Lecture 6: DNN Control

Plan of presentation

- Average cost function
- DNNO (or DNN model of the original system)
- Ideas of Locally adaptive control
- Subgradient
- Pseudoinvers matrix
- Analytical representation of Locally adaptive control

Averaged cost functions

Recal that

$$\delta_t := \hat{x}_t - x_t^*,$$
$$\frac{d}{dt}\hat{x}_t = f_{NN}\left(\hat{x}_t, t\right) + B_{NN}\left(\hat{x}_t, t\right) u_t,$$
$$\frac{d}{dt}\delta_t = f_{NN}\left(\hat{x}_t, t\right) - \dot{x}_t^* + B_{NN}\left(\hat{x}_t, t\right) u_t.$$

Definition

The **average cost function** $\bar{F}_{av,t}$ is defined as

$$\bar{F}_{av,t} = \frac{1}{t}\int_{\tau=0}^{t} F\left(\delta_\tau\right) d\tau, t > 0, \bar{F}_{av,t=0} = 0 \quad (1)$$

where $F : R^n \longrightarrow R^1$ is a **local cost function** defined on the trajectories $\left\{\delta_t\right\}_{t\geq 0}$, controlled by the actions $\left\{u_t\right\}_{t\geq 0}$.

Monotonically decreased local cost function

Remark

If the function $F(\delta_t)$ *is monotonically decreased (non increased) function and is bounded from below, that is,*

$$\inf_{\delta \in R^n} F(\delta) > -\infty,$$

then by the Weiestrass theorem any monotonical subsequence,satisfying

$$F(\delta_{t_{k+1}}) \geq F(\delta_{t_k})$$

has a limit, i.e., there exists a value F_* *such that*

$$\lim_{k \to \infty} F(\delta_{t_k}) = F_*$$

Monotonically decreased local cost function

Recall that we selected the control actions satisfying

$$B_{NN}\left(\hat{x}_t, t\right) u_t = -f_{NN}\left(\hat{x}_t, t\right) - k\partial F\left(\delta_t\right) + \dot{x}_t^* := r_{prop}, \; k > 0,$$

or

$$B_{NN}\left(\hat{x}_t, t\right) u_t = -k\mathrm{SIGN}\left(\partial F\left(\delta_t\right)\right) - f_{NN}\left(\hat{x}_t, t\right) + \dot{x}_t^* := r_{t,sign}, \; k > 0,$$

which guarantee

$$\frac{d}{dt}F\left(\delta_t\right) = -k\left\|\partial F\left(\delta_t\right)\right\|^2 < 0 \text{ or } \frac{d}{dt}F\left(\delta_t\right) = -k\sum_{i=1}^{n}\left|\left[\partial F\left(\delta_t\right)\right]_i\right| < 0.$$

Corollary

But, this leads directly to the monotonicity property for the local cost function $F\left(\delta_t\right)$, and hence, there exists

$$\boxed{\lim_{t\to\infty} F\left(\delta_t\right) := F_*}$$

Main property of Average Cost functions

Lemma

If local cost function converges to some limit point $F(\delta_t) \underset{t\to\infty}{\longrightarrow} F_*$, then the corresponding Average Cost function converge to the same limit, that is,

$$\bar{F}_{av,t} = \frac{1}{t}\int_{\tau=0}^{t} F(\delta_\tau)\, d\tau \underset{t\to\infty}{\longrightarrow} F_*$$

Proof.

For any $\varepsilon > 0$ there exists a time t_0 such that $|F(\delta_\tau) - F_*| \leq \varepsilon$ for all $t \geq t_0$:

$$\bar{F}_{av,t} - F_* = \frac{1}{t}\int_{\tau=0}^{t} [F(\delta_\tau) - F_*]\, d\tau = \frac{1}{t}\int_{\tau=0}^{t_0} [F(\delta_\tau) - F_*]\, d\tau + \frac{1}{t}\int_{\tau=t_0}^{t} [F(\delta_\tau) - F_*]\, d\tau$$

$$= O\left(\frac{1}{t}\right) + \frac{1}{t}\int_{\tau=t_0}^{t} |F(\delta_\tau) - F_*|\, d\tau = O\left(\frac{1}{t}\right) + \varepsilon\left(\frac{t-t_0}{t}\right) \underset{t\to\infty}{\longrightarrow} \varepsilon$$

LQ local functions

Consider the LQ-case when

$$\boxed{F(\delta_t, u_t) := \frac{1}{2}\delta_t^\intercal Q \delta_t + \frac{1}{2}u_t^\intercal R u_t,\, Q = Q^\intercal \geq 0,\, R = R^\intercal > 0}$$

Then selecting $\dot{u}_t = R^{-1}\left(v_t - B_{NN}^\intercal Q \delta_t\right)$, we get

$$\frac{d}{dt}F(\delta_t, u_t) = \partial_\delta^\intercal F(\delta_t)\,\dot{\delta}_t + \partial_u^\intercal F(\delta_t)\,\dot{u}_t =$$
$$\delta_t^\intercal Q\left[f_{NN}(\hat{x}_t, t) \text{-} \mathring{x}_t^* + B_{NN}(\hat{x}_t, t)\, u_t\right] + u_t^\intercal R \dot{u}_t =$$
$$\delta_t^\intercal Q\left[f_{NN}(\hat{x}_t, t) \text{-} \dot{x}_t^*\right] + u_t^\intercal\left(B_{NN}^\intercal Q \delta_t + R\dot{u}_t\right) = \delta_t^\intercal Q\left[f_{NN}(\hat{x}_t, t) \text{-} \dot{x}_t^*\right] + u_t^\intercal v_t$$

Taking

$$v_t = -kRu_t - \underbrace{\frac{u_t}{\|u_t\|^2}}_{\left(u_t^\intercal\right)^+}\left(\delta_t^\intercal Q\left[f_{NN}(\hat{x}_t, t) - \dot{x}_t^*\right]\right)$$

we get for $k > 0$

$$\frac{d}{dt}F(\delta_t, u_t) = \delta_t^\intercal Q\left[f_{NN}(\hat{x}_t, t) - \dot{x}_t^*\right] + u_t^\intercal v_t = -k u_t^\intercal R u_t < 0$$

Final expression for LQ-local DNN controller

Lemma

The control action u_t, governed by the following nonlinear ODE

$$\boxed{\begin{array}{c} \dot{u}_t = -ku_t - R^{-1}\left(B_{NN}^{\intercal} Q\delta_t + \rho_t\right), \; u_{t=0} \text{ - any initial vector, } k > 0, \\ \\ \rho_t = \left(u_t^{\intercal}\right)^{+} \delta_t^{\intercal} Q\left[f_{NN}\left(\hat{x}_t, t\right) - \dot{x}_t^*\right] \end{array}}$$

guarantees local decreasing of the LQ-cost function $F\left(\delta_t, u_t\right)$:

$$\boxed{\frac{d}{dt} F\left(\delta_t, u_t\right) = -ku_t^{\intercal} R u_t < 0}$$

The LQ DNN local adaptive control is a differential feedback!

In some problems by different physical reasons some components of an uncertain dynamic system belongs to *given intervals*, that is,

$$\boxed{x_{i,t} \in \left[x_i^{\min}, x_i^{\max}\right]}$$

For example, they may be positive as concentrations in chemical processes. The DNN components (if the identification is good enough) should also satisfy these constrants, i.e.,

$$\boxed{\hat{x}_{i,t} \in \left[x_i^{\min}, x_i^{\max}\right]}$$

Problem

How do we need to modify the DNN structure to fulfill this requirement?

Definition

Define the **projection operator** $[\cdot]_{-}^{+}$ as follows:

$$[z]_{-}^{+} := \begin{cases} z & \text{if} \quad z \in \left[z_i^{\min}, z_i^{\max}\right] \\ z_i^{\min} & \text{if} \quad z < z_i^{\min} \\ z_i^{\max} & \text{if} \quad z > z_i^{\max} \end{cases}, \; z \in R^1$$

In some sense it is a"cutting" operator. For vector arguments $z \in R^n$

$$[z]_{-}^{+} := \left([z_1]_{-}^{+}, ..., [z_n]_{-}^{+}\right)^{\mathsf{T}}$$

- Non-projectional (original) form of DNN:

$$\frac{d}{dt}\hat{x}_t = f_{NN}(\hat{x}_t, t) + B_{NN}(\hat{x}_t, t)\, u_t$$

- Non-correct format (idea) of the projectional version:

$$\frac{d}{dt}\hat{x}_t = \left[f_{NN}(\hat{x}_t, t) + B_{NN}(\hat{x}_t, t)\, u_t\right]_{-}^{+}$$

  This is velocities projection, but not states!

- Correct DNN model with state projections:

$$\frac{d}{dt}z_t = f_{NN}(\hat{x}_t, t) + B_{NN}(\hat{x}_t, t)\, u_t$$

$$\hat{x}_t = [z_t]_{-}^{+}$$

  where $z_t \in R^n$ is a special (auxilary) variable.

Projectional Observers of Nonlinear Systems

Block-scheme of DNN model with state projections