- Plant and the observer structures
- Main Assumptions: quasi-Lipschitz functions
- Quasi-linear format of the model
- Universal Neuro-Observer structure
- Learning Law designing for weights adaptation

Below we will follows the theory presented in the book:

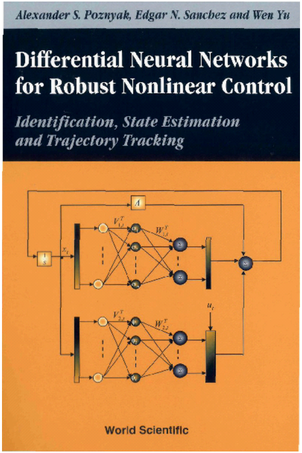

Figure 1: World Scietific, 2001.

Consider the following dynamic system

$$\left.\begin{array}{c} \dot{x}_t = f(x_t, t) + g(x_t, t)\, u_t + \xi_t, \;\; x_0 \text{ is given}, \\ \\ y_t = C x_t + \eta_t, \end{array}\right\} \tag{1}$$

where

$x_t \in R^n$ is the state vector at time $t \geq 0$,
$u_t \in R^r$ is a control action (measurable input) applied to the system at time $t \geq 0$,
$y \in R^m$ is the output of the system at time $t \geq 0$,
$\xi \in R^n$ is an external perturbation acting to the system,
$\eta \in R^m$ is a noise in sensors measurements in the output,
$f : R^n \times R_+ \to R^n$, $g : R^n \times R_+ \to R^{n \times r}$ are given mappings,
$C \in R^{m \times n}$ is a given output matrix.

- y_t and u_t are only **available** at any time $t \geq 0$;
- the matrix $C \in R^{m \times n}$ is **known**;
- $f(x_t, t)$, $g(x_t, t)$ as well as $\xi \in R^n$ and $\eta \in R^m$ are **unknown**;
- the external **perturbations** ξ and η are assumed to be **bounded** with known upper bounds, i.e.,

$$\boxed{\|\xi_t\| \leq \xi_+ < \infty, \|\eta_t\| \leq \eta_+ < \infty,} \tag{2}$$

  and admit the existence of the solutions of ODE (1);
- $f(x_t, t)$ and $g(x_t, t)$ are globally **quasi-Lipschitz** on x_t and measurable on $t \geq 0$, that is, there exist matrices $A \in R^{n \times n}$ and $B \in R^{n \times r}$ such that for all $x \in R^n$ and all admissible $u(x_t, t)$

$$\boxed{\left. \begin{array}{c} \|f(x,t) - Ax\|^2 \leq f_0 + f_1 \|x\|^2, \\ \\ \|(g(x,t) - B)\|^2 \leq g_0 + g_1 \|x\|^2 < \infty, \end{array} \right\}} \tag{3}$$

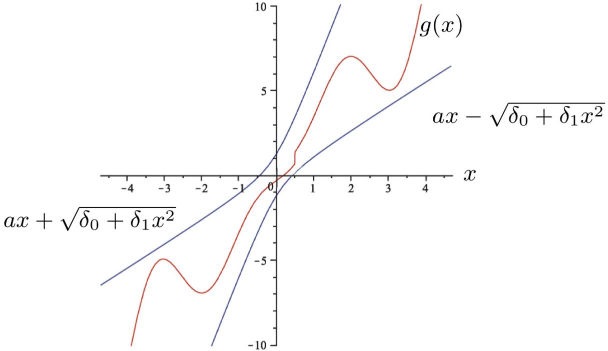

Figure 2: The quasi-Lipschitz function: the single dimensional case n = k = 1, a > c1 > 0.

- the control $u(x_t, t)$ is **measurable** and **bounded**, that is,

$$\|u(x(t), t)\| \leq k, \tag{4}$$

- for any control bounded as in (4) **all trajectory of the systems remain uniformly bounded**, i.e.,

$$\|x\|^2 \leq d_0 + d_1 k^2 \tag{5}$$

Definition

We will referred to this property as **the BIBO-property** (Bounded Input - Bounded Output) or "**heterogeneity**".

Below we will assume that the matrices $A \in R^{n \times n}$, $B \in R^{n \times r}$ as well as the non-negative constants f_0, f_1, g_0, g_1 **are known**.

Under these assumptions the plant (1) can be represented in the **quasi-linear format** as

$$\boxed{\left.\begin{array}{c}\dot{x}_t = Ax_t + Bu_t + \tilde{\xi}_t \\ y_t = Cx_t + \eta_t \\ x_t \in \mathbb{R}^n, \;\; y_t \in \mathbb{R}^m\end{array}\right\}} \tag{6}$$

where the **generalized uncertain term** is

$$\boxed{\tilde{\xi}_t := \left[f\left(x_t, t\right) - Ax_t\right] + \left[g\left(x_t, t\right) - B\right] u_t + \xi_t.} \tag{7}$$

Corollary

Notice that for this new variable $\tilde{\xi}(x_t, t)$ the following property holds:

$$\begin{array}{c} \left\|\tilde{\xi}_t\right\|^2 \leq 4\left\|f(x(t), t) - Ax(t)\right\|^2 + \\ 4\left(k^2\left\|(g(x(t), t) - B)\right\|^2 + \left\|\xi(t)\right\|^2\right) \leq \\ 4\left(f_0 + f_1\left\|x\right\|^2 + k^2 g_0 + k^2 g_1\left\|x\right\|^2 + \xi_+^2\right) = c_0 + c_1\left\|x\right\|^2 \end{array} \quad (8)$$

with

$$c_0 = 4\left(f_0 + k^2 g_0 + \xi_+^2\right), \; c_1 = 4\left(f_1 + k^2 g_1\right).$$

General requirements to a state estimator structure

Definition

We say that a "state estimate" $\hat{x}_t$ is generated by a global (full order) **linear differential observer** (or, a filter) if it satisfies the following three conditions:

1) (ODE property): the function $\hat{x}_t$ is the solution of the following ordinary linear stochastic differential equation

$$\boxed{\frac{d}{dt}\hat{x}_t = G_t\hat{x}_t + Bu_t + L_t y_t,\ \hat{x}_0 \text{ is fixed}} \tag{9}$$

where $G_t \in R^{n\times n}$, $L_t \in R^{n\times m}$ are some deterministic matrices;

Definition

2) (The exact mapping property): the trajectories x_t of the given system (6) and $\hat{x}_t$ (9) coincide for all $t \geq 0$, that is,

$$\boxed{x_t = \hat{x}_t, \ \frac{d}{dt}x_t = \frac{d}{dt}\hat{x}_t,} \qquad (10)$$

if the initial states (9) coincide, i.e., $x_{t=0} = \hat{x}_{t=0}$, and when there are no uncertainties and disturbances at all, that is, when for all

$$\tilde{\xi}\left(x_t, t\right) = 0, \ \eta_t = 0, \ C = I_{m \times n} \text{ for all } t \geq 0; \qquad (11)$$

3) (The asymptotic consistency property): if the initial states of the original model and the estimating model do not coincide, that is, $\left\|x_{t=0} - \hat{x}_{t=0}\right\| > 0$, but still there are no uncertainties (11), then the estimates $\hat{x}_t$ should be asymptotically consistent:

$$\boxed{\Delta x_t := x_t - \hat{x}_t, \ \left\|\Delta x_t\right\| \underset{t \to \infty}{\longrightarrow} 0.} \qquad (12)$$

Gain matrices relation

Lemma

Model (9) satisfies the condition 2 in if and only G_t and L_t are related as

$$\boxed{G_t = A - L_t C \text{ for almost all } t \geq 0.} \quad (13)$$

Proof. Since by the condition 2 $\tilde{\xi}(x_t, t) = 0$, $\eta_t = 0$, it follows

$$\frac{d}{dt}\Delta x_t = (A - L_t C - G_t) x_t - G_t \Delta x_t. \quad (14)$$

a) *Necessity*. Putting $\Delta x_t = 0$ and $\frac{d}{dt}\Delta x_t = 0$, we get $(A - L_t C - G_t) x_t = 0$ for any x_t which implies the identity $A - L_t C - G_t = 0$ for all $t \geq 0$.

b) *Sufficiency*. Suppose that (13) holds. Then by (14) we have $\frac{d}{dt}\Delta x_t = -G_t \Delta x_t$, which, in view of the condition $\Delta x_{t=0} = 0$ implies $\Delta x_t = \Phi_{-G}(t, 0) \Delta x_{t=0} = 0$, where $\Phi_G(t, 0)$ is the fundamental matrix of the last linear vector equation. Lemma is proven.

Substitution (13) with constant parameters in (9) gives

$$\boxed{\frac{d}{dt}\hat{x}_t = A\hat{x}_t + Bu_t + L\left[y_t - C\hat{x}_t\right],\ \hat{x}_0 \text{ is fixed}} \tag{15}$$

which is referred to as the *Luenberger's filter*.

Consider the DNN-observer or *DNN - software sensor* having the following structure

$$\left.\begin{array}{c}\frac{d}{dt}\hat{x}_t = A\hat{x}_t + Bu_t + L\left[y_t - C\hat{x}_t\right] \\ + W_{0,t}\varphi\left(\hat{x}_t\right) + W_{1,t}\psi\left(\hat{x}_t\right)u_t\end{array}\right\} \tag{16}$$

where

$$\hat{x} \in R^n,\ \varphi{:}R^n \to R^{k_\varphi},\ W_0 \in R^{n\times k_\varphi},\ \psi{:}R^r \to R^{k_\psi \times r},\ W_1 \in R^{n\times k_\psi},\ L \in R^{n\times m}$$

and the *sigmoid vector* $\varphi(x)$ and *matrix functions* $\psi(x)$ are supposed to be *bounded,* fulfilling

$$\left\|\varphi(x)\right\| \le \varphi_+,\quad \left\|\psi(x)\right\| = \lambda_{\max}^{1/2}\left(\psi(x)^\intercal \psi(x)\right) \le \psi_+ \tag{17}$$

Neural Observer as a Universal Software Sensor

Time variable parameters in any approximator is a very strong instrument

For nonlinear system

$$\dot{x}_t = f(x_t, t), \; x_{t=0} = \mathring{x}$$

any approximator of the right-hand side

$$f(x_t, t) = A_t \sigma(x_t, t), \; x_{t=0} = \mathring{x}$$

with

$$A_t = \dot{x}_t \sigma^{\intercal}(x_t, t) \left[\sigma(x_t, t) \sigma^{\intercal}(x_t, t) + \varepsilon I\right]^{-1}, \; \varepsilon > 0$$

shows for a bounded dynamics the "nice" $\varepsilon -$*approximating* process

$$\boxed{\frac{d}{dt}\hat{x}_t = \dot{x}_t \sigma^{\intercal}(\hat{x}_t, t) \left[\sigma(\hat{x}_t, t) \sigma^{\intercal}(\hat{x}_t, t) + \varepsilon I\right]^{-1} \sigma(\hat{x}_t, t), \; \hat{x}_{t=0} = \mathring{x}.}$$

Define the **Reinforcement Learning Law** as

$$\left.\begin{array}{c}\dot{W}_0(t) = -\frac{\Lambda_0^{-1}}{k_0}\left(W_{0,t} - W_0^*\right)\varphi\left(\hat{x}_t\right)\varphi^{\intercal}\left(\hat{x}_t\right) - \frac{\alpha}{2}\left(W_{0,t} - W_0^*\right) \\ \\ \dot{W}_1(t) = -\frac{\Lambda_1^{-1}}{k_1}\left(W_{1,t} - W_1^*\right)\psi\left(\hat{x}_t\right)u_t u_t^{\intercal}\psi^{\intercal}\left(\hat{x}_t\right) - \frac{\alpha}{2}\left(W_{1,t} - W_1^*\right)\end{array}\right\} \tag{18}$$

Here W_0^* and W_1^* some matrices which will be defined below as a solution of the corresponding *optimization problem* with a *matrix constraint*.

The analysis of the workability of such DNN is based on *Attractive Ellipsoid Method*.