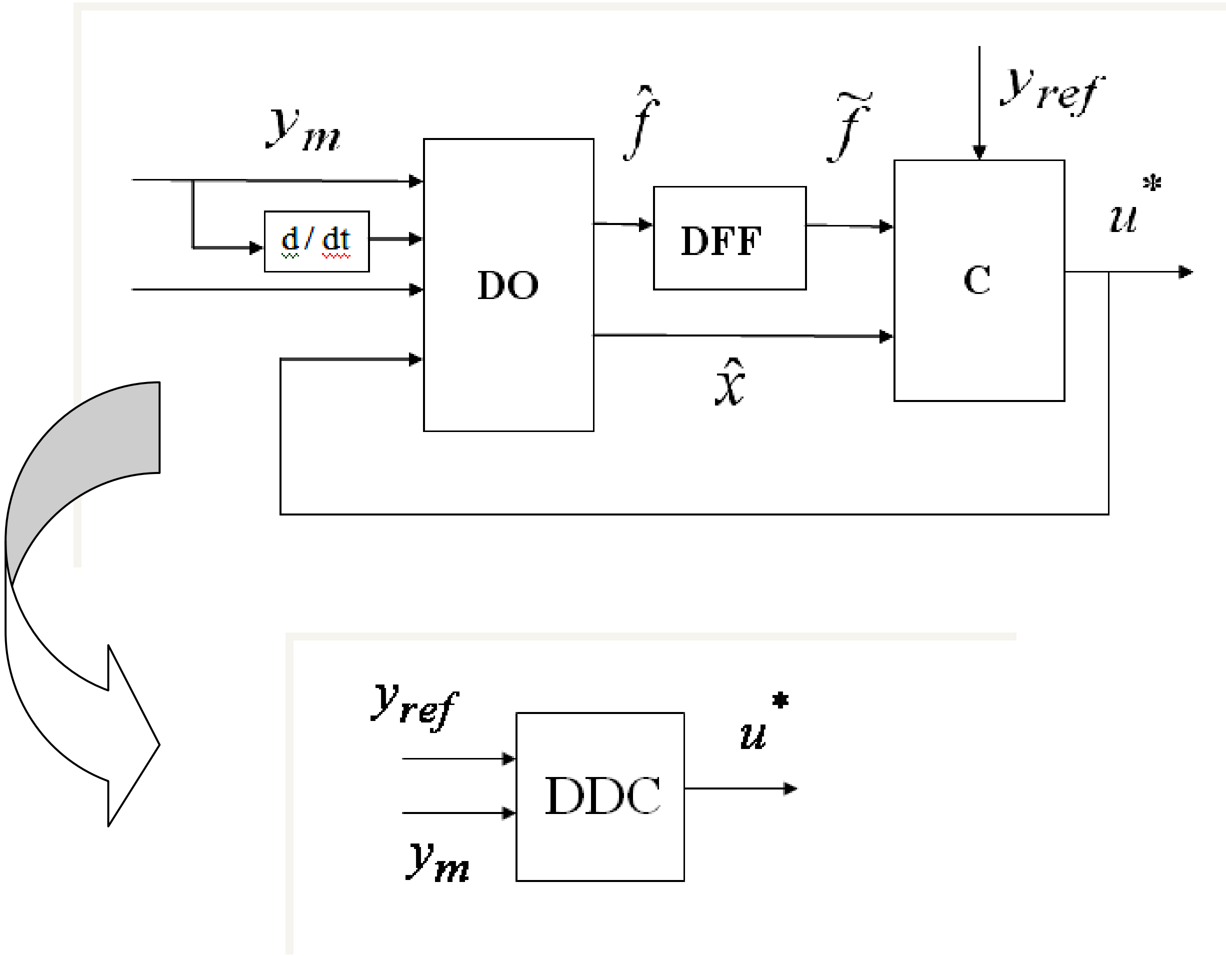
$$\varepsilon\dot{\tilde{f}}(t) = -\tilde{f}(t) + (1-\mu)\hat{f}(t),$$

DDC equations are:

$$\varepsilon\dot{\tilde{u}}(t) = -\mu\tilde{u}(t) + (1-\mu)(\varphi_1(t) + S_{CB}^{-1}S_{CN}\varphi_2(t)), \quad u^*(t) = \tilde{u}(t) + \varphi_1(t),$$

$$\varphi_1(t) = S_{CB}^{-1}(y_{ref}(t) + C_A\hat{x}(t)), \varphi_2(t) = C_N(\dot{y}_m(t) - MAQ\bar{x}(t) - MAPy_m(t)). \qquad (21)$$

Disturbance decoupling controller with fast filter structure



Closed-loop system with disturbance decoupling controller analysis

If system structural matrix $\bar{S}$ is nonsingular, the closed-loop system equation is:

$$\dot{x}(t) = A^0 x(t) + \Pi_B N f(t) + H_B y_{ref}(t) + L e_x(t),$$
$$A^0 = A + H_B C_A = \Pi_B A + H_B A^* C, \qquad (22)$$

The control goal is achieved with $\varepsilon^* = 0$, if closed-loop system (22) is stable, because $e_x(t)$ tends to zero due to properties of UIO.

For nonminimum-phase systems, matrix A^0 is unstable. The problem of closed-loop system stabilizing arises, moreover simple additional state feedback $u(t) = u^*(t) - K\hat{x}(t)$ doesn't change closed-loop matrix spectrum because $\Pi_B(A + BK) = 0$.

In such a case the local optimal control method may be applied.

Local optimal control for disturbance rejection

$$\left\| y_{ref}(t) + C_A A\hat{x}(t) - S_{CB}u(t) - S_{CN}\hat{f}(t) \right\|^2 + \beta \left\| u(t) \right\|^2 \to \min_{u} \qquad (23)$$

The corresponding control law is given by

$$u^*_\beta(t) = D_1 \; \beta \quad y_{ref}(t) + C_A A\hat{x}(t) - S_{CN}\hat{f}(t) \; =$$

$$= D \;\; \beta \;\; S_{CB}u^*(t), \quad D_1 \;\; \beta \; = \; \beta I_m \; + S^{\mathrm{T}}_{CB}S_{CB} \;\; ^{-1} S^{\mathrm{T}}_{CB}, \qquad (24)$$

From (23) the equation of closed-loop system follows

$$\dot{x}(t) = A^0(\beta)x(t) + BD \;\; \beta \;\; y_{ref}(t) + \Pi_B \;\; \beta \;\; Nf(x,t) + L_\beta e_x(t),$$

$$A^0(\beta) = A + BD \;\; \beta \;\; C_A = \Pi_B \;\; \beta \;\; A + BD \;\; \beta \;\; A^*C, \qquad (25)$$

$$\Pi_B \;\; \beta \;\; = I_n - BD \;\; \beta \;\; C.$$

Closed-loop system properties

Using the "combined" control $u(t) = u^{*}(t) - K\hat{x}(t)$ find that

$$A_0(\beta, K) = A_0(\beta) - B_\beta K, \quad B_\beta = \beta B \left(\beta I_m + S_{CB}^{\mathrm{T}} S_{CB}\right)^{-1},$$

Closed-loop system with combined control may be stabilized, if matrix pair $A_0(\beta), B_\beta$ is controllable.

The *control error* is given by

$$\dot{e}_c(t) = A^{*} e_c(t) - \beta S_{CB} \left(\beta I_m + S_{CB}^{\mathrm{T}} S_{CB}\right)^{-1} u^{*}(t) \tag{26}$$

and control goal *is achieved with* $\varepsilon^{*}(\beta)$.

Attainable accuracy of control

Closed-loop two-time-scale system

For the structural singular plant closed-loop system with DDC includes internal filter

Singular perturbation

$$\dot{x}(t) = A^0 x(t) + Nf(x(t),t) - H_B S_{CN} \tilde{f}(t) + H_B y_{ref}(t) + Le_x(t),$$
$$\varepsilon \dot{\tilde{f}}(t) = -\tilde{f}(t) + (1-\mu) f(x(t),t) - (1-\mu) e_f(t) \qquad (27)$$

The closed-loop system (27) is two-time scale system, in which slow motion under $\varepsilon = 0$ coincides with the process in the system with “ideal” DDC and the fast one satisfied the dynamic equation:

$$E(\varepsilon)\dot{\tilde{x}}(t) = \tilde{A}^0 \tilde{x}(t) + \tilde{B}^0 f(\tilde{x}(t)). \qquad (28)$$

$$E(\varepsilon) = \begin{pmatrix} I_n & 0 \\ 0 & \varepsilon I_m \end{pmatrix}, \tilde{A}^0 = \begin{pmatrix} A^0 & -H_B S_{CN} \\ 0_{q,n} & -I_q \end{pmatrix}, \tilde{B}^0 = \begin{pmatrix} N \\ (1-\mu) I_q \end{pmatrix}.$$

Robust decoupling controller design

Fast motion stability problem reduced to the “absolute” stability problem of system (28) with nonlinearities from certain class.

For the particular case of linear state-dependent uncertain disturbance $f(x(t),t)=\Delta_A x(t)$, where $\Delta_A, ||\Delta_A|| \le c_A$ is the system (1) dynamic matrix perturbation

$$\tilde{A}_{\varepsilon}^{0}(\Delta_A)=\begin{pmatrix} A^0-N\Delta_A & -H_B S_{CN} \\ \varepsilon^{-1}(1-\mu)\Delta_A & -\varepsilon^{-1} I_q \end{pmatrix} \tag{29}$$

and fast motion stability analysis reduced to the robust stability problem

$$\operatorname{Re}\lambda(\tilde{A}_{\varepsilon}^{0}(\Delta_A)) \le -\eta, \quad ||\Delta_A|| \le c_A. \tag{30}$$

Disturbance decoupling controller existence conditions

Invertability conditions

$$\begin{aligned} &\text{(a)} \qquad \operatorname{rank} B = \operatorname{rank} S_{CB}(\alpha_1) = r, \\ &\text{(b)} \qquad \operatorname{rank} N \le \operatorname{rank} S_{MN}(\alpha_2) = p. \end{aligned}$$

Structural nonsingularity conditions

$$\operatorname{rank} \bar{S} = m + q, \quad \bar{S} = \begin{pmatrix} I_m & S_{CB}^{-1} S_{CN} \\ C_N S_{MB} & I_q \end{pmatrix}$$

$$\det \Phi \neq 0, \Phi = I_q - C_N S_{MB} S_{CB}^{-1} S_{CN}$$

Input (strong) observability conditions

$\Pi_{\mathrm{N}} A, M$ is observable (detectable)

DD existence conditions extension

IF $CB = 0, \quad CA^{\alpha_1 - 1}B \neq 0$

THAN $y^{*(\alpha_1)}(t) + A^{*}_{\alpha_1 - 1} y^{*(\alpha_1 - 1)}(t) + \ldots + A_0 y^{*}(t) = y_{ref}(t)$

IF $MN = 0, \quad MA^{\alpha_2 - 1}N \neq 0 \quad \alpha_2 > 1$

THAN $Y_m(t) = \left(y_m^{(\alpha_1)}, y_m^{(\alpha_1 - 1)}(t) \right)$

IF $\det \bar{S}(C, B, N, M) = 0$

THAN
$$u^{*}(t) = S_{CB}^{-1}(y^{*}(t) + C_A \hat{x}(t) - S_{CN}\tilde{f}(t)),$$
$$\varepsilon \dot{\tilde{f}}(t) = -\tilde{f}(t) + (1 - \mu)\hat{f}(t)$$

IF $(\Pi_{\mathrm{N}} A, M)$ is non-observable $(p = q)$

THAN observer regularization applied: $(\Pi_{\mathrm{N}}(\nu) A, M)$

IF $\Pi_B A$ is unstable and $r = m$

THAN
$$u(t) = u^{*}(t) - K\hat{x}(t), u^{*}_{\beta}(t) = D(\beta) S_{CB} u^{*}(t),$$
$$D_1(\beta) = \left(\beta I_m + S_{CB}^{\mathrm{T}} S_{CB} \right)^{-1}$$

Example. Magnetic suspension disturbance rejection control

Linearized mathematical model of the system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{f}_1(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ -a_0 & -a_1 & -a_2 & 0 \\ 0 & 0 & 0 & v \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ f_1(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} f_2(t),$$

$$y_c(t) = x_1(t), \quad y_m^1(t) = x_1(t), \quad y_m^2(t) = x_3(t)$$

where input $f_1(t) = \varphi(t)$ and state-dependent disturbances $f_2(t) = f(x(t), u(t))$, characterized the external forces and system's non-stationary parameters variations.

Control problem: using the measurements $y_m^1(t) = x_1(t)$, $y_m^2(t) = x_3(t)$ find the control function $u(t)$ so that the controlled output $y_c(t) = x_1(t)$ (deviation from the desired position) tracks the signal, generated by reference model $\dddot{y}^*(t) + \alpha_2 \ddot{y}^*(t) + \alpha_1 \dot{y}^*(t) + \alpha_0 y^*(t) = 0$.

SYSTEM DESCRIPTION

Plant model

$$\dot{x}_0(t) = A_0 x_0(t) + B_0 u(t) + D_0 \varphi(t) + N_0 f(t)$$

$$y_c(t) = C_0 x_0(t), \quad y_m(t) = M_0 x_0(t)$$

System parameters

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}, \quad D_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad N_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$C_0 = 1 \quad 0 \quad 0 \,, \quad M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

STATE AND DISTURBANCE OBSERVER DESIGN

A. Disturbances model (1-st order)

$$\dot{z}(t) = vz(t), \quad \varphi(t) = hz(t), \quad v \le 0,$$

$$f(t) = \Delta_a^T(t)x(t) + \Delta_B(t)u(t),$$

$$\Delta_a^T(t), \Delta_B(t) \text{ are unknown.}$$

Augmented system model

$$\dot{x}(t) = Ax(t) + Bu(t) + Nf(t), \quad x(t) = \begin{pmatrix} x_0(t) \\ z(t) \end{pmatrix},$$

$$y_m(t) = Mx(t)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ -a_0 & -a_1 & -a_2 & 0 \\ 0 & 0 & 0 & v \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ b \\ 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Minimal-order UIO

$$\dot{\bar{x}}_1(t) = -\pi_1\bar{x}_1(t) + h\bar{x}_2(t) + (\pi_2 h - \pi_1^2)y_m^1(t) + (1 - \pi_1\pi_2)y_m^2(t) - \pi_2\dot{y}_m^2(t),$$

$$\dot{\bar{x}}_2(t) = -\pi_2\bar{x}_1(t) + v\bar{x}_2(t) + (\pi_2 v - \pi_1\pi_2)y_m^1(t) - \pi_2^2 y_m^2(t),$$

$$\hat{x}_1(t) = y_m^1(t),$$

$$\hat{x}_2(t) = \bar{x}_1(t) + \pi_1 y_m^1(t) + \pi_2 y_m^2(t),$$

$$\hat{x}_3(t) = y_m^2(t),$$

$$\hat{x}_4(t) = \bar{x}_2(t) + \pi_2 y_m^1(t).$$

Equivalent form of UIO

$$\dot{\bar{x}}_1(t) = -\pi_1\bar{x}_1(t) + h\bar{x}_2(t) + (\pi_2 h - \pi_1^2)y_m^1(t) + y_m^2,$$

$$\dot{\bar{x}}_2(t) = -\pi_2\bar{x}_1(t) + v\bar{x}_2(t) + (\pi_2 v - \pi_1\pi_2)y_m^1(t),$$

$$\hat{x}_1(t) = y_m^1(t),$$

$$\hat{x}_2(t) = \bar{x}_1(t) + \pi_1 y_m^1(t),$$

$$\hat{x}_3(t) = y_m^2(t),$$

$$\hat{x}_4(t) = \bar{x}_2(t) + \pi_2 y_m^1(t).$$

Disturbances estimators

$$\hat{\varphi}(t) = \hat{x}_4(t) = \overline{x}_2(t) + \pi_2 y_m^1(t),$$

$$\dot{\hat{\varphi}}(t) = -\pi_2 \overline{x}_1(t) + v\overline{x}_2(t) + (\pi_2 v - \pi_1 \pi_2) y_m^1(t) + \pi_2 \dot{y}_m^1(t),$$

$$\hat{f}(t) = \hat{f}_0(t) - bu(t), \quad \hat{f}_0(t) = a_1 \overline{x}_1(t) + (a_0 + a_1 \pi_1) y_m^1 + a_2 y_m^2 + \dot{y}_m^2(t)$$

Observer pole-placement

$$\det(sI_2 - \overline{F}) = s^2 + (\pi_1 - v)s + \pi_2 h - \pi_1 v = s^2 + \lambda_1^* s + \lambda_0^*,$$

$$\pi_1 - v = \lambda_1^*, \quad \pi_2 h - \pi_1 v = \lambda_0^*, \quad \pi_1^* = \lambda_1^* + v, \quad \pi_2^* = h^{-1}(\lambda_0^* + \lambda_1^* v + v^2).$$

PI+UI observer

$(h = 1, \quad v = 0)$

$$\dot{\overline{x}}_1(t) = -\pi_1 \overline{x}_1(t) + \overline{x}_2(t) + (\pi_2 - \pi_1^2) y_m^1(t) + y_m^2,$$

$$\dot{\overline{x}}_2(t) = -\pi_2 \overline{x}_1(t) - \pi_1 \pi_2 y_m^1(t).$$

$$\pi_1 = \lambda_1^*, \quad \pi_2 = \lambda_2^*.$$

Disturbance compensative control law

$$u(t) = -b^{-1}(\tilde{C}_A \hat{x}(t) + \tilde{\varphi}(t) + \hat{f}(t)),$$

$$\tilde{\varphi}(t) = \alpha_2 \hat{\varphi}(t) + \dot{\hat{\varphi}}(t),$$

$$\tilde{C}_A = \alpha_0 C_0 + \alpha_1 C_0 A_0 + \alpha_2 C_0 A_0^2 + C_0 A_0^3 =$$

$$= \; -a_0 + \alpha_0 \quad -a_1 + \alpha_1 \quad -a_2 + \alpha_2$$

System structural matrix

$$S = \begin{pmatrix} 1 & -b^{-1} \\ -b & 1 \end{pmatrix}, \quad \det S = 0$$

"Realizable" form of control law

$$u(t) = -b^{-1}(\tilde{C}_A \hat{x}(t) + \tilde{\varphi}(t) + \tilde{f}(t)),$$

$$\varepsilon \dot{\tilde{f}}(t) = -\tilde{f}(t) + (1-\mu)\hat{f}(t)$$

$$0 < \varepsilon << 1, \quad 0 < \mu << 1$$

- small parameters

Equivalent form of control law

$$\varepsilon \dot{\tilde{u}}(t) = -\mu \tilde{u}(t) - b^{-1}(1-\mu)(r_1(t) + r_2(t)),$$

$$u(t) = \tilde{u}(t) - b^{-1} r_1(t),$$

$$r_1(t) = \tilde{C}_A \hat{x}(t) + \tilde{\varphi}(t), \quad r_2(t) = \hat{f}_0(t)$$

Disturbance decoupling compensator (DD – controller)

(1-st order disturbance model, *h=1, v=0*)

$$\dot{\bar{x}}_1(t) = -\pi_1\bar{x}_1(t) + \bar{x}_2(t) + (\pi_2 - \pi_1^2)y_m^1(t) + y_m^2,$$

$$\dot{\bar{x}}_2(t) = -\pi_2\bar{x}_1(t) - \pi_1\pi_2 y_m^1(t),$$

$$\varepsilon\dot{\tilde{u}}(t) = -\mu\tilde{u}(t) - b^{-1}(1-\mu)[(\alpha_1 - \pi_2)\bar{x}_1(t) + \alpha_2\bar{x}_2(t) + \\ + (\alpha_1\pi_1 + \alpha_2\pi_2 - \pi_1\pi_2 + \alpha_0)y_m^1(t) + \alpha_2 y_m^2(t) - \pi_2\dot{y}_m^1(t) + (1+\pi_2)\dot{y}_m^2(t)],$$

$$u(t) = \tilde{u}(t) - b^{-1}[(\alpha_1 - a_1 - \pi_2)\bar{x}_1(t) + \alpha_2\bar{x}_2(t) + \\ + (\alpha_1\pi_1 + \alpha_2\pi_2 - \pi_1\pi_2 + \alpha_0 - a_0 - a_1\pi_1)y_m^1(t) + \\ + (\alpha_2 - a_2)y_m^2(t) - \pi_2\dot{y}_m^1(t)]$$

Equivalent form of DD - controller

$$\dot{\bar{x}}_1(t) = -\pi_1\bar{x}_1(t) + \bar{x}_2(t) + (\pi_2 - \pi_1^2)y_m^1(t) + y_m^2,$$

$$\dot{\bar{x}}_2(t) = -\pi_2\bar{x}_1(t) - \pi_1\pi_2 y_m^1(t),$$

$$\varepsilon\dot{\tilde{u}}(t) = -\mu\tilde{u}(t) - b^{-1}(1-\mu)[(\alpha_1 - \pi_2)\bar{x}_1(t) + \alpha_2\bar{x}_2(t) +$$

$$+(\alpha_1\pi_1 + \alpha_2\pi_2 - \pi_1\pi_2 - \alpha_0 - \mu\pi_2)y_m^1(t) + (\alpha_2 - \mu)y_m^2(t)],$$

$$u(t) = \tilde{u}(t) - b^{-1}[(\alpha_1 - a_1 - \pi_2)\bar{x}_1(t) + \alpha_2\bar{x}_2(t) +$$

$$+(\alpha_1\pi_1 + \alpha_2\pi_2 - \pi_1\pi_2 - \alpha_0 - a_0 - a_1\pi_1 - \varepsilon^{-1}b^{-1}(1-\mu)\pi_2)y_m^1(t) +$$

$$+(\alpha_2 - a_2 - \varepsilon^{-1}b^{-1}(1-\mu))y_m^2(t) + \pi_2\dot{y}_m^1(t)]$$

Equivalent measurements

$$\dot{y}_m^1(t) = \dot{x}_1(t) = x_2(t),$$

$$\hat{\dot{y}}_m^1(t) = \hat{x}_2(t) = \bar{x}_1(t) + \pi_1 y_m^1(t)$$

NUMERICAL EXAMPLE FOR SIMULATION

System parameters

$a_0 = 1, \quad a_1 = 2, \quad a_2 = 2$

$\alpha_0 = 6, \quad \alpha_1 = 11, \quad \alpha_2 = 6$

$b = 1, \quad h = 1, \quad \nu = 0$

Tuning parameters

$\pi_1 = 1, \quad \pi_2 = 2$

$\varepsilon = 0.1, \quad \mu = 0.01$

Disturbances model

$$\varphi(t) = f_1(t) \quad - \quad step \quad function$$

$$f(t) = f_2(t) = \theta(t)(\sigma_1 x_1(t) + \sigma_1 x_1(t) + \\ + \sigma_2 x_2(t) + \sigma_3 x_3(t) + \sigma_0 u(t),$$

$$\theta(t) = A\,sin(\omega t), \quad A = 1, \quad \omega = 0.5$$

Disturbance decoupling controller

$$\dot{\bar{x}}_1(t) = -\bar{x}_1(t) + \bar{x}_2(t) + y_m^1(t) + y_m^2,$$
$$\dot{\bar{x}}_2(t) = -2\bar{x}_1(t) - 2y_m^1(t),$$
$$0.1\dot{\tilde{u}}(t) = -0.01\tilde{u}(t) - 0.99(9\bar{x}_1(t) + 6\bar{x}_2(t) + 31y_m^1(t) + 6y_m^2(t) - 2\dot{y}_m^1(t) + 3\dot{y}_m^2(t)),$$
$$u(t) = \tilde{u}(t) - (7\bar{x}_1(t) + 6\bar{x}_2(t) + 28y_m^1(t) + 4y_m^2(t) + 2\dot{y}_m^1(t))$$

The designed DD controller has the structure of multivariable PI-controller with small parameters

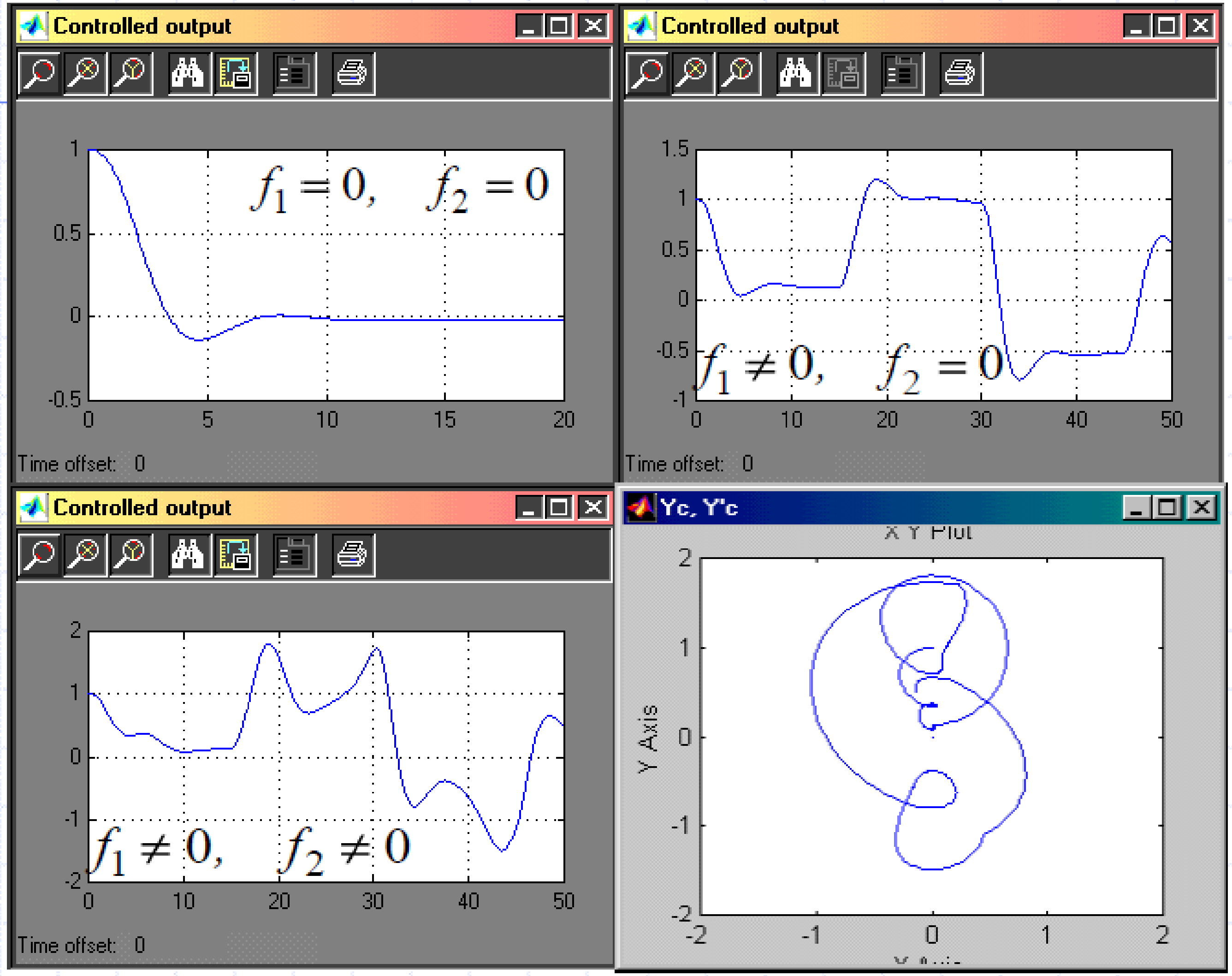
PI - DD controller

$$\varepsilon = 0.1, \quad \mu = 0$$

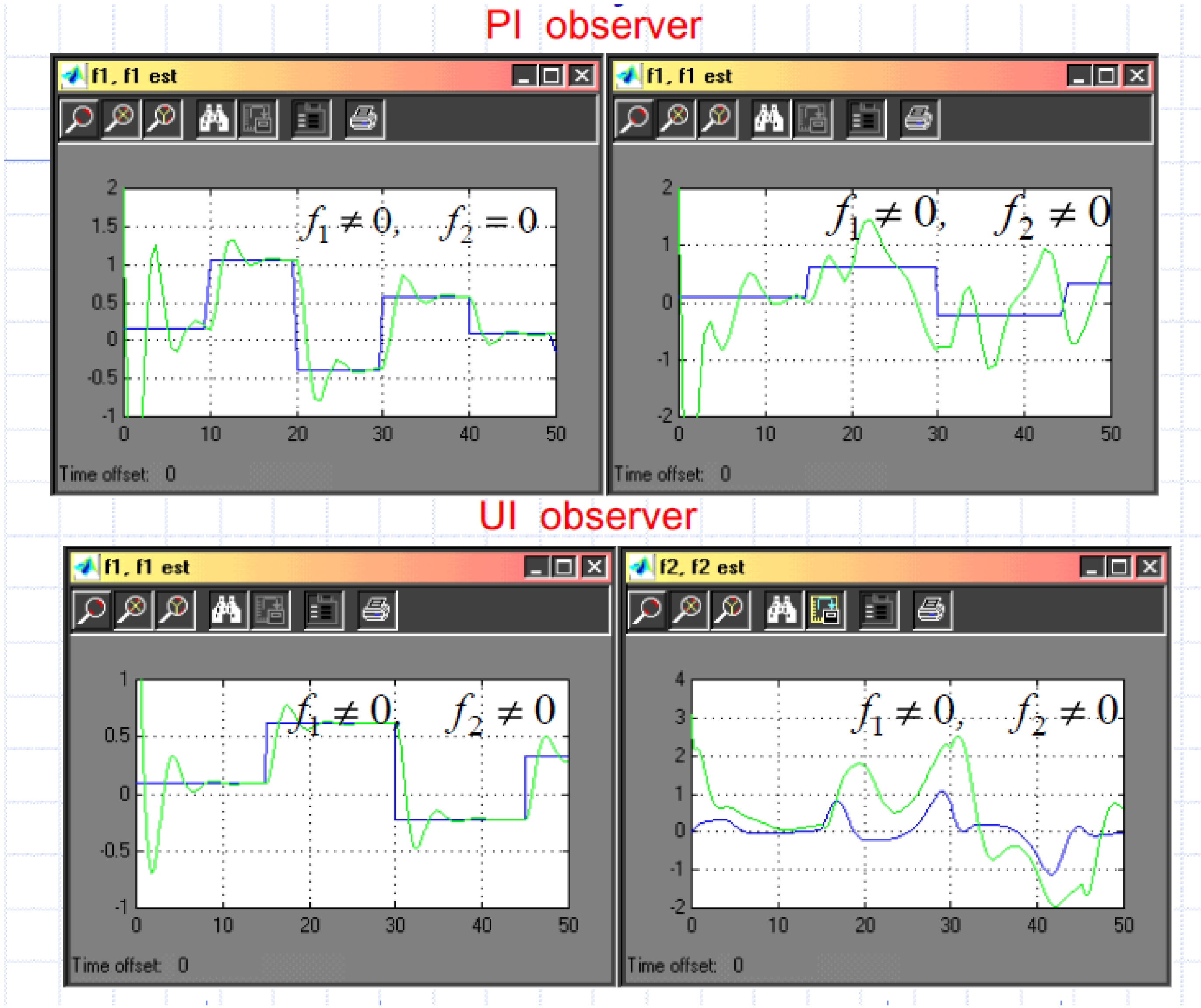
$$\dot{\bar{x}}_1(t) = -\bar{x}_1(t) + \bar{x}_2(t) + y_m^1(t) + y_m^2,$$
$$\dot{\bar{x}}_2(t) = -2\bar{x}_1(t) - 2y_m^1(t),$$
$$u(t) = -10\int_0^t (9\bar{x}_1(\tau) + 6\bar{x}_2(\tau) + 31y_m^1(\tau) + 6y_m^2(\tau))d\tau - (9\bar{x}_1(t) + 6\bar{x}_2(t) - 32y_m^1(t) - y_m^2(t))$$

Simulation results

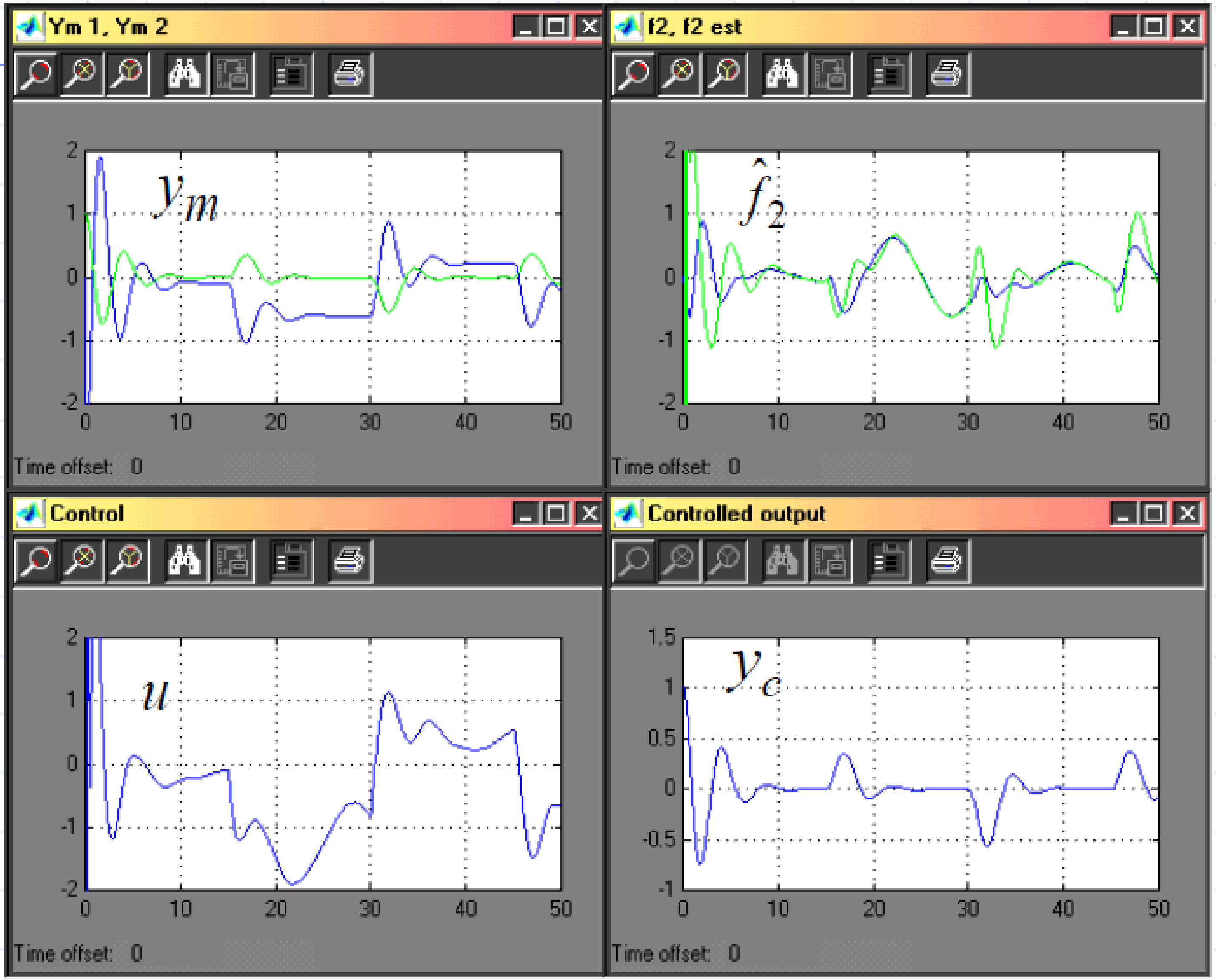
PI – controller



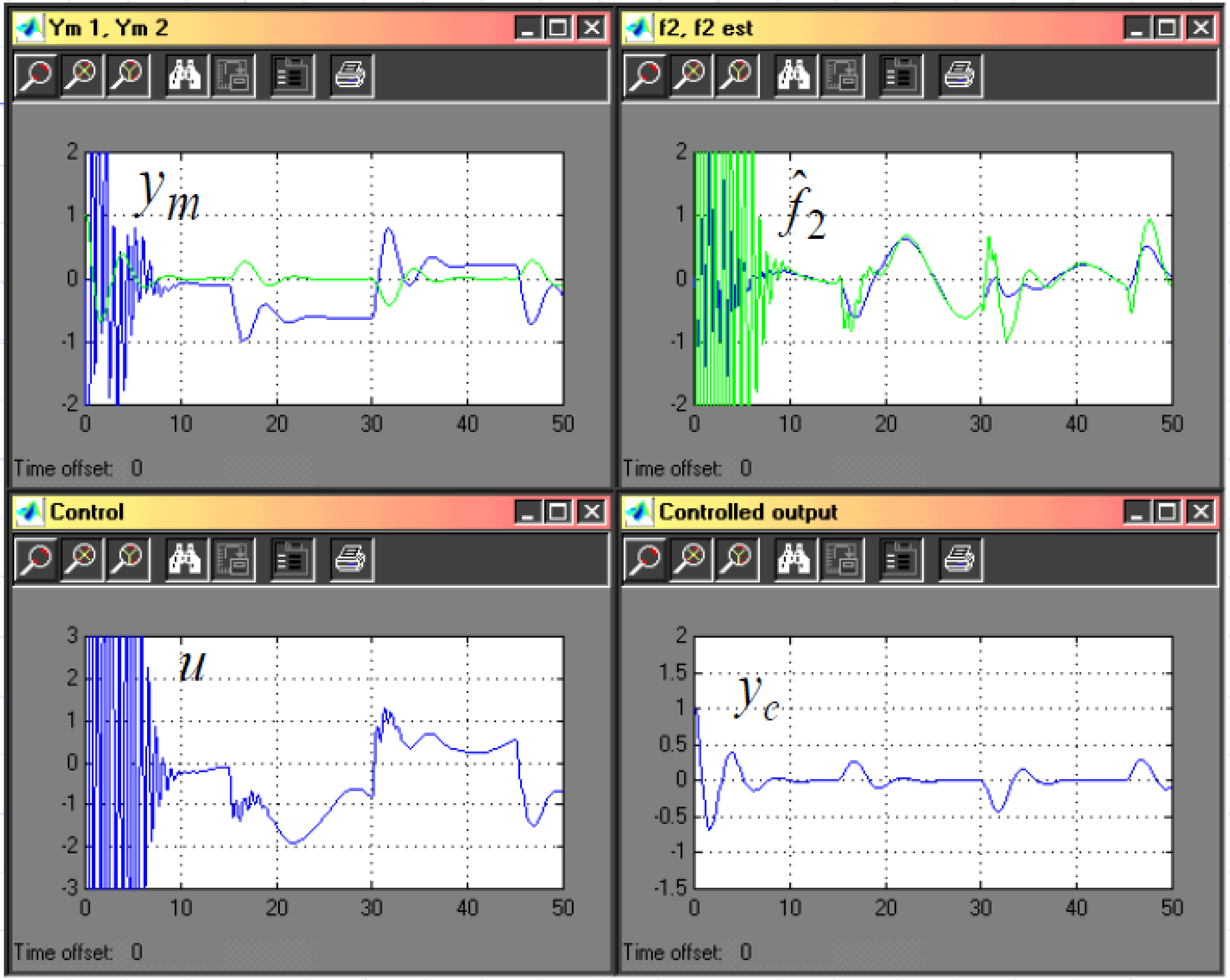
Disturbance estimation by PI and UI observers



UIO – based disturbance compensative controller



Disturbance decoupling controller



Example. Chaotic oscillator synchronisation

Controlled Rösseler attractor

Chaotic system model

$$\dot{x}_1(t) = -x_2(t) - x_3(t),$$
$$\dot{x}_2(t) = x_1(t) + a x_2(t) + u_1(t) + f_1(t),$$
$$\dot{x}_3(t) = -c x_3(t) + u_2(t) + f_1(t) + f_2(x_1, x_3, t).$$

Controlled output $y_c(t) = x_1(t)$

Measurements $y_1(t) = x_1(t), \quad y_2(t) = x_3(t)$

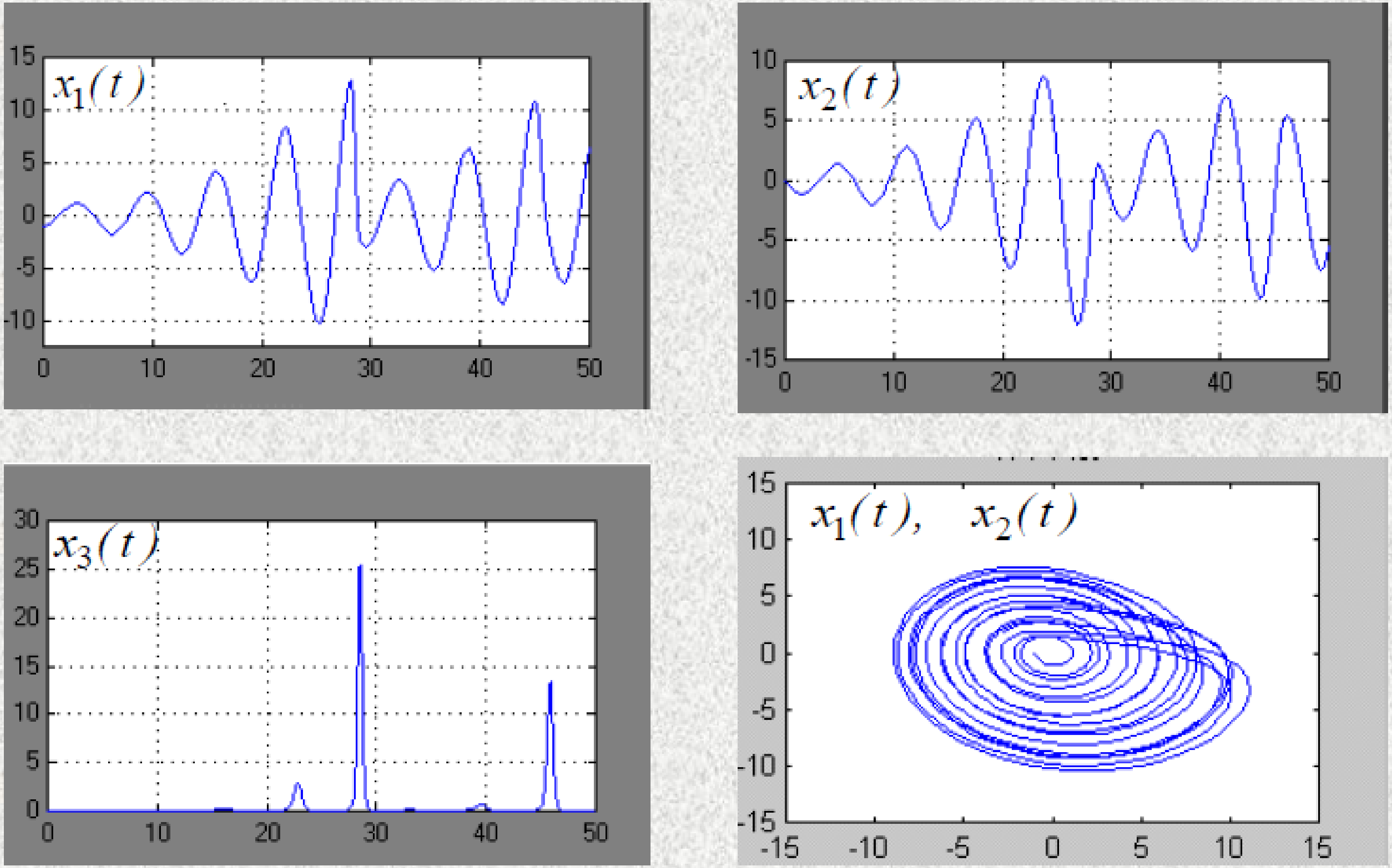
Disturbances $f_1(t) = \delta_f, \quad f_2(x_1, x_3, t) = \delta_c x_3(t) + (1 + \delta_x) x_1(t) x_3(t),$

Parametric Disturbance

Uncertain parameters $\delta_f, \delta_c, \delta_x$

Reference model $\ddot{y}^*(t) + \alpha_1 \dot{y}^*(t) + \alpha_0 y^*(t) = y_{ref}(t)$

Dynamics of Rösseler attractor



UI state and disturbance observer

$$\dot{\bar{x}}_1(t) = \rho_1 \bar{x}_1(t) + \bar{x}_2(t) + (1 + \pi_1 \rho_1 + \pi_2) y_1 + \pi_1 y_2(t),$$

$$\dot{\bar{x}}_2(t) = \pi_2 \bar{x}_1(t) + \pi_1 \pi_2 y_1 + \pi_2 y_2(t),$$

$$\hat{x}_1(t) = y_1(t), \quad \hat{x}_1(t) = \bar{x}_1(t) + \pi_1 y_1(t), \quad \hat{x}_3(t) = y_2(t),$$

$$\rho_1 = (\pi_1 + a - k)$$

$$\hat{f}_1(t) = \bar{x}_2(t) + \pi_2 y_1(t),$$

$$\hat{f}_2(t) = \dot{y}_2(t) + c y_2(t) - \bar{x}_2(t) - \pi_2 y_1(t) - u_2(t)$$

Control law: state feedback and DDC

$$u_1(t) = -k \hat{x}_2(t),$$

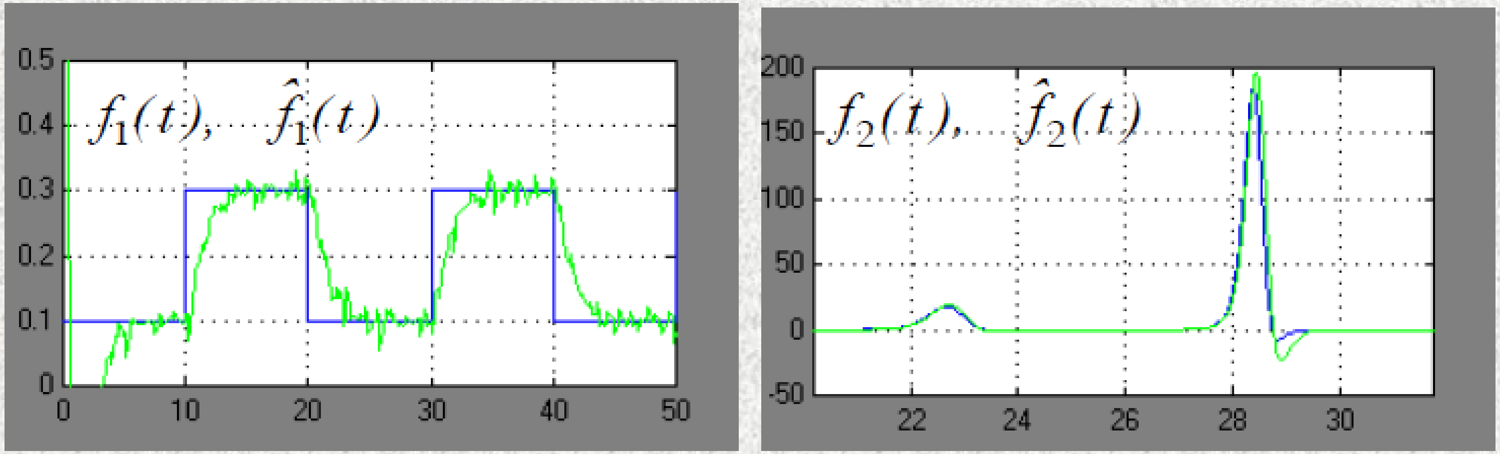
$$\varepsilon \dot{\bar{u}}(t) = \nu_1 \bar{x}_1(t) - \bar{x}_2(t) + (\zeta_1 - \pi_2) y_1 - \alpha_1 y_2(t),$$

$$u_2(t) = \bar{u}(t) + \nu_1 \bar{x}_1(t) - 2\bar{x}_2(t) + (\zeta_1 - 2\pi_2) y_1 + (c - \alpha_1 - \varepsilon^{-1}) y_2(t),$$

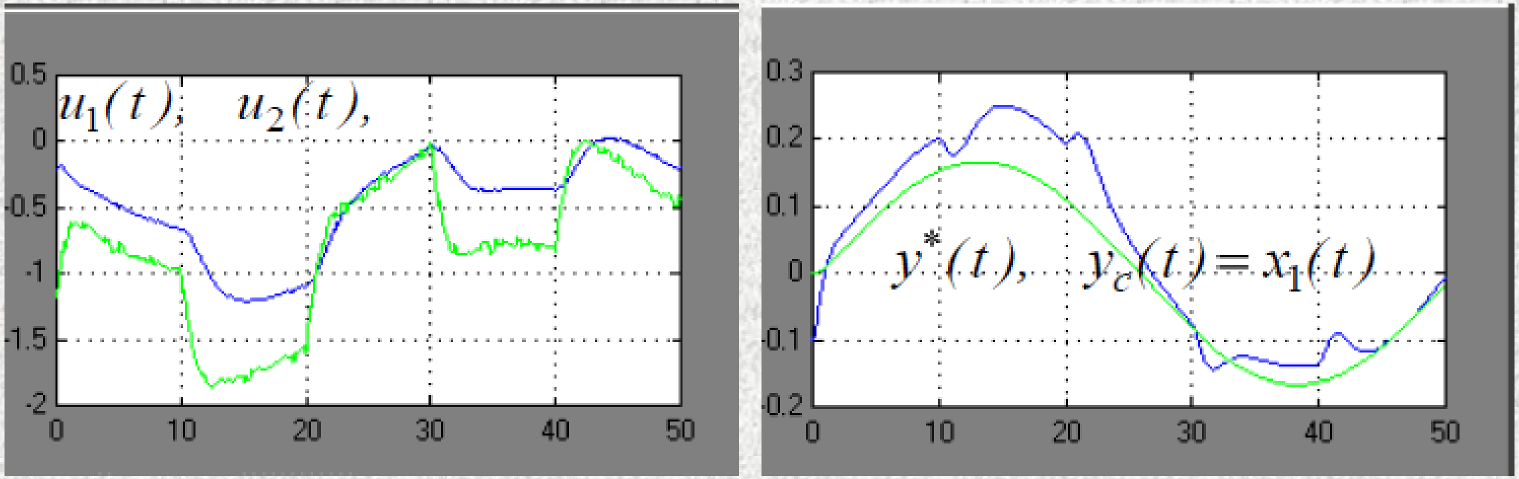
$$\zeta_1 = \alpha_1 + \nu_1 \pi_1 - 1, \quad \nu_1 = k - \alpha_1 - a$$

/

Disturbance estimation



Disturbance decoupling control



Conclusion

- Disturbance decoupling compensator (DDC) design method for multivariable systems measurements is proposed using the UIO technique.
- The design procedure includes state and disturbance observer design and disturbance compensator design.
- If system structure non-singularity conditions take place, the disturbance estimation may be eliminated from the control law and DDC equations are obtained in the explicit form.
- For the case when such a conditions are violated the realizable form of the DDC should be included additional internal dynamic filter with small time constant.
- For two-time-scale closed-loop system if the fast motion is stable the slow one coincides with the processes in the system with ideal compensator.

Thank you for your attention!