- Separation principle
- DNNO (or DNN model of the original system)
- Ideas of Locally adaptive control
- Subgradient
- Pseudoinvers matrix
- Analytical representation of Locally adaptive control

Separation principle

To realize the control of uncertain plants (when we do not know exactly the model of the process or can not measure on-line all coordinated of the process to be controlled) let us apply the, so-called, *Separation Principle* which is based on the following inequality

$$\boxed{\begin{array}{c} \left\| x_t - x_t^* \right\| = \left\| (x_t - \hat{x}_t) + (\hat{x}_t - x_t^*) \right\| \leq \\ \\ \left\| \hat{x}_t - x_t \right\| + \left\| \hat{x}_t - x_t^* \right\|, \end{array}} \tag{1}$$

where x_t is the state vector of the controlled plant, $\hat{x}_t$ is its estimate and x_t^* is a desired trajectory which we are intended to track.

Separation principle

Corollary

If we are able to realize a good enough state estimations, namely, fulfilling

$$\|\hat{x}_t - x_t\| \leq \varepsilon_1 \text{ for all } t \geq T_1,$$

and we can realize a good tracking of our model (generating $\hat{x}_t$) to the desired trajectory x_t^*, fulfilling

$$\|\hat{x}_t - x_t^*\| \leq \varepsilon_2 \text{ for all } t \geq T_2$$

then we can guarantee a good enough control of our uncertain plant, that is,

$$\boxed{\|x_t - x_t^*\| \leq \varepsilon_1 + \varepsilon_2, \text{ for all } t \geq T := \max\{T_1, T_2\}.}$$

DNNO representation

DNNO (or DNN model of the original system) is

$$\left.\begin{array}{c}\frac{d}{dt}\hat{x}_t = A\hat{x}_t + Bu_t + L\left[y_t - C\hat{x}_t\right] \\ \\ + \; W_{0,t}\varphi\left(\hat{x}_t\right) + \; W_{1,t}\psi\left(\hat{x}_t\right)u_t\end{array}\right\}$$

which can be represented as

$$\boxed{\frac{d}{dt}\hat{x}_t = f_{NN}\left(\hat{x}_t, t\right) + B_{NN}\left(\hat{x}_t, t\right)u_t,} \tag{2}$$

with some initial conditions $\hat{x}_0$, where

$$\boxed{\begin{array}{c}f_{NN}\left(\hat{x}_t, t\right) := A\hat{x}_t + L\left[y_t - C\hat{x}_t\right] + W_{0,t}\varphi\left(\hat{x}_t\right), \\ \\ B_{NN}\left(\hat{x}_t, t\right) := B + W_{1,t}\psi\left(\hat{x}_t\right).\end{array}}$$

The system (2) is **completely defined** and **does not contain any uncertainties**.

Important remark

Fact

The functions $f_{NN}(\hat{x}_t, t)$ and $B_{NN}(\hat{x}_t, t)$ are available on-line only in time t (or earlier $\tau < t$), but not in future. So, Optimal Control Methods are not applicable in this situation. Only versions of a feedback control are admitted.

Cost function

To realize "a good" tracking on-line, using DNNO, we need to make smaller the difference $\delta_t := \hat{x}_t - x_t^*$, minimizing the corresponding *convex cost function* $F(\delta_t)$. For example, such functions may be as follows:

- quadratic

$$F(\delta_t) = \|\delta_t\|^2 \text{ or } F(\delta_t) = \delta_t^\intercal G \delta_t, \; G = G^\intercal > 0,$$

- norm

$$F(\delta_t) = \|\delta_t\| = \sqrt{\sum_{i=1}^{n} \delta_{i,t}^2},$$

- sum of modules

$$F(\delta_t) = \sum_{i=1}^{n} |\delta_{i,t}|,$$

- dead-zone

$$F(\delta_t) = \sum_{i=1}^{n} |\delta_{t,i}|_\varepsilon^+, \quad |z|_\varepsilon^+ := \begin{cases} z - \varepsilon & \text{if} \quad z \geq \varepsilon \\ -z - \varepsilon & \text{if} \quad z \leq -\varepsilon \\ 0 & \text{if} \quad |z| < \varepsilon \end{cases}.$$

Local optimization

Since for small enough $\tau > 0$

$$\frac{\hat{x}_{t+\tau} - \hat{x}_t}{\tau} \simeq \frac{d}{dt}\hat{x}_t = f_{NN}\left(\hat{x}_t, t\right) + B_{NN}\left(\hat{x}_t, t\right) u_t$$

we have

$$\boxed{\hat{x}_{t+\tau} \simeq \hat{x}_t + \tau \left[f_{NN}\left(\hat{x}_t, t\right) + B_{NN}\left(\hat{x}_t, t\right) u_t \right],}$$

$$\frac{F\left(\delta_{t+\tau}\right) - F\left(\delta_t\right)}{\tau} \simeq \partial^{\intercal} F\left(\delta_t\right) \frac{\left(\delta_{t+\tau} - \delta_t\right)}{\tau} =$$

$$\tau^{-1} \partial^{\intercal} F\left(\delta_t\right) \left(\hat{x}_{t+\tau} - \hat{x}_t - \left(x^*_{t+\tau} - x^*_t\right)\right) =$$

$$\tau^{-1} \partial^{\intercal} F\left(\delta_t\right) \left(\tau \left[f_{NN}\left(\hat{x}_t, t\right) + B_{NN}\left(\hat{x}_t, t\right) u_t \right] - \left(x^*_{t+\tau} - x^*_t\right)\right) \simeq$$

$$\partial^{\intercal} F\left(\delta_t\right) \left(f_{NN}\left(\hat{x}_t, t\right) + B_{NN}\left(\hat{x}_t, t\right) u_t - \dot{x}^*_t\right)$$

and

$$\boxed{F\left(\delta_{t+\tau}\right) \simeq F\left(\delta_t\right) + \tau \partial^{\intercal} F\left(\delta_t\right) \left[f_{NN}\left(\hat{x}_t, t\right) - \dot{x}^*_t + B_{NN}\left(\hat{x}_t, t\right) u_t \right]}$$

Sub-gradient

Definition

Recall that a vector $a(x) \in \mathbb{R}^n$, satisfying the inequality

$$F(x+y) \geq F(x) + a^{\intercal}(x)y$$

for all $y \in \mathbb{R}^n$, is called t**he sub-gradient** of the function $F(x)$ at the point $x \in \mathbb{R}^n$ and is denoted by $a(x) \in \partial F(x)$ which is the set of all sub-gradients of F at the point x.

- If $F(x)$ is differentiable at a point x, then $a(x) = \nabla F(x)$.
- In the minimal point x^* we have $0 \in \partial F(x^*)$.

How realize the local optimization?

To make the cost function $F(\delta_{t+\tau})$ in a nearest future less then $F(\delta_t)$ in a current time we need to select control u_t which guarantees

$$\boxed{\partial^{\intercal} F(\delta_t)\left[f_{NN}(\hat{x}_t, t) - \dot{x}_t^* + B_{NN}(\hat{x}_t, t) u_t\right] < 0}$$

This may be done by selection u_t satisfying

- $$f_{NN}(\hat{x}_t, t) - \dot{x}_t^* + B_{NN}(\hat{x}_t, t) u_t = -k\partial F(\delta_t), \ \ k > 0$$

  providing $-k \left\|\partial F(\delta_t)\right\|^2 < 0$, or, equivalently,

  $$\boxed{B_{NN}(\hat{x}_t, t) u_t = -f_{NN}(\hat{x}_t, t) - k\partial F(\delta_t) + \dot{x}_t^*}$$

- or

  $$f_{NN}(\hat{x}_t, t) - \dot{x}_t^* + B_{NN}(\hat{x}_t, t) u_t = -k\mathrm{SIGN}(\partial F(\delta_t)), \ \ k > 0$$

  providing $-k \sum_{i=1}^{n} \left|[\partial F(\delta_t)]_i\right| < 0$, or, equivalently,

  $$\boxed{B_{NN}(\hat{x}_t, t) u_t = -f_{NN}(\hat{x}_t, t) - k\mathrm{SIGN}(\partial F(\delta_t)) + \dot{x}_t^*, k > 0}$$

On SIGN function

Definition

$$\text{Sign}\left(s_t\right) := \left(\text{sign}\left(s_{1,t}\right), \ldots, \text{sign}\left(s_{n,t}\right)\right)^{\intercal},$$

$$\text{sign}\left(s_{i,t}\right) \begin{cases} = +1 & \text{if} \quad s_{i,t} > 0 \\ = -1 & \text{if} \quad s_{i,t} < 0 \\ \in [-1, +1] & \text{if} \quad s_{i,t} = 0 \end{cases}.$$

How to find the control vector?

Fact

So, if we select u_t satifying

$$B_{NN}\left(\hat{x}_t, t\right) u_t = -f_{NN}\left(\hat{x}_t, t\right) - k\partial F\left(\delta_t\right) + \dot{x}_t^* := r_{prop}, \;\; k > 0,$$

we guarantee

$$\frac{d}{dt} F\left(\delta_t\right) = -k\left\|\partial F\left(\delta_t\right)\right\|^2 < 0,$$

selecting u_t satifying

$$B_{NN}\left(\hat{x}_t, t\right) u_t = -k\mathrm{SIGN}\left(\partial F\left(\delta_t\right)\right) - f_{NN}\left(\hat{x}_t, t\right) + \dot{x}_t^* := r_{t,sign}, \;\; k > 0,$$

we guarantee

$$\frac{d}{dt} F\left(\delta_t\right) = -k\sum_{i=1}^{n}\left|\left[\partial F\left(\delta_t\right)\right]_i\right| < 0$$

How to find the control vector?

In any case we need to resolve the linear algebraic equation

$$\boxed{B_{NN}\left(\hat{x}_t, t\right) u_t = r_t, r_t = \left(r_{t,prop} \text{ or } r_{t,sign}\right)}$$

or equivalently, in more extended format,

$$\boxed{\left\| B_{NN}\left(\hat{x}_t, t\right) u_t - r \right\|^2 \to \min_{u_t}}$$

On Pseudo-inversion

Theorem

For any real $(n \times m)$*-matrix* H *the limit*

$$\boxed{H^{+} := \lim_{\delta \to 0} \left(H^{\intercal} H + \delta^{2} I_{m \times m} \right)^{-1} H^{\intercal} = \lim_{\delta \to 0} H^{\intercal} \left(H H^{\intercal} + \delta^{2} I_{n \times n} \right)^{-1}} \quad (3)$$

always exists. Matrix H^{+} *is referred to as the pseudo-inverse matrix to the matrix* H*. For any vector* $z \in R^{n}$ *the vector*

$$\boxed{\hat{x} = H^{+} z}$$

is the vector of the minimal norm among those which minimize $\|z - Hx\|^{2}$*, namely,*

$$\boxed{\hat{x} = H^{+} z = \arg\min_{x} \|z - Hx\|^{2}}$$

and has the minimal norm $\|\hat{x}\|$ *among any other possible minimizing points.*

Corollary

For any real $n \times m$ matrix H

1. $$\boxed{H^{+} = \left(H^{\intercal}H\right)^{+} H^{\intercal}} \tag{4}$$
2. $$\boxed{\left(H^{\intercal}\right)^{+} = \left(H^{+}\right)^{\intercal}} \tag{5}$$
3. $$\boxed{H^{+} = H^{\intercal}\left(HH^{\intercal}\right)^{+}} \tag{6}$$
4. $$\boxed{H^{+} = H^{-1}} \tag{7}$$

   if H is square and nonsingular.

In MATLAB H^{+} calculate using the operator

$$H^{+} = \mathrm{pinv}(H)$$

Some properties of Pseudo-inversion operator

1. $$\boxed{\left(\mathrm{O}_{m\times n}\right)^{+} = \mathrm{O}_{n\times m}}$$

2. For any $x \in R^{n}$ $(x \neq 0)$
$$\boxed{x^{+} = \frac{x^{\intercal}}{\|x\|^{2}}}$$

3. $$\boxed{\left(H^{+}\right)^{+} = H}$$

4. In general,
$$\boxed{(AB)^{+} \neq B^{+}A^{+}}$$
The identity takes place if
$$\boxed{\begin{array}{c} A^{\intercal}A = I, \text{ or } BB^{\intercal} = I, \text{ or } B = A^{\intercal}, \text{ or } B = A^{+} \text{ or} \\ \text{both } A \text{ and } B \text{ are of full rank, or } \mathrm{rank}A = \mathrm{rank}B \end{array}}$$

Analytical representation of locally adaptive control

Corollary

$$\boxed{u_t = B_{NN}^{+}\left(\hat{x}_t, t\right) r_t}$$

where $\delta_t = \hat{x}_t - x_t^*$, $k > 0$ *and*

$$B_{NN}\left(\hat{x}_t, t\right) := B^* + W_{1,t}\psi\left(\hat{x}_t\right),$$

$$r_t = r_{t,prop} = -\left[A^*\hat{x}_t + L^*\left[y_t - C\hat{x}_t\right] + W_{0,t}\varphi\left(\hat{x}_t\right)\right] - k\partial F\left(\delta_t\right) + \dot{x}_t^*,$$

or

$$r_t = r_{t,sign} = -\left[A^*\hat{x}_t + L^*\left[y_t - C\hat{x}_t\right] + W_{0,t}\varphi\left(\hat{x}_t\right)\right] \text{-}k\mathrm{SIGN}\left(\partial F\left(\delta_t\right)\right) + \dot{x}_t^*.$$

Weight Matrices $W_{0,t}$ *and* $W_{1,t}$ *move according to Learning Laws (ODE's) containing* $W_0 = W_0^*$, $W_1 = W_1^*$.

Control under addional constraints

Theorem (LS problem with constrants)

Suppose the set

$$\mathcal{J} = \{x : Gx = v\}$$

is not empty. Then the vector x_0 *minimizes* $\|z - Hx\|^2$ *over* $\mathcal{J}$ *if and only if*

$$\boxed{\begin{array}{c} x_0 = G^+ v + \bar{H}^+ z + (I - G^+ G)(I - \bar{H}^+ \bar{H}) w \\ \bar{H} := H(I - G^+ G) \end{array}} \quad (8)$$

where $w \in R^n$ *is any vector and among all solutions*

$$\boxed{\bar{x}_0 = G^+ v + \bar{H}^+ z} \quad (9)$$

has the minimal Euclidian norm.

Corollary (DNN Control under addional constraints)

Under the additional constraints

$$\boxed{Gu = v}$$

the DNN local adaptive control is

$$\boxed{u_t = G^{+} v + B_{NN}^{+} \left(\hat{x}_t, t \right) r_t}$$

Block scheme of Local Adaptive control

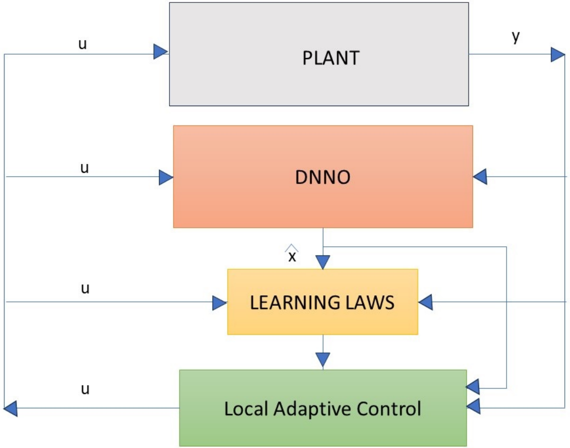

Figure 1: Local Adaptive Control