Lecture 14: Home task problems

Plan of presentation

- Nonholonomic Autonomous Vehicle (AV) models
- Coordinate Systems
- Kinematic constraints of the differential-drive mobil AV
- The Euler-Lagrange equation for nonholonomic systems
- Problem formulation as ASG-DNN backstepping control
- Another more simple problem

Reference

Hector Vargas, Jesús A. Meda, Alexander Poznyak.
ASG version of integral sliding mode robust controller for AV nonholonomic 2D models avoiding obstacles.
Nonlinear Dynamics, April, 2022.
https://doi.org/10.1007/s11071-022-07408-4

Two different coordinate systems (frames) need to be defined (see Fig.1).

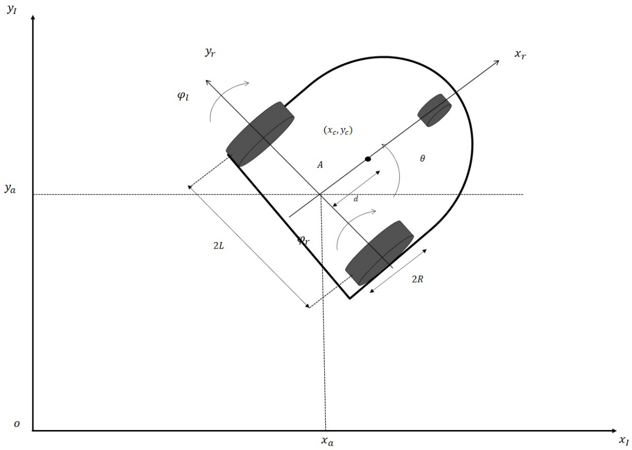

Figure 1: AV description.

1. *Inertial Coordinate System:* This coordinate system is a global frame which is fixed in the environment or plane in which the AV moves in. Moreover, this frame is considered as the reference frame and is denoted as $\{x_I, y_I\}$.
2. *Relative* (or *proper) Coordinate System*: This coordinate system is a local frame attached to the considered AV, and thus, moving with it. This frame is denoted as $\{x_r, y_r\}$. The origin of the relative frame is defined to be the mid-point A on the axis between the wheels. The center of mass C of the AV is assumed to be on the axis of symmetry, at a distance d from the origin A.

As shown in Fig.1, the robot position and orientation in the Inertial Frame can be defined as

$$\boxed{q^I := (x_a, y_a, \theta)^{\intercal}} \tag{1}$$

The important issue that needs to be explained at this stage is the mapping between these two frames. The position of any point on the AV can be defined in the inertial frame and the relative frame as follows:

$$\boxed{\Theta^{I} := \left(x^{I}, y^{I}, \theta^{I}\right)^{\intercal}, \, \Theta^{r} := \left(x^{r}, y^{r}, \theta^{r}\right)^{\intercal}} \tag{2}$$

Then, the two coordinates are related by the following transformation:

$$\boxed{\Theta^{I} = R\left(\theta\right)\Theta^{r}} \tag{3}$$

where $R\left(\theta\right)$ is the orthogonal $\left(R^{\intercal}\left(\theta\right) = R^{-1}\left(\theta\right)\right)$ rotation matrix

$$\boxed{R\left(\theta\right) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}} \tag{4}$$

This transformation is enable also the handling of motion between frames.

Kinematic constraints of the differential-drive mobil AV

The motion of a differential-drive mobile AV is characterized by two non-holonomic constraint equations, which are obtained by two main assumptions:

- *No lateral slip motion:* This constraint simply means that the mobil AV can move only in a curved motion (forward and backward) but not sideward. In the relative frame, this condition means that the velocity of the center-point A is zero along the lateral axis, namely, for any time $t \geq 0$

$$\boxed{y^r = 0, \dot{y}^r = 0} \tag{5}$$

which can be expressed as

$$\boxed{-\dot{x}_a \sin\theta + \dot{y}_a \cos\theta = 0} \tag{6}$$

- *Pure rolling constrain:* The pure rolling constraint represents the fact that each wheel maintains a one contact point p with the ground. There is **no slipping** of the wheel in its longitudinal axis (x^r) and **no skidding** in its orthogonal axis (y^r). The velocities of the contact points in the relative frame are related to the **right** (r) and **left** (l) wheel velocities by:

$$\boxed{v_p = R\dot{\varphi}_r, v_l = R\dot{\varphi}_l} \tag{7}$$

In the inertial frame, these velocities can be calculated as a function of the velocities of the AV center-point A:

$$\boxed{\left.\begin{array}{c} \dot{x}_{pR} = \dot{x}_a + L\theta\cos\theta \\ \dot{y}_{pR} = \dot{y}_a + L\theta\sin\theta \end{array}\right\}} \tag{8}$$

and

$$\boxed{\left.\begin{array}{c} \dot{x}_{pL} = \dot{x}_a + L\dot{\theta}\cos\theta \\ \dot{y}_{pL} = \dot{y}_a + L\dot{\theta}\sin\theta \end{array}\right\}} \tag{9}$$

Using the rotation matrix $R(\theta)$ (4) and (7), the rolling constraint equations (6) are formulated as follows:

$$\left.\begin{array}{c} \dot{x}_{pR}\cos\theta + \dot{y}_{pR}\sin\theta = R\dot{\varphi}_R\,, \\ \\ \dot{x}_{pL}\cos\theta + \dot{y}_{pL}\sin\theta = R\dot{\varphi}_L\,. \end{array}\right\} \quad (10)$$

Applying the contact points velocities equation (8) - (9), and , the three constraint equations (6) and (10) can be rewritten in the following matrix form:

$$\boxed{\Lambda(q)\,\dot{q} = 0} \quad (11)$$

where $q = (x_a, y_a, \theta, \varphi_R, \varphi_L)^\intercal$ is the generalized position vector and $\Lambda(q)$ is

$$\Lambda(q) = \begin{bmatrix} -\sin\theta & \cos\theta & 0 & 0 & 0 \\ \cos\theta & \sin\theta & L & -R & 0 \\ \cos\theta & \sin\theta & -L & 0 & -R \end{bmatrix}. \quad (12)$$

The Hamilton's principle

To derive equations of motion for nonholonomic systems (with constrains depending on the derivatives $\dot{q}$ of the generalized coordinate) we will apply the, so-called, Lagrangian approach using the **Hamilton's principle** and the Euler-Lagrange equations, respectively. In case when we have no any constrains for dynamic trajectories, according to the Hamiltonian principle, the equations of motion for the considered system provides the extremal value for the **Hamiltonian action**, that is,

$$\boxed{\int_{t=a}^{b} L\left(q, \dot{q}, t\right) dt \longrightarrow \underset{q,\dot{q}\in\mathbb{R}^n}{\operatorname{extr}}} \qquad (13)$$

where $L = T - \Pi$ is the Lagrange function with the **kinetic** T and **potential** Π **energies**, respectively. This corresponds the condition

$$\boxed{\delta \int_{t=a}^{b} L\left(q, \dot{q}, t\right) dt = \int_{t=a}^{b} \left(\frac{\partial^{\intercal} L}{\partial q}\delta q + \frac{\partial^{\intercal} L}{\partial \dot{q}}\delta \dot{q}\right) dt = 0} \qquad (14)$$

Here we choose virtual (under fixed t) variations δq_t of the curve q_t $\left(t \in [a, b]\right)$ in such a way that $\delta q_{t=a} = \delta q_{t=b} = 0$.

Integrating by parts with fixed endpoints we get

$$\boxed{\int_{t=a}^{b} \left(\frac{\partial^\intercal L}{\partial q} - \frac{d}{dt}\frac{\partial^\intercal L}{\partial \dot{q}} \right) \delta q dt = 0}$$

which gives us, from the *arbitrariness* of δq, the Euler-Lagrange equation (in the vector form)

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0} \tag{15}$$

Unfortunately, for the nonholonomic systems when there are constrains for the admissible trajectories we can conclude (15) since the admissible virtual variations δq are not arbitrary. They should satisfy (11) for all $t \geq 0$, or equivalently,

$$\boxed{\Lambda\left(q\right)\delta q = 0} \tag{16}$$

To apply correctly in our case the Hamiltonian principle we need to take into account the relation (11) considering the following extremal problem:

$$\boxed{\begin{array}{c} \int\limits_{t=a}^{b} L\left(q, \dot{q}, t\right) dt \rightarrow \underset{q,\dot{q}\in\mathbb{R}^n}{\mathrm{extr}} \\ \text{under constrains (16)} \end{array}} \tag{17}$$

The Hamilton's principle

Using the Lagrange multipliers approach we may conclude that the variation of the Hamiltonian action (13) around the admissible extremal curve q, satisfying the constrains (16), corresponds to the condition (the extended version of D'Alembert's principle)

$$\boxed{\delta \int_{t=a}^{b} L\left(q, \dot{q}, t\right) dt + \int_{t=a}^{b} \lambda^{\intercal} \Lambda\left(q\right) \delta q dt = 0} \tag{18}$$

where λ is the vector of Lagrange multipliers depending on t and the virtual variations now are arbitrary. Following the analogous procedure as before we conclude that (18) implies

$$\boxed{\int_{t=a}^{b} \left(\frac{\partial^{\intercal} L}{\partial q} \delta q + \frac{\partial^{\intercal} L}{\partial \dot{q}} \delta \dot{q} + \lambda^{\intercal} \Lambda\left(q\right) \delta q \right) dt = 0}$$

And again the integration by parts of the second term leads to

$$\boxed{\int_{t=a}^{b}\left(\frac{\partial^{\intercal}L}{\partial q}-\frac{d}{dt}\left(\frac{\partial^{\intercal}L}{\partial \dot{q}}\right)+\lambda^{\intercal}\Lambda\left(q\right)\right)\delta q dt=0}$$

which, by the arbitrariness of δq, is possible if and only if

$$\boxed{\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}+\Lambda^{\intercal}\left(q\right)\lambda=0}$$

or equivalently,

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}-\frac{\partial L}{\partial q}=\Lambda^{\intercal}\left(q\right)\lambda} \tag{19}$$

In the presence of external *nonpotential forces* Q_{nonpot} the Euler-Lagrange equation for nonholonomic systems leads to

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q_{nonpot} + \Lambda^{\intercal}(q)\,\lambda} \tag{20}$$

Kinetic energy calculation fo AV

The *total kinetic energy* T for the considered AV contains three parts:

- the kinetic energy of the AV platform

$$T_c = \frac{1}{2} m_c v_c^2 + \frac{1}{2} I_c \dot{\theta}^2$$

- the kinetic energy of the right wheel

$$T_{wR} = \frac{1}{2} m_w v_{wR}^2 + \frac{1}{2} I_m \dot{\theta}^2 + \frac{1}{2} I_w \dot{\varphi}_R^2$$

- the kinetic energy of the left wheel

$$T_{wL} = \frac{1}{2} m_w v_{wL}^2 + \frac{1}{2} I_m \dot{\theta}^2 + \frac{1}{2} I_w \dot{\varphi}_L^2$$

where m_c is the mass of the AV without the driving wheels and actuators (DC motors), m_w is the mass of each driving wheel (with actuator), I_c is the moment of inertia of the AV with respect to the vertical axis passing through the center of mass, I_w is the moment of inertia of each driving wheel with a motor with respect to the wheel axis, and I_m is the moment of inertia of each driving wheel with a motor with repect to the wheel diameter.

Kinetic energy calculation fo AV

Taking into account that all velocities may be expressed as a function of the generalized coordinates using the general velocity equation in the inertial frame, namely, as $v_i^2 = \dot{x}_i^2 + \dot{y}_i^2$, and in view of the relations

$$\begin{aligned} x_c &= x_a + d\cos\theta, \;\; y_c = y_a + d\sin\theta \\ x_{wR} &= x_a + L\sin\theta, \;\; y_{wR} = y_a + L\cos\theta \\ x_{wL} &= x_a - L\sin\theta, \;\; y_{wL} = y_a + L\cos\theta \end{aligned}$$

we may conclude that

$$\boxed{\begin{gathered} T = T_c + T_{wR} + T_{wL} = \frac{1}{2}m\left(\dot{x}_a^2 + \dot{y}_a^2\right) - \\ m_c d\dot{\theta}\left(\dot{y}_a\cos\theta - \dot{x}_a\sin\theta\right) + \frac{1}{2}I_w\left(\dot{\varphi}_R^2 + \dot{\varphi}_L^2\right) + \frac{1}{2}I\dot{\theta}^2 \end{gathered}}$$

where $m = m_c + 2m_w$ is the total mass of AV and

$$\boxed{I = I_c + m_c d^2 + 2m_w L^2 + 2I_m}$$

is the total equivalent of the moment of inertia.

Since during 2D movements $\Pi = \text{const}$, the dynamic model of the nonholonomic Euler-Lagrange system (20) with $q = (x_a, y_a, \theta, \varphi_R, \varphi_L)^{\intercal}$ becomes

$$\boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = Q_{nonpot} + \Lambda^{\intercal}(q)\,\lambda} \tag{21}$$

or, in the open format,

$$\boxed{M(q)\ddot{q} + V(q, \dot{q})\dot{q} = Q_{nonpot} + \Lambda^{T}(q)\lambda} \tag{22}$$

The Euler-Lagrange model in open format

Here:

$$M(q)=\begin{bmatrix} m & 0 & \text{-}md\sin\theta & 0 & 0 \\ 0 & m & \text{-}md\cos\theta & 0 & 0 \\ \text{-}md\sin\theta & \text{-}md\cos\theta & I & 0 & 0 \\ 0 & 0 & 0 & I_w & 0 \\ 0 & 0 & 0 & 0 & I_w \end{bmatrix}, \quad B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Q_{nonpot} = B\tau + \mathcal{F}(q,\dot{q}), \quad V(q,\dot{q}) = \begin{bmatrix} 0 & 0 & -md\dot{\theta}\cos\theta & 0 & 0 \\ 0 & 0 & -md\dot{\theta}\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$F(q,\dot{q})$ is the centripetal and Coriolis forces, τ is the control action vector (torques of the right and left motors).

Consider the kinematic relation

$$\boxed{\dot{q} = S\dot{\varphi},\ \varphi := \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix}} \tag{23}$$

where

$$\boxed{S(q) = \frac{1}{2}\begin{bmatrix} R\cos\theta & R\cos\theta \\ R\sin\theta & R\sin\theta \\ \frac{R}{L} & -\frac{R}{L} \\ 2 & 0 \\ 0 & 2 \end{bmatrix}} \tag{24}$$

Modified Euler-Lagrange equation

It can be verified that the matrix $S(q)$ (24) acts in the null space of the constraint matrix $\Lambda^{T}(q)$, that is,

$$\boxed{\Lambda(q)S(q) = 0,\ S^{\intercal}(q)\,\Lambda^{\intercal}(q) = 0} \tag{25}$$

and therefore, differentiation of (23), leads to

$$\boxed{\ddot{q} = \dot{S}(q)\,\dot{\varphi} + S(q)\,\ddot{\varphi}} \tag{26}$$

Substitution (26) into (22) gives

$$\boxed{\begin{array}{c} M(q)\left(\dot{S}(q)\,\dot{\varphi} + S(q)\,\ddot{\varphi}\right) + V(q,\dot{q})S(q)\,\dot{\varphi} \\ = B\tau + \mathcal{F}(q,\dot{q}) + \Lambda^{T}(q)\lambda \end{array}} \tag{27}$$

Modified Euler-Lagrange equation

Multiplying both sides of (27) from left by $S^{\intercal}(q)$ and using the property (25), we finally get

$$\boxed{D(q)\ddot{\varphi} + \bar{V}(q,\dot{q})\dot{\varphi} = \bar{B}\tau + \bar{\mathcal{F}}(q,\dot{q})} \tag{28}$$

where

$$\left.\begin{array}{c} D(q) = S^{\intercal}(q)\, M(q) S(q)\,, \\ \\ \bar{V}(q,\dot{q}) = S^{\intercal}(q)\left[M(q)\dot{S}(q) + V(q,\dot{q})S(q)\right], \\ \\ \bar{B}(q) = S^{\intercal}(q)\, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \\ \bar{\mathcal{F}}(q,\dot{q}) = S^{\intercal}(q)\, \mathcal{F}(q,\dot{q})\,. \end{array}\right\} \tag{29}$$

Recuperation of dynamics in initial variables

Using (26) and (28)

$$\ddot{q} = \dot{S}\left(q\right)\dot{\varphi} + S\left(q\right)\ddot{\varphi},$$

$$\ddot{\varphi} = -D^{-1}(q)\bar{V}(q,\dot{q})\dot{\varphi} + D^{-1}(q)\tau + D^{-1}(q)\bar{\mathcal{F}}(q,\dot{q}),$$

we may get

$$\boxed{\ddot{q} = G(q,\dot{q},t + D^{-1}(q)\tau + \xi(q,\dot{q}),} \tag{30}$$

where

$$\boxed{\begin{array}{c} G(q,\dot{q},t) := \left[\dot{S}\left(q\right) - S\left(q\right)D^{-1}(q)\bar{V}(q,\dot{q})\right]\dot{\varphi} \\ \\ \xi(q,\dot{q}) := D^{-1}(q)S^{\intercal}\left(q\right)\mathcal{F}(q,\dot{q}) \end{array}}$$

Problem formulation

Problem

Using dynamics (28), (30) and backstepping approach to design a robust feedback $\tau = \tau(q, \dot{q}, t)$ *which provides the arriving of the coordinates*

$$\boxed{(x_a, y_a)^{\intercal} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{C_0} (x_a, y_a, \theta, \varphi_R, \varphi_L)^{\intercal} = C_0 q}$$

to a given point $(x_a^*, y_a^*,)^{\intercal}$ *avoiding obstacle of ellipsoidal type (see reference in the beginning of the lecture).*

Another more simple problem

Problem

To stabilize the inverted pendulum in vertical position with horizontal perturbations using DNN controller and DC-motor as an actuator.

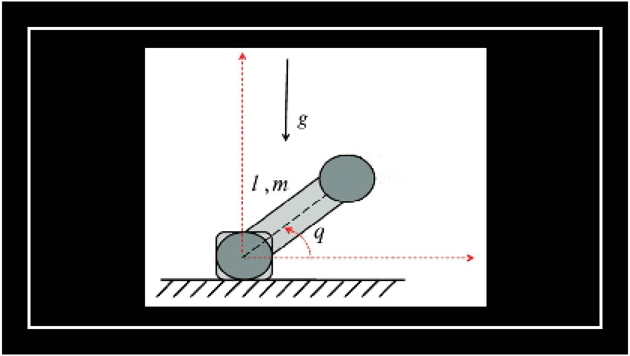

Figure 2: Inverted pendulum.