- DNNN model
- Desired dynamics
- Problem formulation
- Main theorem on the desired dynamics
- Theorem on ASG-DNN robust controller
- Guidance Control of Underwater Autonomous Vehicle

DNN model

Consider again the following DNN model of the mechanical system

$$\left.\begin{array}{c} \frac{d}{dt}\hat{x}_{1,t} = \hat{x}_{2,t} \\ \frac{d}{dt}\hat{x}_{2,t} = f_{NN}\left(\hat{x}_t, t\right) + B_{NN}\left(\hat{x}_t, t\right) u_t, \\ \\ f_{NN}\left(\hat{x}_t, t\right) := A\hat{x}_t + L\left[y_t - C\hat{x}_t\right] + W_{0,t}\varphi\left(\hat{x}_t\right), \\ B_{NN,t} := B + W_{1,t}\psi\left(\hat{x}_t\right). \end{array}\right\} \quad (1)$$

and in variables $\Delta_{1,t} = \hat{x}_{1,t} - x_t^*$, $\Delta_{2,t} = \hat{x}_{2,t} - \dot{x}_t^*$ we get

$$\left.\begin{array}{c} \dot{\Delta}_{1,t} = \Delta_{2,t} \\ \dot{\Delta}_{2,t} = f_{NN}\left(\hat{x}_t, t\right) - \ddot{x}_t^* + B_{NN}\left(\hat{x}_t, t\right) u_t \end{array}\right\} \quad (2)$$

Define the vector function $s(t) \in \mathbb{R}^n$, which from now on and throughout this lecture will be referred to as *"sliding variable"*:

$$\left. \begin{array}{c} s(t) = \Delta_{2,t}(t) + \dfrac{\Delta_{1,t}(t) + \eta}{t+\theta} + \tilde{G}(t), \ \eta = \mathrm{const} \in \mathbb{R}^n, \\ \tilde{G}(t) := \dfrac{1}{t+\theta} \int\limits_{\tau=t_0}^{t} a(\Delta_{1,t}(\tau))\, d\tau, \ \theta > 0, \\ a(\Delta_{1,t}(\tau)) \in \partial F(\Delta_{1,t}(\tau)) \end{array} \right\} \quad (3)$$

Desired dynamics

Define the desired ASG dynamics as

$$\boxed{s(t) = \dot{s}(t) = 0, t \geq t_0,} \tag{4}$$

which corresponds exactly to the situation when the sliding variable $s(t)$ is equal to zero for all $t \geq t_0$. Since

$$\left.\boxed{\begin{array}{c} (t+\theta)\, s(t) = (t+\theta)\, \Delta_{2,t}(t) + \Delta_{1,t}(t) + \eta = \zeta(t), \\ \dot{\zeta}(t) = -a(\Delta_{1,t}(t)), \ \zeta(t_0) = 0, \end{array}}\right\}$$

in the desired regime (4) when $s(t) = 0$ we have

$$\boxed{\left.\begin{array}{c} (t+\theta)\, \Delta_{2,t}(t) + \Delta_{1,t}(t) + \eta = \zeta(t), \quad t \geq t_0 \geq 0, \\ t_0 \text{ is the moment when the desired dynamics may begin.} \end{array}\right\}} \tag{5}$$

Problem formulation

Problem

We need to design a control strategy $u(t)$ as a feedback in (2), which provides the **functional convergence** *of the cost function $F(\delta(t))$ to its minimum value F^* that is, to guarantee*

$$\boxed{F(\Delta_{1,t}(t)) \underset{t \longrightarrow \infty}{\longrightarrow} \inf_{\Delta_{1,t} \in \mathbb{R}^n} F(\Delta_{1,t}) = F^*,} \tag{6}$$

supposing that the current **sub-gradient** *$a(\Delta_{1,t}(t))$ of the convex function $F(\Delta_{1,t})$, to be optimized, is available on-line.*

Main theorem on the desired dynamics

Lemma

For the variable $\Delta_{1,t}(t)$, *satisfying the ideal dynamics (2), with any* $\theta > 0$ *and* η, *for all* $t \geq t_0 \geq 0$ *the following inequality is guaranteed:*

$$\boxed{F(\Delta_{1,t}(t)) - F^* \leq \frac{\Phi(t_0)}{t+\theta} \underset{t\to\infty}{\to} 0,} \quad (7)$$

where

$$\boxed{\Phi(t_0) = \Phi(\Delta_{1,t}(t_0), \theta, \eta) := (t_0 + \theta) F(\Delta_{1,t}(t_0)) - F^* + \frac{1}{2} \left\| \Delta_1^* - \eta \right\|^2,} \quad (8)$$

and

$$\boxed{\begin{array}{c} \Delta_1^* \in \mathrm{Arg} \underset{\Delta_1 \in \mathbb{R}^n}{\inf} F(\Delta_1) \\ F^* := \underset{\Delta_1 \in \mathbb{R}^n}{\inf} F(\Delta_1), \ (\Delta_1^* \textit{ may be not unique}). \end{array}} \quad (9)$$

Some corollaries (1)

Remark

The parameter η will be chosen below in such a way that the desired optimization regime starts from the beginning of the process, namely, when, $t_0 = 0$.

Some corollaries (2)

Corollary

In the partial case when

$$\Delta_1^* = 0,\ \ t_0 = 0 \text{ and } F^* = 0$$

the formula (8) becomes

$$\boxed{\Phi\left(t_0\right) = \Phi\left(\Delta_1^*\left(t_0\right), \theta, \eta\right) := \theta F\left(\Delta_1^*\left(0\right)\right) + \frac{1}{2}\left\|\eta\right\|^2.} \tag{10}$$

Theorem on ASG-DNN robust controller

Theorem

Under assumptions 1-5 the ISM robust controller

$$\boxed{u(t) = B_{NN}^{\intercal}(\hat{x}_t, t)\, S_{\varepsilon}(\hat{x}_t, t)\left[-k_t \mathrm{SIGN}(s(t)) + u_{comp}(t)\right]} \tag{11}$$

where $k_t = \rho_0 > 0$, *and*

$$\boxed{\begin{gathered} S_{\varepsilon}(\hat{x}_t, t) := \left[B_{NN}(\hat{x}_t, t)\, B_{NN}^{\intercal}(\hat{x}_t, t) + \varepsilon I_{n\times n}\right]^{-1}, \\ u_{comp}(t) = \left(I_{n\times n} + \varepsilon S_{\varepsilon}(\hat{x}_t, t)\right)^{-1}\left[\varepsilon k_t S_{\varepsilon}(\hat{x}_t, t)\, \mathrm{SIGN}(s(t)) - p_t^{reali}\right], \\ p_t^{reali} := f_{NN}(\hat{x}_t, t) - \ddot{x}_t^{*} + \frac{1}{t+\theta}\left(\Delta_{2,t} - \frac{\Delta_{1,t} + \eta}{t+\theta} - \tilde{G}(t) + a(\Delta_{1,t})\right), \end{gathered}}$$

$$\boxed{\eta = -\theta\delta_{2,0} - \delta_{1,0}} \tag{12}$$

guarantees the functional convergence (7) from $t_0 = 0$.

Proof.

By (2) we have

$$\left.\begin{array}{c}\dot{\Delta}_{1,t}=\Delta_{2,t}\\ \dot{\Delta}_{2,t}=f_{NN}\left(\hat{x}_t,t\right)-\ddot{x}_t^{*}+B_{NN}\left(\hat{x}_t,t\right)u_t\end{array}\right\}$$

For the Lyapunov function $V\left(s\right)=\frac{1}{2}s^{\intercal}s$ we have

$$\left.\begin{array}{c}\dot{V}\left(s\left(t\right)\right)=s^{\intercal}\left(t\right)\dot{s}\left(t\right)=\\ s^{\intercal}\left(t\right)\left(\dot{\Delta}_{2,t}+\frac{\Delta_{2,t}}{t+\theta}-\frac{\Delta_{1,t}+\eta}{\left(t+\theta\right)^{2}}-\frac{1}{t+\theta}\tilde{G}\left(t\right)+\frac{1}{t+\theta}a\left(\Delta_{1,t}\right)\right)=\\ s^{\intercal}\left(t\right)\left(f_{NN}\left(\hat{x}_t,t\right)-\ddot{x}_t^{*}+B_{NN}\left(\hat{x}_t,t\right)u_t\right)+\\ s^{\intercal}\left(t\right)\left(\frac{\Delta_{2,t}}{t+\theta}-\frac{\Delta_{1,t}+\eta}{\left(t+\theta\right)^{2}}-\frac{1}{t+\theta}\tilde{G}\left(t\right)+\frac{1}{t+\theta}a\left(\Delta_{1,t}\right)\right)=\\ s^{\intercal}\left(t\right)p_t^{\text{reali}}+s^{\intercal}\left(t\right)B_{NN}\left(\hat{x}_t,t\right)u_t.\end{array}\right\}\quad(13)$$

□

Proof of Main Theorem (2)

Proof.

Selecting $u(t)$ as in (11) for the second term in (13) we get

$$\left.\begin{array}{c}\dot{V}(s(t)) = s^{\intercal}(t)\,p_t^{\text{reali}} + s^{\intercal}(t)\,B_{NN}(\hat{x}_t,t)\,u_t = s^{\intercal}(t)\,p_t^{\text{reali}} + \\ s^{\intercal}(t)\,B_{NN}(\hat{x}_t,t)\,B_{NN}^{\intercal}(\hat{x}_t,t)\,S_{\varepsilon}(\hat{x}_t,t)\left[\text{-}k_t\mathrm{SIGN}(s(t)) + u_{comp}(t)\right] \\ = s^{\intercal}(t)\,p_t^{\text{reali}} + s^{\intercal}(t)\left[-k_t\mathrm{SIGN}(s(t)) + u_{comp}(t)\right] - \\ \varepsilon s^{\intercal}(t)\,S_{\varepsilon}(\hat{x}_t,t)\left[-k_t\mathrm{SIGN}(s(t)) + u_{comp}(t)\right] = \\ -k_t\sum_{i=1}^{n}|s_i(t)| + s^{\intercal}(t)\left[p_t^{\text{reali}} - \varepsilon k_t S_{\varepsilon}(\hat{x}_t,t)\,\mathrm{SIGN}(s(t))\right] \\ + s^{\intercal}(t)\left(I_{n\times n} + \varepsilon S_{\varepsilon}(\hat{x}_t,t)\right)u_{comp}(t) = -k_t\sum_{i=1}^{n}|s_i(t)|\end{array}\right\} \qquad (14)$$

□

Proof of Main Theorem (3)

Proof.

Taking into account that $\sum_{i=1}^{n} |s_i(t)| \geq \|s(t)\|$ and, in view of (14), we derive

$$\dot{V}(s(t)) \leq -\rho_0 \|s(t)\| = -\sqrt{2}\rho_0 \sqrt{V(s(t))},$$

implying $2\left(\sqrt{V(s(t))} - \sqrt{V(s(t_0))}\right) \leq -\sqrt{2}\rho_0 t$ and

$$0 \leq \sqrt{V(s(t))} \leq \sqrt{V(s(t_0))} - \frac{\rho_0}{\sqrt{2}} t,$$

which leads to the conclusion that for all

$$t \geq t_{reach} := \frac{1}{\rho_0}\sqrt{2V(s_{t_0})} = \frac{\|s_{t_0}\|}{\rho_0}.$$

□

Proof.

To make the reaching time $t_{reach} = 0$ it is sufficient to gurantee that $s_{t_0=0} = 0$. But since

$$(t+\theta)\, s(t) = (t+\theta)\, \dot{\delta}(t) + \delta(t) + \eta = \zeta(t),$$
$$(t_0+\theta)\, s(t_0) = (t_0+\theta)\, \dot{\delta}(t_0) + \delta(t_0) + \eta = \zeta(t_0)$$
$$s_{t_0} = \dot{\delta}_{t_0} + \frac{\delta_{t_0} + \eta}{t_0 + \theta},$$

we need to fulfill the condition

$$\boxed{s_{t_0=0} = \dot{\delta}_{t_0=0} + \frac{\delta_{t_0=0} + \eta}{\theta} = 0,}$$

which is possible if take η as in (12), providing

$$t_{reach} = \rho_0^{-1} \left\| s_{t_0=0} \right\| = 0.$$

1. Hernandez-Sanchez, A., Chairez, I., Poznyak, A. and Olga Andrianova. Dynamic Motion Backstepping Control of Underwater Autonomous Vehicle Based on Averaged Sub-gradient Integral Sliding Mode Method. *J Intell Rob Syst*, 103, 48 (2021). https://doi.org/10.1007/s10846-021-01466-3
2. Alejandra Hernandez-Sanchez, Alexander Poznyak, Isaac Chairez. Robust proportional–integral control of submersible autonomous robotized vehicles by backstepping-averaged sub-gradient sliding mode control. *Ocean Engineering*, 263 (2022) 112196. https://doi.org/10.1016/j.oceaneng.2022.112196

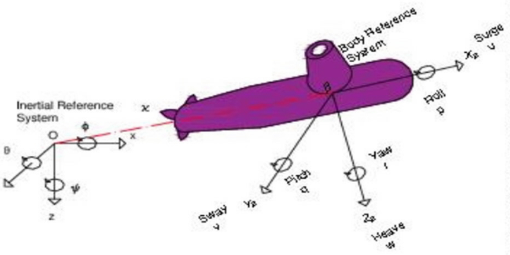

Figure 1: UV and its coordinates

Main coordinates are (see Fig.1):

- $\varkappa = \begin{bmatrix} x & y & z \end{bmatrix}^{\intercal}$ is the vector of the UV centre of mass position;
- $\boldsymbol{\eta} = \begin{bmatrix} \theta & \psi \end{bmatrix}^{\intercal}$ defines the orientation angles;
  (the position and orientation coordinates are given with respect to the inertial framework attached to the origin).
- $\boldsymbol{v} = \begin{bmatrix} u & v & w \end{bmatrix}^{\intercal}$ is the vector of translation velocity of the UV centre of mass;
- $\boldsymbol{\omega} = \begin{bmatrix} q & r \end{bmatrix}^{\intercal}$ is the vector of the angular velocity with respect to the body framework attached to the center of mass.

The mathematical model of the UV contains both the *kinematic* and *dynamic* parts.

It is as follows:

$$\boxed{\frac{d}{dt}\varkappa = \Theta\left(\boldsymbol{\eta}\right)\boldsymbol{v} + \boldsymbol{\zeta}_{\varkappa}(\varkappa, \boldsymbol{\eta}, t),} \tag{15}$$

$$\boxed{\frac{d}{dt}\boldsymbol{\eta} = \begin{bmatrix} q \\ r \\ \hline c_{\theta_t} \end{bmatrix} + \boldsymbol{\zeta}_{\eta}(\varkappa, \boldsymbol{\eta}, t),} \tag{16}$$

where $\zeta_{\varkappa}$ and ζ_{η} are the perturbations vector satisfying

$$\boxed{\left\|\boldsymbol{\zeta}_{\varkappa}\right\| \leq \zeta_{\varkappa}^{+}, \left\|\frac{d}{dt}\boldsymbol{\zeta}_{\varkappa}\right\| \leq \left(\frac{d}{dt}\zeta_{\varkappa}\right)^{+}, \left\|\boldsymbol{\zeta}_{\eta}\right\| \leq \zeta_{\eta}^{+}} \tag{17}$$

and $\Theta : \mathbb{R}^3 \to R^{3\times 3}$ is the rotation matrix (given in the Euler angles):

$$\left.\begin{array}{c} \Theta\left(\boldsymbol{\eta}\right) = \begin{bmatrix} c_\theta c_\psi & -s_\psi & s_\theta c_\psi \\ c_\theta s_\psi & c_\psi & s_\theta s_\psi \\ -s_\theta & 0 & c_\theta \end{bmatrix}, \\ s_\theta = \sin(\theta),\ c_\theta = \cos(\theta),\ \theta \in (-\pi/2, \pi/2). \end{array}\right\} \tag{18}$$

It is given by the following system of ODE:

$$\boxed{\frac{d}{dt}\boldsymbol{v} = f_v(\boldsymbol{v}, \boldsymbol{\omega}) + B_v \boldsymbol{\tau} + \boldsymbol{\zeta}_v(\varkappa, \boldsymbol{\eta}, t),} \tag{19}$$

$$\boxed{\frac{d}{dt}\boldsymbol{\omega} = f_\omega(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\eta}) + B_\omega \boldsymbol{\tau} + \boldsymbol{\zeta}_\omega(\varkappa, \boldsymbol{\eta}, t),} \tag{20}$$

where f_v and f_ω describe the *drift* and the *rotation* effects:

$$f_v(\boldsymbol{v}, \boldsymbol{\omega}) = \begin{bmatrix} -\frac{d_1 u}{I_1} + \frac{I_2 v}{I_1} r - \frac{I_3 w}{I_1} q \\ -\frac{I_1 u}{I_2} r - \frac{d_2 v}{I_2} \\ \frac{I_1 u}{I_3} q - \frac{d_3 w}{I_3} \end{bmatrix}, \quad f_\omega(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\eta}) = \begin{bmatrix} \frac{I_3 - I_1}{I_5} uw - \frac{d_5}{m} q - \frac{mghs_\theta}{I_5} \\ \frac{I_1 - I_2}{I_6} uv - \frac{d_6}{I_6} r \end{bmatrix}.$$

The control vector $\boldsymbol{\tau} = \begin{bmatrix} \tau_u & \tau_q & \tau_r \end{bmatrix}^\intercal \in \mathbb{R}^3$, and

$$B_v = \begin{bmatrix} \frac{1}{I_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_\omega = \begin{bmatrix} 0 & \frac{1}{I_5} & 0 \\ 0 & 0 & \frac{1}{I_6} \end{bmatrix},$$

Actuator dynamics

The dynamic of the vector τ is given by

$$\boxed{\frac{d}{dt}\boldsymbol{\tau} = Z_E \left(\mathbf{g}(\varkappa, \boldsymbol{\eta}, t) + \boldsymbol{\nu} \right),} \tag{21}$$

where $g : \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3$ corresponds to the contra-electromotive forces, satisfying the constrains

$$\boxed{\left\| \frac{d}{dt}\mathbf{g}(\varkappa, \boldsymbol{\eta}, t) \right\| \leq \dot{g}^+ < \infty,} \tag{22}$$

$Z_E \in R$ is the matrix of contra-electromotive gains which assumed to be invertable and known, the vector ν corresponds to the voltages in the actuators, realizing torques τ_u, τ_q and τ_r:

$$\boxed{\boldsymbol{\nu} = \begin{bmatrix} \nu_1 & \nu_2 & \nu_3 \end{bmatrix}^{\intercal} \in U_{v,adm},} \tag{23}$$

where the admissible set $U_{v,adm}$ which may include discontinuos control actions (voltages).

Taking into account the expression of the UV kinematics, translation and orientation dynamics in equations (15),(16),(19) and (20) respectively , and considering the actuators dynamic in expression (21), the complete dynamic system can be described as the following system no ODE:

$$\left.\begin{array}{c}
\frac{d}{dt}\varkappa = \Theta\left(\boldsymbol{\eta}\right)\boldsymbol{v} + \zeta_{\varkappa}(\varkappa, \boldsymbol{\eta}, t), \\
\frac{d}{dt}\boldsymbol{\eta} = \left[\begin{array}{c} q \\ r \\ \hline c_{\theta} \end{array}\right] + \zeta_{\boldsymbol{\eta}}(\varkappa, \boldsymbol{\eta}, t), \\
\frac{d}{dt}\boldsymbol{v} = f_{v}\left(\boldsymbol{v}, \boldsymbol{\omega}\right) + B_{v}\boldsymbol{\tau} + \zeta_{v}(\varkappa, \boldsymbol{\eta}, t), \\
\frac{d}{dt}\boldsymbol{\omega} = f_{\omega}\left(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{\eta}\right) + B_{\omega}\boldsymbol{\tau} + \zeta_{\omega}(\varkappa, \boldsymbol{\eta}, t), \\
\frac{d}{dt}\boldsymbol{\tau} = Z_{E}\left[g(\varkappa, \boldsymbol{\eta}, t) + \boldsymbol{\nu}\right].
\end{array}\right\} \tag{24}$$

Problem statement in descriptive form

The problem of the interest here is to design the control ν (23), realizing the behaviour of the dynamic system (24) fulfilling

$$\boxed{\|\boldsymbol{\varphi}_1(t)\| \underset{t\to\infty}{\rightarrow} \min,\ \boldsymbol{\varphi}_1(t) := \varkappa(t) - \varkappa^*(t)} \tag{25}$$

where $\varkappa^*(t) \in \mathbb{R}^3$ is the vector of reference trajectory satisfying:

$$\left\| \frac{d}{dt}\varkappa^* \right\| \leq \left(\frac{d}{dt}\varkappa \right)^+ = \underset{t}{\text{const}},\ \left\| \frac{d^2}{dt^2}\varkappa^* \right\| \leq \left(\frac{d^2}{dt^2}\varkappa \right)^+ = \underset{t}{\text{const}}. \tag{26}$$

An alternative formulation of this tracking trajectory statement as an optimization, realized by an uncertain controlable dynamic plant, looks as follows:

$$\boxed{\begin{array}{c} J(\boldsymbol{\varphi}_1) = \sum_{i=1}^{3} \left| \varphi_{1,i} \right| \underset{t\to\infty}{\rightarrow} \underset{\boldsymbol{\nu}(\cdot) \in U_{adm}}{\min} \\ \text{subjected to (24).} \end{array}} \tag{27}$$

At the *first stage* let us consider the translation kinematic (15) only where the vector υ will be treated as an auxilary intermediate "*pseudo-control*", defining it as $\mathbf{u}_1 = \upsilon$, which implies

$$\boxed{\frac{d}{dt}\varkappa = \Theta\left(\eta\right)\mathbf{u}_1 + \zeta_{\varkappa}(\varkappa,\eta,t),} \tag{28}$$

The corresponding optimization problem realized by the dynamic plant (28) can be formulated as

$$\boxed{\begin{array}{c} J_1(\boldsymbol{\varphi}_1) = J(\boldsymbol{\varphi}_1) = \sum_{i=1}^{3}\left|\varphi_{1,i}\right| \underset{t\to\infty}{\to} \min_{\mathbf{u}_1(\cdot)\,\in\, U_{1,adm}} \\ \text{subjected to (28),} \end{array}} \tag{29}$$

where $U_{1,adm}$ is a set of **differentiable functions**.

Theorem

Under the accepted assumptions the intermediate pseudo-control u_1^*, realizing the soltion of the problem (29), satisfies the following ODE's

$$\left.\begin{array}{l} \frac{d}{dt}\left(\Theta \mathbf{u}_1^*\right) + \mathbf{g}_1 = -k_1 \mathrm{Sign}\left(\mathbf{s}_1\right), \ \mathbf{u}_1^*(0) = \mathbf{u}_{1,0}^*, \ k_1 > \dot{\zeta}_{\varkappa}^+, \\ \mathrm{Sign}\left(\mathbf{s}_1\right) := \left[\ \mathrm{sign}\left(\mathbf{s}_{1,1}\right), \quad \mathrm{sign}\left(\mathbf{s}_{1,2}\right), \quad \mathrm{sign}\left(\mathbf{s}_{1,3}\right), \ \right]^{\mathsf{T}} \end{array}\right\} \quad (30)$$

$$\mathbf{g}_1 := -\frac{d^2}{dt^2}\varkappa^* + \frac{\frac{d}{dt}\varkappa - \frac{d}{dt}\varkappa^*}{t+\theta} - \frac{\varkappa - \varkappa^* + \boldsymbol{\alpha}_1}{(t+\theta)^2} - \frac{1}{t+\theta}\Gamma_1 + \frac{1}{t+\theta}\partial J_1(\boldsymbol{\varphi}_1) \quad (31)$$

Theorem (continuation)

where the integral sliding variable s_1 is defined as

$$\left.\begin{array}{c}
\mathbf{s}_1 = \frac{d}{dt}\boldsymbol{\varphi}_1 + \frac{\boldsymbol{\varphi}_1 + \boldsymbol{\alpha}_1}{t+\theta} + \Gamma_1, \ \Gamma_1 = \frac{1}{t+\theta} \int\limits_{\tau=0}^{t} \partial J_{\mathbf{1}}(\boldsymbol{\varphi}_1) d\tau, \ t \geq 0, \\
\partial J_{\mathbf{1}}(\boldsymbol{\varphi}_1) = \left[\ \text{sign}\left(\varphi_{1,1}\right), \ \ \text{sign}\left(\varphi_{1,2}\right), \ \ \text{sign}\left(\varphi_{1,3}\right) \ \right]^{\intercal} \\
\boldsymbol{\alpha}_1 = -\theta \frac{d}{dt}\boldsymbol{\varphi}_1(0) - \boldsymbol{\varphi}_1(0), \ \theta > 0,
\end{array}\right\} \tag{32}$$

It guarantees that

$$J_1(\boldsymbol{\varphi}_1(t)) \leq \frac{\Phi_1}{t+\theta} \underset{t\to\infty}{\to} 0, \ \Phi_1 = \theta J_{\mathbf{1}}(\boldsymbol{\varphi}_1(0)) + \frac{1}{2}\|\boldsymbol{\alpha}_1\|^2. \tag{33}$$

Proof.

For $V(\mathbf{s}) = \frac{1}{2}\|\mathbf{s}\|^2$ with $\mathbf{s} := \mathbf{s}_1$, we get

$$\left.\begin{array}{c} \dot{V}(\mathbf{s}_1) = \mathbf{s}_1^{\intercal}\dot{s}_1 = \mathbf{s}_1^{\intercal}\left[\ddot{\varphi}_1 + \frac{\dot{\varphi}_1}{t+\theta} - \frac{\varphi_1+\alpha_1}{(t+\theta)^2} - \frac{1}{t+\theta}\Gamma_1 + \frac{1}{t+\theta}\partial J_{\mathbf{1}}(\boldsymbol{\varphi}_1)\right] \\ = \mathbf{s}_1^{\intercal}\left[\ddot{\varkappa} - \ddot{\varkappa}^* + \frac{\dot{\varkappa}-\dot{\varkappa}^*}{t+\theta} - \frac{\varkappa-\varkappa^*+\alpha_1}{(t+\theta)^2} - \frac{1}{t+\theta}\Gamma_1 + \frac{1}{t+\theta}\partial J_{\mathbf{1}}(\boldsymbol{\varphi}_1)\right] \\ = \mathbf{s}_1^{\intercal}\left[\frac{d}{dt}\left(\Theta\mathbf{u}_1^*\right) + \dot{\zeta}_{\varkappa} + \mathbf{g}_1\right]. \end{array}\right\} \quad (34)$$

Select $\boldsymbol{v} = \mathbf{u}_1^*$ satisfying (30). Then from (34) we get

$$\left.\begin{array}{c} \dot{V}(\mathbf{s}_1) = \mathbf{s}_1^{\intercal}\left[-k_1\operatorname{sign}(\mathbf{s}_1) + \dot{\zeta}_{\varkappa}\right] \leq \left(-k_1\sum_{i=1}^{3}|s_{1,i}| + \|\mathbf{s}_1\|\dot{\zeta}_{\varkappa}^{+}\right) \leq \\ \|\mathbf{s}_1\|\left(-k_1 + \dot{\zeta}_{\varkappa}^{+}\right) = -\mathring{\rho}\|\mathbf{s}_1\| = -\mathring{\rho}\sqrt{2V(\mathbf{s}_1)},\ k_1 - \dot{\zeta}_{\varkappa}^{+} = \mathring{\rho} > 0 \end{array}\right\}$$

Proof.

which leads to the following relations

$$\left.\begin{array}{c}\dfrac{dV(\mathbf{s}_1)}{\sqrt{V(\mathbf{s}_1)}} \leq -\mathring{\rho}\sqrt{2}dt \rightarrow 2\left(\sqrt{V(\mathbf{s}_1)} - \sqrt{V(\mathbf{s}_1(0))}\right) \leq -\mathring{\rho}\sqrt{2}t, \\ 0 \leq \sqrt{V(\mathbf{s}_1)} \leq \sqrt{V(\mathbf{s}_1(0))} - \dfrac{\mathring{\rho}}{\sqrt{2}}t, \end{array}\right\}$$

implying that $V(\mathbf{s}_1(t)) = 0$ for all

$$t \geq t_{reach} := \frac{1}{\mathring{\rho}}\sqrt{2V(\mathbf{s}_1(0))} = \frac{\|\mathbf{s}_1(0)\|}{\mathring{\rho}}. \quad (35)$$

But by (32), $\mathbf{s}_1(0) = \mathbf{0}$, and hence from the begginning of the proces

$$\mathbf{s}_1(t) = \dot{s}_1(t) = \mathbf{0}. \quad (36)$$

Proof.

b) Defining $\mu(t) := t + \kappa$, let represent (36) as

$$\mu(t)\mathbf{s}_1 = \mu(t)\dot{\varphi}_1(t) + \varphi_1(t) + \alpha_1 + \gamma(t) = 0,$$
$$\dot{\gamma}(t) = \partial J_{\mathbf{1}}(\boldsymbol{\varphi}_1(t)), \ \gamma(0) = 0,$$

or, equivalently,

$$\boxed{\mu(t)\dot{\varphi}_1(t) + \varphi_1(t) + \alpha_1 = -\gamma(t),}$$

which gives

$$\begin{aligned} \frac{d}{dt}\left[\frac{1}{2}\|\gamma\|^2\right] &= \dot{\gamma}^{\intercal}\gamma = -\partial^{\intercal} J_{\mathbf{1}}(\boldsymbol{\varphi}_1)\left[\mu\dot{\boldsymbol{\varphi}}_1 + \boldsymbol{\varphi}_1 + \boldsymbol{\alpha}_1\right] \\ &= -\partial^{\intercal} J_{\mathbf{1}}(\boldsymbol{\varphi}_1)\boldsymbol{\varphi}_1 - \partial^{\intercal} J_{\mathbf{1}}(\boldsymbol{\varphi}_1)\left(\mu\dot{\boldsymbol{\varphi}}_1 + \boldsymbol{\alpha}_1\right). \end{aligned}$$

Proof.

By the relations

$$\partial^{\mathsf{T}} J_{\mathbf{1}}(\varphi_1)\varphi_1 \geq J_{\mathbf{1}}(\varphi_1) - J_{\mathbf{1}}(\varphi_1^*) = J_{\mathbf{1}}(\varphi_1),\ J_{\mathbf{1}}(\varphi_1^*) = 0,\ \partial^{\mathsf{T}} J_{\mathbf{1}}(\varphi_1)\dot{\varphi}_1 = \frac{d}{dt} J_{\mathbf{1}}(\varphi_1)$$

$$\Longrightarrow \frac{d}{dt}\left[\frac{1}{2}\|\gamma\|^2\right] \leq -J_{\mathbf{1}}(\boldsymbol{\varphi}_1) - \mu\frac{d}{dt}J_{\mathbf{1}}(\boldsymbol{\varphi}_1) - \partial^{\mathsf{T}} J_{\mathbf{1}}(\boldsymbol{\varphi}_1)\boldsymbol{\alpha}_1.$$

Then, integrating this inequality on interval $[0, t]$, we get

$$\int_{\tau=0}^{t} J_{\mathbf{1}}(\varphi_1(\tau))d\tau \leq \frac{1}{2}\left(\underbrace{\|\gamma(0)\|^2}_{0} - \|\gamma\|^2\right) -$$

$$\int_{\tau=0}^{t} \mu(\tau)\tfrac{d}{dt}J_{\mathbf{1}}(\varphi_1(\tau))d\tau - \left(\int_{\tau=0}^{t} \partial J_{\mathbf{1}}(\boldsymbol{\varphi}_1)d\tau\right)^{\mathsf{T}} \alpha_1$$

Proof.

Since $\dot{\mu}_{\tau} = 1$, using of the integration by parts we get

$$\int_{\tau=0}^{t} \mu(\tau) \frac{d}{dt} J_{\mathbf{1}}(\varphi_1(\tau)) d\tau =$$

$$[\mu(\tau) J_{\mathbf{1}}(\boldsymbol{\varphi}_1(\tau))]_{\tau=0}^{\tau=t} - \int_{\tau=0}^{t} \dot{\mu}(\tau) J_{\mathbf{1}}(\varphi_1(\tau)) d\tau =$$

$$\mu J_{\mathbf{1}}(\varphi_1) - \theta J_{\mathbf{1}}(\varphi_1(0)) \text{-} \int_{\tau=0}^{t} J_{\mathbf{1}}(\varphi_1(\tau)) d\tau = \mu J_{\mathbf{1}}(\varphi_1) \text{-} \theta J_{\mathbf{1}}(\varphi_1(0)) \text{-} \gamma,$$

which leads to

$$\int_{\tau=0}^{t} J_{\mathbf{1}}(\varphi_1(\tau)) d\tau \leq -\frac{1}{2} \|\gamma\|^2 - \mu J_{\mathbf{1}}(\varphi_1) +$$

$$\theta J_{\mathbf{1}}(\varphi_1(0)) + \int_{\tau=0}^{t} J_{\mathbf{1}}(\varphi_1(\tau)) d\tau - \gamma^{\mathsf{T}} \alpha_1,$$

Proof.

or equivalently,

$$\begin{gathered}\mu J_{\mathbf{1}}(\boldsymbol{\varphi}_1) \leq \theta J_{\mathbf{1}}(\boldsymbol{\varphi}_1(0)) - \frac{1}{2}\|\gamma\|^2 - \gamma^{\intercal}\boldsymbol{\alpha}_1 = \\ \theta J_{\mathbf{1}}(\boldsymbol{\varphi}_1(0)) - \frac{1}{2}\left(\|\gamma\|^2 + 2\gamma^{\intercal}\boldsymbol{\alpha}_1\right) = \\ \theta J_{\mathbf{1}}(\boldsymbol{\varphi}_1(0)) - \frac{1}{2}\left(\|\gamma\|^2 + 2\gamma^{\intercal}\boldsymbol{\alpha}_1 + \|\boldsymbol{\alpha}_1\|^2\right) + \frac{1}{2}\|\boldsymbol{\alpha}_1\|^2 \\ = \theta J_{\mathbf{1}}(\boldsymbol{\varphi}_1(0)) - \frac{1}{2}\|\gamma + \boldsymbol{\alpha}_1\|^2 + \frac{1}{2}\|\boldsymbol{\alpha}_1\|^2 \\ \leq \theta J_{\mathbf{1}}(\boldsymbol{\varphi}_1(0)) + \frac{1}{2}\|\boldsymbol{\alpha}_1\|^2\end{gathered}$$

that gives (33). Theorem is proven. □