# Lecture 8: Sliding Mode DNN Control for Mechanical Systems

## Plan of presentation

- Optimization of Attractive ellipsoid
- Feedback design
- Taking into account the PC-motor as an activating device
- How calculate derivatives on-line: super-twist differentiator

# Attractive ellipsoid minimization

## Problem

*To minimizate the Attractive Ellipsoid we need to resolve the following optimiation with Nonlinear Matrix Constraints:*

$$\operatorname{tr}\left\{\frac{\alpha}{\varepsilon c_0}\begin{bmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{bmatrix}\right\} \to \max_{P_1>0, P_2>0, A, L, W_0^*, \alpha>0, \varepsilon>0}$$

*subject to the matrix constraint*

$$S := \begin{bmatrix} (\alpha-\varepsilon)P_1 + 2\varepsilon c_1 I_{n\times n} & P_1 - P_2 LC & 0_{n\times n} & 0_{n\times n} \\ P_1 - C^\intercal L^\intercal P_2 & \alpha P_2 + P_2 A + A^\intercal P - P_2 LC - C^\intercal L^\intercal P_2 + 2\varepsilon c_2 I_{n\times n} & P_2 W_0^* & P_2 \\ 0_{n\times n} & (W_0^*)^\intercal P_2 & -\varepsilon I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & P_2 & 0_{n\times n} & -\varepsilon I_{n\times n} \end{bmatrix} \leq 0$$

# Necessary condition to fulfil NMI

## Fact

*One of necessary requirements to fulfil NMI is*

$$\left(\alpha-\varepsilon\right)P_1+2\varepsilon c_1 I_{n\times n}\leq 0$$

*or equivalently*

$$\boxed{\left.\begin{array}{c}\varepsilon>\alpha,\\ P_1\geq 2\dfrac{\varepsilon c_1}{\varepsilon-\alpha}I_{n\times n}\end{array}\right\}}\qquad(1)$$

# Representation of the optimization aim

$$\operatorname{tr}\left\{\frac{\alpha}{\varepsilon c_0}\begin{bmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{bmatrix}\right\} = \frac{\alpha}{\varepsilon c_0}\left(\operatorname{tr}\{P_1\} + \operatorname{tr}\{P_2\}\right) \to \max_{P_1>0, P_2>0, A, L, W_0^*, \alpha>0, \varepsilon>0}$$

or equivalently,

$$\operatorname{tr}\left\{\left(\frac{\alpha}{\varepsilon c_0}\begin{bmatrix} P_1 & 0_{n\times n} \\ 0_{n\times n} & P_2 \end{bmatrix}\right)^{-1}\right\} = \frac{\varepsilon c_0}{\alpha}\left(\operatorname{tr}\{P_1^{-1}\} + \operatorname{tr}\{P_2^{-1}\}\right) \to \min_{P_1>0, P_2>0, A, L, W_0^*, \alpha>0, \varepsilon>0}$$

# NMI as LMI in new variables

Introduce new variables

$$\boxed{X_1 := P_1,\ X_2 := P_2,\ Y := P_2 A,\ Z := P_2 L,\ H := P_2 W_0^*}$$

and use the Schur's complement obtain the following equivalent representation:

$$\boxed{\begin{array}{c} 0 < P_1^{-1} = X_1^{-1} \leq Q_1 \Leftrightarrow \begin{bmatrix} Q_1 & I_{n\times n} \\ I_{n\times n} & X_1 \end{bmatrix}, \\ \\ 0 < P_2^{-1} = X_2^{-1} \leq Q_2 \Leftrightarrow \begin{bmatrix} Q_2 & I_{n\times n} \\ I_{n\times n} & X_2 \end{bmatrix} \end{array}}$$

# NMI as LMI in new variables: problem formulation

## Problem

$$\frac{\varepsilon c_0}{\alpha}\left(\operatorname{tr}\{Q_1\}+\operatorname{tr}\{Q_2\}\right)\to \min_{X_1>0,X_2>0,Y,Z,Q_1>0,Q_2>0,\alpha>0,\varepsilon>0}$$

*subject to the LMI constraints*

$$\begin{bmatrix} Q_1 & I_{n\times n} \\ I_{n\times n} & X_1 \end{bmatrix} > 0, \quad \begin{bmatrix} Q_2 & I_{n\times n} \\ I_{n\times n} & X_2 \end{bmatrix} > 0,$$
$$S=\begin{bmatrix} \begin{matrix}-(\varepsilon-\alpha)X_1\\ +2\varepsilon c_1 I_{n\times n}\end{matrix} & X_1-ZC & 0_{n\times n} & 0_{n\times n} \\ X_1-C^{\intercal}Z^{\intercal} & \begin{matrix}\alpha X_2+Y+Y^{\intercal}\\ -ZC-C^{\intercal}Z^{\intercal}\\ +2\varepsilon c_2 I_{n\times n}\end{matrix} & H & X_2 \\ 0_{n\times n} & H^{\intercal} & -\varepsilon I_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & X_2 & 0_{n\times n} & -\varepsilon I_{n\times n} \end{bmatrix}\leq 0 \tag{2}$$

# Recuperation of original variables

If the solution of the optimization problem (2) is

$$X_1^* > 0,\ X_2^* > 0,\ Y^*,\ Z^*,\ Q_1^* > 0,\ Q_2^* > 0,\ H^*,\ \varepsilon^* > \alpha^* > 0,$$

then the the original parameters of the DNNO are as follows:

$$\boxed{\begin{array}{c} P_1^* = X_1^*, P_2^* = X_2^*, A^* = \left(X_2^*\right)^{-1} Y^*, \\ \\ L^* = \left(X_2^*\right)^{-1} Z^*, W_0^* = \left(X_2^*\right)^{-1} H^*, \end{array}}$$

and learning law becomes to be as

$$\boxed{\begin{array}{c} \dot{W}_{0,t} = -\dfrac{\alpha^*}{2}\left(W_{0,t} - W_0^*\right) - \varepsilon^* \Lambda^{-1}\left(W_{0,t} - W_0^*\right)\varphi\left(\hat{q}_{1,t}\right)\varphi^{\intercal}\left(\hat{q}_{1,t}\right) \\ \\ -\dfrac{\varepsilon^* \rho_t \left(W_{0,t} - W_0^*\right)}{\operatorname{tr}\left\{\left(W_{0,t} - W_0^*\right)^{\intercal} \Lambda \left(W_{0,t} - W_0^*\right)\right\}}, \end{array}}$$

# Optimal DNNO

$$\left.\boxed{\begin{aligned}\frac{d}{dt}\hat{q}_{1,t} &= \hat{q}_{2,t}, \\ \frac{d}{dt}\hat{q}_{2,t} &= f_{NN}^{*}\left(\hat{q}_{1,t}, \hat{q}_{2,t}\right) + u_t\ ,\end{aligned}}\right\} \quad (3)$$

where

$$\boxed{\begin{gathered} f_{NN}^{*}\left(\hat{q}_{1,t}, \hat{q}_{2,t}\right) = A^{*}\hat{q}_{2,t} + L^{*}\left[y_t - C\begin{pmatrix}\hat{q}_{1,t} \\ \hat{q}_{2,t}\end{pmatrix}\right] + W_{0,t}\varphi\left(\hat{q}_{1,t}\right) \\ \text{with Learning Law} \\ \left.\begin{gathered}\dot{W}_{0,t} = -\frac{\alpha^{*}}{2}\left(W_{0,t} - W_{0}^{*}\right) - \varepsilon^{*}\Lambda^{-1}\left(W_{0,t} - W_{0}^{*}\right)\varphi\left(\hat{q}_{1,t}\right)\varphi^{\intercal}\left(\hat{q}_{1,t}\right) \\ - \frac{\varepsilon^{*}\rho_t\left(W_{0,t} - W_{0}^{*}\right)}{\operatorname{tr}\left\{\left(W_{0,t} - W_{0}^{*}\right)^{\intercal}\Lambda\left(W_{0,t} - W_{0}^{*}\right)\right\}},\end{gathered}\right\}\end{gathered}}$$

For DNNO-tracking error $\Delta_t = \hat{q}_{1,t} - q^*_{1,t} \in R^{2n}$ and for the Lyapunov function $V_t = \frac{1}{2}\Delta_t^\intercal \Delta_t$ we have

$$\dot{V}_t = \Delta_t^\intercal \dot{\Delta}_t = \Delta_t^\intercal P\left(\tfrac{d}{dt}\hat{q}_{1,t} - \dot{q}^*_{1,t}\right) =$$

$$\Delta_t^\intercal \left(\hat{q}_{2,t} - \dot{q}^*_{1,t}\right) = \Delta_t^\intercal \left( \int\limits_{\tau=0}^{t} f^*_{NN}\left(\hat{q}_{1,\tau}, \hat{q}_{2,\tau}\right) d\tau + \int\limits_{\tau=0}^{t} u_\tau d\tau - \dot{q}^*_{1,t} \right)$$

Select $u_\tau$ satisfying

$$\int\limits_{\tau=0}^{t} f^*_{NN}\left(\hat{q}_{1,\tau}, \hat{q}_{2,\tau}\right) d\tau + \int\limits_{\tau=0}^{t} u_\tau d\tau - \dot{q}^*_{1,t} = -\alpha_0 \Delta_t,\ \alpha_0 > 0$$

or, equivalently,

$$f^*_{NN}\left(\hat{q}_{1,t}, \hat{q}_{2,t}\right) + u_t - \dot{q}^*_{2,t} = -\alpha_0 \dot{\Delta}_t = -\alpha_0 \left(\hat{q}_{2,t} - q^*_{2,t}\right),$$

$$\boxed{u_t = -\alpha_0 \left(\hat{q}_{2,t} - q^*_{2,t}\right) - f^*_{NN}\left(\hat{q}_{1,t}, \hat{q}_{2,t}\right) + \dot{q}^*_{2,t}\,.}$$

# Feedback with DC-motor as actuator

If we take into account that

$$\boxed{\begin{array}{c} u_t := -\int\limits_{\tau=t_0}^{t} \tilde{v}_{a\tau} d\tau, \\ \\ \tilde{v}_{at} = \dot{u}_t = -\alpha_0 \left( \frac{d}{dt} \hat{q}_{2,t} - \dot{q}^*_{2,t} \right) - \frac{d}{dt} f^*_{NN} \left( \hat{q}_{1,t}, \hat{q}_{2,t} \right) + \ddot{q}^*_{2,t} , \end{array}} \quad (4)$$

then the motor-voltage $v_{at}$ will be

$$\boxed{\begin{array}{c} v_{at} = -R_a K_a^{-1} W^{-1} \int\limits_{\tau=t_0}^{t} \tilde{v}_{a\tau} d\tau + K_e W^{\intercal} \hat{q}_{2,t} + L_a K_a^{-1} W^{-1} \tilde{v}_{at} = \\ \\ R_a K_a^{-1} W^{-1} \left[ -\alpha_0 \left( \hat{q}_{2,t} - q^*_{2,t} \right) - f^*_{NN} \left( \hat{q}_{1,t}, \hat{q}_{2,t} \right) + \dot{q}^*_{2,t} \right] + K_e W^{\intercal} \hat{q}_{2,t} \\ \\ + L_a K_a^{-1} W^{-1} \left[ -\alpha_0 \left( \frac{d}{dt} \hat{q}_{2,t} - \dot{q}^*_{2,t} \right) - \frac{d}{dt} f^*_{NN} \left( \hat{q}_{1,t}, \hat{q}_{2,t} \right) + \ddot{q}^*_{2,t} \right] \end{array}}$$

# How calculate derivatives on-line: super-twist differentiator

The problem consists in estimating the *first derivative* of asignal $\phi(t)$ based on its noisy measurement

$$\boxed{y(t) = \phi(t) + \eta(t).}$$

Only two *assumption* will be made:
- the second derivative $\ddot{\phi}(t)$ of the base signal $\phi(t)$ is uniformly bounded by a known constant $L$, i.e.,

$$\boxed{|\ddot{\phi}(t)| \leq L,}$$

- the measurement noise $\eta(t)$ is uniformly bounded by $\delta$, i.e.

$$\boxed{|\eta(t)| \leq \delta.}$$

# Super-twist differentiator

Setting

$$x_1(t) := \phi(t), \; x_2(t) := \dot{\phi}(t),$$

the problem is transformed into the design of an observer for the system

$$\boxed{\left.\begin{array}{c} \dot{x}_1(t) = x_2(t), \; \dot{x}_2(t) = \ddot{\phi}(t), \\ \\ y(t) = \phi(t) + \eta(t), \end{array}\right\}} \qquad (5)$$

based on the measured output $y(t)$ only. The signal $\ddot{\phi}(t)$ is unknown and should be considered as a perturbation. Designing the state estimates $(\hat{x}_1(t), \hat{x}_2(t))$ using the *supert-twist observer,* we may conclude that $\hat{x}_2(t)$ may be considered as an estimate of $\dot{\phi}(t)$ :

$$\boxed{x_2(t) \simeq \dot{\phi}(t)}$$

# Super-twist differentiator with low-pass filter

$$\left.\begin{array}{c}\frac{d}{dt}\hat{x}_1(t) = \hat{x}_2(t) - \alpha\left\|\phi(t) - \hat{x}_1(t)\right\|^{1/2}\mathrm{SIGN}\left(\hat{x}_1(t) - y(t)\right), \\ \frac{d}{dt}\hat{x}_2(t) = -\beta\mathrm{SIGN}\left(\hat{x}_1(t) - y(t)\right), \ \ |\dot{\phi}(t)| \leq \beta, \ \alpha > 4\beta, \\ \text{and low-pass filter:} \\ \mu\dot{v}(t) + v(t) = \hat{x}_2(t), \ \mu = 0.01, \end{array}\right\} \tag{6}$$

so that,

$$\boxed{v(t) \simeq \dot{\phi}(t).}$$