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Uso de Modelos Implícitos para el Análisis y Control de Sistemas de Estructura Variable

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A mis estimados hermanos:

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Índice General

Notación

$\mathcal{V}$, $\mathcal{W}$, ...	Espacios lineales
v, w, ...	Elementos de los espacios lineales
$\dim(\mathcal{V})$	La dimensión de un espacio $\mathcal{V}$
$\frac{\mathcal{W}}{\mathcal{V}}$	Espacio cociente $\mathcal{W}$ módulo $\mathcal{V}$ ($\mathcal{V} \subset \mathcal{W}$)
$\mathcal{V} \approx \mathcal{W}$	Se utiliza para indicar $\dim(\mathcal{V}) = \dim(\mathcal{W})$
$\oplus$	Símbolo para representar la suma directa de espacios independientes
$\operatorname{Im} X = X\mathcal{V}$	Imagen del mapa lineal $X : \mathcal{V} \to \mathcal{W}$
$\mathcal{K}_X$, $\ker \mathrm{X}$	Kernel del mapa lineal $X : \mathcal{V} \to \mathcal{W}$
$X^{-1}\mathcal{T}$	Imagen inversa del subespacio $\mathcal{T}$ por el mapa lineal X
$X^{(-1)}$	Mapa inverso de X (cuando este existe) para evitar confusiones con $X^{-1}\mathcal{T}$
$\mathcal{E}$	Imagen del mapa $E : \mathcal{X} \to \underline{\mathcal{X}}$
$\mathcal{B}$	Imagen del mapa $B : \mathcal{U} \to \underline{\mathcal{X}}$
$\{x, y, z\}$	Subespacio generado por los vectores x, y y z
e_i	Vector con 1 en su $i-ésima$ componente y 0 en las demás
$\mathcal{L}^{-1}\{\cdot\}$	Transformada Inversa de Laplace
s	Variable compleja de Laplace
p	Operador derivada d/dt
$\underline{\chi}_k^i$	Vector de dimensión kx1 cuya i-ésima componente es 1 y las otras cero
$\underline{1}_k$	Vector de dimensión kx1 cuyas componentes son 1
I_k	Matriz identidad kxk, o simplemente I
$U\left\{v^T\right\}$	Matriz triangular Toeplitz cuyo primer renglón es el vector v^T
$L\{v\}$	Matriz triangular inferior Toeplitz cuya primer columna es el vector v
$D\{X_1, ..., X_k\}$	Matriz diagonal a bloques cuyos elementos en la diagonal son las matrices $X_1, ..., X_k$

Ejemplo 1

$$\underline{\chi}_2^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \underline{1}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad U\left\{\begin{bmatrix} a & b \end{bmatrix}\right\} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix},$$
$$L\left\{\begin{bmatrix} a \\ b \end{bmatrix}\right\} = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}, \quad D\{X_1, X_2\} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}.$$

Capítulo 1

Introducción

El tema del presente trabajo se puede considerar como una extensión de los sistemas *clásicos* estrictamente propios:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{1.1}$$

Rosenbrock [42] fue el primero en introducir este tipo de *descripciones implícitas* (conocidos también como sistemas generalizados, sistemas diferenciales y algebraicos, sistemas singulares), $\Sigma(E, A, B, C)$:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t); \\ y(t) &= Cx(t) \end{aligned} \tag{1.2}$$

donde $E : \mathcal{X} \to \underline{\mathcal{X}}$, $A : \mathcal{X} \to \underline{\mathcal{X}}$, $B : \mathcal{U} \to \underline{\mathcal{X}}$ y $C : \mathcal{X} \to \mathcal{Y}$ son operadores lineales de dimensiones apropiadas. A partir de esta introducción, mucha gente ha puesto especial atención a esta clase amplia de sistemas con puntos de vista diferentes:

i) Enfoque Geométrico (e.g. [2, Armentano 1986], [23, Cobb 1984], [33, Lewis 1992], [39, Malabre 1989], [41, Ozcaldiran 1986], [48, Wong 1974]),

ii) Enfoque en el dominio del tiempo (e.g. [49, Yip y Sincovec 1981]),

iii) Transformada de Laplace (e.g. [3, Bernhard 1982], [29, Kuijper y Schumacher 1989], [44, Verghese et al. 1981]),

iv) Teoría de Kronecker (e.g. [35, Loiseau 1985], [36, Loiseau y Lebret 1990]),

v) Enfoque polinomial (e.g. [30, Kucera y Zagalak 1988]),

vi) Algebra diferencial (e.g. [24, Fliess 1989]), y

vii) Técnicas de inclusión diferencial (e.g. [26, Frankowska 1990]).

No sólo se han resuelto muchos problemas interesantes de investigación sino también han resurgido algunos aspectos prácticos de la teoría de sistemas a través de esta visión. En efecto, ahora se reconoce que las descripciones implícitas son capaces de describir comportamientos más numerosos y más ricos que para el caso clásico estrictamente propio (ver [44, Verghese et al. 1981], [22, Campbell 1982], [23, Cobb 1984], [1, Aplevich 1991], [32, Lewis 1986]). Se puede, por ejemplo, mencionar:

- Sistemas estrictamente propios ($E = I$)
- Sistemas que tienen comportamientos impulsionales (debido a derivadores)
- Sistemas controlados por leyes de control proporcional y derivativa
- Sistemas con restricciones en la entrada
- Sistemas con restricciones algebraicas sobre el estado
- sistemas con restricciones sobre la ley de control, etc..

En este contexto generalizado, estamos interesados particularmente en modelos implícitos en los cuales el número de variables internas es superior al número de ecuaciones de estado. Estos *sistemas implícitos rectangulares* tienen la característica particular que para una condición inicial y una ley de control dadas, la solución en términos de trayectoria de estado no es única, es decir, poseen un grado de libertad interno (en algunas salidas en el origen el comportamiento no es *único*).

De esta manera se pueden describir *sistemas de estructura variable* (por ejemplo, donde el comportamiento depende de la posición de ciertos conmutadores). Más precisamente, se puede describir una familia de n sistemas donde los cambios provienen de una ecuación de restricción algebraica del tipo $D_i x = 0$ $(i = 1, ..., n)$ en superposición con un conjunto *constante* de ecuaciones dinámicas del tipo $E\dot{x} = Ax + Bu$. Esta clase de sistemas se describe en detalle en el Capítulo 2 donde se considera el problema de controlar un conjunto de sistemas lineales con el uso de un modelo implícito rectangular, caracterizado por una cierta estructura común.

Esta familia de sistemas puede ser descrito por un modelo implícito único conocido con el nombre de *sistema global*, Σ_i^g:

$$\begin{aligned} \underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix}}_{\mathbb{E}} \dot{x} &= \underbrace{\begin{bmatrix} A \\ D_i \end{bmatrix}}_{\mathbb{A}_i} x + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\mathbb{B}} u \\ y &= Cx \end{aligned} \tag{1.3}$$

donde $\mathbb{E} : \mathcal{X} \to \underline{\mathcal{X}}_{g_i}$, $\mathbb{A}_i = \mathcal{X} \to \underline{\mathcal{X}}_{g_i}$, $\mathbb{B} : \mathcal{U} \to \underline{\mathcal{X}}_{g_i}$ y $C : \mathcal{X} \to \mathcal{Y}$ son operadores lineales, con $\underline{\mathcal{X}}_{g_i} = \underline{\mathcal{X}} \oplus \underline{\mathcal{X}}_i$. Esta representación global es importante ya que bajo ciertas condiciones (poco restrictivas, principalmente en lo que respecta a la ley de control) es posible controlar esta familia de sistemas con la ayuda de leyes de control de tipo proporcional y derivativa, para obtener una respuesta única en lazo cerrado sin importar el valor de los parámetros de las ecuaciones algebraicas $D_i x = 0$ $(i = 1, ..., n)$.

Como ejemplo ilustrativo se puede mencionar una clase de sistemas particularmente interesantes conocidos como *sistemas escalera* (descritos en el Capitulo 2) que describen los cambios de posición de varios conmutadores. Una aplicación reciente de estos sistemas escalera ha permitido describir, a través de un modelo lineal con conmutadores (con tres componentes lineales) el proceso de crecimiento vegetal descrito por la función (no lineal) logística (muy utilizada en este dominio, ver por ejemplo [19]).

En lo que sigue se va enfatizar sobre la riqueza de estos modelos implícitos de estructura variable y se presentaran algunas contribuciones que muestran como se pueden controlar este tipo de sistemas.

1.1 Résumé en français

Dans ce travail, on considère des descriptions implicites décrites par des modèles du type $E\dot{x}(t) = Ax(t) + Bu(t)$, $y = Cx(t)$. Ces descriptions peuvent être vues comme une extension des systèmes classiques, strictement propres, pour lesquels E est inversible.

Les descriptions implicites permettent de décrire des comportements beaucoup plus nombreux et plus riches que dans le cas classique strictement propre. On peut par exemple mentionner: des systèmes ayant des comportements impulsionnels (liés par exemple à des dérivations), des systèmes contrôlés par des lois de commande proportionnelles et dérivées (P-D), des systèmes avec des contraintes algébriques sur l'état, des systèmes avec des restrictions sur les commandes, etc ...

Dans ce contexte généralisé, on s'intéresse tout particulièrement à des modèles implicites pour lesquels le nombre des variables internes est supérieur au nombre d'équations d'état. Ces *Systèmes Implicites Rectangulaires* ont ceci de particulier que pour une condition initiale donnée, et pour une loi de commande donnée, la solution en termes de trajectoire d'état n'est pas unique, c'est à dire, ils possèdent un degré de liberté interne.

Avec ces modèles, on peut décrire des systèmes à structure variable. C'est le cas notamment pour une famille de n systèmes dont les changements potentiels proviennent d'une équation de contrainte algébrique en superposition avec un ensemble constant d'équations dynamiques. Cette classe de systèmes est décrite en détail dans le Chapitre 2, où l'on s'intéresse au problème de la commande d'un tel ensemble de systèmes linéaires, en utilisant comme intermédiaire le modèle implicite rectangulaire, caractérisé par la structure commune.

Cette famille de systèmes peut être décrite par le modèle implicite englobant l'ensemble des restrictions algébriques, $\begin{bmatrix} E \\ 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} A \\ D_i \end{bmatrix} x(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t)$, $y(t) = Cx(t)$, et désigné par *Système Global.* Cette représentation est très importante puisque sous certaines conditions (assez peu restrictives, principalement de type commandabilité), il est possible de commander cette famille de systèmes à l'aide de lois de commande du type P-D, de manière à obtenir une réponse bouclée unique, à la dynamique imposée, quelle que soit la valeur des paramètres liés à la partie algébrique des équations dans l'ensemble des modèles possibles ainsi décrits.

A titre d'exemple illustratif, une classe particulièrement intéressante dans la famille de tels modèles est celle des *Systèmes en Escalier* (décrites dans le Chapitre 2) qui décrit les changements liés à la position de plusieurs commutateurs. Une application récente de ces Systèmes en Escalier a permis de décrire par un modèle linéaire à commutations, (en utilisant trois systèmes linéaires), le processus de croissance végétale décrit par la fonction (non linéaire) dite fonction logistique (et classique dans ce domaine).

Dans la Section 1.2 nous donnons un exemple illustratif pour décrire plus précisément notre contexte de travail. Dans cet exemple, le système régi par les (mêmes) équations dynamiques : $\dot{x}_1(t) = x_2(t) - x_3(t)$, $\dot{x}_2(t) = x_1(t) - x_3(t) + u(t)$, $y(t) = x_3(t)$, mais contraint par la relation $\begin{bmatrix} \alpha & \beta & 1 \end{bmatrix} x(t) = 0$ peut avoir des comportements très différents : par exemple, avoir un degré relatif non constant (égal à 1 ou à 2), posséder ou pas des zéros finis, avoir un gain statique positif ou négatif, suivant les valeurs des paramètres (α et β) sur un ensemble fini. D'abord, on obtient une réalisation implicite globale du système complet et ensuite, on propose une loi de commande P-D qui rend inobservable sur la sortie le changement de structure provoqué par les variations des paramètres α et β. Le système en boucle fermée se comporte alors comme un système du premier ordre, avec constante de temps fixée, quelles que soient les valeurs des paramètres α et β.

La mise en œuvre de telles lois de commande P-D présuppose que les variables internes et ses dérivées sont disponibles, ce qui n'est généralement pas le cas, et qui oblige à les observer. Malheureusement, il n'existait pas jusqu'à ce jour, de résultats pour construire de tels observateurs de la variable interne pour ce type de systèmes à structure variable.

Pour pallier ce manque, on propose d'utiliser les résultats donnés dans le Chapitre 3 où l'on introduit un détecteur de structure s'appuyant sur un algorithme adaptatif à base de gradient normalisé, et dont la finalité est de déterminer quelle est la structure interne active parmi celles qui ont été décrites dans le modèle. Après cette phase de détection, on peut directement utiliser des schémas classiques d'observation.

Une deuxième difficulté concerne le caractère impropre de la loi de commande (due à l'action dérivée). Pour résoudre ce problème, il est nécessaire de passer par une phase d'approximation. Ceci fait l'objet du Chapitre 4, qui concerne un aspect pratique de la mise en oeuvre de compensateurs impropres, à partir d'une analyse simplifiée de systèmes implicites, à savoir, des approximations par des systèmes propres à grand gain. Ces résultats sont particulièrement bien adaptés dans le contexte de systèmes classiques quand on utilise des lois de commande généralisées (P-D).

Le Chapitre 5 décrit des résultats récents sur une alternative qui consiste à chercher une loi de commande proportionnelle à partir d'une approximation du problème original. Ce développement est particulièrement adapté à la commande des systèmes à structure variable. Il est intéressant de souligner un ingrédient de la solution qui y est proposée : on donne également une interprétation, en termes de la théorie des systèmes, de l'intégration par parties, formulée en termes des changements de bases généralisés.

Dans le chapitre 6, on propose une loi de commande sous-optimale du système global, obtenue comme une combinaison linéaire des n lois de commande optimale de chaque système particulier. Lorsque le *i-eme* système est actif, ce compromis global est évidement moins *bon* que sa commande optimale associée ; mais ceci permet un réglage global plus simple et malgré tout satisfaisant (comme montré en simulation). Ce schèma est opportun quand on ne sait pas exactement quel système est actif.

1.2 Ejemplo ilustrativo

Para ilustrar con más precisión nuestro contexto de trabajo, se va a considerar un sistema descrito por las ecuaciones dinámicas:[1]

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{x} &= \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \end{aligned} \tag{1.4}$$

sujeto a la restricción:

$$\begin{bmatrix} \alpha & \beta & 1 \end{bmatrix} x = 0 \tag{1.5}$$

Según los valores de los parámetros α y β, este sistema puede tener comportamientos muy diferentes:

- Si $(\alpha, \beta) = (-1, -1)$, i.e., $x_3 = x_1 + x_2$, se tiene:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \end{aligned} \tag{1.6}$$

  con la descripción entrada-salida $\dot{y}(t) + y(t) = u(t)$.

- Si $(\alpha, \beta) = (-1, 0)$, i.e., $x_3 = x_1$, se tiene:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \end{aligned} \tag{1.7}$$

  con la descripción entrada-salida $\ddot{y}(t) + \dot{y}(t) = u(t)$.

- Si $(\alpha, \beta) = (-1, -5)$, i.e., $x_3 = x_1 + 5x_2$, se tiene:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & -4 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \end{aligned} \tag{1.8}$$

  con la descripción entrada-salida $\ddot{y} + 6\dot{y} + 5y = 5\dot{u} + u$.

- Finalmente, si $(\alpha, \beta) = (1, 1)$, i.e., $x_3 = -(x_1 + x_2)$, se tiene:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \end{aligned} \tag{1.9}$$

  con la descripción entrada-salida $\dot{y} - 3y = -u$.

[1] Este ejemplo se trata con más detalle en el Capítulo 2.

Una realización implícita global de (1.4) y (1.5) está dada por la siguiente expresión:

$$\begin{aligned}\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 1 & 0\\ \hline 0 & 0 & 0\end{array}\right]\dot{x} &= \left[\begin{array}{ccc}0 & 1 & -1\\ 1 & 0 & -1\\ \hline \alpha & \beta & 1\end{array}\right]x + \left[\begin{array}{c}0\\ 1\\ \hline 0\end{array}\right]u\\ y &= \left[\begin{array}{ccc}0 & 0 & 1\end{array}\right]x\end{aligned}$$

Aplicando el procedimiento propuesto en el Capítulo 2, se obtiene una ley de control proporcional derivativa que hace inobservable sobre la salida el cambio de estructura provocado por las variaciones de los parámetros (α, β): $F^*_{d_1} = \left[\begin{array}{ccc}0 & 1 & -1\end{array}\right]$ y $F^*_{p_1} = \left[\begin{array}{ccc}-1 & 0 & 0\end{array}\right]$. Enseguida se pueden colocar los polos de la dinámica externa tomando una segunda retroalimentación proporcional: $F^*_{p_2} = \left[\begin{array}{ccc}0 & 0 & (1-1/\tau_o)\end{array}\right]$. De esta manera, la ley de control $u = F^*_{d_1}\dot{x} + \left(F^*_{p_1} + F^*_{p_2}\right)x + R/\tau_o$ conduce al sistema en lazo cerrado:

$$\begin{aligned}\left[\begin{array}{ccc}1 & 0 & 0\\ 0 & 0 & 1\end{array}\right]\dot{x} &= \left[\begin{array}{ccc}0 & 1 & -1\\ 0 & 0 & -1/\tau_o\end{array}\right]x + \left[\begin{array}{c}0\\ 1/\tau_o\end{array}\right]R\\ y &= \left[\begin{array}{ccc}0 & 0 & 1\end{array}\right]x\end{aligned}$$

El grado de libertad interno (debido a los cambios posibles de los valores de los parámetros) se ha hecho inobservable. En otros términos, la *variación interna de estructura* ya no está presente sobre la salida. El sistema en lazo cerrado se comporta como el sistema de primer orden $\tau_o\dot{y}+y = R$, sin importar la restricción activa $0 = \left[\begin{array}{ccc}\alpha & \beta & 1\end{array}\right]x$, i.e., sin importar los valores de los parámetros α y β.

La implementación de estas leyes de control proporcional derivativa presuponen que las variables internas x y su derivada $\dot{x}$ están disponibles, lo que no es generalmente el caso, por lo que hay que observarla. A la fecha no se han desarrollado observadores de la variable descriptora para este tipo de sistemas con estructura variable.

Para solucionar este problema se pueden utilizar los resultados presentados en el Capitulo 3 donde se presenta un *detector de estructura* que se basa en un algoritmo adaptable de gradiente normalizado, cuya finalidad es determinar cual es la estructura *activa* entre las que han sido descritas en el modelo.[2] Después se pueden utilizar esquemas clásicos de observación.

Una segunda dificultad concierne al carácter impropio de la ley de control (acción derivativa). Para resolver este problema es necesario pasar por una fase de aproximación. Ese es el tema del Capítulo 4 que concierne a un aspecto práctico sobre la implementación de compensadores impropios deducidos de un análisis simplificado de sistemas implícitos, a saber, aproximaciones por sistemas propios de gran ganancia. Estos resultados son particularmente bien adaptados en el contexto de sistemas clásicos cuando se utilizan leyes de control generalizadas (proporcionales y derivativas).

El Capítulo 5 presenta los resultados recientes sobre una alternativa consistente en buscar una ley de control proporcional aproximada del problema original. Este desarrollo nos parece particularmente adaptado al control de sistemas de estructura variable. También se enfatiza sobre un ingrediente de la solución que nos parece interesante: una *interpretación en término de sistemas* del procedimiento matemático de *integración por partes* formulada en términos de cambios de base generalizados (considerando derivadas de las variables).

[2] Sobre el ejemplo precedente, precisar si el comportamiento es de primer orden, de segundo orden o de segundo orden con un cero dominante.

En el Capítulo 6 se obtiene una ley de control *sub-óptima*, u, del sistema global (1.3), como una combinación lineal de los controles óptimos, u_i^* $(i = 1, ..., n)$, de cada sistema, que es, obviamente, menos óptima, cuando el sistema i está activo que su control óptimo asociado, pero que conduce a un compromiso global satisfactorio. Este esquema es conveniente cuando no se sabe con suficiente precisión que sistema está activo (artículo en proceso de elaboración).

La presentación de este trabajo termina con algunas perspectivas que prosiguen a los resultados obtenidos.

1.3 Subespacios y Algoritmos

Relacionado con el sistema implícito(1.2), están los siguientes subespacios (ver [43, Verghese 1981], [41, Ozcaldiran 1986] y [38, Malabre 1987]):

- El subespacio supremo (A, E, B) invariante contenido en el subespacio $\mathcal{T}$, $\mathcal{V}^*_{\mathcal{T},\Sigma} := \sup\{\mathcal{V} \subset \mathcal{T} | A\mathcal{V} \subset E\mathcal{V} + \mathcal{B}\}$, es el limite del algoritmo:[3]

$$\mathcal{V}^0_{\mathcal{T},\Sigma} = A^{-1} \operatorname{Im} A; \quad \mathcal{V}^{\mu+1}_{\mathcal{T},\Sigma} = \mathcal{T} \cap A^{-1}(E\mathcal{V}^{\mu}_{\mathcal{T},\Sigma} + \mathcal{B}), \qquad \mu \geq 0. \tag{1.10}$$

- El subespacio supremo (A, E, B) invariante contenido en el subespacio $\mathcal{K}_C$, $\mathcal{V}^*_{\mathcal{K}_C,\Sigma}$, se obtiene de (1.10) y generalmente se representa con $\mathcal{V}^*$ (para simplificar la notación), es decir:

$$\mathcal{V}^0 = \mathcal{X}; \quad \mathcal{V}^{\mu+1} = \mathcal{K}_C \cap A^{-1}(E\mathcal{V}^{\mu} + \mathcal{B}), \qquad \mu \geq 0. \tag{1.11}$$

- El subespacio $\mathcal{V}^*_{\mathcal{X}}$ caracteriza (junto con $E\mathcal{V}^*_{\mathcal{X}} + \mathcal{B}$) el conjunto de todas las posibles trayectorias que no son idénticamente cero para cualquier entrada u. Este subespacio es el limite del algoritmo:

$$\mathcal{V}^o_{\mathcal{X}} = \mathcal{X} \ ; \quad \mathcal{V}^{\mu+1}_{\mathcal{X}} = A^{-1}\left(E\mathcal{V}^{\mu}_{\mathcal{X}} + \mathcal{B}\right). \tag{1.12}$$

- El subespacio $\mathcal{V}^*_o$ caracteriza (junto con $E\mathcal{V}^*_o$) el conjunto de todas las trayectorias exponenciales que no son observables en la salida y. Este subespacio es el limite del algoritmo:

$$\mathcal{V}^o_o = \mathcal{X} \ ; \quad \mathcal{V}^{\mu+1}_o = \mathcal{K}_C \cap A^{-1}E\mathcal{V}^{\mu}_o.$$

- El subespacio $\mathcal{V}^*_{\mathcal{X}_o}$ caracteriza (junto con $E\mathcal{V}^*_{\mathcal{X}_o}$) las trayectorias exponenciales y $\mathcal{S}^*_{\mathcal{X}_o}$ caracteriza (junto con $A\mathcal{S}^*_{\mathcal{X}_o}$) el conjunto de todas las trayectorias debido a acciones diferenciales puras (ver [48, Wong 1974] y [2, Armentano 1986]); note que $\mathcal{S}^*_{\mathcal{X}_o} \supset \mathcal{R}^*_{a0}$.

- El subespacio supremo (E, A, B) de alcanzabilidad contenido en el subespacio $\mathcal{T}$, $\mathcal{R}^*_{\mathcal{T},\Sigma}$ es el límite del algoritmo:

$$\mathcal{R}^0_{\mathcal{T},\Sigma} = \mathcal{V}^*_{\mathcal{T},\Sigma} \cap \mathcal{K}_E; \quad \mathcal{R}^{\mu+1}_{\mathcal{T},\Sigma} = \mathcal{V}^*_{\mathcal{T},\Sigma} \cap E^{-1}(A\mathcal{R}^{\mu}_{\mathcal{T},\Sigma} + \mathcal{B}). \tag{1.13}$$

[3] Se escribe $A^{-1} \operatorname{Im} A$ en lugar del espacio vectorial del dominio puesto que se trabaja con dominios de definición diferentes. Si $A : \mathcal{V} \to \underline{\mathcal{V}}$ entonces $A^{-1} \operatorname{Im} A = A^{-1}A\mathcal{V} = \mathcal{V} + \mathcal{K}_A = \mathcal{V}$.

- El subespacio $\mathcal{R}^*_{\mathcal{X}}$ se obtiene de (1.13) con $\mathcal{T} = \mathcal{X}$, es decir:

$$\mathcal{R}^o_{\mathcal{X}} = \mathcal{V}^*_{\mathcal{X}} \cap \mathcal{K}_E; \quad \mathcal{R}^i_{\mathcal{X}} = \mathcal{V}^*_{\mathcal{X}} \cap E^{-1}\left(\mathcal{B} + A\mathcal{R}^{i-1}_{\mathcal{X}}\right), \; i \geq 1, \tag{1.14}$$

- El subespacio supremo (A, E) de semi-controlabilidad contenido en $\mathcal{K}_C$, $\mathcal{R}_{a0} := \inf\{\mathcal{R} \subset \mathcal{K}_C | \mathcal{R} = \mathcal{K}_C \cap E^{-1}(A\mathcal{R})\}$, caracteriza (junto con $A\mathcal{R}_{ao}$) el conjunto de todas las trayectorias debido a acciones diferenciales que no tienen influencia en las trayectorias entrada-salida. Este conjunto se conoce como variables descriptoras diferencialmente redundantes. Este subespacio es el límite del algoritmo:

$$\mathcal{R}^0_{ao} = \mathcal{K}_C \cap \mathcal{K}_E; \quad \mathcal{R}^{\mu+1}_{ao} = \mathcal{K}_C \cap E^{-1}(A\mathcal{R}^{\mu}_{ao}) \quad \text{para } \mu \geq 0 \tag{1.15}$$

- El subespacio ínfimo (A, E, B) invariante que contiene a $\mathcal{B}$, $\mathcal{S}^*_{\mathcal{T},\Sigma} := \inf\{\mathcal{S} \subset \mathcal{X} | \mathcal{S} = E^{-1}(A(\mathcal{T} \cap \mathcal{S}) + \mathcal{B})\}$, es el limite del algoritmo:

$$\mathcal{S}^0_{\mathcal{T},\Sigma} = \mathcal{K}_E; \quad \mathcal{S}^{\mu+1}_{\mathcal{T},\Sigma} = E^{-1}(A(\mathcal{T} \cap \mathcal{S}^{\mu}_{\mathcal{T},\Sigma}) + \mathcal{B}), \qquad \mu \geq 0. \tag{1.16}$$

Sea $F(\mathcal{V}^*_{\Sigma})$ el conjunto de todas las retroalimentaciones proporcional y derivativa, (F_p, F_d), tal que $(A + BF_p)\mathcal{V}^*_{\Sigma} \subset (E - BF_d)\mathcal{V}^*_{\Sigma}$. Dicha (F_p, F_d) se conoce como un par amigo de $\mathcal{V}^*_{\Sigma}$.

Ahora veamos el siguiente resultado (relacionado con $\mathcal{V}^*_{\mathcal{T},\Sigma}$).

Hecho 1 ***i)*** *Para el sistema en lazo cerrado $\Sigma_F(E - BF_d, A + BF_p, B, C)$ y para cualquier par de retroalimentaciones proporcional y derivativa, (F_p, F_d), siempre se cumple que $\mathcal{V}^*_{\tau,\Sigma} = \mathcal{V}^*_{\tau,\Sigma_F}$; entonces, se escribe $\mathcal{V}^*$ para representar $\mathcal{V}^*_{\Sigma}$ o $\mathcal{V}^*_{\Sigma_F}$*
ii) *Para cualquier retroalimentación derivativa, F_d, existe una retroalimentación proporcional, F^*_p tal que $(A + BF^*_p)\mathcal{V}^*_{\tau,\Sigma} \subset (E - BF^*_d)\mathcal{V}^*_{\tau,\Sigma}$, es decir, $(F^*_p, F^*_d) \in F(V^*_{\Sigma})$*

Los siguientes algoritmos están asociados con el pencil $\Psi = [\lambda\mathbb{F} - \mathbb{G}]$ (ver por ejemplo [39, Malabre] y las referencias mencionadas en dicho artículo) con $\mathbb{F}, \mathbb{G} : \mathcal{V} \rightarrow \underline{\mathcal{V}}$:

$$\begin{cases} \mathcal{A}^0_{1,\Psi} = \{0\}; \quad \mathcal{A}^{\mu+1}_{1,\Psi} = \mathbb{F}^{-1}\mathbb{G}\mathcal{A}^{\mu}_{1,\Psi} \quad \text{que converge a } \mathcal{A}^*_{1,\Psi} \\ \mathcal{A}^0_{2,\Psi} = \mathcal{V}; \quad \mathcal{A}^{\mu+1}_{2,\Psi} = \mathbb{G}^{-1}\mathbb{F}\mathcal{A}^{\mu}_{2,\Psi} \quad \text{que converge a } \mathcal{A}^*_{2,\Psi} \\ \mathcal{B}^0_{1,\Psi} = \underline{\mathcal{V}}; \quad \mathcal{B}^{\mu+1}_{1,\Psi} = \mathbb{F}\mathbb{G}^{-1}\mathcal{B}^{\mu}_{1,\Psi} \quad \text{que converge a } \mathcal{B}^*_{1,\Psi} \\ \mathcal{B}^0_{2,\Psi} = \{0\}; \quad \mathcal{B}^{\mu+1}_{2,\Psi} = \mathbb{G}\mathbb{F}^{-1}\mathcal{B}^{\mu}_{2,\Psi} \quad \text{que converge a } \mathcal{B}^*_{2,\Psi} \end{cases} \tag{1.17}$$

Capítulo 2

Sistemas Implícitos Rectangulares

Los sistemas implícitos rectangulares se pueden utilizar para modelar una clase muy amplia de sistemas lineales, incluyendo sistemas con interruptores internos. Existen condiciones necesarias y suficientes (expresados en términos de un modelo implícito global) para controlarlos de tal manera que presenten un comportamiento único, sin importar las variaciones de estructura interna.

En este capítulo se muestra como considerar la variación de estructura interna, presente en las descripciones implícitas rectangulares, dentro de un subespacio (A, E, B) invariante contenido en el kernel del mapa de salida. Gracias a esto, se hace inobservable la variación de estructura interna, obteniendo de esta manera un sistema propio en lazo cerrado con una estructura pre-establecida controlable.

2.1 Résumé en français

Les systèmes implicites rectangulaires peuvent être utilisés pour modéliser une classe très large de systèmes linéaires, et notamment des systèmes avec des interrupteurs internes. Pour de tels systèmes, des conditions nécessaires et suffisantes, (exprimées sur un système implicite global), garantissent l'existence d'une commande assurant un comportement unique à la sortie, quelles que soient les variations de structure interne.

Dans ce chapitre, on montre comment rendre inobservable sur la sortie, la variation de structure interne, présente dans les descriptions implicites rectangulaires. On obtient de cette manière un système propre en boucle fermé, avec une structure pré-établie commandable.

Dans la section 2.2, on étudie le problème (Problème 1) qui consiste à commander un ensemble de systèmes linéaires par un retour d'état P-D, tout en assignant à chacun d'entre eux le même comportement externe en boucle fermée. Dans la section 2.3, on donne un bref descriptif de certains outils existants pour les systèmes implicites et qui seront utilisés dans le développement de ce travail. Ceci englobe notamment des thèmes tels que propriétés structurelles, atteignabilité, rejet de la structure variable, et atteignabilité du système en boucle fermée. On montre sur un exemple (Exemple 3) une illustration de ces résultats. Cet exemple est le même que celui montré dans la section 1.2 du Chapitre 1, mais plus détaillé.

Dans la Section 2.4, sont introduits les *Systèmes en Escalier*, dont le comportement entré-sortie peut varier depuis un système du premier ordre jusqu'à un système du $n^{ième}$ ordre (en fonction de la position d'interrupteurs internes) ; la description implicite rectangulaire globale est unique. Quand certains conditions sont satisfaites, il est possible de commander le système global

de telle façon que son comportement externe soit décrit par un modèle fixe pré établi. Cette présentation est principalement motivée par la modélisation du procédé de croissance végétale dans des systèmes agronomiques.

On considère dans la sous-section 2.4.1 un système issu de commutations sur un ensemble de n systèmes linéaires, le tout décrit par un système implicite global ; puis, dans les sous-sections 2.4.2, 2.4.3 et 2.4.4 des cas particuliers sont traités (voir Figures 2.1a, 2.1b, 2.1c) pour lesquels (voir tables 2.1, 2.2 et 2.3) les modes inobservables sont bien stables.

2.2 Introducción

Considere la descripción implícita $\Sigma(E, A, B, C)$ dada en (1.2). En [8, Bonilla y Malabre 1991] se mostró que cuando $\dim \underline{\mathcal{X}} \leq \dim \mathcal{X}$, es posible describir sistemas lineales con una estructura interna variable. En efecto, cuando $\dim \underline{\mathcal{X}} < \dim \mathcal{X}$ y si el sistema tiene solución (i.e., posee al menos una solución), las soluciones generalmente no son únicas. En algún sentido existe un grado de libertad en (1.2) que se puede usar, por ejemplo, para tomarlo en cuenta como una posible *variación de estructura* de una manera implícita.

Problema 1 ([16, Bonilla y Malabre 2002], [18, Bonilla y Malabre 2003]) *Considere un conjunto de sistemas lineales estrictamente propios, con entrada $u(t)$ y salida $y(t)$:*

$$F_i(p)y(t) = G_i(p)u(t). \tag{2.1}$$

Si este conjunto de sistemas se puede incluir en el siguiente modelo de sistemas implícitos (sistema global), Σ_i^g:

$$\begin{aligned} \begin{bmatrix} E \\ 0 \end{bmatrix} \dot{x} &= \begin{bmatrix} A \\ D_i \end{bmatrix} x + \begin{bmatrix} B \\ 0 \end{bmatrix} u; \\ y &= Cx \end{aligned} \tag{2.2}$$

para $i = 1, ..., n$, donde $E : \mathcal{X} \to \underline{\mathcal{X}}$, $A : \mathcal{X} \to \underline{\mathcal{X}}$, $B : \mathcal{U} \to \underline{\mathcal{X}}$, $D_i : \mathcal{X} \to \underline{\mathcal{X}}_i$ y $C : \mathcal{X} \to \mathcal{Y}$ son operadores lineales de dimensiones apropiadas; $\underline{\mathcal{X}}_g = \underline{\mathcal{X}} \oplus \underline{\mathcal{X}}_i$ $(i = 1, ..., n)$; E y D_i $(i = 1, ..., n)$ son mapas épicos, i.e., $\mathcal{E} = \underline{\mathcal{X}}$ y $\operatorname{Im} D_i = \underline{\mathcal{X}}_i$.

Pregunta: *¿Bajo qué condiciones se puede controlar este conjunto de sistemas lineales por una retroalimentación de estado proporcional derivativa fija, $u(t) = F_p x(t) + F_d \dot{x}(t)$, asignando a todos ellos el mismo comportamiento externo en lazo cerrado, cuya síntesis se base en la estructura interna común, descrita por el sistema implícito rectangular $E\dot{x}(t) = Ax(t) + Bu(t)$?*

Este problema fue considerado de manera muy general en [6, Bonilla et al. 1994]. En dicho artículo, el problema fue tratado utilizando la generalización de la forma canónica de Morse ([40, Morse 1973]) conocida con el nombre de *forma canónica para sistemas descriptores* introducida en [34, Lebret y Loiseau 1994], la cual, en algunas ocasiones, no es fácil de obtener.

En este capítulo se estudia este problema y se muestra cual es la estructura interna común del conjunto de sistemas lineales descrito por el sistema global (2.2) que resuelve el Problema 1. De esta manera es posible saber como considerar un conjunto de sistemas lineales en un sistema implícito de la forma (1.2) con la propiedad de asignamiento dinámico de salida. Algo muy importante es el procedimiento presentado (muy simple) para sintetizar retroalimentaciones proporcional derivativa. Para esto, se estudian las consecuencias estructurales de las condiciones geométricas presentadas en [6, Bonilla et al. 1994] cuando se aplican para resolver el Problema 1. Gracias a esto, se presenta un procedimiento para sintetizar retroalimentaciones proporcional derivativa que hacen inobservable la variación de la estructura y asignan la dinámica de salida en lazo cerrado.

2.3 Propiedades Estructurales Básicas

Definición 1 ([27, Gantmacher]) *Un pencil* $[\lambda\mathbb{F}-\mathbb{G}]$ *es regular si es cuadrado y además* $\det(\lambda\mathbb{F}-\mathbb{G}) \neq 0$.

Hecho 2 ([3, Bernhard], [2, Armentano]) *Un pencil* $\Psi = [\lambda\mathbb{F} - \mathbb{G}]$ *es regular si y solamente si* $\mathbb{F}^{-1}\operatorname{Im}\mathbb{F} = \mathcal{A}^*_{1,\Psi} \oplus \mathcal{A}^*_{2,\Psi}$.

Definición 2 ([3, Bernhard], [2, Armentano]) *El sistema* $\mathbb{F}\dot{x} = \mathbb{G}x + v$ *es internamente propio si y solamente si el pencil* $[\lambda\mathbb{F} - \mathbb{G}]$ *es regular y no tiene ceros infinitos de orden mayor que uno (no existen derivadores).*

Uno de los conceptos más estudiados en la teoría de sistemas es el relacionado con la alcanzabilidad, debido a que caracteriza, de cierta manera, todos los *estados*[1] que pueden ser controlados. Más precisamente, el concepto de alcanzabilidad está generalmente asociado con el *conjunto de vectores del espacio de estado, el cual puede ser alcanzable desde el origen en un tiempo finito, siguiendo trayectorias (soluciones del sistema) generadas por una entrada externa.*

Ozcaldiran [41] extendió la caracterización geométrica de Wonham de los subespacios de alcanzabilidad ([47, Wonham 1985]) de los sistemas clásicos (1.1) donde el pencil $[\lambda E - A]$ es regular y $E = I$ para obtener el subespacio supremo (E, A, B) de alcanzabilidad $\mathcal{R}^*_{\mathcal{T},\Sigma}$.

Ozcaldiran [41] también mostró que para cualquier $\mu \geq 0$ se tiene que:

$$\mathcal{R}^{\mu}_{\mathcal{T},\Sigma} = \mathcal{V}^*_{\mathcal{T},\Sigma} \cap \mathcal{S}^{\mu}_{\mathcal{T},\Sigma}.$$

En el caso general (cuando las matrices E y A no son necesariamente cuadradas), Frankowska [26, Frankowska 1990] extendió la caracterización geométrica del subespacio alcanzable, $\mathcal{R}^*_{\mathcal{X},\Sigma}$, usando técnicas de inclusión diferencial.

Teorema 1 ([26, Frankowska 1990]) *Considere el sistema implícito (1.2) y suponga que se satisface la condición:*

$$\operatorname{Im} A \subset \mathcal{E} + \mathcal{B}. \tag{2.3}$$

El sistema es alcanzable, es decir, para cada x_0, $x_1 \in \mathcal{X}$ *y* $T > 0$ *existe una ley de control* $u : [0, T] \to \mathcal{U}$ *y una trayectoria* $x : [0, T] \to \mathcal{X}$ *de clase* C^1 *tal que* $x(0) = x_0$ *y* $x = x_1$, *si y solamente si* $\mathcal{R}^*_{\mathcal{T},\Sigma} = \mathcal{X}$.

[1] Cuando se trabaja con sistemas implícitos se habla de la variable descriptora en lugar del *estado*.

Cabe mencionar que cuando las matrices A y E no son cuadradas, se pueden tener sistemas alcanzables, aún en la ausencia de entradas; esto es posible debido a la existencia de variables descriptoras libres (grado de libertad) que actúan como controles internos. En efecto, para el sistema autónomo:

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \end{bmatrix} x + [0]\, u$$

se tiene que $\mathcal{R}^*_{\mathcal{I},\Sigma} = \mathcal{X}$ (ver las expresiones 1.10, 1.16, 2.3).

Para evitar tales patologías se introdujo en [31, Lebret] una segunda condición que garantiza alcanzabilidad por medio de una entrada de control externa.

Teorema 2 ([6, Bonilla et al. 1994]) *El sistema implícito (1.2) es alcanzable con asignamiento dinámico de salida, i.e., es alcanzable y la dinámica de salida, **y**, puede asignarse de manera arbitraria por medio de la ley de control $u(t) = F_d\dot{x}(t) + F_p x(t)$, si y solamente si*

$$\mathcal{R}^*_{\mathcal{X}} = \mathcal{X}, \quad y \tag{2.4}$$

$$\dim\left(\mathcal{V}^*_{\mathcal{X}} \cap \mathcal{K}_E\right) - \dim\left(\frac{\mathcal{B}}{\mathcal{B}\cap\mathcal{E}}\right) \leq \dim \mathcal{V}^* \cap E^{-1}\mathcal{B} \tag{2.5}$$

donde $\mathcal{R}^*_{\mathcal{X}}$, $\mathcal{V}^*_{\mathcal{X}}$ y $\mathcal{V}^*$ son respectivamente los límites de los algoritmos (1.14), (1.12) y (1.11).

Definición 3 ([9, Bonilla y Malabre 1991]) *Dada una descripción implícita, $\Sigma(Y,X,Z,W)$:*

$$\begin{aligned} Y\dot{x} &= Xx + Zu, \\ y &= Wx, \end{aligned}$$

donde **u**, **y**, **x** *son la entrada, la salida y la variable descriptora, respectivamente; $Y : \mathcal{X} \to \underline{\mathcal{X}}_g$, $X : \mathcal{X} \to \underline{\mathcal{X}}_g$, $Z : \mathcal{U} \to \underline{\mathcal{X}}_g$ y $W : \mathcal{X} \to \mathcal{Y}$ son mapas lineales de dimensiones apropiadas. Se le llama* ***redundancia algebraica*** *a cualquier variable descriptora $\xi(t) \in \mathcal{X}$ la cual (para toda t) es una combinación lineal constante de algunas otras variables descriptoras y se puede eliminar sin modificar el comportamiento externo, i.e., el conjunto de todas las posibles trayectorias de entrada-salida (u,y) del sistema $\Sigma(Y,X,Z,W)$, permanecen inalteradas (ver [45, Willems 1983]).*

Proposición 1 ([10, Bonilla y Malabre 1995]) *Dada la descripción implícita (1.2), se puede restringir el sistema a $\mathcal{V}^*_{\mathcal{X}}$ en el dominio y a $E\mathcal{V}^*_{\mathcal{X}} + \mathcal{B}$ en el codominio sin modificar el comportamiento externo (es decir, el conjunto de todas las posibles trayectorias de entrada-salida, (u,y), ver [45, Willems]). Además el sistema inducido, $\hat{\Sigma}(\hat{E},\hat{A})$:*

$$\hat{E}\,\dot{\hat{\xi}} = \hat{A}\hat{\xi},$$

con $\hat{E} : \mathcal{X}/\mathcal{V}^*_{\mathcal{X}} \to \underline{\mathcal{X}}/(E\mathcal{V}^*_{\mathcal{X}} + \mathcal{B})$ y $\hat{A} : \underline{\mathcal{X}}/\mathcal{V}^*_{\mathcal{X}} \to \underline{\mathcal{X}}/(E\mathcal{V}^*_{\mathcal{X}} + \mathcal{B})$ representa la parte más grande del sistema que caracteriza las relaciones algebraicas, las cuales pueden reducirse por simple manipulaciones algebraicas. La parte del sistema restringido a $\mathcal{V}^*_{\mathcal{X}}$ y $E\mathcal{V}^*_{\mathcal{X}} + \mathcal{B}$ se le llama parte algebraica no redundante; la variable descriptora, $\hat{\xi}$, del sistema inducido, $\hat{\Sigma}(\hat{E},\hat{A})$, se le llama variable descriptora algebraica redundante.

Proposición 2 ([10, Bonilla y Malabre 1995]) *Dada una descripción implícita* $\Sigma(E, A, B, C)$, $\mathcal{R}_{ao}$ *caracteriza las variables descriptoras diferencialmente redundantes. El sistema* $\widehat{\Sigma}(\hat{E}, \hat{A}, \hat{B}, \hat{C})$:

$$\begin{aligned} \hat{E}\,\dot{\widehat{x}} &= \hat{A}\widehat{x} + \hat{B}u; \\ y &= \hat{C}\widehat{x} \end{aligned}$$

con $\hat{E} : \mathcal{X}/\mathcal{R}^*_{ao} \rightarrow \underline{\mathcal{X}}/A\mathcal{R}^*_{ao}$, $\hat{A} : \mathcal{X}/\mathcal{R}^*_{ao} \rightarrow \underline{\mathcal{X}}/A\mathcal{R}^*_{ao}$, $\hat{B} : \mathcal{U} \rightarrow \underline{\mathcal{X}}/A\mathcal{R}^*_{ao}$ *y* $\hat{C} : \mathcal{X}/\mathcal{R}^*_{ao} \rightarrow \mathcal{Y}$, *no tiene variables descriptoras diferencialmente redundantes. Por lo tanto, los sistemas* $\Sigma(E, A, B, C)$ *y* $\widehat{\Sigma}(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ *son externamente equivalentes.*

Corolario 1 *Una descripción implícita* $\Sigma(E, A, B, C)$ *es externamente propia si su sistema inducido* $\widehat{\Sigma}(\hat{E}, \hat{A}, \hat{B}, \hat{C})$ *(definido en la Proposición 2) es internamente propio.*

Proposición 3 ([31, Lebret 1991], [33, Lewis 1992]) *El sistema implícito,* $\Sigma(E, A, B, C)$ *dado por la expresión (1.2) admite al menos una solución en la salida,* $\mathbf{y}$, *(en* c^∞*) para toda entrada,* $\mathbf{u}$, *(en* C^∞*) si y solamente si:*

$$\mathcal{B} \subset \mathcal{B}^*_{1,\Psi} + \mathcal{B}^*_{2,\Psi} \Leftrightarrow \mathcal{B} \subset \operatorname{Im}\Psi \tag{2.6}$$

donde Ψ *es el pencil* $[\lambda E - A]$. *En este caso, se dice que el sistema acepta todas las entradas. La solución,* $\mathbf{y}$, *es única si y solamente si:*

$$\mathcal{A}^*_{1,\Psi} \cap \mathcal{A}^*_{2,\Psi} \subset \mathcal{K}_C \Leftrightarrow \mathcal{K}_\Psi \subset \mathcal{K}_C \tag{2.7}$$

Ahora se presentarán las propiedades estructurales para las descripciones implícitas globales, Σ^g_i, dada por la expresión (2.2). Para reducir la complejidad en la notación, se escribirá D en lugar de D_i para $i = 1, 2, ..., n$ fija.

Considere, como en (2.2), la siguiente descripción implícita global, $\Sigma^g\left(\mathbb{E}, \mathbb{A}, \mathbb{B}, C\right)$:

$$\begin{aligned} \mathbb{E}\dot{x}(t) &= \mathbb{A}x(t) + \mathbb{B}u(t); \\ y &= Cx(t) \end{aligned} \tag{2.8}$$

con:

$$\mathbb{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}; \quad \mathbb{A} = \begin{bmatrix} A \\ D \end{bmatrix}; \quad \mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

y donde $x(t)$, $u(t)$, $y(t)$ son, respectivamente, la variable descriptora, la entrada y la salida; $\mathbb{E} : \mathcal{X} \rightarrow \underline{\mathcal{X}}_g$, $\mathbb{A} : \mathcal{X} \rightarrow \underline{\mathcal{X}}_g$, $\mathbb{B} : \mathcal{U} \rightarrow \underline{\mathcal{X}}_g$, y $C : \mathcal{X} \rightarrow \mathcal{Y}$ son operadores lineales, tal que:

$$\operatorname{Im} A + \mathcal{B} \subset \mathcal{E}; \quad \mathcal{E} \oplus \operatorname{Im} D = \underline{\mathcal{X}}_g \tag{2.9}$$

La parte dinámica del sistema global, Σ^g, que se activa por cualquier elección de D_i, se denotará por $\Sigma(E, A, B, C)$, tal como en la expresión (1.2).

Proposición 4 ([16, Bonilla y Malabre 2002], [18, Bonilla y Malabre 2003]) *La descripción implícita global (2.8) es internamente propia si y solamente si:*

$$\mathcal{K}_D \oplus \mathcal{K}_E = \mathcal{X} \tag{2.10}$$

En base a la prueba de la Proposición 4, se definen:

$$\underline{P} : \underline{\mathcal{X}}_g \to \mathcal{E}, \quad \text{la proyección natural a lo largo de } \operatorname{Im} D \tag{2.11}$$

$$V : \mathcal{K}_D \to \mathcal{X}, \text{ y } \underline{V} : \mathcal{E} \to \underline{\mathcal{X}}_g, \quad \text{mapas de inserción.} \tag{2.12}$$

Entonces, los mapas (E, A, B) y $(\mathbb{E}, \mathbb{A}, \mathbb{B})$ están relacionados por:

$$\underline{P}\mathbb{E} = E; \quad \underline{P}\mathbb{A} = A; \quad \underline{P}\mathbb{B} = B.$$

Además, existen mapas únicos $\left(\overline{E}, \overline{A}, \overline{B}, \overline{C}\right)$ tal que:

$$\mathbb{E}V = \underline{V}\overline{E}; \quad \mathbb{A}V = \underline{V}\overline{A}; \quad \mathbb{B} = \underline{V}\overline{B}; \quad CV = \overline{C} \tag{2.13}$$

Proposición 5 ([16, Bonilla y Malabre 2002], [18, Bonilla y Malabre 2003]) *El sistema global (2.8), $\Sigma^g(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$, restringido a $\mathcal{K}_D$ en el dominio y $E\mathcal{K}_D$ en el codominio, no contiene partes algebraicas redundantes. Además, el grado de libertad, caracterizado por la variable descriptora algebraica redundante, esta localizada en $\mathcal{K}_E$. El sistema global (2.8), es externamente equivalente a la descripción de espacio de estado reducida, $\Sigma^s : \left(I, A_o, B_o, \overline{C}\right)$:*

$$\begin{aligned} \dot{\bar{x}} &= A_o\bar{x} + B_o u; \\ y &= \overline{C}\bar{x} \end{aligned} \tag{2.14}$$

con:

$$A_o = \bar{E}^{(-1)}\bar{A} \quad y \quad B_o = \bar{E}^{(-1)}\bar{B}$$

$$\begin{gathered} \bar{E} = \underline{P}\mathbb{E}V = EV; \quad \bar{A} = \underline{P}\mathbb{A}V = AV \\ \bar{B} = \underline{P}\mathbb{B} = B; \quad \bar{C} = CV \end{gathered} \tag{2.15}$$

Tomando en cuenta (2.9a) en la Proposición (3) se obtiene el siguiente Lema.

Lema 1 ([16, Bonilla y Malabre 2002], [18, Bonilla y Malabre 2003]) *Las siguientes aseveraciones son verdaderas:*

1. *Si la condición geométrica (2.9a) se cumple, entonces el sistema implícito (1.2), $\Sigma(E, A, B, C)$, admite al menos una solución en la salida $\mathbf{y}$ (en C^∞) para todas las entradas $\mathbf{u}$ (en C^∞).*

2. *Si la condición geométrica (2.9b) se satisface, entonces la descripción implícita global (2.8), $\Sigma^g(\mathbb{E}, \mathbb{H}, \mathbb{B}, C)$, admite al menos una solución en la salida $\mathbf{y}$ (en C^∞) para todas las entradas $\mathbf{u}$ (en C^∞).*

3. *Si además de la condición geométrica (2.9a), se toma una retroalimentación proporcional derivativa, $u = F_d\dot{x} + F_p x + v$, tal que:*

$$\operatorname{Im}(E - BF_d) = \mathcal{E} \tag{2.16}$$

entonces el sistema en lazo cerrado, $\Sigma^i_F(E - BF_d, A + BF_p, B, C)$, admite al menos una solución en la salida $\mathbf{y}$ (en C^∞) para todas las entradas $\mathbf{u}$ (en C^∞).

Teorema 3 ([16, Bonilla y Malabre 2002], [18, Bonilla y Malabre 2003]) *Dado el sistema implícito (1.2) satisfaciendo la condición geométrica (2.9a), se toma una retroalimentación proporcional derivativa,* $u = F_d\dot{x} + F_p x + v$, *con* $(F_p, F_d) \in \mathbf{F}(\mathcal{V}^*)$ *y tal que el sistema en lazo cerrado* $\Sigma_F\,(E - BF_d, A + BF_p, B, C)$ *satisfaga la condición geométrica (2.16). Entonces, la salida* $\boldsymbol{y}$ *del sistema en lazo cerrado es única si y solamente si se satisfacen las condiciones geométricas:*

$$\ker(E - BF_d) \subset \mathcal{V}^* \tag{2.17}$$

$$\dim(\mathcal{K}_E) \leq \dim\left(\mathcal{V}^* \cap E^{-1}\mathcal{B}\right) \tag{2.18}$$

El Teorema 3 se basa en la existencia de por lo menos un par amigo, $(F_p, F_d) \in \mathbf{F}(\mathcal{V}^*)$ que satisface (2.16). Posteriormente se mostrará que dicho par existe y se proporcionará un procedimiento para construirlo.

2.3.1 Alcanzabilidad

Ahora se presentará un resultado muy importante que caracteriza geométricamente el subespacio de alcanzabilidad de la descripción implícita global, $\Sigma^g(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$. Esta caracterización muestra que la alcanzabilidad del sistema global se puede separar en la alcanzabilidad obtenida a través de la ley de control de la entrada y la alcanzabilidad debido al grado de libertad interno.

Teorema 4 ([16, Bonilla y Malabre 2002], [18, Bonilla y Malabre 2003]) *Dado el sistema (2.8), los algoritmos del subespacio de alcanzabilidad* $\mathcal{R}^*_{\mathcal{X}}$ *del sistema implícito rectangular* $\Sigma\,(E, A, B, C)$, *el subespacio de alcanzabilidad* $\mathcal{R}^*_{\mathcal{K}_D,\Sigma^s}$ *del sistema reducido,* $\Sigma^s\left(I, A_o, B_o, \bar{C}\right)$ *y la alcanzabilidad del subespacio* $\mathcal{R}^*_{\mathcal{X},\Sigma_0}$ *del sistema homogéneo,* $\Sigma_0(E, A, 0, C)$, *satisfacen:*

$$\begin{aligned} \mathcal{R}^{\mu+1}_{\mathcal{K}_D,\Sigma^s} &= \Sigma^{\mu}_{j=0} A_0^j \operatorname{Im} B_0 \\ \mathcal{R}^{\mu+1}_{\mathcal{X}} &= (E^{-1}A)^{\mu+1}\mathcal{K}_E + V\Sigma^{\mu}_{j=0} A_0^j \operatorname{Im} B_0 \\ \mathcal{R}^{\mu+1}_{\mathcal{X},\Sigma_0} &= (E^{-1}A)^{\mu+1}\mathcal{K}_E \end{aligned} \tag{2.19}$$

y están relacionados por la expresión:

$$\mathcal{R}^*_{\mathcal{X}} = \mathcal{R}^*_{\mathcal{X},\Sigma_0} + V\mathcal{R}^*_{\mathcal{K}_D,\Sigma^s}$$

Además, $\mathcal{R}^*_{\mathcal{X},\Sigma_0}$ *es el subespacio alcanzable por el grado de libertad, es decir,* $\mathcal{R}^*_{\mathcal{X},\Sigma_0} = \mathcal{R}^*_{\mathcal{X},\Sigma_0^g}$ *donde* $\Sigma_0^g(\mathbb{E}, \mathbb{A}, \underline{W}D, C)$ *es el sistema implícito:*

$$\begin{aligned} \mathbb{E}\dot{x} &= \mathbb{A}x + \underline{W}D\bar{x} \\ y &= Cx \end{aligned}$$

donde $\bar{x}$ *caracteriza el grado de libertad y* $\underline{W}$: $\operatorname{Im} D \rightarrow \underline{\mathcal{X}}_g$ *es el mapa de inserción natural de* $\operatorname{Im} D$ *en* $\underline{\mathcal{X}}_g$.

La importancia del Teorema 4 es evidente cuando se considera la ubicación de polos entrada-salida. En efecto, para forzar que cualquier sistema Σ_i^g en (2.2) tenga los mismos polos de entrada-salida, se debe cumplir la condición geométrica $\mathcal{X} = \mathcal{R}^*_{\mathcal{X},\Sigma_0} + V_i\mathcal{R}^*_{\mathcal{K}_{D_i},\Sigma_i^s}$ para $i = 1, 2, ..., n$. Esto restringe la clase de modelos globales Σ_i^g a un conjunto de sistemas lineales estrictamente propios. Pero esta restricción no es tan fuerte como se muestra, por ejemplo, para un conjunto de sistemas escalera introducidos en la Sección 2.4.

Ahora se mostrarán las propiedades estructurales aplicadas al sistema (1.4) tratado en el Capitulo 1.

Ejemplo 2 *Considere el sistema (1.4). Su realización implícita global (2.8) está dada por la expresión:*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ \hline \alpha & \beta & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ \hline 0 \end{bmatrix} u \qquad (2.20)$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x.$$

Ahora se expresa el espacio descriptor en las bases $\mathcal{X} = \mathcal{K}_D \oplus \mathcal{K}_E$ y el espacio de ecuaciones en las bases $\underline{\mathcal{X}}_g = E\mathcal{K}_D \oplus \operatorname{Im} D$. Para esto, se hace $x = T_R\xi$ y se premultiplica la ecuación diferencial (2.20) por T_L, donde:

$$T_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -\beta & -(\alpha+\beta) & 1 \end{bmatrix}, \quad T_L = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

para obtener:

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 \quad 0 & 0 \end{bmatrix} \dot{\xi} = \begin{bmatrix} \boxed{\begin{matrix} -1 & 0 \\ \rho_1 & \rho_2 \end{matrix}} & \begin{matrix} 0 \\ -1 \end{matrix} \\ \hline 0 \quad 0 & 1 \end{bmatrix} \xi + \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} \\ \hline 0 \end{bmatrix} u \qquad (2.21)$$
$$y = \begin{bmatrix} \boxed{\begin{matrix} -\beta & -\rho_3 \end{matrix}} & 1 \end{bmatrix} \xi$$
$$\rho_1 = 1+\beta, \quad \rho_2 = 1+\alpha+\beta, \quad \rho_3 = \alpha+\beta.$$

En el sistema global $\Sigma^g(\mathbb{E},\mathbb{A},\mathbb{B},C)$ mostrado en (2.21) sobre la línea continua se localiza el sistema implícito rectangular $\Sigma(E,A,B,C)$ y dentro de los cuadros se localiza la descripción de estado reducida $\Sigma^s\left(I,A_o,B_o,\overline{C}\right)$. Debajo de la línea continua de $\mathbb{A}$, se localiza la matriz D .
Note de (2.21) que:

1. $\mathcal{K}_E \oplus \mathcal{K}_D = \mathcal{X}$;
2. $V\mathcal{R}^*_{\mathcal{K}_D,\Sigma^s} = Span\left\{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & (1+\beta) & 0 \end{bmatrix}^T\right\}$ *(ver la expresión 2.19a)*;
3. $\mathcal{R}^*_{\mathcal{X},\Sigma_0} = Span\left\{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T\right\}$ *(ver la expresión 2.19c)*;
4. $\mathcal{R}^*_{\mathcal{X},\Sigma^i} = \mathcal{X}$ *(ver la expresión 2.19b), i.e., el sistema implícito rectangular $\Sigma(E,A,B,C)$ es alcanzable sin importar los valores de (α,β). Esta alcanzabilidad se logra tanto por la acción de entrada a través de $\underline{V}\operatorname{Im} B_o$, y el grado de libertad, a través de $\underline{V}\operatorname{Im} D$;*

5. La matriz de observabilidad:

$$\begin{bmatrix} \overline{C} \\ \overline{C}A_o \end{bmatrix} = \begin{bmatrix} -\beta & -(\alpha+\beta) \\ \beta-(1+\beta)(\alpha+\beta) & -(1+\alpha+\beta)(\alpha+\beta) \end{bmatrix}$$

y

$$\det(\lambda I - A_o) = (\lambda+1)(\lambda-1-\alpha-\beta).$$

Observe lo siguiente:

- *Para el caso del sistema de primer orden,* $\dot{y}(t)+y(t)=u(t)$, *obtenido con* $(\alpha,\beta)=(-1,-1)$, *se obtiene del segundo y quinto punto que hay un modo estable no controlable y no observable,* $(\lambda+1)$, *en el sistema reducido* $\Sigma^s\left(I,A_o,B_o,\bar{C}\right)$.
- *Para el caso del sistema de primer orden con ganancia negativa,* $\dot{y}(t)-3y(t)=-u(t)$, *obtenido con* $(\alpha,\beta)=(1,1)$, *se tiene del segundo y quinto punto que hay un modo estable controlable y no observable,* $(\lambda+1)$, *en el sistema reducido* $\Sigma^s\left(I,A_o,B_o,\bar{C}\right)$.
- *Los otros sistemas reducidos obtenidos con* $(\alpha,\beta)=(-1,0)$ *y* $(\alpha,\beta)=(-1,-5)$ *son completamente controlables y observables.*

2.3.2 Rechazo de Estructura

Primero note que $\mathcal{V}^*_{\mathcal{K}_C,\Sigma_0}\subset\mathcal{V}^*$. En base al Hecho 1 (Capitulo 1), el comportamiento entrada-salida se puede fijar simplemente haciendo inobservable $\mathcal{V}^*$ con un par amigo.

A continuación se muestra la utilidad del subespacio (A,E,B) invariante contenido en $\mathcal{K}_C$, $\mathcal{V}^*$, que hace inobservable la variación de estructura interna en el comportamiento entrada-salida del sistema en lazo cerrado.

Empecemos considerando el siguiente teorema, el cual muestra como incluir el grado de libertad en $\mathcal{V}^*$.

Teorema 5 ([17, Bonilla y Malabre 2002], [18, Bonilla y Malabre 2003]) *Si (2.9a) y (2.18) se cumplen, entonces existen retroalimentaciones proporcional y derivativa* $(F_p^*,F_d^*)\in\mathbf{F}(\mathcal{V}^*)$ *tal que (2.16) y (2.17) se satisfacen.*

En la prueba de este último teorema primero se descompone el espacio $\mathcal{X}$ y los subespacios $\mathcal{V}^*$, $E^{-1}\mathcal{B}$, y $\mathcal{K}_E$ de la siguiente manera:

$$\begin{aligned} \mathcal{X} &= \left(\mathcal{V}^*+E^{-1}\mathcal{B}\right)\oplus\mathcal{X}_0; \quad \mathcal{V}^*=\mathcal{X}_{\mathcal{V}^*}\oplus\left(\mathcal{V}^*\cap E^{-1}\mathcal{B}\right) \\ E^{-1}\mathcal{B} &= \left(\left(\mathcal{V}^*\cap E^{-1}\mathcal{B}\right)+\mathcal{K}_E\right)\oplus\mathcal{X}_3 \\ \mathcal{K}_E &= \left(\mathcal{V}^*\cap\mathcal{K}_E\right)\oplus\mathcal{X}_{\mathcal{K}_E} \end{aligned} \tag{2.22}$$

donde $\mathcal{X}_0$, $\mathcal{X}_{\mathcal{V}^*}$, $\mathcal{X}_3$ y $\mathcal{X}_{\mathcal{K}_E}$ son subespacios complementarios cualesquiera y se muestra que:

$$\mathcal{V}^*\cap E^{-1}\mathcal{B}=\mathcal{X}_1\oplus\mathcal{X}_2\oplus\left(\mathcal{V}^*\cap\mathcal{K}_E\right), \quad \text{con } \mathcal{X}_2\approx\mathcal{X}_{\mathcal{K}_E} \tag{2.23}$$

por lo tanto:

$$\begin{aligned}\mathcal{X} &= \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \\ \mathcal{V}^* &= \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \\ E^{-1}\mathcal{B} &= \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3.\end{aligned} \tag{2.24}$$

Después, a partir de la proyección canónica $\hat{N} : \mathcal{X} \to \mathcal{E}/\mathcal{B}$, se tiene:

$$\hat{N}A\mathcal{V}^* \subset \hat{N}E\mathcal{V}^* \tag{2.25}$$

y para cualquier par $(\bar{F}_p, \bar{F}_d) \in \mathbf{F}(\mathcal{V}^*)$, $(A + B\bar{F}_p)\mathcal{V}^* \subset (E - B\bar{F}_d)V^*$ y $\mathcal{B} \subset \mathcal{K}_{\hat{N}}$

También se definen las siguientes proyecciones naturales (note que $\mathcal{E} = E\mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{B} \oplus E\mathcal{X}_0$ y $\mathcal{B} = E\mathcal{X}_1 \oplus E\mathcal{X}_2 \oplus E\mathcal{X}_3$):[2]

$$\begin{aligned}P &: \mathcal{E} \to \mathcal{B}, \quad \text{a lo largo de } E(\mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_0), \quad \text{tal que } PB = I \\ Q_{\mathcal{V}^*} &: \mathcal{X} \to \mathcal{V}^*; \quad Q_{\mathcal{X}_{\mathcal{V}^*}} : \mathcal{X} \to \mathcal{X}_{\mathcal{V}^*}; \\ Q_{\mathcal{X}_2} &: \mathcal{X} \to \mathcal{X}_2; \quad Q_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X} \to \mathcal{X}_{\mathcal{K}_E}\end{aligned} \tag{2.26}$$

y los mapas de inserción:

$$\begin{aligned}R_{\mathcal{V}^*} &: \mathcal{V}^* \to \mathcal{X}; \quad R_{\mathcal{X}_{\mathcal{V}^*}} : \mathcal{X}_{\mathcal{V}^*} \to \mathcal{X} \\ R_{\mathcal{X}_2} &: \mathcal{X}_2 \to \mathcal{X}; \quad R_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \to \mathcal{X}\end{aligned} \tag{2.27}$$

tal que $Q_{\mathcal{X}_i} R_{\mathcal{X}_i} = I$ y donde $Q_{\mathcal{X}_i} R_{\mathcal{X}_j} = 0$ para todo $i \neq j$. De igual manera, se define el isomorfismo:

$$T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \to \mathcal{X}_2 \tag{2.28}$$

y los mapas $F_p^* : \mathcal{X} \to \mathcal{U}$ y $F_d^* : \mathcal{X} \to \mathcal{U}$:

$$\begin{aligned}BF_d^* &= -PE\left(R_{\mathcal{X}_2} T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} - R_{\mathcal{X}_2} Q_{\mathcal{X}_2}\right) \\ BF_p^* &= -PAR_{\mathcal{V}^*} Q_{\mathcal{V}^*}\end{aligned} \tag{2.29}$$

En el siguiente teorema se muestra que el sistema cociente, $\hat{\Sigma}_F^*(E_*, A_*, B_*, C_*)$, es estrictamente propio y descrito en el espacio de estados.

Teorema 6 ([17, Bonilla y Malabre 2002], [18, Bonilla y Malabre 2003]) *Sea* $(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*)$ *como en el Teorema 5. Entonces existen mapas únicos* (E_*, A_*, B_*, C_*) *que satisfacen:*

$$\Pi A_{F^*} = A_* \Phi; \quad \Pi E_{F^*} = E_* \Phi; \quad \Pi B = B_*; \quad C = C_* \Phi \tag{2.30}$$

donde:

$$\begin{aligned}E_{F^*} &= (E - BF_d^*); \quad A_{F^*} = (A + BF_p^*); \quad y \\ \Pi &: \mathcal{E} \to \mathcal{E}/E_{F^*}\mathcal{V}^*; \quad \Phi : \mathcal{X} \to \mathcal{X}/\mathcal{V}^*\end{aligned}$$

[2] Las proyecciones naturales $Q_{\mathcal{X}_i}$ son proyectadas a lo largo de los subespacios complementarios de $\mathcal{X}_i$, mostrado en (2.24).

son proyecciones canónicas y el mapa inducido E_* es un isomorfismo. Además, cuando el sistema (2.8) es retroalimentado con el par (F_p^*, F_d^*), es decir, con la ley de control $u = F_p^* x + F_d^* \dot{x} + v$, se hace externamente equivalente al sistema cociente:

$$\begin{aligned} \dot{\hat{x}} &= E_*^{(-1)} A_* \hat{x} + E_*^{(-1)} B_* v; \\ y &= C_* \hat{x} \end{aligned} \qquad (2.31)$$

donde $\hat{x} = \Phi x$.

2.3.3 Alcanzabilidad en Lazo Cerrado

En el siguiente teorema se muestra que la condición de alcanzabilidad del Teorema 2 (ecuación (2.4)) implica la alcanzabilidad del sistema en lazo cerrado.

Teorema 7 ([17, Bonilla y Malabre 2002], [18, Bonilla y Malabre 2003]) *Dado cualquier par (F_p^*, F_d^*) como en el Teorema 6, y el sistema en lazo cerrado, Σ_{F^*} :*

$$E_{F^*} \dot{x} = A_{F^*} x + B v.$$

Si $\mathcal{R}_{\mathcal{X}}^* = \mathcal{X}$, entonces:

$$\begin{aligned} \mathcal{R}^*_{\mathcal{X}, \Sigma_{F^*}} &= \mathcal{X} \\ \Sigma_{j=0}^{n-1} \left(E_*^{(-1)} A_* \right)^j \operatorname{Im} \left(E_*^{(-1)} B \right) &= \mathcal{X} / \mathcal{V}^* \end{aligned} \qquad (2.32)$$

Apliquemos estos resultados a nuestro ejemplo.

Ejemplo 3 *Considere nuevamente el sistema (1.4) donde:*

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$D = \begin{bmatrix} \alpha & \beta & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Sea $\{e_1, e_2, e_3\}$ una base en $\mathcal{X}$. Note que:

$$\mathcal{V}^* = \mathcal{K}_C = \{e_1, e_2\}, \quad \mathcal{K}_E = \{e_3\}, \quad E^{-1}\mathcal{B} = \{e_2, e_3\},$$

entonces (ver las expresiones (2.22)-(2.24)):

$$\mathcal{X}_{\mathcal{V}^*} = \{e_1\}, \quad \mathcal{X}_1 = \{0\}, \quad \mathcal{X}_2 = \{e_2\}, \quad \mathcal{V}^* \cap \mathcal{K}_E = \{0\},$$
$$\mathcal{X}_{\mathcal{K}_E} = \{e_3\}, \quad \mathcal{X}_3 = \{0\}, \quad \mathcal{X}_0 = \{0\}.$$

De (2.26)-(2.28) se tiene:

$$\begin{aligned} P &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad Q_{\mathcal{V}^*} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = R^T_{\mathcal{V}^*}, \quad Q_{\mathcal{X}_{\mathcal{V}^*}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = R^T_{\mathcal{X}_{\mathcal{V}^*}}, \\ Q_{\mathcal{X}_2} &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} = R^T_{\mathcal{X}_2}, \quad Q_{\mathcal{X}_{\mathcal{K}_E}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} = R^T_{\mathcal{X}_{\mathcal{K}_E}}, \quad T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} = [1] \end{aligned}$$

que junto con (2.29) implica:

$$F_{d_1}^* = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}, \quad F_{p_1}^* = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}.$$

En vista del Teorema 7, la dinámica del sistema cociente estrictamente propio (2.31) se puede asignar, por ejemplo, con $F_{p_2}^* = \begin{bmatrix} 0 & 0 & (1-1/\tau_0) \end{bmatrix}$. *De esta manera, con* $u = F_{d_1}^* \dot{x} + (F_{p_1}^* + F_{p_2}^*)x + R/\tau_0$, *el sistema en lazo cerrado está dado por:*

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{x} &= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -1/\tau_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/\tau_0 \end{bmatrix} R; \\ y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x. \end{aligned}$$

El grado de libertad se ha hecho inobservable, es decir, la variación de estructura ya no está presente en la salida. El sistema en lazo cerrado se comporta como el sistema de primer orden $\tau_0 \dot{y} + y = R$ *sin importar la restricción activa* $0 = \begin{bmatrix} \alpha & \beta & 1 \end{bmatrix} x$.

La estrategia de control desarrollada usa directamente la *variable descriptora* $x(t)$ y se necesita tener algún procedimiento para estimarla; en el Capítulo 3 se presenta una opción.

2.4 Sistemas Escalera

En esta sección se introducen los *sistemas escalera*, en los cuales, su comportamiento entrada-salida puede variar desde un sistema de primer orden hasta un sistema de n-ésimo orden (dependiendo de la posición de los interruptores internos); la descripción implícita rectangular es única. Cuando se satisfacen ciertas condiciones, es posible controlar dicho sistema de tal manera que se comporte como un modelo pre-establecido. Esta presentación está principalmente orientada hacia el modelado, con un ejemplo particular extraído de aplicaciones agronómicas (ver, por ejemplo, [28, Hunt 1982]).

2.4.1 Introducción

En [13] se introducen los sistemas escalera como una herramienta para sintetizar sistemas implícitos rectangulares, con la propiedad de asignamiento dinámico de salida, el cual es capaz de describir sistemas lineales con una estructura interna variable.

Considere un sistema cuyo comportamiento externo (u, y) se puede describir por uno de los siguientes 2^{n-1} sistemas lineales:

$$\left\{ \begin{array}{rlll} (pa_1+1)y(t) & = u(t) & & \\ (pa_i+1)(pa_1+1)y(t) & = u(t) & i \in \{2,\dots,n\} & \\ (pa_j+1)(pa_i+1)(pa_1+1)y(t) & = u(t) & i,j \in \{2,\dots,n\} & \& \ i \neq j \\ (pa_k+1)(pa_j+1)(pa_i+1)(pa_1+1)y(t) & = u(t) & i,j,k \in \{2,\dots,n\} & \& \ i \neq j \neq k \neq i \\ \dots\dots\dots\dots & \dots & \dots\dots & \dots\dots \\ (pa_n+1)\dots\dots\dots(pa_1+1)y(t) & = u(t) & & \end{array} \right.$$

donde los a_i $(i = 1, ..., n)$ son números reales.

Todos estos 2^{n-1} sistemas lineales se pueden describir por el siguiente sistema implícito global:

$$\left\{\begin{array}{lllll} \dot{x}_2(t) = & & -a_1(x_1(t)+x_2(t)) & +(x_3(t)+x_4(t)) & \\ \dot{x}_4(t) = & (a_n-1)x_1(t) & -a_n(x_3(t)+x_4(t)) & +(x_5(t)+x_6(t)) & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots \\ \dot{x}_{2n-4}(t) = & (a_4-1)x_{2n-7}(t) & -a_4(x_{2n-5}(t)+x_{2n-4}(t)) & +(x_{2n-3}(t)+x_{2n-2}(t)) & \\ \dot{x}_{2n-2}(t) = & (a_3-1)x_{2n-5}(t) & -a_3(x_{2n-3}(t)+x_{2n-2}(t)) & +(x_{2n-1}(t)) & \\ \dot{x}_{2n-1}(t) = & (a_2-1)x_{2n-3}(t) & -a_2x_{2n-1}(t) & & +u(t) \end{array}\right. \tag{2.33}$$

$$\left\{\begin{array}{lll} 0 = & x_1(t) & -\theta_0(x_3(t)+x_4(t)) \\ 0 = & x_3(t) & -\theta_1(x_5(t)+x_6(t)) \\ \cdots & \cdots\cdots & \cdots\cdots \\ 0 = & x_{2n-5}(t) & -\theta_{n-3}(x_{2n-3}(t)+x_{2n-2}(t)) \\ 0 = & x_{2n-3}(t) & -\theta_{n-2}(x_{2n-1}(t)) \end{array}\right. \tag{2.34}$$

$$y(t) = (x_1(t)+x_2(t)) \tag{2.35}$$

donde los parámetros θ_i, $i = 0, 1, ..., n-2$ conmutan entre 1 y 0.

A continuación se consideran los siguientes tres casos:

1. Factores irreducibles de orden 1, $(p + a_i)$,
2. Factores irreducibles de orden 2, $(p^2 + 2\rho\omega_o p + \omega_o^2)$, y
3. Red de compensación de atraso/adelanto, $\alpha(\alpha s + 1)/(s + \alpha)$.

2.4.2 Caso 1: Factores Irreducibles de Orden 1

Primero considere un sistema cuyo comportamiento entrada-salida se puede describir por una de las siguientes ocho ecuaciones diferenciales:

$$\left(\Pi_{i=0}^{n-2}\left((1-\theta_i)(p+a_{n-i})+\theta_i\right)\right)(p+a_1)y(t) = \left(\Pi_{i=0}^{n-2}\left((1-\theta_i)a_{n-i}+\theta_i\right)\right)a_1u(t)$$

con $n = 4$, y donde $\theta_i \in \{0, 1\}$ para $i = 0, 1, 2$. De (2.33)-(2.35) se obtiene la descripción implícita (2.36)-(2.37) (ver también la Figura 2.1a)

$$\begin{aligned} \begin{bmatrix} 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}\dot{x}(t) &= \begin{bmatrix} -\bar{a}_1 & -a_1 & k_4 & 1 & 0 & 0 & 0\\ (a_4-1) & 0 & -\bar{a}_4 & -a_4 & k_3 & 1 & 0\\ 0 & 0 & (a_3-1) & 0 & -\bar{a}_3 & -a_3 & 1\\ 0 & 0 & 0 & 0 & (a_2-1) & 0 & -a_2 \end{bmatrix}x(t) + \\ &+ \begin{bmatrix} 0 & 0 & 0 & k_2 \end{bmatrix}u(t) \qquad (2.36)\\ y(t) &= \begin{bmatrix} k_1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}x(t) \end{aligned}$$

$$0 = \begin{bmatrix} 1 & 0 & -\theta_0 k_4 & -\theta_0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & -\theta_1 k_3 & -\theta_1 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & -\theta_2 \end{bmatrix}x(t). \tag{2.37}$$

θ_0	θ_1	θ_2	Comportamiento Entrada-Salida	Comportamiento Interno
1	1	1	$y=(x_2+k_1(x_4+k_4(x_6+k_3x_7)))$ $\boxed{(p+a_1)y=a_1u}$	$(p+a_1)x_2=k_1(1-a_1)(x_4+k_4(x_6+k_3x_7))$ $(p+1)x_4=0;\ (p+1)x_6=0;\ (p+1)x_7=k_2u$
0	1	1	$y=(x_2+k_1(x_4+k_4x_6))$ $\boxed{(p+a_2)(p+a_1)y=a_2a_1u}$	$(p+a_1)x_2=k_1(1-a_1)(x_4+k_4x_6)$ $(p+1)x_4=0;\ (p+1)x_6=x_7;\ (p+a_2)x_7=k_2u$
1	0	1	$y=(x_2+k_1x_4)$ $\boxed{(p+a_3)(p+a_1)y=a_3a_1u}$	$(p+a_1)x_2=k_1(1-a_1)x_4;\ (p+1)x_4=(x_6+k_3x_7)$ $(p+a_3)x_6=k_3(1-a_3)x_7;\ (p+1)x_7=k_2u$
1	1	0	$y=x_2$ $\boxed{(p+a_4)(p+a_1)y=a_4a_1u}$	$(p+a_1)x_2=(x_4+k_4(x_6+k_3x_7))\,;\ (p+1)x_6=0$ $(p+a_4)x_4=k_4(1-a_4)(x_6+k_3x_7)\,;\ (p+1)x_7=k_2u$
0	0	1	$y=(x_2(t)+k_1x_4)$ $\boxed{(p+a_3)(p+a_2)(p+a_1)y=a_3a_2a_1u}$	$(p+a_1)x_2=k_1(1-a_1)x_4;\ (p+1)x_4=x_6$ $(p+a_3)x_6=x_7;\ (p+a_2)x_7=k_2u$
0	1	0	$y=x_2$ $\boxed{(p+a_4)(p+a_2)(p+a_1)y=a_4a_2a_1u}$	$(p+a_1)x_2=(x_4+k_4x_6)\,;\ (p+1)x_6=x_7$ $(p+a_4)x_4=k_4(1-a_4)x_6;\ (p+a_2)x_7=k_2u$
1	0	0	$y=x_2$ $\boxed{(p+a_4)(p+a_3)(p+a_1)y=a_4a_3a_1u}$	$(p+a_1)x_2=x_4;\ (p+a_4)x_4=(x_6+k_3x_7)$ $(p+a_3)x_6=k_3(1-a_3)x_7;\ (p+1)x_7=k_2u$
0	0	0	$y=x_2$ $\boxed{\left(\Pi_{i=1}^4(p+a_i)\right)y=\left(\Pi_{i=1}^4 a_i\right)u}$	$(p+a_1)x_2=x_4;\ (p+a_4)x_4=x_6$ $(p+a_3)x_6=x_7;\ (p+a_2)x_7=k_2u$

Tabla 2.1: Posibles descripciones del sistema (2.36)-(2.37)

Los parámetros $\bar{a}_i$ y las ganancias k_i tienen los siguientes valores:

$$\begin{cases} \bar{a}_i = k_i(a_i-1)+1; \quad i=1,3,4 \\ k_2=\hat{a}_4\hat{a}_3\hat{a}_2\hat{a}_1 \;\; ; \;\; (k_1,k_3,k_4)=\left(\frac{1}{\hat{a}_4},\frac{1}{\hat{a}_2},\frac{1}{\hat{a}_3}\right) \;\; ; \\ \text{si } a_i\neq 0 \text{ entonces } \hat{a}_i=|a_i|, \text{ de otra manera } \hat{a}_i=1. \end{cases}$$

En la Tabla 2.1 se muestran los comportamientos internos involucrados.

2.4.3 Caso 2: Factores Irreducibles de orden 2

Ahora tomemos en cuenta un sistema cuyo comportamiento entrada-salida se puede describir por una de las siguientes cuatro ecuaciones diferenciales:

$$\begin{aligned} &((1-\theta_1)(p+a_2)+\theta_1))\left((1-\theta_0)(p^2+2\rho\omega_o p+\omega_o^2)+\theta_0)\right)(p+a_1)y(t)= \\ &=((1-\theta_1)a_2+\theta_1)\left((1-\theta_0)\omega_o^2+\theta_0\right)a_1u(t) \end{aligned}$$

donde θ_1, $\theta_o\in\{0,1\}$. Para este caso (2.36)-(2.37) se modifica como sigue (ver también la Figura 2.1b):

$$\begin{aligned} \begin{bmatrix} 0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}\dot{x}(t) &= \begin{bmatrix} -\bar{a}_1 & -a_1 & k_4 & 1 & 0 & 0 & 0\\ (a_4-1) & 0 & -\bar{a}_4 & -a_4 & k_3 & 1 & 0\\ \omega_o^2 & 0 & -\hat{a}_3 & -\omega_o^2 & -\bar{a}_3 & -a_3 & 1\\ 0&0&0&0&(a_2-1)&0&-a_2 \end{bmatrix}x(t)+ \\ &+\begin{bmatrix} 0 & 0 & 0 & k_2 \end{bmatrix}^T u(t) \qquad (2.38) \\ y(t) &= \begin{bmatrix} k_1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}x(t) \end{aligned}$$

θ_0	θ_1	Comportamiento Entrada-Salida	Comportamiento Interno
1	1	$y = x_2 + k_1\left(x_4 + k_4\left(x_6 + k_3 x_7\right)\right)$ $\boxed{(p+a_1)y = a_1 u}$	$(p+a_1)x_2 = k_1(1-a_1)\left(x_4 + k_4\left(x_6 + k_3 x_7\right)\right)$ $(p+1)x_4 = 0;\ (p+1)x_6 = 0;\ (p+1)x_7 = k_2 u$
1	0	$y = x_2 + k_1(x_4 + k_4 x_6)$ $\boxed{(p+a_2)(p+a_1)y = a_2 a_1 u}$	$(p+a_1)x_2 = k_1(1-a_1)\left(x_4 + k_4 x_6\right)$ $(p+1)x_4 = 0;\ (p+1)x_6 = x_7;\ (p+a_2)x_7 = k_2 u$
0	1	$y = x_2$ $\boxed{(p^2 + 2\rho\omega_o p + \omega_o^2)(p+a_1)y = \omega_o^2 a_1 u}$	$(p+a_1)x_2 = k_1(1-a_1)x_4;\ px_6 = k_3 x_7 - \omega_o^2$ $(p+2\rho\omega_o)x_4 = x_6 + k_3 x_7;\ (p+1)x_7 = k_2 u$
0	0	$y = x_2$ $\boxed{(p^2 + 2\rho\omega_o p + \omega_o^2)\left(\Pi_{i=1}^2 (p+a_i)\right) y = \omega_o^2 a_2 a_1 u}$	$(p+a_1)x_2 = x_4;\ (p+2\rho\omega_o)x_4 = x_6$ $px_6 = x_7 - \omega_o^2 x_4;\ (p+a_2)x_7 = k_2 u$

Tabla 2.2: Posibles descripciones del sistema (2.38)-(2.39)

$$0 = \begin{bmatrix} 1 & 0 & -\theta_0 k_4 & -\theta_0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\theta_1 k_3 & -\theta_1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\theta_1 \end{bmatrix} x(t). \tag{2.39}$$

Los parámetros a_3, a_4 $\bar{a}_i$ $(i = 1, 3, 4)$, $\bar{a}_3$ y las ganancias k_i tienen los siguientes valores:

$$\begin{cases} a_3 = \omega_o^2 \ \ , \ \ a_4 = 2\rho\omega_o \ \ , \ \ \tilde{a}_3 = k_4\omega_o^2 + 1, \\ \bar{a}_1 = k_1(a_1 - 1) + 1 \ \ , \ \ \bar{a}_3 = 1 - k_3 \ \ , \ \ \bar{a}_4 = k_4(2\rho\omega_o - 1) + 1 \\ k_2 = \hat{a}_3\hat{a}_2\hat{a}_1 \ \ , \ \ (k_1,\ k_3,\ k_4) = \left(1/\sqrt{\hat{a}_3},\ 1/\hat{a}_2,\ 1/\sqrt{\hat{a}_3}\right) \\ if\ \ a_i \neq 0 \text{ entonces } \hat{a}_i = |a_i|, \text{ de otra manera, } \hat{a}_i = 1 \ , \ \ (i = 1, 2, 3) \end{cases}$$

En la Tabla 2.2 se muestran los comportamientos internos.

2.4.4 Caso 3: Red de Compensación de Atraso/Adelanto

Finalmente, tomando en cuenta un sistema cuyo comportamiento entrada-salida puede ser descrito por una de las siguientes cuatro ecuaciones diferenciales:

$$\begin{aligned} &\left((1-\theta_1)(p+a_2) + \theta_1\right)\left((1-\theta_0)(p+\alpha) + \theta_0\right)(p+a_1)y(t) = \\ &= \left((1-\theta_0)(\alpha p + 1) + \theta_0\right)\left((1-\theta_1)a_2 + \theta_1\right)\left((1-\theta_0)\alpha + \theta_0\right)a_1 u(t) \end{aligned}$$

donde θ_1, $\theta_o \in \{0, 1\}$. Para este caso (2.36)-(2.37) se modifica como sigue (ver también la Figura 2.1c):

$$\begin{aligned} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \dot{x}(t) &= \begin{bmatrix} -\bar{a}_1 & -a_1 & (k_3 - \alpha) & 1 & \alpha \\ (a_2 - 1) & 0 & -\bar{a}_2 & -a_2 & (1 - a_2\alpha) \\ 0 & 0 & (\alpha - 1) & 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ k_2 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} k_1 & 1 & 0 & 0 & 0 \end{bmatrix} x(t) \end{aligned} \tag{2.40}$$

$$0 = \begin{bmatrix} 1 & 0 & -\theta_0(k_3 - \alpha) & -\theta_0 & 0 \\ 0 & 0 & 1 & 0 & -\theta_1 \end{bmatrix} x(t). \tag{2.41}$$

θ_0	θ_1	**Comportamiento Entrada-Salida**	**Comportamiento Interno**
1	1	$y = x_2 + k_1 (x_4 + k_3 x_5)$ $\boxed{(p+a_1)y = a_1 u}$	$(p+a_1)x_2 = k_1(1-a_1)(x_4 + k_3 x_5)$ $(p+1)x_4 = 0;\ (p+1)x_5 = k_2 u$
1	0	$y = x_2 + k_1(x_4 + \alpha x_5)$ $\boxed{(p+\alpha)(p+a_1)y = (\alpha p + 1)\alpha a_1 u}$	$(p+a_1)x_2 = k_1(1-a_1)(x_4 + \alpha x_5)$ $(p+1)x_4 = (1-\alpha)x_5;\ (p+\alpha)x_5 = k_2 u$
0	1	$y = x_2$ $\boxed{(p+a_2)(p+a_1)y = a_2 a_1 u}$	$(p+a_2)x_4 = k_3(1-a_2)x_5$ $(p+a_1)x_2 = x_4 + k_3 x_5;\ (p+1)x_5 = k_2 u$
0	0	$y = x_2$ $\boxed{(p+\alpha)(p+a_2)(p+a_1)y = (\alpha p + 1)\alpha a_2 a_1 u}$	$(p+a_2)x_4 = (1-a_2\alpha)x_5$ $(p+a_1)x_2 = x_4 + \alpha x_5;\ (p+\alpha)x_5 = k_2 u$

Tabla 2.3: Posibles descripciones del sistema (2.40)-(2.41)

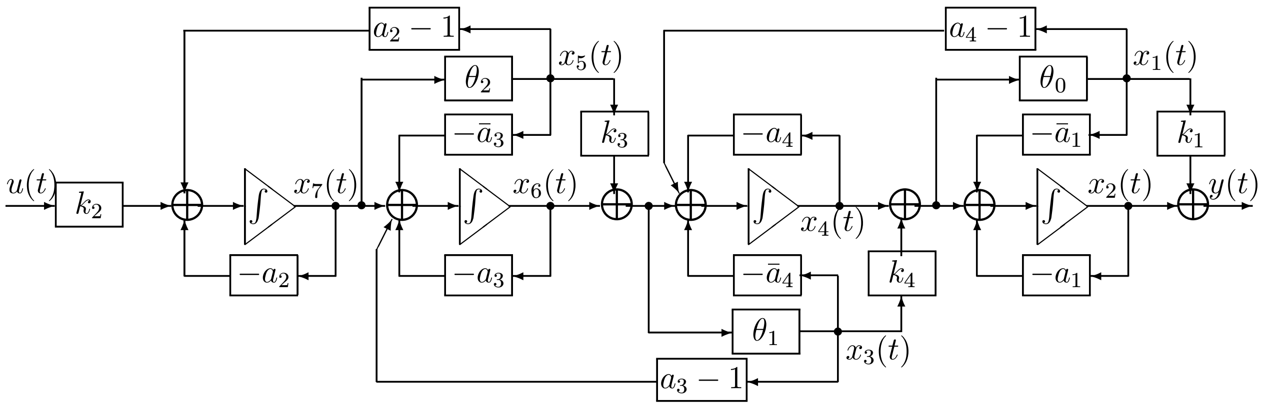
Los parámetros $\bar{a}_1$, $\bar{a}_2$ y las ganancias k_i $(i = 1, 2, 3)$ tienen los siguientes valores:

$$\left\{ \begin{array}{c} \bar{a}_1 = k_1(a_1 - 1) + 1 \quad , \quad \bar{a}_2 = k_3(a_2 - 1) + (1 - a_2\alpha) \quad ; \quad a_3 = \alpha \\ k_2 = \hat{a}_3\hat{a}_2\hat{a}_1 \quad , \quad (k_1,\ k_3) = (1/\hat{a}_2,\ 1/\hat{a}_3) \\ \text{Si } a_i \neq 0 \text{ entonces } \hat{a}_i = |a_i|, \text{ de otra manera } \hat{a}_i = 1 \ , \quad (i = 1, 2, 3) \end{array} \right.$$

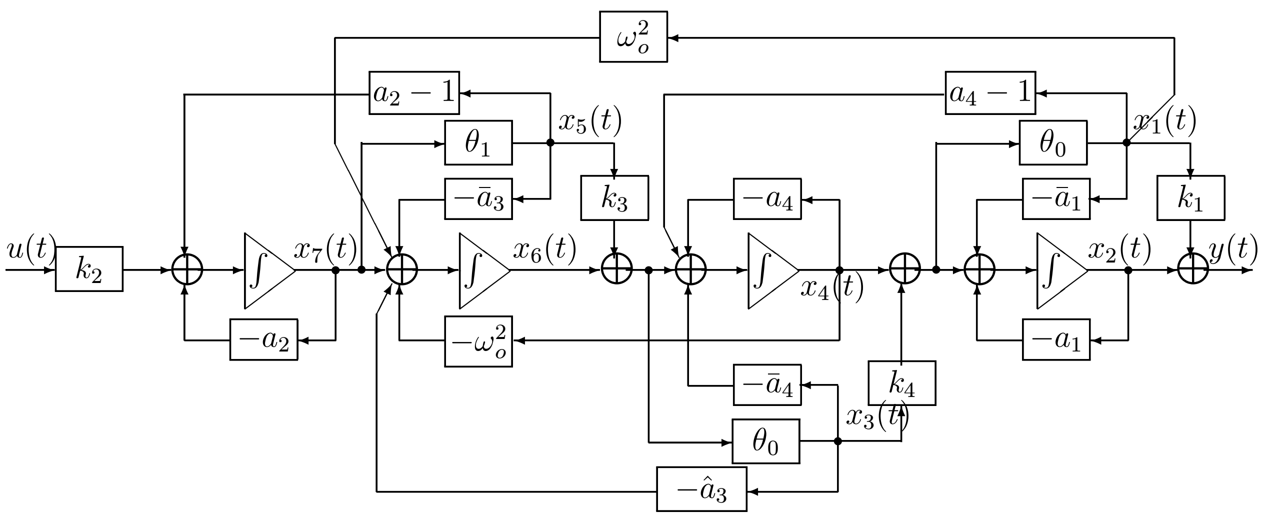
En la Tabla 2.3 se muestran los comportamiento internos.

Las descripciones implícitas rectangulares anteriores tienen la propiedad de *asignamiento dinámico de salida* y por lo tanto se pueden controlar por una ley de control externa.

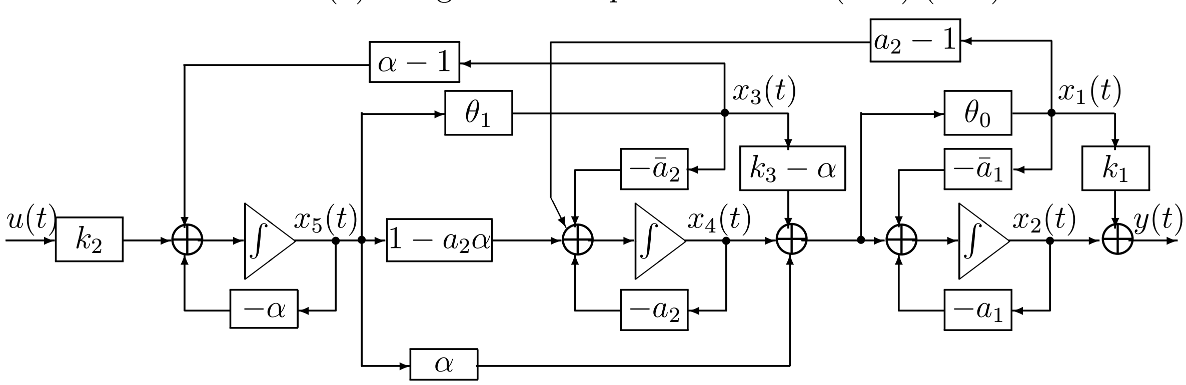
Comentario 1 *Se puede notar de la Tabla 2.1 que para cada una de las posiciones de los interruptores (primera columna), los modos no observables (tercera columna) son estables. Lo mismo sucede en las Tablas 2.2 y 2.3.*



(a) Diagrama a bloques del sistema (2.36)-(2.37).



(b) Diagrama a bloques del sistema (2.38)-(2.39)



(c) Diagrama a bloques del sistema (2.40)-(2.41).

Figura 2.1: Diagrama a bloques de los *Sistemas Escalera.*

Capítulo 3

Detección de la Estructura

En este capítulo se presenta un detector de estructura adaptable para un sistema cuya estructura interna cambia entre n sistemas lineales diferentes (ver Capítulo 2). El detector de estructura está basado en un algoritmo de gradiente normalizado proyectado a lo largo de una hiper-esfera dada, cuya convergencia en tiempo finita es probada.

Como se mencionó en la Introducción, la síntesis del reconstructor de la *variable descriptora* requiere del conocimiento total sobre cual estructura interna está activa, i.e., si el sistema es de primer orden, (1.6), de segundo orden (1.7) o (1.8), con o sin cero dominante; i.e., se tiene que sintetizar un *detector de estructura interna*.

3.1 Résumé en français

Dans ce chapitre un détecteur adaptatif de structure est proposé pour un système dont la structure interne change entre n systèmes linéaires différents (voir Chapitre 2). Ce détecteur s'appuie sur un algorithme de gradient normalisé, projeté à travers une hyper -sphère et sa convergence en temps fini est prouvée (un article est en cours de rédaction).

On considère dans la section 3.2 des filtres discriminants qui servent à estimer la structure interne active. Si le système a le comportement du *i-ème* ordre, la sortie du *i-ème* filtre discriminant est égale à zéro. Une question cruciale à ce niveau est de savoir si le zéro du filtre, à l'instant t, est un *vrai* zéro ou bien un simple passage par zéro, ou encore un signal très petit mais pas nul. Pour résoudre ce problème il est nécessaire de disposer d'un procédé discriminant faisant la distinction entre ces différentes éventualités. Ces dilemmes sont montrés dans le Problème 2 et sont re-écrits dans le Problème 3.

Le Problème 3 est un problème d'optimisation avec critère quadratique et contraintes inégalités. Des conditions nécessaires sont obtenues, en appliquant le Théorème de Kuhn et Tucker.

On propose dans la Section 3.3 un détecteur de structure adaptatif basé sur un algorithme de gradient normalisé activé par un interrupteur de type hystérésis et dont le bon comportement est démontré dans les Lemmes 3, 4 et 5.

On montre dans la sous-section 3.3.1 que ce détecteur de structure adaptatif converge en un temps fini (Théorème 8). Dans la sous-section 3.3.3, on donne un exemple illustratif (voir figures 3.3 et 3.4). La section 3.4 (Annexe) est consacrée à la démonstration des résultats énoncés au long de ce chapitre.

3.2 Introducción

Una manera simple de estimar que estructura interna está activa es sintetizando los siguientes filtros:

$$\begin{cases} (\varepsilon_d p + 1)\, y_1 = ((p+1)\, y - u) + \chi_1, \\ (p + \beta_d)\, \chi_1 = -\varepsilon_d^2 y_1. \end{cases} \tag{3.1}$$

$$\begin{cases} (\varepsilon_d p + 1)^2\, y_2 = (p\,(p+1)\, y - u) + \chi_2, \\ (p + \beta_d)\, \chi_2 = -\varepsilon_d^3 y_2. \end{cases} \tag{3.2}$$

$$\begin{cases} (\varepsilon_d p + 1)^2\, y_3 = ((p+5)\,(p+1)\, y - (5p+1)\, u) + \chi_3, \\ (p + \beta_d)\, \chi_3 = -\varepsilon_d^3 y_3. \end{cases} \tag{3.3}$$

donde $\beta_d > 0$, $\varepsilon_d > 0$. Estos filtros satisfacen el siguiente Lema.

Lema 2 ([7, Bonilla y Lozano]) *Sean $X(t)$ y $Y(t)$, soluciones de:*

$$\begin{aligned} (\varepsilon p + 1)^j\, Y(t) &= Y^*(t) + X(t) \\ (p + \beta)\, X(t) &= -\varepsilon^{j+1} Y(t), \end{aligned}$$

donde $Y^*(t)$ es una señal de tiempo medible. Entonces:

$$\lim_{\varepsilon \to 0} (Y(t) - Y^*(t)) = e^{-\beta t} X(0).$$

En efecto, si el sistema tiene un comportamiento de i-ésimo orden ((1.6), (1.7) o (1.8)), la salida y_i del i-ésimo filtro discriminante ((3.1), (3.2) o (3.3)) es idénticamente cero. En este punto se podría pensar que es suficiente observar el comportamiento de las salidas acotadas y_i, con $i = 1, 2, 3$, para decidir que *estructura* está activa. Pero existe un problema: *cuando la salida de uno de estos filtros discriminantes es cero, ¿cómo sabemos si es una señal cruzando por cero o un cero permanente?. Además, ¿cómo podemos distinguir una señal de nivel bajo de un cero?.* Claramente se necesita un procedimiento discriminante para distinguir entre estos fenómenos diferentes.

Estos dilemas son formalmente establecidos en el siguiente problema.

Problema 2 *Sea el siguiente conjunto de funciones continuas:*

$$\mathcal{F} = \begin{cases} y_i \in C\,[0, T]\,, \ i = 1, ..., n | \exists! k \in \{1, ..., n\} \\ \textit{tal que } y_k \equiv 0 \ \& \ y_{i \neq k} \neq 0 \ \textit{(casi en todas partes)} \end{cases}$$

Encontrar el índice k para el cual $y_k(t) = 0$, con $0 \leq t \leq T$.

Note que el problema debe ser resuelto en línea, porque el resultado se usa en la estrategia de control mencionada anteriormente. Para encontrar una solución analítica, se formula el problema anterior de la siguiente manera:

Problema 3 *Minimizar el criterio de costo*

$$J(\theta) = \sum_{i=1}^{n} \theta_i^2 z_i(t) \tag{3.4}$$

con:

$$z_i(t) = e^{-t/\tau}|y_i(0)| + \int_0^t e^{-(t-\sigma)/\tau}|y_i(\sigma)|d\sigma$$

sujeto a la restricción $g(\theta) \leq 0$, *donde*

$$g(\theta) = r^2 - \sum_{i=n}^{n} \theta_i^2. \tag{3.5}$$

El término $z_i(t)$ actúa como un término de integración, sobre los valores absolutos de las salidas de los *filtros discriminantes*, con un factor de olvido dado por el término exponencial. Por otra parte, gracias a la restricción (3.5) el conjunto de soluciones son aquellas que satisfacen $\theta_{i\neq k} = 0$ y $\theta_k \geq r$, y sólo tenemos que buscar el parámetro mayor o igual al radio r de la hiper-esfera definida por la restricción.

Este es un problema de optimización con un criterio cuadrático y restricciones con desigualdades, entonces, del Teorema de Kuhn-Tucker (ver [37, Luenberger 1969]) se obtienen las condiciones necesarias de primer orden:

$$\bigtriangledown J(\bar{\theta}) + \mu \bigtriangledown g(\bar{\theta}) = 0, \quad \mu \geq 0, \quad \text{y} \quad \mu g(\bar{\theta}) = 0,$$

i.e.,

$$\bar{\theta}_i \left(z_i(t) - \bar{\mu}^2\right) = 0, \quad i = 1, ..., n \quad \text{y} \quad \bar{\mu}^2 \left(r^2 - \sum_{i=1}^{n} \bar{\theta}_i^2\right) = 0;$$

con $\mu = \bar{\mu}^2$.

Como condiciones necesarias de segundo orden, este teorema establece:

$$\frac{\partial^2 J(\bar{\theta})}{\partial\theta_i \partial\theta_j} + \frac{\partial^2 g(\bar{\theta})}{\partial\theta_i \partial\theta_j} = 2\left(z_i(t) - \mu\right) \triangle \left(i - j\right) \geq 0,$$

donde:

$$\triangle(0) = 1 \quad \text{y} \quad \triangle\left(\neq 0\right) = 0,$$

i.e.,

$$\bar{\mu}^2 \leq \min\left\{z_i(t),\ i = 1, ..., n\right\} = 0.$$

Y así, se cumplen las siguientes condiciones necesarias:

$$\bar{\theta}_i z_i(t) = 0,\ i = 1, .., n;\ \text{sujeto a } g(\bar{\theta}) \leq 0. \tag{3.6}$$

3.3 Detector de Estructura Adaptable

Ahora, en vista de (3.6), se propone la siguiente versión modificada del *detector de estructura adaptable,* propuesto en [5, Bonilla et al. 2000], que está basado en el siguiente algoritmo de gradiente normalizado con una proyección que se activa por un interruptor de histéresis:[1]

$$\tau \dot{z}_i(t) + z_i(t) = |y_i(t)|, \quad i = 1, ..., n \tag{3.7}$$

$$\dot{\hat{\theta}}_i\,(t) = -\rho_1 \hat{\theta}_i(t) \frac{z_i^2(t)}{\alpha^2 + z_i^2(t)} + \gamma(\hat{\theta}(t))\rho_2 e^{\sqrt{|g(\hat{\theta}(t))|}}, \quad i = 1, ..., n \tag{3.8}$$

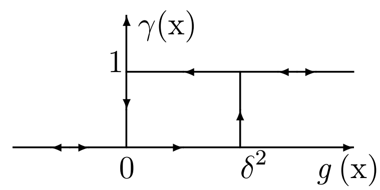


Figura 3.1: Interruptor de Histéresis (3.9).

$$\gamma(x) = \begin{cases} 0 \text{ si: } \ (g(x) \le 0) \ \text{ o } \ \big((0 < g(x) < \delta^2) \ \text{ y } \gamma(x) = 0\big), \\ 1 \text{ si: } \ \big(g(x) \ge \delta^2\big) \ \text{ o } \ \big((0 < g(x) < \delta^2) \ \text{ y } \gamma(x) = 1\big), \end{cases} \tag{3.9}$$

donde los parámetros ρ_1 y ρ_2 son las ganancias positivas del algoritmo y su proyección; α es una constante positiva (la razón de cambio de la hipérbola rectangular); δ es el ancho de la ventana de histéresis ($0 < \delta < r$); γ(x) es la salida de un interruptor de histéresis que se activa en el punto δ^2 y se desactiva en el punto 0 (ver la Figura 3.1).

Las condiciones iniciales se escogen de tal manera que:

$$g(\hat{\theta}(0)) < 0 \quad \text{y} \quad \hat{\theta}_i(0) > 0, \quad \text{para } i = 1, ..., n.$$

Note que la derivada de $g(\hat{\theta}(t))$ es (ver (3.5), (3.7) y (3.8)):

$$\dot{g}(\hat{\theta}(t)) = 2 \sum_{i=1}^{n} \left(\rho_1 \hat{\theta}_i^2(t) \frac{z_i^2(t)}{\alpha^2 + z_i^2(t)} - \gamma(\hat{\theta}(t))\rho_2 e^{\sqrt{|g(\hat{\theta}(t))|}} \hat{\theta}_i(t) \right) \tag{3.10}$$

Comentario 2 *Los siguientes lemas están relacionados con los algoritmos (3.7), (3.8) y (3.9) (ver la Figura 3.2).*[2]

Denotemos la hyper-esfera centrada en el orígen y de radio ρ con la expresión $S_0(\rho) = \{x \in \mathbb{R}^n : \|(x - \rho)\| \le 0\}$.

[1] La ecuación de error es: $e_i(t) = \hat{\theta}(t) z_i(t)$ y $\rho \frac{z_i(t)}{\alpha^2 + z_i^2(t)} e_i(t) = \left(\frac{\rho}{\alpha^2}\right) \left(\frac{z_i(t)}{1 + (1/\alpha^2) z_i^2(t)}\right) e_i(t)$.

[2] Las pruebas se encuentran en el apéndice de este capítulo

Lema 3 *Cuando los parámetros no-negativos $\hat{\theta}_i(t)$, $i = 1, \ldots, n$, están fuera de la hiper-esfera $\mathcal{S}_0(r)$ (en $t = t_0$) con radio r y centrada en el origen, la restricción $g(\hat{\theta}(t))$ es negativa, y el algoritmo forza a $\hat{\theta}_i(t)$ a entrar en la hiper-esfera $\mathcal{S}_0(\sqrt{r^2 - \delta^2})$ (en $t = t_1$). Además, durante este periodo de tiempo ($t_0 \le t \le t_1$) el valor positivo del parámetro $\hat{\theta}_\kappa(t)$ permanece constante y las ecuaciones (3.8) y (3.10) toman la siguiente forma:*

$$\begin{cases} \dot{\hat{\theta}}_\kappa(t) = 0 \quad y \quad \dot{\hat{\theta}}_{i\neq\kappa}(t) = -\rho_1 \hat{\theta}_i(t) \frac{z_i^2(t)}{\alpha^2 + z_i^2(t)} \le 0 \\ \dot{g}(\hat{\theta}(t)) = 2\rho_1 \sum\limits_{i=1}^{n} \hat{\theta}_i^2(t) \frac{z_i^2(t)}{\alpha^2 + z_i^2(t)} \end{cases} \tag{3.11}$$

Lema 4 *Los parámetros $\hat{\theta}_i(t)$, con $i = 1, \ldots, n$, son siempre no negativos. Además, $g(\hat{\theta}(t)) < r^2$ $\forall\ t \ge 0$.*

Lema 5 *Cuando los parámetros no negativos $\hat{\theta}_i(t)$, con $i = 1, \ldots, n$, entran en la hiper-esfera $\mathcal{S}_0(\sqrt{r^2 - \delta^2})$, existe un tiempo finito t_1' en el cual $\dot{g}(\hat{\theta}(t_1'))$ cambia a un valor negativo, y los parámetros $\hat{\theta}_i(t)$ abandonan la hiper-esfera $\mathcal{S}_0(\sqrt{r^2 - \delta^2})$ para salir de $\mathcal{S}_0(r)$ (en $t = t_3$). Además, durante este periodo de tiempo el parámetro $\hat{\theta}_\kappa(t)$ incrementa su valor positivo y las ecuaciones (3.8) y (3.10) toman la siguiente forma:*

$$\begin{cases} \dot{\hat{\theta}}_\kappa(t) = \rho_2 e^{\sqrt{g(\hat{\theta}(t))}} > 0 \quad y \quad \dot{\hat{\theta}}_{i\neq\kappa}(t) = -\rho_1 \hat{\theta}_i(t) \frac{z_i^2(t)}{\alpha^2 + z_i^2(t)} + \rho_2 e^{\sqrt{g(\hat{\theta}(t))}} \\ \dot{g}(\hat{\theta}(t)) = 2 \sum_{i=1}^{n} \left(\rho_1 \hat{\theta}_i^2(t) \frac{z_i^2(t)}{\alpha^2 + z_i^2(t)} - \rho_2 \hat{\theta}_i(t) e^{\sqrt{g(\hat{\theta}(t))}} \right) \end{cases} \tag{3.12}$$

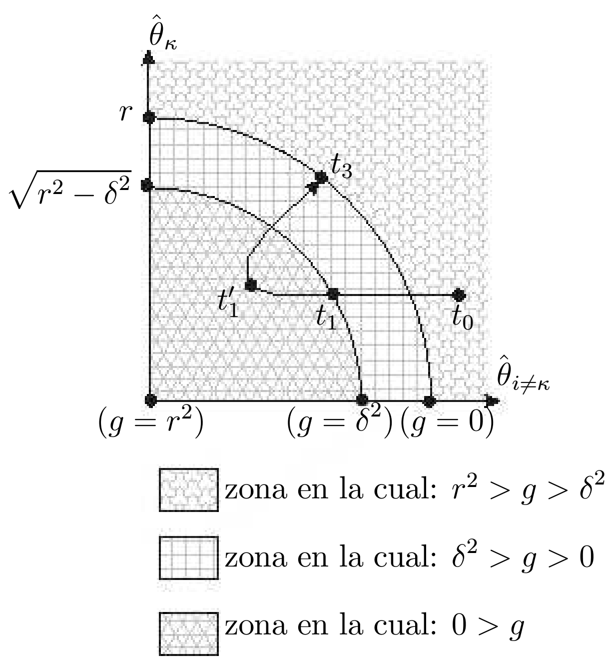


Figura 3.2: Hyper-esfera

3.3.1 Convergencia en Tiempo Finito

En esta subsección se prueba que el *esquema del detector de estructura adaptable* converge en un tiempo finito.

Para un intervalo de tiempo cerrado, $[T_{i,1},\ T_{i,2}]$, se define el conjunto $\Gamma(\mu; z_i)$ y su longitud asociada $\Delta(\Gamma(\mu; z_i))$:

$$\Gamma(\mu; z_i) = \{[t_{i,(2j-1)}, t_{i,2j}] : \quad j \in \{1, \dots, l^*\}, \quad |z_i(t)| \le \mu \quad \forall t_{i,(2j-1)} \le t \le t_{i,2j}, \\ |z_i(t)| > \mu \quad \forall t_{i,2j} < t < t_{i,(2j+1)}\} \tag{3.13}$$

$$\Delta(\Gamma(\mu; z_i)) = \sum_{j=1}^{l^*} \left(t_{i,2j} - t_{i,(2j-1)}\right) \tag{3.14}$$

donde $t_{i,1}, \dots, t_{i,2l^*}$ es una partición finita[3] de $[T_{i,1},\ T_{i,2}]$, con $l^* \in \mathbb{Z}^+$ y μ es un número real positivo dado.

Teorema 8 *Suponga que:*

- *Las condiciones iniciales son $z_i(0) = 0$ y $\hat{\theta}_i(0) = \frac{r}{\sqrt{n}}$ para $i = 1, \dots, n$.*
- *Se cumplen las siguientes hipótesis:*

  H1: *Dada $\tau^* \in \mathbb{R}^+\ \exists\ \mu^* \in \mathbb{R}^+$ tal que $\Delta(\Gamma(\mu^*; z_i)) \le \tau^*\ \forall\ i \ne \kappa$.*

  H2: *Se seleccionan r y δ de tal manera que $r \ge \delta\sqrt{2n/(n-1)}$ con $n > 1$.*

  H3: *Sea β cualquier número real positivo tal que $2 \le \beta \le (r/\delta)^2(n-1)/n$. El algoritmo se detiene cuando cualquier parámetro satisface $\hat{\theta}_i(t) > \sqrt{r^2 - \beta\delta^2}$.*

Entonces existe un tiempo finito t^*,

$$t^* \le \frac{r}{\rho_2} + \frac{1}{\rho_1}\left(1 + \frac{2ne^{r\sqrt{1-(\rho_2/r\rho_1)^2}}}{(\delta/r)^2}\right)\left(1 + \frac{1}{4}(\delta/r)^2\right), \tag{3.15}$$

tal que $\hat{\theta}_\kappa(t_*) \ge r$ y $\hat{\theta}_{i\ne\kappa} = 0$; además $\dot{\hat{\theta}}_i(t^*) = 0$, para $i = 1, \dots, n$. Más aún, el valor máximo g^* alcanzado por la restricción 3.5 está acotada de la siguiente manera:

$$g^* \le r^2 - (\rho_2/\rho_1)^2 \tag{3.16}$$

[3] La partición es finita debido a que las salidas y_i son acotadas y las z_i son salidas de filtros de primer órden de frecuencia baja (ver la ecuación (3.7)).

3.3.2 Criterio de Diseño

De acuerdo al Teorema 8, se necesita la prueba de paro:

$$\widehat{\theta}_i \geq \sqrt{r^2 - \beta\delta^2} \text{ con } i = 1, ..., n.$$

donde el discriminante debe cumplir $r^2 - \beta\delta^2 > 0$. Esto implica que:

$$\boxed{\delta < \tfrac{r}{\sqrt{\beta}}}. \tag{3.17}$$

Ahora bien, $\widehat{\theta}_k(t_1) \geq \widehat{\theta}_k(0) = r/\sqrt{n}$, entonces $\widehat{\theta}_k^2(t_1) \geq r^2/n$, pero $\widehat{\theta}_k^2(t_1) \leq r^2 - \beta\delta^2$; por lo tanto $r^2 - \beta\delta^2 \geq r^2/n$. De esta manera

$$\boxed{2 \leq \beta \leq r^2 - r^2/n = \tfrac{n-1}{n} r^2}, \tag{3.18}$$

esto implica que:

$$\boxed{r \geq \sqrt{\tfrac{2n}{n-1}}}. \tag{3.19}$$

Dada la cota superior (3.16) de g^*, se propone la siguiente relación entre ρ_1 y ρ_2:

$$\rho_2 = r\rho_1\sqrt{1 - (1-a)(\delta/r)^2} \text{ con } 0 \leq a < 1. \tag{3.20}$$

Con esta relación se tiene que la cota superior de g^* está dada por la expresión (sustituyendo (3.20) en (3.16)):

$$g^* \leq (1-a)\delta^2 \tag{3.21}$$

De (3.21) se observa que con esta selección, el comportamiento del algoritmo de adaptación evolucionará en las cercanias de la franja de histéresis de las hyper-esferas (ver Figura 3.2). La cota superior del tiempo de convergencia será entonces:

$$t^* = \frac{1}{\rho_1}\left[\frac{1}{\sqrt{1-(1-a)(\delta/r)}} + \left(1 + \frac{2ne^{\sqrt{1-a}\delta}}{(\delta/r)^2}\right)\left(1 + \frac{1}{4}(\delta/r)^2\right)\right]. \tag{3.22}$$

En síntesis el procedimiento es el siguiente:

1. Para n sistemas, seleccionar el radio de la hiper-esfera, r, de acuerdo a la ecuación (3.19).
2. Seleccionar $\beta \in \mathbb{R}^+$ como se indica en la ecuación (3.18).
3. Calcular el ancho de la ventana de histéresis, δ, dado por la ecuación (3.17).
4. Proponer un tiempo de convergencia ,t^*, y se calculan las ganancias (positivas) del algoritmo y su proyección, ρ_1 y ρ_2, con las expresiones (3.20) y (3.22).

3.3.3 Ejemplo Ilustrativo

Ahora se aplicará el criterio de diseño al sistema (1.4) sujeto a la restricción (1.5), para los siguientes valores de α y β (ver Capítulo 1):

1. Si $(\alpha, \beta) = (-1, -1)$ se tiene:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \end{aligned} \tag{3.23}$$

2. Si $(\alpha, \beta) = (-1, 0)$ se tiene:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \end{aligned} \tag{3.24}$$

Como se tienen dos sistemas, entonces, $n = 2$. Ahora se toma $r = 10$ y $\beta = 2$. Con estos valores de r y β se escoge $\delta = 3$. Finalmente, si se desea un tiempo de convergencia $t^* = 1$ segundo, se obtiene $\rho_1 = 119.39$ y $\rho_2 = 1188.5$ (con $a = 0.9$).

Resultados

1. Para el sistema (3.23) se obtienen los resultados mostrados en la Figura 3.3. En esa figura se puede observar que el algoritmo efectivamente converge en un tiempo menor a 1 segundo y también se observa la zona de convergencia.
2. Para el sistema (3.24) se obtienen los resultados mostrados en la Figura 3.4. Como en el punto anterior, se observa que el algoritmo converge en un tiempo inferior a 1 segundo y de igual manera se puede observar la zona de convergencia.

3.4 Apéndice

3.4.1 Prueba del Lema 3

Puesto que $g(\hat{\theta}(t)) < 0$, de (3.8) y (3.9) se tiene que:

$$\dot{\hat{\theta}}_i(t) = -\frac{\rho_1 \hat{\theta}_i(t) z_i^2(t)}{\alpha^2 + z_i^2(t)} < 0 \text{ (casi en todas partes)}, \quad \forall i \neq \kappa, \text{y },$$

además:

$$\begin{aligned} \text{de (3.7) se tiene que } y_\kappa &= 0 \Rightarrow z_\kappa = 0 \\ \text{de la Figura 3.1 se tiene } g &< 0 \Rightarrow \gamma = 0 \\ \text{y (3.8)} &\Rightarrow \dot{\hat{\theta}}_\kappa(t) = 0 \end{aligned}$$

lo cual implica que[4] $\hat{\theta}_{i\neq\kappa}(t) \to 0$ y $\hat{\theta}_\kappa(t) = \hat{\theta}_\kappa(0)$. Si $\hat{\theta}_\kappa(0) \geq r$, el algoritmo (3.7)-(3.9) converge, por lo tanto, asumiremos de ahora en adelante que $0 < \hat{\theta}_\kappa(t) < r$. La expresión mostrada en (3.11) se obtiene directamente de (3.8) y (3.9) (ver también la Figura 3.1). ∎

3.4.2 Prueba del Lema 4

Cuando cualquier parámetro no negativo $\hat{\theta}_j(t)$ alcanza el origen, en el tiempo t^o, de (3.8) y (3.9) se observa que pueden ocurrir los siguientes dos casos:

i) $\dot{\hat{\theta}}_j(t^o) = 0$ si $g(\hat{\theta}(t^o)) < \delta^2$ y $\gamma(\hat{\theta}(t^o)) = 0$, y

ii) $\dot{\hat{\theta}}_j(t^o) = \rho_2 e^{\sqrt{g(\hat{\theta}(t^o))}} > 0$ si $0 < g(\hat{\theta}(t^o)) \leq r^2$ y $\gamma(\hat{\theta}(t^o)) = 1$,
entonces $\hat{\theta}_j(t) \geq \hat{\theta}_j(t^o)$ para una vecindad de t^o. En vista de que todos los parámetros $\hat{\theta}_i(t)$ son no negativos se tiene que $g(\hat{\theta}(t)) \leq r^2$; pero puesto que $\hat{\theta}_\kappa(t) > 0$ entonces $g(\hat{\theta}(t)) \neq r^2$. ∎

3.4.3 Prueba del Lema 5

Para realizar esta prueba suponga que los parámetros $\hat{\theta}_i(t)$ nunca abandonan $\mathcal{S}_0(\sqrt{r^2-\delta^2})$, i.e. $g(\hat{\theta}(t)) \geq \delta^2$. Sea t_1 el primer tiempo en el cual los parámetros $\hat{\theta}_i(t)$ entran en $\mathcal{S}_0(\sqrt{r^2-\delta^2})$, es decir $g(\hat{\theta}(t)) \geq \delta^2$ para todo $t \geq t_1$. Entonces la derivada de $g(\hat{\theta}(t))$ para todo $t \geq t_1$ (ver (3.10)) está dada por la expresión:

$$\dot{g}(\hat{\theta}(t)) = 2\sum_{i\neq\kappa}\left(\rho_1\hat{\theta}_i^2(t)\frac{z_i^2(t)}{\alpha^2+z_i^2(t)} - e^{\sqrt{g(\hat{\theta}(t))}}\hat{\theta}_i(t)\right) - 2\rho_2 e^{\sqrt{g(\hat{\theta}(t))}}\left(\hat{\theta}_\kappa(0) + \rho_2\int_{t_1}^{t} e^{\sqrt{g(\hat{\theta}(\sigma))}}d\sigma\right)$$

como

$$\frac{z_i^2(t)}{\alpha^2+z_i^2(t)} \leq 1 \Rightarrow \dot{g}(\hat{\theta}(t)) \leq 2\rho_1\sum_{i\neq\kappa}\hat{\theta}_i^2(t) - 2\rho_2^2\int_{t^{(i)}}^{t} d\sigma \leq 2\rho_1(n-1)\max_{i\neq\kappa}\{\theta_i\} - 2\rho_2^2(t-t_1)$$

lo cual implica que $\dot{g}(\hat{\theta}(t)) \leq 2\rho_1(n-1)(r^2-\delta^2) - 2\rho_2^2(t-t_1)$. Integrando esto implica que:

$$g(\hat{\theta}(t)) \leq \delta^2 + 2\rho_1(n-1)(r^2-\delta^2)(t-t_1) - \rho_2^2(t-t_1)^2 \qquad \forall t \geq t_1.$$

Entonces existe un tiempo finito $t_1'' > t_1$ tal que $g(\hat{\theta}(t)) < 0$ para todo $t > t_1''$, lo cual contradice la suposición de que $\hat{\theta}_i(t)$ nunca abandona $\mathcal{S}_0(r)$. La expresión (3.12) se obtiene directamente de (3.9) (ver también la Figura 3.1). ∎

[4] Recuerde que los $\hat{\theta}_i$ son siempre no negativos (ver Lema 4) y la condición inicial seleccionada es $\hat{\theta}_i > 0$ $\forall i \in \{1, ..., n\}$.

3.4.4 Prueba del Teorema 8

Antes de probar este lema, note que para cualquier espacio Euclideano, $\Re^n$, se satisface:

$$n\sqrt{\sum_{i=1}^{n}|x_i|^2} \geq \sum_{i=1}^{n}|x_i| \geq \sqrt{\sum_{i=1}^{n}|x_i|^2}.$$

Entonces, de (3.5) se tiene:

$$n\sqrt{r^2 - g(\hat{\theta})} \geq \sum_{i=1}^{n}\hat{\theta}_i \geq \sqrt{r^2 - g(\hat{\theta})} \tag{3.25}$$

Prueba del Teorema 8:

i) $t \in [t_0,\ t_1)$**:** Para este periodo de tiempo, $\gamma(\hat{\theta}(t)) = 0$, entonces de (3.8) y (3.10), se tiene que (recuerde que $g(\hat{\theta}(t_1)) = \delta^2$ y $g(\hat{\theta}(t_0)) = 0$):

$$\begin{aligned} \hat{\theta}_\kappa(t_1) &= \hat{\theta}_\kappa(t_0) \\ \hat{\theta}_{i\neq\kappa}(t_1) &\leq \hat{\theta}_{i\neq\kappa}(t_0) \end{aligned} \tag{3.26}$$

$$\delta^2 = g(\hat{\theta}(t_1)) - g(\hat{\theta}(t_0)) = 2\rho_1 \int_{t_0}^{t_1} \sum_{i=1}^{n} \hat{\theta}_i^2(t) \frac{z_i^2(t)}{\alpha^2 + z_i^2(t)} dt \tag{3.27}$$

De (3.27) y (3.5) se tiene:

$$\delta^2 \leq 2\rho_1 \int_{t_0}^{t_1} \sum_{i=1}^{n} \hat{\theta}_i^2(t) dt = 2\rho_1 \int_{t_0}^{t_1} \left(r^2 - g(\hat{\theta}(t))\right) dt \leq 2\rho_1 r^2 (t_1 - t_0).$$

De esta manera, se tiene la siguiente cota inferior:

$$(t_1 - t_0) \geq \frac{1}{2\rho_1}(\delta/r)^2 \quad \text{para } (t_1 - t_0) \tag{3.28}$$

Ahora, para obtener una cota superior de $(t_1 - t_0)$ primero note que la hipótesis **H1** implica que existe una $\alpha^* \in \mathbb{R}^+$ tal que:

$$\Delta(\Gamma(\alpha^*; z_i)) \leq \frac{1}{4\rho_1}(\delta/r)^2 \quad \forall i \neq \kappa. \tag{3.29}$$

Entonces, de (3.27) y (3.26) se obtiene:[5]

$$\begin{aligned} \delta^2 &= 2\rho_1 \sum_{i=1}^{n} \int_{t_0}^{t_1} \hat{\theta}_i^2(t) \tfrac{z_i^2(t)}{\alpha^2 + z_i^2(t)} dt \geq 2\rho_1 \sum_{i=1}^{n} \hat{\theta}_i^2(t_1) \int_{t_0}^{t_1} \tfrac{z_i^2(t)}{\alpha^2 + z_i^2(t)} dt \\ &\geq 2\rho_1 \sum_{i=1}^{n} \hat{\theta}_i^2(t_1) \int_{[t_0,\ t_1]\setminus\Gamma(\alpha^*; z_i)} \tfrac{z_i^2(t)}{\alpha^2 + z_i^2(t)} dt = 2\rho_1 \sum_{i=1(i\neq\kappa)}^{n} \hat{\theta}_i^2(t_1) \int_{[t_0,\ t_1]\setminus\Gamma(\alpha^*; z_i)} \tfrac{z_i^2(t)}{\alpha^2 + z_i^2(t)} dt \\ &\geq \rho_1 \sum_{i=1(i\neq\kappa)}^{n} \hat{\theta}_i^2(t_1) \int_{[t_0,\ t_1]\setminus\Gamma(\alpha^*; z_i)} dt \end{aligned} \tag{3.30}$$

[5] Note que $|z_i(t)| > \alpha^*$ $\forall t \in [t_0,\ t_1]\setminus\Gamma(\alpha; z_i)$, *i.e.* $z_i^2(t)/(\alpha^2 + z_i^2(t)) \geq 1/2$ $\forall\ t \in [t_0,\ t_1]\setminus\Gamma(\alpha; z_i)$.

Ahora, en vista de (3.29) se tiene que:[6]

$$\int_{[t_0,\ t_1]\backslash\Gamma(\alpha^*;z_i)} dt = (t_1 - t_0) - \Delta(\alpha^*; z_i) \geq (t_1 - t_0) - (\delta/r)^2/(4\rho_1) \geq (\delta/r)^2/(4\rho_1) \forall i \neq \kappa.$$

De esta manera, de (3.5) y (3.30) se obtiene (recuerde que $g(\hat{\theta}(t_1)) = \delta^2$ y $\hat{\theta}_\kappa(t_1) \leq \sqrt{r^2 - \delta^2}$):

$$\begin{aligned}
\delta^2 \; & \geq \rho_1 \sum_{i=1(i\neq\kappa)}^{n} \hat{\theta}_i^2(t_1)\left((t_1 - t_0) - \tfrac{(\delta/r)^2}{4\rho_1}\right) \\
& = \rho_1 \left(r^2 - g(\hat{\theta}(t_1)) - \hat{\theta}_\kappa^2(t_1)\right)\left((t_1 - t_0) - \tfrac{(\delta/r)^2}{4\rho_1}\right) \\
& = \rho_1 \left(r^2 - \delta^2 - \hat{\theta}_\kappa^2(t_1)\right)\left((t_1 - t_0) - \tfrac{(\delta/r)^2}{4\rho_1}\right) \\
& \geq \rho_1 \left(r^2 - \delta^2 - r^2 + \beta\delta^2\right)\left((t_1 - t_0) - \tfrac{1}{4\rho_1}\left(\tfrac{\delta}{r}\right)^2\right) \\
& = \rho_1 \left(\beta - 1\right)\delta^2\left((t_1 - t_0) - \tfrac{1}{4\rho_1}\left(\tfrac{\delta}{r}\right)^2\right) \\
& \geq \rho_1\delta^2\left((t_1 - t_0) - \tfrac{1}{4\rho_1}\left(\tfrac{\delta}{r}\right)^2\right)
\end{aligned} \tag{3.31}$$

De (3.28) y (3.31) se tiene:

$$\frac{1}{2\rho_1}(\delta/r)^2 \leq (t_1 - t_0) \leq \frac{1}{\rho_1}\left(1 + \frac{1}{4}(\delta/r)^2\right) \tag{3.32}$$

ii) $t \in [t_1,\ t_3)$: Para este periodo de tiempo $\gamma(\hat{\theta}(t)) = 0$. Entonces de (3.8) y (3.10), se tiene:

1. Primero se calcula una cota superior para el valor máximo de $g(\hat{\theta}(t))$. Para esto, se tiene que resolver $\dot{g}(\hat{\theta}(t_1')) = 0$, es decir:

$$\sum_{i=1}^{n} \rho_1\hat{\theta}_i^2(t_1')\frac{z_i^2(t_1')}{\alpha^2 + z_i^2(t_1')} = \sum_{i=1}^{n} \rho_2\hat{\theta}_i(t_1')e^{\sqrt{g^*}} = \rho_2 e^{\sqrt{g^*}}\sum_{i=1}^{n}\hat{\theta}_i(t_1'). \tag{3.33}$$

   Trabajemos con el miembro izquierdo de la ecuación (3.33) (recuerde (3.25)):

$$\rho_2 e^{\sqrt{g^*}}\sum_{i=1}^{n}\hat{\theta}_i(t_1') \geq \rho_2 e^{\sqrt{g^*}}\sqrt{\sum_{i=1}^{n}\hat{\theta}_i^2(t_1')} = \rho_2 e^{\sqrt{g^*}}\sqrt{r^2 - g^*} \geq \rho_2\sqrt{r^2 - g^*}. \tag{3.34}$$

   Ahora consideremos el miembro izquierdo de (3.33) (recuerde (3.25)):

$$\sum_{i=1}^{n} \rho_1\hat{\theta}_i^2(t_1')\frac{z_i^2(t_1')}{\alpha^2 + z_i^2(t_1')} \leq \sum_{i=1}^{n} \rho_1\hat{\theta}_i^2(t_1') \leq \rho_1(r^2 - g^*) \tag{3.35}$$

   De (3.34) y (3.35) se obtiene 3.16.

[6] *Note that (3.28) implica* $(t_1 - t_0) - (\delta/r)^2/(4\rho_1) \geq (\delta/r)^2/(4\rho_1) > 0$.

2. Ahora se calcula una cota inferior para el periodo de tiempo $(t_3 - t_1)$. De (3.12) se obtiene (recuerde (3.5) y (3.16)):

$$
\begin{aligned}
&\hat{\theta}_\kappa(t_3) - \hat{\theta}_\kappa(t_1) = 0 - \delta^2 = 2\sum_{i=1}^{n}\int_{t_1}^{t_3}\left(\rho_1\hat{\theta}_i^2(t)\tfrac{z_i^2(t)}{\alpha^2+z_i^2(t)} - \rho_2\hat{\theta}_i(t)e^{\sqrt{g(\hat{\theta}(t))}}\right)dt \\
&\geq -2\rho_2\int_{t_1}^{t_3}\sum_{i=1}^{n}\hat{\theta}_i(t)e^{\sqrt{g(\hat{\theta}(t))}}dt \geq -2\rho_2 e^{\sqrt{g^*}}\int_{t_1}^{t_3}\sum_{i=1}^{n}\hat{\theta}_i(t)dt \\
&\geq -2\rho_2 n e^{\sqrt{g^*}}\int_{t_1}^{t_3}\sqrt{r^2 - \left(\hat{\theta}(t)\right)}dt \geq -2\rho_2 n e^{\sqrt{g^*}} r(t_3 - t_1) \geq -2\rho_2 n e^{\sqrt{r^2-(\rho_2/\rho_1)^2}} r(t_3 - t_1),
\end{aligned}
$$

es decir:

$$
(t_3 - t_1) \geq \frac{\delta^2}{2\rho_2 n r e^{\sqrt{r^2-(\rho_2/\rho_1)^2}}} \tag{3.36}
$$

3. Ahora se calcula una cota inferior para el incremento del parámetro $\hat{\theta}_\kappa$. De (3.12) y (3.36) se obtiene:

$$
\hat{\theta}_\kappa(t_3) - \hat{\theta}_\kappa(t_1) = \rho_2\int_{t_1}^{t_3} e^{\sqrt{g(\hat{\theta}(t))}}dt \geq \rho_2(t_3 - t_1),
$$

es decir:

$$
\frac{\hat{\theta}_\kappa(t_3) - \hat{\theta}_\kappa(t_0)}{r} = \frac{\hat{\theta}_\kappa(t_3) - \hat{\theta}_\kappa(t_1)}{r} \geq \frac{(\delta/r)^2}{2ne^{r\sqrt{1-(\rho_2/r\rho_1)^2}}} \tag{3.37}
$$

- Sea $\lambda^* \in \mathbb{Z}^+$ tal que:

$$
\frac{2ne^{r\sqrt{1-(\rho_2/r\rho_1)^2}}}{(\delta/r)^2} \leq \lambda^* < 1 + \frac{2ne^{r\sqrt{1-(\rho_2/r\rho_1)^2}}}{(\delta/r)^2}, \tag{3.38}
$$

entonces, de (3.37) y (3.38) se tiene:

$$
\lambda^*\left(\frac{\hat{\theta}_\kappa(t_3) - \hat{\theta}_\kappa(t_0)}{r}\right) \geq \lambda^*\frac{(\delta/r)^2}{2ne^{r\sqrt{1-(\rho_2/r\rho_1)^2}}} \geq 1 \tag{3.39}
$$

- Sea $\bar{\lambda} \in \mathbb{Z}^+$ el número de ciclos necesarios para asegurar la convergencia de $\hat{\theta}_\kappa$ en r, es decir, $1 \leq \bar{\lambda} \leq \lambda^*$.

- Sea $t \in [t_{1,j}, t_{3,j}]$, el j-*ésimo* ciclo; entonces:

$$
\hat{\theta}_\kappa(t_{3,j}) - \hat{\theta}_\kappa(t_{1,j}) \geq \rho_2(t_{3,j} - t_{1,j})
$$

lo cual implica:

$$
1 \geq \sum_{j=1}^{\bar{\lambda}}\frac{\hat{\theta}_\kappa(t_{3,j}) - \hat{\theta}_\kappa(t_{1,j})}{r} \geq \frac{\rho_2}{r}\sum_{j=1}^{\bar{\lambda}}(t_{3,j} - t_{1,j}).
$$

Entonces se tiene:

$$\sum_{j=1}^{\bar{\lambda}}(t_{3,j} - t_{1,j}) \leq r/\rho_2. \tag{3.40}$$

También, de (3.32) y (3.38) se tiene:

$$\sum_{j=1}^{\bar{\lambda}}(t_{1,j} - t_{0,j}) \leq \lambda^* \frac{1}{\rho_1}\left(1 + \frac{1}{4}(\delta/r)^2\right) \leq \frac{1}{\rho_1}\left(1 + \frac{2ne^{r\sqrt{1-(\rho_2/r\rho_1)^2}}}{(\delta/r)^2}\right)\left(1 + \frac{1}{4}(\delta/r)^2\right) \tag{3.41}$$

Por lo tanto, (3.15) se obtiene de (3.40) y (3.41). ■

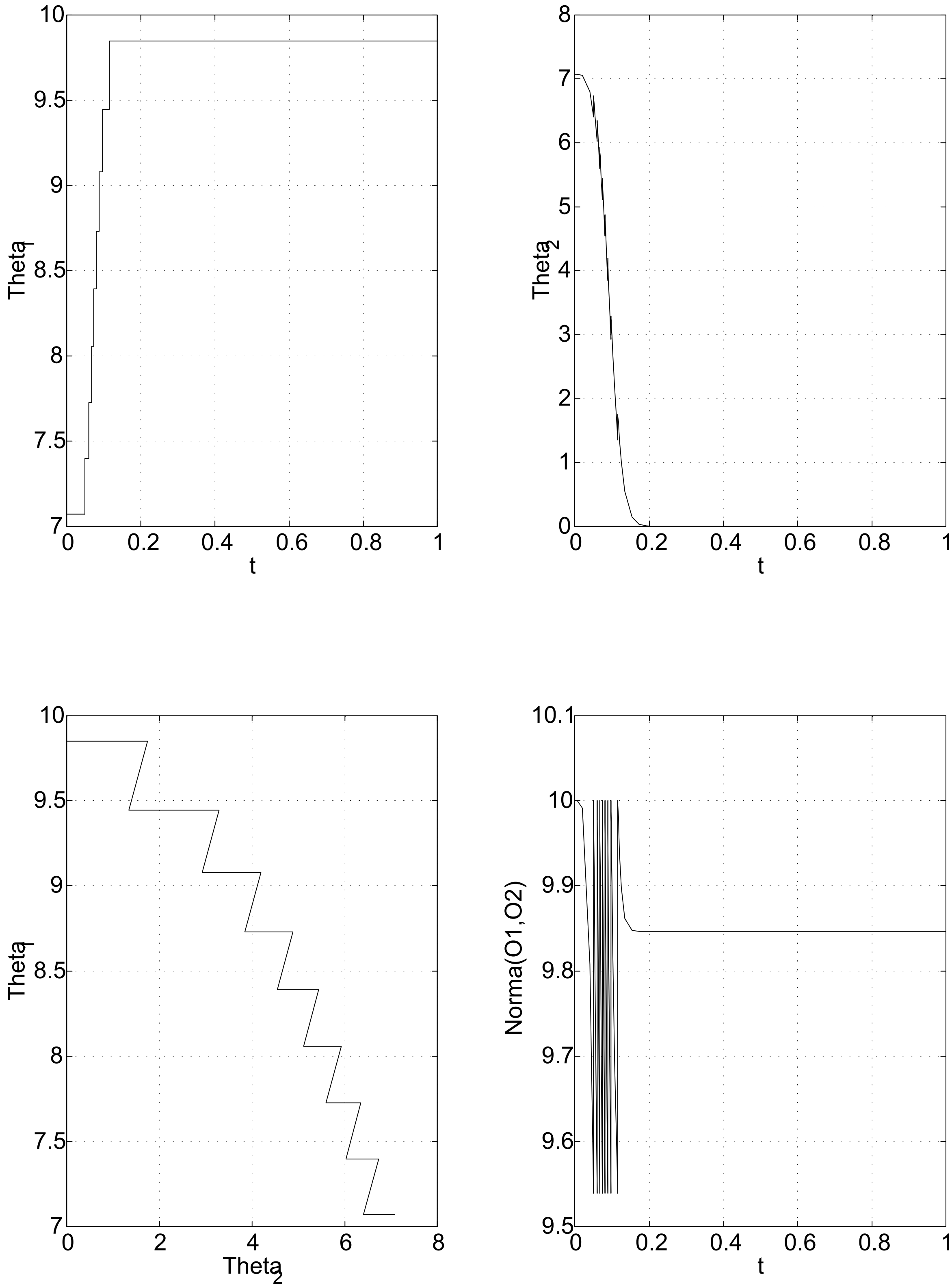


Figura 3.3: Resultados de la simulación para el sistema (3.23).

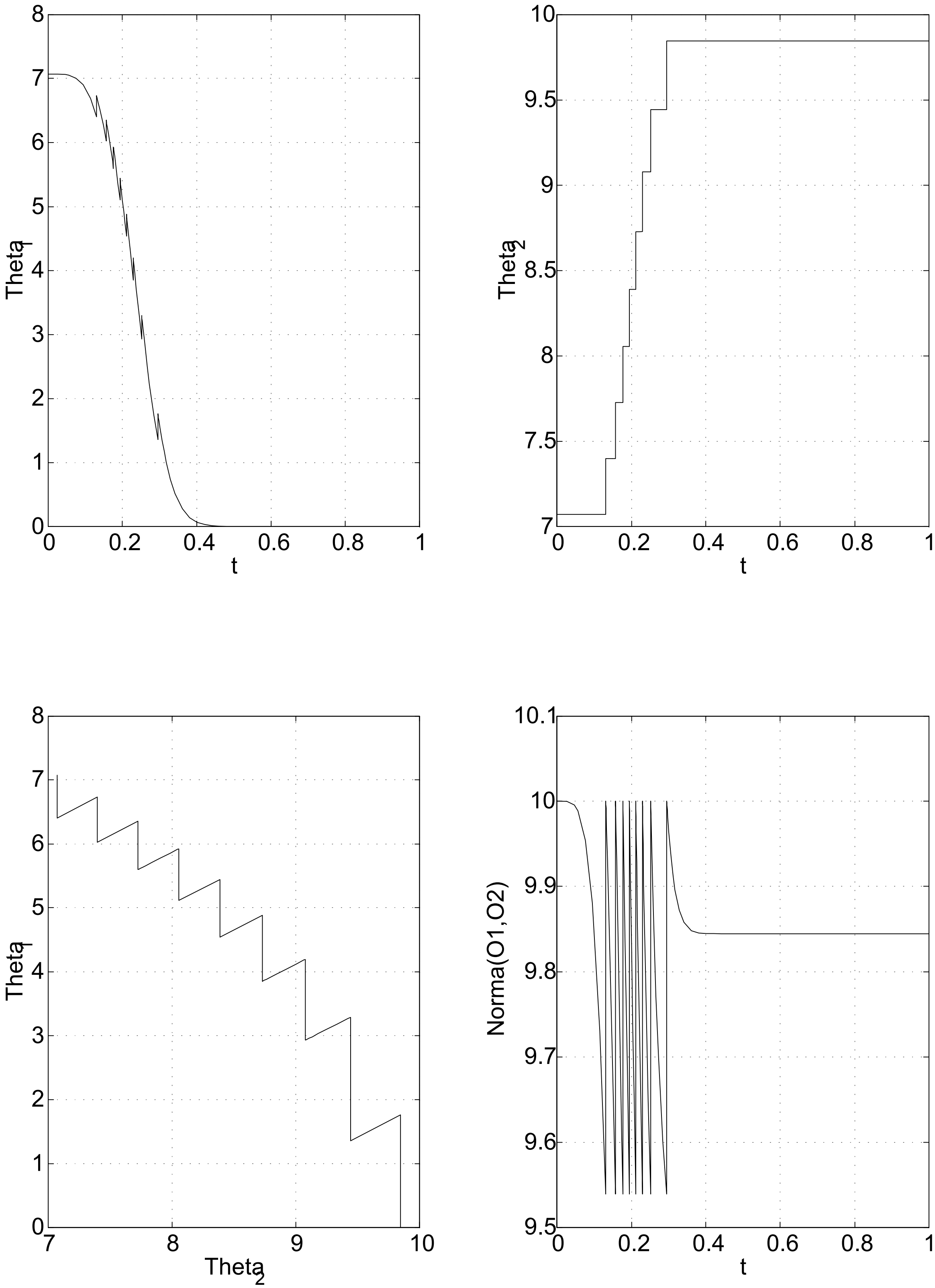


Figura 3.4: Resultados de la simulación para el sistema (3.24).

Capítulo 4

Aproximación Exponencial Propia de Compensadores Impropios: Caso Multivariable

En algunos problemas de control puede ser conveniente, al menos desde un punto de vista analítico, usar compensadores impropios. Sin embargo, en lo que respecta a su implementación, se tienen que realizar aproximaciones propias. En este capítulo, primero se muestra como se pueden realizar aproximaciones exponenciales de manera sencilla. Después, se caracteriza, en términos geométricos, la propiedad externa de una descripción implícita. Finalmente, la combinación de estos dos resultados resuelve el problema de aproximación exponencial propia y generaliza al caso multivariable un resultado previo de Bonilla y Lozano [7].

4.1 Résumé en français

Dans certains problèmes de commande, il peut être utile, ne serait-ce que du point de vue de l'analyse, d'utiliser des compensateurs impropres. Néanmoins, en ce qui concerne leur réalisation pratique, il est alors impératif de passer par des approximations propres que ne modifient pas (trop) les objectifs de commande, c'est à dire, on veut obtenir un filtre strictement propre, disons Σ^f, qui soit internement stable, et qui approxime de manière exponentielle le compensateur impropre , disons Σ^c, et tel que le système global $\Sigma^f \circ \Sigma^c$, soit externement propre. Ce problème (Problème 4), est défini dans la section 4.2.

Dans la section 4.3, on donne une solution au premier point du Problème 4 (Lemme 6), à savoir une réalisation strictement propre et internement stable dont la sortie soit une approximation exponentielle de celle du compensateur impropre.

On met en évidence, dans la section 4.4, les conditions géométriques nécessaires et suffisantes (Théorème 9) qui garantissent que le comportement externe d'une description implicite soit propre. Sous certaines hypothèses supplémentaires, une condition géométrique plus compacte et d'un intérêt pratique est obtenue comme conséquence du Théorème 9 (Corollaire 2).

Dans la section 4.5 on propose les changements de basse (Lemme 7), qui permettent d'identifier aisément les sous-espaces impliqués dans le Corollaire 2. On montre, également que, pour que le système global soit externement propre, il est nécessaire et suffisant (Théorème 10) que les ordres des zéros à l'infini du filtre soient respectivement supérieurs ou égaux à la longueur des chaînes de dérivateurs du compensateur, ce qui généralise au cas multivariable un précédent résultat de Bonilla et Lozano [7].

Finalement, dans la section 4.6 on donne trois exemples illustratifs, dont deux correspondent à des systèmes monovariables (première et deuxième approximations), et un à des systèmes multivariables. Pour le cas monovariable, on considère un compensateur impropre de type double dérivateur. Dans la première approximation, le filtre a un zéro à l'infini d'ordre 1 (voir Figure4.1), et bien que le système global soit bien externement équivalent à celui du compensateur, on ne satisfait pas le Corollaire 2, et par conséquent la propreté externe n'est pas satisfaite ; ceci est du au fait que le filtre proposé ne satisfait pas le Théorème 10. Pour la deuxième approximation, le filtre a un zéro à l' infini d'ordre 2 (voir Figure 4.2), et le système global est externement équivalent à celui du compensateur et en plus, satisfait le Corollaire 2 ; par conséquent, il est externement propre (voir Figure 4.3); dans ce cas là, le Théorème 10 est satisfait.

Pour l'exemple multivariable, on considère un compensateur impropre plus complexe avec plusieurs *blocs* dérivateurs. Ce système est mis sous forme canonique de Kronecker, à partir de laquelle est proposé le filtre pour faire l'approximation. Le système global est externement équivalent à celui du compensateur, satisfait le Corollaire 2, et donc est externement propre.

4.2 Introducción

En muchos problemas de control (por ejemplo: desacoplamiento, rechazo de perturbaciones, etc.), frecuentemente se tienen que considerar compensadores impropios. Esto se debe, ya sea a que las soluciones exactas propias no existen o su obtención (basadas frecuentemente en técnicas de inversión) es mucho más fácil (ver por ejemplo [12, Bonilla y Malabre 1999], [14, Bonilla y Malabre 2000] y [15, Bonilla y Malabre 2001]). Entonces, para su efectiva implementación, se deben realizar aproximaciones propias que no alteren mucho los objetivos de control. En este Capítulo se propone un procedimiento sistemático para obtener una aproximación exponencial. Más precisamente, se da solución al siguiente problema:

Problema 4 *Dado el compensador no propio, $\Sigma^c : \mathcal{U} \to \mathcal{Y}$, con realización:*

$$\begin{aligned} N\dot{\omega}(t) &= \omega(t) + \Gamma u(t); \\ y^*(t) &= \Delta\omega(t) \end{aligned} \tag{4.1}$$

donde $N\colon \mathcal{W} \to \mathcal{W}$, $\Gamma : \mathcal{U} \to \mathcal{W}$, $\Delta : \mathcal{W} \to \mathcal{Y}$ son operadores lineales; N es un operador nilpotente; $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{W}$ son los espacios de la entrada, la salida y la variable descriptora respectivamente, obtener un filtro estrictamente propio, $\Sigma^f : \mathcal{Y} \to \mathcal{Y}$, con realización:

$$\begin{aligned} \dot{z}(t) &= A(\varepsilon)\, z(t) + B(\varepsilon)\, y^*(t); \\ y(t) &= Cz(t) \end{aligned} \tag{4.2}$$

tal que:

1. $\lim_{\varepsilon \to 0} \|y(t) - y^*(t)\| \leq Ke^{-\beta t}$*, con* $K, \beta > 0$ *y* Σ^f *internamente estable para todo* $\varepsilon > 0$

2. *La función de transferencia matricial del sistema global,* $\Sigma^f \circ \Sigma^c$*, sea propia.*

Se asumirá que el compensador no propio (4.1) es completamente observable, y por lo tanto su forma canónica de Kronecker tiene solamente bloques de índices mínimos por renglones (ver [27], [35] y [39]), es decir, tiene la forma:

$$\begin{array}{c} N = D\left\{N_1, \ldots, N_n\right\}, \quad \Delta = D\left\{\Delta_1^T, \ldots, \Delta_n^T\right\} \\ N_i = L\left\{\underline{\chi}^2_{(k_i+1)}\right\}, \quad \Delta_i = \underline{\chi}^{(k_i+1)}_{(k_i+1)}, \quad \text{con } i = 1, \ldots, n \end{array} \tag{4.3}$$

donde $k_i \geq 0$, $i = 1, \ldots, n$, denota los órdenes de los polos al infinito del compensador (4.1).

En la Sección 4.5 se propone una realización estrictamente propia e internamente estable, Σ^f, cuyo comportamiento externo se aproxime de manera exponencial al compensador impropio, Σ^c (solución de la Parte 1 del Problema 4). En la Sección 4.4 se caracteriza la propiedad externa a través de algunos resultados geométricos agradables (Teorema 9 y su Corolario 2). En la Sección 4.5 se muestra que la construcción de Σ^f hace que la función de transferencia matricial, $\left(\Sigma^f \circ \Sigma^c\right)(s)$, sea externamente propia (solución de la Parte 2 del Problema 4). Esto conduce al Teorema Estructural 10, el cual generaliza al caso multivariable un resultado previo para sistemas monovariables presentados por Bonilla y Lozano [7]. En la Sección 4.6 se detallan ejemplos ilustrativos.

Las demostraciones de los resultados de este capítulo se encuentran en el Apéndice A.

4.3 Aproximación Exponencial

En el siguiente resultado se propone la generalización de la aproximación propuesta en [7, Bonilla y Lozano 1998] y se caracteriza su comportamiento.

Lema 6 *Considere el siguiente sistema,* $\Sigma^f : \mathcal{Y} \rightarrow \mathcal{Y}$*,*

$$\begin{array}{rcl} \dot{\bar{x}}(t) & = & A_\beta \bar{x}(t) - \varepsilon^{k+1} y(t); \\ \varepsilon \dot{\hat{x}}(t) & = & A_o \hat{x}(t) + B_o\left(\bar{x}(t) + y^*(t)\right); \\ y(t) & = & C_o \hat{x}(t) \end{array} \tag{4.4}$$

i.e.

$$\begin{array}{rcl} \dot{z}(t) & = & \begin{bmatrix} A_\beta & -\varepsilon^{k+1} C_o \\ (1/\varepsilon) B_o & (1/\varepsilon) A_o \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ (1/\varepsilon) B_o \end{bmatrix} y^*(t) \\ y(t) & = & \begin{bmatrix} 0 & C_o \end{bmatrix} z(t) \end{array}$$

con $z(t) = \begin{bmatrix} \overline{x} & \widehat{x} \end{bmatrix}$, *y donde* $\hat{x} \in \widehat{\mathcal{X}}$; $\bar{x}, y, y^* \in \mathcal{Y}$, $\varepsilon > 0$ *son tal que:*

H1: A_β *y* A_o *son Hurwitz,*

H2: $\mathcal{L}^{-1}\left\{(\mathrm{sI} - (1/\varepsilon) A_o)^{-1}\right\} = \overline{A}_o(t, \varepsilon) e^{-t/\varepsilon}$,

H3: *Los elementos de* $\overline{A}_o(t, \varepsilon)$ *son polinomios en la variable* t/ε*, con grado menor o igual a un entero* $\kappa \geq 2$*,*

H4: $\int_0^\infty C_o \overline{A}_o(\lambda) e^{-\lambda} B_o \mathrm{d}\lambda = \mathrm{I}$*, donde* $\overline{A}_o(\lambda) = \overline{A}_o(\varepsilon\lambda, \varepsilon)$*,*

H5: *$\det\left[I+\varepsilon^{\kappa}C_o(sI-(1/\varepsilon)A_o)^{-1}B_o(sI-A_\beta)^{-1}\right]=\Pi_{i=1}^{n}\left(1+\varepsilon\frac{h_i(\varepsilon)f_i(s)}{g_i(s,\varepsilon)}\right)$, donde ($\varepsilon$ es un número real positivo) $f_i(s)$, $g_i(s)\in\mathbb{R}[s]$, the $g_i(s)$ son polinomios Hurwitz and $h_i(\varepsilon)\in\mathbb{R}[\varepsilon]$.*

H6: *$y^*(t)$ es continua, acotada y Lypchitz.*

Entonces:

$$\lim_{\varepsilon\to 0}\left(y(t)-y^*(t)\right)=e^{A_\beta t}\bar{x}\left(0\right); \quad t>0 \tag{4.5}$$

$$\det\begin{bmatrix}(\mathrm{sI}-A_\beta) & \varepsilon^{k+1}C_o\\ -(1/\varepsilon)B_o & (\mathrm{sI}-(1/\varepsilon)A_o)\end{bmatrix}\ \textit{es Hurwitz} \tag{4.6}$$

Es importante mencionar que la siguiente elección de A_o, B_o y C_o satisfacen los requerimientos del Lema 6:

$$A_o=D\left\{A_1,\ldots,A_n\right\},\quad B_o=D\left\{b_1,\ldots,b_n\right\},\quad C_o=D\left\{c_1^T,\ldots,c_n^T\right\} \tag{4.7}$$

$$\begin{gathered}A_i=-\mathrm{I}_{k_i}+U\{(\underline{\chi}_{k_i}^{2})^T\},\quad b_i=\underline{\chi}_{k_i}^{k_i},\ c_i=\underline{\chi}_{k_i}^{1},\\ \text{con } i=1,...,n,\ \kappa=\max\{k_1,\ldots,k_n\}\end{gathered} \tag{4.8}$$

En efecto,

$$\mathcal{L}^{-1}\{(\varepsilon\mathrm{sI}-A_i)^{-1}\}=\frac{1}{\varepsilon}U\left\{\begin{bmatrix}1 & \frac{t/\varepsilon}{1!} & \cdots\frac{(t/\varepsilon)^{k_i-1}}{(k_i-1)!}\end{bmatrix}\right\}e^{-t/\varepsilon},$$

entonces:

$$\int_0^\infty c_i^T\overline{A}_i(\varepsilon\lambda,\varepsilon)e^{-\lambda}b_i\mathrm{d}\lambda=c_i^TU\{\underline{1}_{k_i}^T\}b_i=1.$$

Además, los parámetros de Markov de cada subsistema $\left\{A_i,b_i,c_i^T\right\}$ satisfacen:

$$\begin{aligned}h_{i,j+1}&=c_i^TA_i^jb_i=0,\quad \text{para } j=0,1,\ldots,k_i-2,\\ i&=1,\ldots,n\ \text{ y }\ h_{i,k_i}=1,\text{para } i=1,\ldots,n\end{aligned} \tag{4.9}$$

y

$$\det\left[\mathrm{I}+\varepsilon^{\kappa}C_o(\mathrm{sI}-(1/\varepsilon)A_o)^{-1}B_o(\mathrm{sI}-A_\beta)^{-1}\right]=\Pi_{i=1}^{n}\left(1+\varepsilon\frac{\varepsilon^{\kappa-k_i}}{(\mathrm{s}+1/\varepsilon)^{k_i}(\mathrm{s}+\beta)}\right).$$

4.4 Propiedad Externa

En el siguiente teorema se dan las condiciones geométricas necesarias y suficientes que garantizan que el comportamiento externo de una descripción implícita sea propia.

Teorema 9 *Si (1.2) es observable y no tiene trayectorias identicamente nulas, independientemente de la acción de entrada, es decir,* $\mathcal{V}_o^* = \{0\}$ *y* $\mathcal{V}_\mathcal{X}^* = \mathcal{X}$*, (1.2) es externamente propia si y solamente si:*

$$\mathcal{V}_{\mathcal{X}_o}^* + \mathcal{S}_{\mathcal{X}_o}^* = \mathcal{X}; \quad \mathcal{V}_{\mathcal{X}_o}^* \cap \mathcal{S}_{\mathcal{X}_o}^* \subset \mathcal{R}_{a0}^* \tag{4.10}$$

$$\dim \left(\frac{\mathcal{V}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^* + \overline{\mathcal{T}}_1^2}{\mathcal{V}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^* + \overline{\mathcal{T}}_1^1} \right) = 0 \tag{4.11}$$

donde $\mathcal{V}_{\mathcal{X}_o}^*$ *y* $\mathcal{S}_{\mathcal{X}_o}^*$ *son, respectivamente, los limites de los dos algoritmos siguientes:*

$$\begin{aligned} \mathcal{V}_{\mathcal{X}_o}^o &= \mathcal{X}; \quad \mathcal{V}_{\mathcal{X}_o}^{\mu+1} = A^{-1}E\mathcal{V}_{\mathcal{X}_o}^\mu \\ \mathcal{S}_{\mathcal{X}_o}^o &= \mathcal{K}_E; \quad \mathcal{S}_{\mathcal{X}_o}^{\mu+1} = E^{-1}A\mathcal{S}_{\mathcal{X}_o}^\mu \end{aligned} \tag{4.12}$$

$\overline{\mathcal{T}}_1^\mu$ *y* $\overline{\mathcal{T}}_2^\mu$ *se obtienen de los algoritmos:*

$$\begin{aligned} \overline{\mathcal{T}}_1^o &= \mathcal{R}_{a0}^*; \quad \overline{\mathcal{T}}_1^{\mu+1} = E^{-1}A\left(\overline{\mathcal{T}}_1^\mu + \mathcal{R}_{a0}^*\right) \\ \overline{\mathcal{T}}_2^o &= \mathcal{X}; \quad \overline{\mathcal{T}}_2^{\mu+1} = A^{-1}E\overline{\mathcal{T}}_2^\mu + \mathcal{R}_{a0}^*. \end{aligned}$$

De este teorema se obtiene el siguiente corolario de interés práctico.

Corolario 2 *Si el sistema implícito (1.2) es exponencialmente observable, no tiene trayectorias identicamente nulas, sin importar la acción de entrada, y tiene únicamente acciones integrales y derivativas, es decir:*

$$\mathcal{V}_o^* = \{0\}, \quad \mathcal{V}_\mathcal{X}^* = \mathcal{X}, \quad y \quad \mathcal{X} = \mathcal{V}_{\mathcal{X}_o}^* \oplus \mathcal{S}_{\mathcal{X}_o}^*$$

entonces (1.2) es externamente propio si y solamente si:

$$E^{-1}A\mathcal{R}_{a0}^* = \mathcal{S}_{\mathcal{X}_o}^* \tag{4.13}$$

4.5 Aproximación Exponencial Propia

Considerando, en conjunto, el compensador impropio (4.1) y el filtro estrictamente propio (4.4) en una descripción implícita, se tiene que el sistema global, $\left(\Sigma^f \circ \Sigma^c\right)$, está dado por la expresión:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t); \\ y(t) &= Cx(t) \end{aligned} \tag{4.14}$$

donde:

$$E = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & N \end{bmatrix}, \ A = \begin{bmatrix} A_p & B_p \\ 0 & \mathrm{I} \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}, C = \begin{bmatrix} C_p & 0 \end{bmatrix} \tag{4.15}$$

con:

$$A_p = \begin{bmatrix} A_\beta & -\varepsilon^{k+1} C_o \\ (1/\varepsilon) B_o & (1/\varepsilon) A_o \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ (1/\varepsilon) B_o \Delta \end{bmatrix}, \tag{4.16}$$
$$C_p = \begin{bmatrix} 0 & C_o \end{bmatrix}, \quad x^T = \begin{bmatrix} \overline{x}^T & \widehat{x}^T & \omega^T \end{bmatrix}.$$

Tomando en cuenta las formas particulares de Σ^c y Σ^f, de (4.3), (4.7) y (4.15) se obtiene:

$$\begin{aligned} B_p &= [B_{p_1} | B_{p_2} | \cdots | B_{p_n}]; \quad B_{p_i} = [0 | \cdots | 0 | b_{p_i}] \\ b_{p_i}^T &= \left[0 | \, 0 | \cdots | 0 | \, (1/\varepsilon) b_i^T | \, 0 | \cdots | 0 \right], \quad \text{con } i = 1, ..., n. \end{aligned} \tag{4.17}$$

Lema 7 *Se definen los siguientes dos cambios de base:*

$$R = \begin{bmatrix} \mathrm{I} & R_p \\ 0 & \mathrm{I} \end{bmatrix}, \quad L = \begin{bmatrix} \mathrm{I} & L_p \\ 0 & \mathrm{I} \end{bmatrix},$$

donde:

$$\begin{aligned} R_p &= \begin{bmatrix} R_{p_1} | & R_{p_2} | & \cdots & | R_{p_n} \end{bmatrix} \\ R_{p_i} &= \begin{bmatrix} A_p^{k_i - 1} b_{p_i} | & \cdots & | A_p b_{p_i} & | b_{p_i} & | 0 \end{bmatrix} \\ L_p &= - \left(A_p R_p + B_p \right). \end{aligned} \tag{4.18}$$

Entonces:

$$R_p + L_p N = 0 \tag{4.19}$$

$$\begin{aligned} C_p A_p^j b_{p_i} &= 0, \quad \textit{para } j = 0, 1, \ldots, k_i - 2, \quad i = 1, \ldots, n \\ C_p A_p^{k_i - 1} b_{p_i} &= \frac{1}{\varepsilon^{k_i}} \underline{\chi}_n^i, \quad \textit{para } i = 1, \ldots, n \end{aligned} \tag{4.20}$$

Ahora, en vista del Lema 7, se tiene:

$$LER = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & N \end{bmatrix}, \quad LAR = \begin{bmatrix} A_p & 0 \\ 0 & \mathrm{I} \end{bmatrix}, \quad CR = \begin{bmatrix} C_p & C_n \end{bmatrix} \tag{4.21}$$

donde:

$$C_n = D \left\{ \nu_1^T, ..., \nu_n^T \right\} \; ; \quad \nu_i = \frac{1}{\varepsilon^{k_i}} \underline{\chi}^1_{(k_i+1)}, \quad \text{para } i = 1, ..., n. \tag{4.22}$$

De esta manera, por simple calculo, se puede verificar de (4.21) y (4.22) que:

$$E^{-1} A \mathcal{R}^*_{a0} = \mathcal{S}^*_{\mathcal{X}_o},$$

es decir, el sistema global (4.14)-(4.15) es externamente propio y satisface el Lema 6.

De (4.3), se puede ver fácilmente que el entero n corresponde al número de cadenas de derivadores del compensador impropio Σ^c en (4.1), cada cadena de longitud k_i. Por otra parte, los parámetros de Markov igual a cero que aparecen en (4.20) expresan el hecho que el orden de los ceros al infinito del filtro estrictamente propio Σ^f son mayores o iguales a k_i. Esto se expresa en el siguiente teorema.

Teorema 10 *Considere el filtro propio, Σ^f, diseñado como en el Lema 6 para aproximar de manera exponencial el compensador impropio Σ^f. Entonces el sistema global $\left(\Sigma^f \circ \Sigma^c\right)$ es externamente propio si y solamente si los ordenes de los ceros al infinito de Σ^f son respectivamente mayor o igual a k_i.*

Este resultado generaliza al caso *multivariable* el filtro introducido en [7, Bonilla y Lozano 1998] para sistemas monovariables.

Comentario 3 *Como se mencionó en el Capítulo 1, una aplicación importante de estos resultados se encuentra en la síntesis práctica de compensadores impropios, entre los que podemos mencionar la retroalimentación proporcional derivativa (ver por ejemplo [12, Bonilla y Malabre 1999], [14, Bonilla y Malabre 2000] y [15, Bonilla y Malabre 2001]). Note que se puede utilizar idénticamente el mismo procedimiento para implementar otro tipo de sistemas impropios, por ejemplo observadores de estados derivativos, es decir, sistemas que proporcionan una estimación exponencial de $\dot{x}(t)$ y $x(t)$ (ecuación (4.5) con $y^*(t) = \left[\begin{array}{cc} \dot{x}^T(t) & x^T(t) \end{array}\right]^T$ y el Teorema 10).*

4.6 Ejemplos Ilustrativos

4.6.1 Sistema Monovariable

Para aclarar las ideas principales de este Capítulo, considere el siguiente sistema impropio que tiene un polo al infinito de orden dos:

$$\begin{aligned} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{\xi} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} u \\ y^* &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \xi \end{aligned} \tag{4.23}$$

y con función de transferencia $G(s) = s^2$.

En las siguientes dos subsecciones se va a considerar la aproximación propuesta en el Lema 6. En la primera aproximación se toma un sistema con un cero al infinito de orden uno y en el segundo uno de orden dos.

Primera Aproximación

Considere el sistema de la Figura 4.1, cuya forma descriptor está dada por la expresión:

$$\begin{aligned} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \dot{\overline{x}} &= \left[\begin{array}{cc|ccc} -\beta & -\varepsilon^2 & 0 & 0 & 0 \\ 1/\varepsilon & -1/\varepsilon & 0 & 0 & 1/\varepsilon \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \overline{x} + \left[\begin{array}{c} 0 \\ 0 \\ \hline -1 \\ 0 \\ 0 \end{array}\right] u \\ y &= \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \end{array}\right] \overline{x} \end{aligned} \tag{4.24}$$

donde $\overline{x} = \left[\begin{array}{ccc} x & z & \xi^T \end{array}\right]^T$ y con función de transferencia $G(s) = s^2 \frac{s+\beta}{s^2\varepsilon + (1+\beta\varepsilon)s + \beta + \varepsilon^2}$ que es impropia y por lo tanto no resuelve nuestro problema. Aún asi, se puede verificar que $\lim_{\varepsilon \to 0} G(s) = s^2$, i.e., el sistema global (4.24) es externamente equivalente al compensador (4.23) cuando $\varepsilon \to 0$.

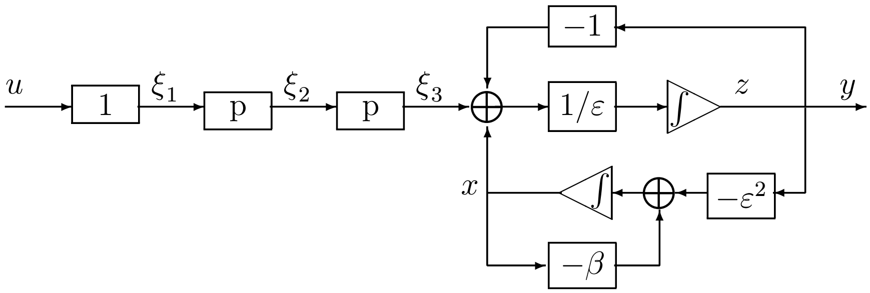


Figura 4.1: Primera aproximación

A continuación se mostrará que el sistema global no es externamente propio.

Comparando el sistema (4.24) con (4.14)–(4.17), se obtiene:

$$A_p = \begin{bmatrix} -\beta & -\varepsilon^2 \\ 1/\varepsilon & -1/\varepsilon \end{bmatrix}, \quad b_{p1} = \begin{bmatrix} 0 \\ 1/\varepsilon \end{bmatrix}, \quad B_p = B_{p1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\varepsilon \end{bmatrix}.$$

Entonces,

$$R_p = \begin{bmatrix} -\varepsilon & 0 & 0 \\ -1/\varepsilon^2 & 1/\varepsilon & 0 \end{bmatrix}, \quad L_p = \begin{bmatrix} -\varepsilon\beta - 1 & \varepsilon & 0 \\ 1 - 1/\varepsilon^3 & 1/\varepsilon^2 & -1/\varepsilon \end{bmatrix},$$

por lo tanto:

$$R = \begin{bmatrix} \mathrm{I}_2 & R_p \\ 0 & \mathrm{I}_3 \end{bmatrix}, \quad L = \begin{bmatrix} \mathrm{I}_2 & L_p \\ 0 & \mathrm{I}_3 \end{bmatrix}.$$

Multiplicando por la izquierda (4.24) por L y haciendo $\overline{x} = R\widetilde{x}$, se obtiene:

$$\begin{aligned} \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \dot{\widetilde{x}} &= \left[\begin{array}{cc|ccc} -\beta & -\varepsilon^2 & 0 & 0 & 0 \\ 1/\varepsilon & -1/\varepsilon & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \widetilde{x} + \left[\begin{array}{c} (\varepsilon\beta + 1) \\ (1/\varepsilon^3 - 1) \\ \hline -1 \\ 0 \\ 0 \end{array}\right] u \\ y &= \left[\begin{array}{cc|ccc} 0 & 1 & -1/\varepsilon^2 & 1/\varepsilon & 0 \end{array}\right] \widetilde{x} \end{aligned} \qquad (4.25)$$

con función de transferencia $G(s) = s^2 \frac{s+\beta}{s^2\varepsilon + (1+\beta\varepsilon)s + \beta + \varepsilon^2}$.

Haciendo la comparación de (4.25) con (4.21), se pueden identificar fácilmente las matrices N, A_p, C_p y C_n. Aplicando, ahora, los algoritmos (1.15) y (4.12) a (4.25), se obtiene:

$$\mathcal{R}^*_{a0} = \{e_5\}\,; \quad E^{-1}A\mathcal{R}^*_{a0} = \{e_4, e_5\}\,; \quad \mathcal{S}^*_{\mathcal{X}_o} = \{e_3, e_4, e_5\}$$

de donde se observa que $\mathcal{S}^*_{\mathcal{X}_o} \neq E^{-1}A\mathcal{R}^*_{a0}$. De esta manera, del Corolario 2 se tiene que el sistema (4.24) no es externamente propio.

Segunda Aproximación

Considere el sistema de la Figura 4.2, cuya forma descriptor está dada por la expresión:

$$
\begin{aligned}
\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \dot{\overline{x}} &= \left[\begin{array}{ccc|ccc} -\beta & -\varepsilon^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon & 0 & 0 & 1/\varepsilon^2 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \overline{x} + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline -1 \\ 0 \\ 0 \end{array}\right] u \qquad (4.26) \\
y &= \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \overline{x}
\end{aligned}
$$

con $\overline{x} = \left[\begin{array}{cccc} x & z_1 & z_2 & \xi^T \end{array}\right]^T$ y con función de transferencia $G(s) = s^2 \frac{s+\beta}{\varepsilon^2 s^3 + (\beta\varepsilon^2 + 2\varepsilon)s^2 + (2\beta\varepsilon + 1)s + \beta + \varepsilon^2}$ que es propia. Se puede verificar que $\lim_{\varepsilon \to 0} G(s) = s^2$, i.e., el sistema global (4.26) es externamente equivalente al compensador (4.23) cuando $\varepsilon \to 0$.

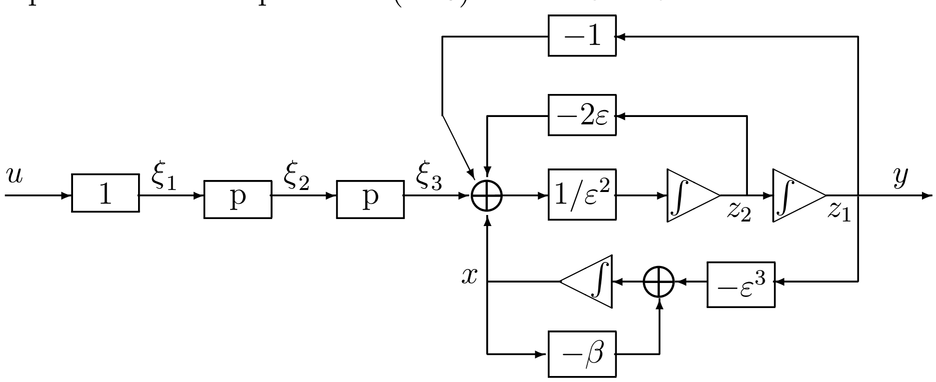


Figura 4.2: Segunda aproximación

Para verificar que el sistema global es externamente propio se procede de la siguiente manera. Comparando el sistema (4.26) con (4.14)–(4.17), se obtiene:

$$
A_p = \begin{bmatrix} -\beta & -\varepsilon^2 & 0 \\ 0 & 0 & 1 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon \end{bmatrix}, \quad b_{p1} = \begin{bmatrix} 0 \\ 0 \\ 1/\varepsilon^2 \end{bmatrix}, \quad B_p = B_{p1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\varepsilon^2 \end{bmatrix}.
$$

Entonces:

$$
R_p = \begin{bmatrix} 0 & 0 & 0 \\ 1/\varepsilon^2 & 0 & 0 \\ -2/\varepsilon^3 & 1/\varepsilon^2 & 0 \end{bmatrix}, \quad L_p = \begin{bmatrix} 1 & 0 & 0 \\ 2/\varepsilon^3 & -1/\varepsilon^2 & 0 \\ -3/\varepsilon^4 & 2/\varepsilon^3 & -1/\varepsilon^2 \end{bmatrix},
$$

por lo tanto:

$$
R = \begin{bmatrix} \mathrm{I}_3 & R_p \\ 0 & \mathrm{I}_3 \end{bmatrix}, \quad L = \begin{bmatrix} \mathrm{I}_3 & L_p \\ 0 & \mathrm{I}_3 \end{bmatrix}.
$$

Multiplicando por la izquierda (4.26) por L y haciendo $\overline{x} = R\widetilde{x}$, se obtiene:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \dot{\widetilde{x}} = \left[\begin{array}{ccc|ccc} -\beta & -\varepsilon^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \widetilde{x} + \left[\begin{array}{c} -1 \\ -2/\varepsilon^3 \\ 3/\varepsilon^4 \\ \hline -1 \\ 0 \\ 0 \end{array}\right] u \qquad (4.27)$$

$$y = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1/\varepsilon^2 & 0 & 0 \end{array}\right] \widetilde{x}.$$

con función de transferencia $G(s) = s^2 \frac{s+\beta}{\varepsilon^2 s^3 + (\beta\varepsilon^2 + 2\varepsilon)s^2 + (2\beta\varepsilon + 1)s + \beta + \varepsilon^2}$.

Haciendo la comparación de (4.27) con (4.21), se pueden identificar fácilmente las matrices N, A_p, C_p y C_n. Aplicando nuevamente los algoritmos (1.15) y (4.12) a (4.27), se obtiene:

$$\mathcal{R}^*_{a0} = \{e_5, e_6\}\,; \quad E^{-1}A\mathcal{R}^*_{a0} = \{e_4, e_{5,}e_6\}\,; \quad \mathcal{S}^*_{\mathcal{X}_o} = \{e_4, e_5, e_6\}$$

donde se observa que $\mathcal{S}^*_{\mathcal{X}_o} = E^{-1}A\mathcal{R}^*_{a0}$. De esta manera, del Corolario 2 se tiene que el sistema (4.26) es externamente propio. En efecto, es externamente equivalente al siguiente sistema propio (ver la Figura 4.3):

$$\begin{aligned} \dot{\widehat{x}} &= \begin{bmatrix} -\beta & -\varepsilon^2 & 0 \\ 0 & 0 & 1 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon \end{bmatrix} \widehat{x} + \begin{bmatrix} -1 \\ -2/\varepsilon^3 \\ 3/\varepsilon^4 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \widehat{x} + (1/\varepsilon^2)u \end{aligned} \qquad (4.28)$$

cuya función de transferencia es $G(s) = s^2 \frac{s+\beta}{\varepsilon^2 s^3 + (\beta\varepsilon^2 + 2\varepsilon)s^2 + (2\beta\varepsilon + 1)s + \beta + \varepsilon^2}$

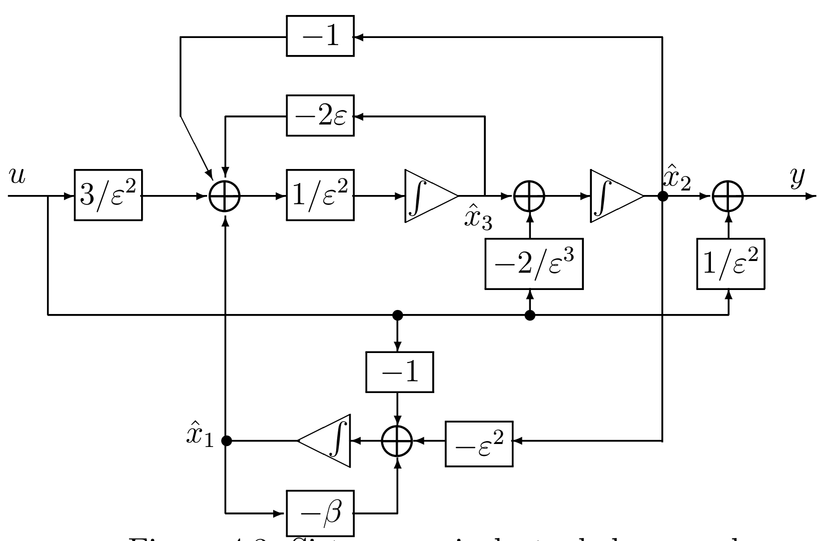


Figura 4.3: Sistema equivalente de la segunda aproximación

4.6.2 Sistema Multivariable

Considere el siguiente sistema multivariable no propio:

$$
\begin{aligned}
\left[\begin{array}{ccc|cc|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right]\dot{\xi} &= \left[\begin{array}{ccc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & -1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]\xi + \left[\begin{array}{ccc}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\ \hline
2 & 0 & 0 \\
-2 & 0 & 0 \\ \hline
-1 & -1 & -1 \\
0 & 0 & 0
\end{array}\right]u \\
z &= \left[\begin{array}{ccc|cc|cc}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & -1 & 1
\end{array}\right]\xi
\end{aligned} \tag{4.29}
$$

cuyo comportamiento entrada-salida se describe por la siguiente ecuación diferencial[1]:

$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & (\mathrm{p}+1) & 0 \\ 0 & 0 & (\mathrm{p}+1) \end{bmatrix} z = \begin{bmatrix} \mathrm{p} & \mathrm{p}^2 & 0 \\ -\mathrm{p}(\mathrm{p}+2) & -\mathrm{p}^3 & 0 \\ \mathrm{p}^2 & \mathrm{p}^2 & (\mathrm{p}^2-1) \end{bmatrix} u \tag{4.30}
$$

Ahora llevemos el sistema (4.29) a la forma canónica de Kronecker. Para esto, se define $\bar{z} = T_{L_o} z$, $\bar{\xi} = T_R^{(-1)}\xi$ y se multiplica, por la izquierda, la ecuación descriptora por T_L, donde:

$$
T_{L_o} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad T_L = \left[\begin{array}{ccc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\ \hline
-1 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & -1 & 0 & 0
\end{array}\right], \quad T_R = \left[\begin{array}{ccc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline
0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0
\end{array}\right],
$$

es decir:

$$
\begin{aligned}
\left[\begin{array}{ccc|cc|c|c}
0 & 0 & 0 & & & & \\
1 & 0 & 0 & & & & \\
0 & 1 & 0 & & & & \\ \hline
 & & & 0 & 0 & & \\
 & & & 1 & 0 & & \\ \hline
 & & & & & 0 & \\ \hline
 & & & & & & 1
\end{array}\right]\dot{\bar{\xi}} &= \left[\begin{array}{ccc|cc|c|c}
1 & 0 & 0 & & & & \\
0 & 1 & 0 & & & & \\
0 & 0 & 1 & & & & \\ \hline
 & & & 1 & 0 & & \\
 & & & 0 & 1 & & \\ \hline
 & & & & & 1 & \\ \hline
 & & & & & & -1
\end{array}\right]\bar{\xi} + \left[\begin{array}{ccc}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\ \hline
-1 & -1 & -1 \\
1 & 1 & 1 \\ \hline
2 & 1 & 0 \\ \hline
1 & 1 & 0
\end{array}\right]u \\
\bar{z} &= \left[\begin{array}{ccc|cc|c|c}
0 & 0 & 1 & & & & \\ \hline
 & & & 0 & 1 & & 1 \\ \hline
0 & 1 & 0 & & & 1 & 1
\end{array}\right]\bar{\xi}
\end{aligned} \tag{4.31}
$$

donde $\bar{\xi} = [\ \bar{\xi}_{i,1}^T \quad \bar{\xi}_{i,2}^T \quad \bar{\xi}_{p,1}^T \quad \bar{\xi}_{p,2}^T\ ]^T$. Para obtener la forma particular propuesta en (4.3), se descompone (4.31) como sigue (recuerde que $z = T_{L_o}^{(-1)}\bar{z}$):

$$
\begin{aligned}
\left[\begin{array}{ccc|cc|cc}
0 & 0 & 0 & & & & \\
1 & 0 & 0 & & & & \\
0 & 1 & 0 & & & & \\ \hline
 & & & 0 & 0 & & \\
 & & & 1 & 0 & & \\ \hline
 & & & & & 0 & 0 \\
 & & & & & 1 & 0
\end{array}\right]\dot{w} &= \left[\begin{array}{ccc|cc|cc}
1 & 0 & 0 & & & & \\
0 & 1 & 0 & & & & \\
0 & 0 & 1 & & & & \\ \hline
 & & & 1 & 0 & & \\
 & & & 0 & 1 & & \\ \hline
 & & & & & 1 & 0 \\
 & & & & & 0 & 1
\end{array}\right]w + \underbrace{\left[\begin{array}{ccc}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\ \hline
-1 & -1 & -1 \\
1 & 1 & 1 \\ \hline
0 & -1 & 0 \\
-1 & 0 & 0
\end{array}\right]}_{\Gamma}u \\
y^* &= \left[\begin{array}{ccc|cc|cc}
0 & 0 & 1 & & & & \\ \hline
 & & & 0 & 1 & & \\ \hline
 & & & & & 0 & 1
\end{array}\right]w
\end{aligned} \tag{4.32}
$$

[1]Recuerde que p es el operador derivada (ver la notación en la página 4).

$$\dot{\bar{\xi}}_{p,2} = -\bar{\xi}_{p,2} + \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} u \tag{4.33}$$

$$z = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} y^* + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \bar{\xi}_{p,2} + \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} u \tag{4.34}$$

Note que la descripción entrada-salida de (4.32) es:

$$y^* = \begin{bmatrix} \mathrm{p} & \mathrm{p}^2 & 0 \\ (\mathrm{p}-1) & (\mathrm{p}-1) & (\mathrm{p}-1) \\ 1 & \mathrm{p} & 0 \end{bmatrix} u \tag{4.35}$$

Ahora se procede a aproximar el sistema no propio (4.32).

1. Comparando (4.32) con (4.1) y (4.3), se obtiene: $n = 3$, $k_1 = 2$, y $k_2 = k_3 = 1$; de esta manera $\kappa = 2$. La matriz Γ se indica en (4.32).

2. Usando (4.7) y (4.8), se obtiene el siguiente sistema (c.f. (4.14)-(4.16)):

$$\left[\begin{array}{ccc|cccc|ccc|cc|cc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{array}\right] \dot{x} = \left[\begin{array}{ccc}
0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\ \hline
0&-1&0\\-1&0&0\\0&0&0\\ \hline
-1&-1&-1\\1&1&1\\ \hline
0&-1&0\\-1&0&0
\end{array}\right] u$$

$$+ \left[\begin{array}{ccc|cccc|ccc|cc|cc}
-\beta&0&0&-\varepsilon^3&0&0&0&0&0&0&0&0&0&0\\
0&-\beta&0&0&0&-\varepsilon^3&0&0&0&0&0&0&0&0\\
0&0&-\beta&0&0&0&-\varepsilon^3&0&0&0&0&0&0&0\\ \hline
0&0&0&-1/\varepsilon&1/\varepsilon&0&0&0&0&0&0&0&0&0\\
1/\varepsilon&0&0&0&-1/\varepsilon&0&0&0&0&1/\varepsilon&0&0&0&0\\
0&1/\varepsilon&0&0&0&-1/\varepsilon&0&0&0&0&0&1/\varepsilon&0&0\\
0&0&1/\varepsilon&0&0&0&-1/\varepsilon&0&0&0&0&0&0&1/\varepsilon\\ \hline
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right] x \tag{4.36}$$

$$y = \left[\begin{array}{ccc|cccc|ccccccc}
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0
\end{array}\right] x$$

3. Las matrices R_p y L_p definidas en el Lema 7 son (see (4.18)):

$$R_p = \left[\begin{array}{ccc|cc|cc}
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\
0&0&0&0&0&0&0\\ \hline
1/\varepsilon^2&0&0&0&0&0&0\\
-1/\varepsilon^2&1/\varepsilon&0&0&0&0&0\\
0&0&0&1/\varepsilon&0&0&0\\
0&0&0&0&0&1/\varepsilon&0
\end{array}\right], \quad L_p = \left[\begin{array}{ccc|cc|cc}
\varepsilon&0&0&0&0&0&0\\
0&0&0&\varepsilon^2&0&0&0\\
0&0&0&0&0&\varepsilon^2&0\\ \hline
2/\varepsilon^3&-1/\varepsilon^2&0&0&0&0&0\\
-1/\varepsilon^3&1/\varepsilon^2&-1/\varepsilon&0&0&0&0\\
0&0&0&1/\varepsilon^2&-1/\varepsilon&0&0\\
0&0&0&0&0&1/\varepsilon^2&-1/\varepsilon
\end{array}\right];$$

note que (4.19) y (4.20) se satisfacen.

4. Definiendo $\bar{x} = \begin{bmatrix} I & R_p \\ 0 & I \end{bmatrix}^{(-1)} x$ y premultiplicando (4.36) por $\begin{bmatrix} I & L_p \\ 0 & I \end{bmatrix}$, se obtiene (c.f. (4.21)):

$$\left[\begin{array}{ccc|cccc|ccc|cc|cc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ \hline
0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{array}\right]\dot{\bar{x}} = \left[\begin{array}{ccc}
0 & -\varepsilon & 0\\
-\varepsilon^2 & -\varepsilon^2 & -\varepsilon^2\\
0 & -\varepsilon^2 & 0\\ \hline
1/\varepsilon^2 & -2/\varepsilon^3 & 0\\
-1/\varepsilon^2 & 1/\varepsilon^3 & 0\\
-(1/\varepsilon^2+1/\varepsilon) & -(1/\varepsilon^2+1/\varepsilon) & -(1/\varepsilon^2+1/\varepsilon)\\
1/\varepsilon & -1/\varepsilon^2 & 0\\ \hline
0 & -1 & 0\\
-1 & 0 & 0\\
0 & 0 & 0\\ \hline
-1 & -1 & -1\\
1 & 1 & 1\\ \hline
0 & -1 & 0\\
-1 & 0 & 0
\end{array}\right]u$$

$$+\left[\begin{array}{ccc|cccc|ccc|cc|cc}
-\beta&0&0&-\varepsilon^3&0&0&0&0&0&0&0&0&0&0\\
0&-\beta&0&0&0&-\varepsilon^3&0&0&0&0&0&0&0&0\\
0&0&-\beta&0&0&0&-\varepsilon^3&0&0&0&0&0&0&0\\ \hline
0&0&0&-1/\varepsilon&1/\varepsilon&0&0&0&0&0&0&0&0&0\\
1/\varepsilon&0&0&0&-1/\varepsilon&0&0&0&0&0&0&0&0&0\\
0&1/\varepsilon&0&0&0&-1/\varepsilon&0&0&0&0&0&0&0&0\\
0&0&1/\varepsilon&0&0&0&-1/\varepsilon&0&0&0&0&0&0&0\\ \hline
0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ \hline
0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}\right]\bar{x}$$

$$y = \left[\begin{array}{ccc|cccc|ccc|cc|cc}
0&0&0&1&0&0&0&1/\varepsilon^2&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1/\varepsilon&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&1/\varepsilon&0
\end{array}\right]\bar{x} \tag{4.37}$$

5. Aplicando los algoritmos (1.15) y (4.12) a (4.37), se obtiene: $\mathcal{R}^*_{a0} = \{e_9,\ e_{10},\ e_{12},\ e_{14}\}$

y $E^{-1}A\mathcal{R}^*_{a0} = \{e_8,\ e_9,\ e_{10},\ e_{11},\ e_{12},\ e_{14}\} = \mathcal{S}^*_{\mathcal{X},0}$. Por lo tanto, del Corolario 2 se tiene que el sistema (4.37) es externamente propio y es externamente equivalente a (sólo aplique el algoritmo de minimización matricial propuesto en [11]):

$$\dot{\hat{x}} = \left[\begin{array}{ccc|cccc}
-\beta&0&0&-\varepsilon^3&0&0&0\\
0&-\beta&0&0&0&-\varepsilon^3&0\\
0&0&-\beta&0&0&0&-\varepsilon^3\\ \hline
0&0&0&-1/\varepsilon&1/\varepsilon&0&0\\
1/\varepsilon&0&0&0&-1/\varepsilon&0&0\\
0&1/\varepsilon&0&0&0&-1/\varepsilon&0\\
0&0&1/\varepsilon&0&0&0&-1/\varepsilon
\end{array}\right]\hat{x} + \left[\begin{array}{ccc}
0 & -\varepsilon & 0\\
-\varepsilon^2 & -\varepsilon^2 & -\varepsilon^2\\
0 & -\varepsilon^2 & 0\\ \hline
1/\varepsilon^2 & -2/\varepsilon^3 & 0\\
-1/\varepsilon^2 & 1/\varepsilon^3 & 0\\
-(1/\varepsilon^2+1/\varepsilon) & -(1/\varepsilon^2+1/\varepsilon) & -(1/\varepsilon^2+1/\varepsilon)\\
1/\varepsilon & -1/\varepsilon^2 & 0
\end{array}\right]u \tag{4.38}$$

$$y = \left[\begin{array}{ccc|cccc}
0&0&0&1&0&0&0\\
0&0&0&0&0&1&0\\
0&0&0&0&0&0&1
\end{array}\right]\hat{x} + \begin{bmatrix}
0 & 1/\varepsilon^2 & 0\\
1/\varepsilon & 1/\varepsilon & 1/\varepsilon\\
0 & 1/\varepsilon & 0
\end{bmatrix}u$$

Note que la descripción entrada-salida de (4.38) es:

$$
\begin{aligned}
F(\mathrm{p})y &= (\mathrm{p}+\beta)\begin{bmatrix} \mathrm{p} & \mathrm{p}^2 & 0 \\ (\mathrm{p}-1) & (\mathrm{p}-1) & (\mathrm{p}-1) \\ 1 & \mathrm{p} & 0 \end{bmatrix} u && (4.39)\\
F(\mathrm{p}) &= \begin{bmatrix} (\varepsilon \mathrm{p}+1)^2(\mathrm{p}+\beta)+\varepsilon^3 & 0 & 0 \\ 0 & (\varepsilon \mathrm{p}+1)(\mathrm{p}+\beta)+\varepsilon^3 & 0 \\ 0 & 0 & (\varepsilon \mathrm{p}+1)(\mathrm{p}+\beta)+\varepsilon^3 \end{bmatrix} && (4.40)
\end{aligned}
$$

Entonces, de (4.39) y (4.35), se tiene que:

$$F(\mathrm{p})(y) = (\mathrm{p}+\beta)y^*(t).$$

Por lo tanto existe un número real positivo, ε^*, tal que $\det F(\mathrm{p})$ es Hurwitz para todo $\varepsilon \in (0, \varepsilon^*)$. También se observa que los polos dominantes estan en $-\beta$ y los demás estan muy cercanos a $-1/\varepsilon$ (para ε muy pequeña). Además,

$$y(t) \approx y^*(t) + e^{-\beta t}(y(0) - y^*(0)) \text{ para } \varepsilon \text{ muy pequeño.}$$

El filtro buscado está dado por (4.38), (4.33) y (cf (4.34)):

$$z = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \bar{\xi}_{p,2} + \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} u \tag{4.41}$$

Capítulo 5

Semi-rechazo de Variaciones Estructurales Internas en Sistemas Lineales

En este capítulo se consideran sistemas lineales con variaciones de estructura interna. Bajo algunas ligeras suposiciones, el control de tales sistemas es, en efecto, posible gracias al uso de modelos implícitos y a retroalimentaciones proporcional y derivativa. Sin embargo, el diseño de tales leyes de control usualmente requieren aproximaciones adecuadas para los derivadores, por lo tanto, se necesitan aproximaciones proporcionales de la ley de control proporcional derivativa correspondiente a la solución exacta. La principal contribución de este capítulo es obtener una versión *cercana* del problema a través de una retroalimentación proporcional pura (de alta ganancia). El diseño es diferente, pero muy cercano al procedimiento utilizado para obtener retroalimentaciones proporcional derivativa. Un punto muy interesante es la interpretación teórica de sistemas del proceso clásico de la *integración por partes* que equivale a un cambio de base particular.

5.1 Résumé en français

Comme spécifié dans le Chapitre 2, la commande de systèmes à structure variable est possible grâce à l'utilisation de modèles implicites et aux lois de commande proportionnelles et dérivées (P-D). La contribution principale de ce chapitre, consiste en l'obtention d'une version approchante du problème, au travers d'un retour d'état proportionnel pur (à grand gain). La synthèse est différente, mais très proche de celle du procédé utilisé pour obtenir la loi de commande P-D. Un produit annexe de cette démarche alternative est l'interprétation, du point de vue de la théorie des systèmes, du procédé classique de l'intégration par parties, et qui est traduit ici par un changement de base (généralisé) particulier.

Dans la section 5.2 on propose le problème (Problème 5) d'approximer le retour d'état P-D par un retour d'état purement proportionnel, de manière à conserver les propriétés structurelles décrites dans le Chapitre 2.

Dans la section 5.3, on obtient l'équivalence, dans le cadre de la théorie des systèmes, de l'intégration par parties. Dans cette section, on montre qu'il est nécessaire d'appliquer plusieurs fois de suite l'intégration par parties, (deux fois sur l'exemple simple d'un dérivateur d'ordre 1) pour obtenir des informations utiles, et aboutir aux conclusions (désirées), c'est à dire, observer ce qui se passe avec le comportement externe quand un certain ε tend vers zéro. Dans la sous-section 5.3.1 on montre l'équivalence entre l'intégration par parties et les descriptions généralisées de type Fliess (i.e. avec des dérivées explicites des entrées). Ceci est illustré sur l'exemple simple choisi, tant pour les changements de variable sous-jacents que pour les conclusions sur la propreté interne et externe, respectivement, des descriptions implicites.

Dans la section 5.4 on obtient une série de décompositions géométriques des espaces, qui permettent de proposer une approximation propre de la loi de commande P-D. Pour celle-ci, on montre que si certaines conditions géométriques sont satisfaites, alors le système implicite peut être exprimé dans la forme donnée par le Lemme 8.

Dans la section 5.5, on montre la contribution principale du chapitre. D'abord, on analyse le retour d'état dérivé dans les bases choisies et à partir de là, on propose le retour d'état proportionnel qui l'approxime. On utilise alors un résultat du Chapitre 2 (voir Théorème 5) pour établir l'existence d'un retour d'état proportionnel solution du Problème 5 (Théorème 11).

Finalement, dans la section 5.6, on montre les contributions du chapitre sur un exemple illustratif où l'on pourra observer que l'exemple de la section 1.2 du Chapitre 1 ne satisfait pas la condition géométrique requise mais que ce problème peut être résolu en ajoutant un intégrateur externe. Néanmoins, il faut noter que la région des variations qui préservent la stabilité interne est réduite quand on fait une approximation proportionnelle du retour d'état P-D.

5.2 Introducción

Ya se ha mostrado (ver Capitulo 2) que la solución al problema de controlar un conjunto de sistemas lineales está basada en el uso de una ley de control proporcional derivativa. La parte derivativa de la ley de control, F_d, es crucial para resolver el problema, es decir, una retroalimentación estática no es, generalmente, suficiente. Surge ahora una cuestión natural desde un punto de vista práctico: *¿qué se puede hacer si estamos restringidos a utilizar únicamente retroalimentación proporcional?.* Más precisamente, *¿cómo aproximar la retroalimentación proporcional derivativa obtenida, de manera que se conserven (tan cercanas como sea posible) las propiedades estructurales agradables mencionadas en el Capítulo 2?.* Esta cuestión se establece en el siguiente problema ARISV.[1]

Las demostraciones de los resultados de este Capítulo se encuentran en el Apéndice B.

Problema 5 (ARISV) *Considere la descripción implícita global (1.3). Sea $u^* = F_p^* x + F_d^* \dot{x} + \bar{u}$ una ley de control que resuelve el Problema 1 (ver Capítulo 2) y sea y^* la salida obtenida con esta retroalimentación proporcional derivativa. Para una $\delta \in \mathbb{R}^+$ dada, encontrar una retroalimentación de estado proporcional $u = F_p x + \bar{u}$ tal que la salida en lazo cerrado, $\mathbf{y}$, satisfaga ($t^*(\delta)$ es un tiempo fijo que depende de δ, la cual es dada):*

$$|y - y^*| \leq \delta, \qquad \forall t \geq t^*(\delta)$$

En este Capítulo se resuelve este problema. En la Sección 5.4 se obtienen algunas descomposiciones geométricas básicas y en la Sección 5.5 se resuelve el problema.

[1]De las siglas en inglés: Almost Rejection of the Internal Structural Variations

5.3 Integración por Partes

En esta Sección se obtendrá la equivalencia, en el marco de la *teoría de sistemas*, de una poderosa herramienta de análisis funcional: la *integración por partes*. Para esto considere el siguiente sistema propio:

$$\begin{aligned} \dot{x} &= [-1/\varepsilon]x + [1/\varepsilon]f; \\ y &= [-1/\varepsilon]x + [1/\varepsilon]f \end{aligned} \tag{5.1}$$

donde ε es un parámetro positivo; y es la salida y f es una entrada que se considera al menos dos veces diferenciable y tal que $f, \dot{f}, \ddot{f} \in L_\infty$, con $x(0) = x_o$, $f(0) = f_o$ y $\dot{f}(0) = \dot{f}_o$.

Ahora analicemos el comportamiento externo cuando ε tiende a cero. Para esto, primero se obtiene la solución de (5.1):

$$\begin{aligned} x(t) &= e^{-t/\varepsilon}x_o + \frac{1}{\varepsilon}\int_0^t e^{-(t-\tau)/\varepsilon} f(\tau)d\tau \\ y(t) &= -\frac{1}{\varepsilon}e^{-t/\varepsilon}x_o + \frac{1}{\varepsilon}f(t) - \frac{1}{\varepsilon^2}\int_0^t e^{-(t-\tau)/\varepsilon} f(\tau)d\tau \end{aligned}$$

Entonces $|y(t)| \leq \frac{1}{\varepsilon}e^{-t/\varepsilon}|x_o| + \frac{1}{\varepsilon}|f(t)| + \frac{1}{\varepsilon}||f||_\infty$. En estos momentos no se puede concluir nada cuando $\varepsilon \rightarrow 0$.Por lo tanto integramos por partes:

$$\begin{aligned} x(t) - f(t) &= e^{-t/\varepsilon}(x_o - f_o) - \int_0^t e^{-(t-\tau)/\varepsilon} \dot{f}(\tau)d\tau \\ y(t) &= -\frac{1}{\varepsilon}e^{-t/\varepsilon}(x_o - f_o) + \frac{1}{\varepsilon}\int_0^t e^{-(t-\tau)/\varepsilon} \dot{f}(\tau)d\tau \end{aligned}$$

de donde se observa que $|y(t)| \leq \frac{1}{\varepsilon}e^{-t/\varepsilon}|x_o - f_o| + ||\dot{f}||_\infty$ y solamente se puede concluir que y está acotada cuando $\varepsilon \rightarrow 0$ para $t > 0$.

Ahora, nuevamente integremos por partes:

$$\begin{aligned} x(t) - f(t) + \varepsilon\dot{f}(t) &= e^{-t/\varepsilon}(x_o - f_o + \varepsilon\dot{f}_o) + \varepsilon\int_0^t e^{-(t-\tau)/\varepsilon} \ddot{f}(\tau)d\tau \\ y(t) - \dot{f}(t) &= -\frac{1}{\varepsilon}e^{-t/\varepsilon}(x_o - f_o + \varepsilon\dot{f}_o) - \int_0^t e^{-(t-\tau)/\varepsilon} \ddot{f}(\tau)d\tau \end{aligned}$$

entonces $|y(t) - \dot{f}(t)| \leq \frac{1}{\varepsilon}e^{-t/\varepsilon}|x_o - f_o + \varepsilon\dot{f}_o| + \varepsilon||\ddot{f}||_\infty$. Por lo tanto $y(t) \rightarrow \dot{f}(t)$ cuando $\varepsilon \rightarrow 0$ para toda $t > 0$.

De este análisis simple, observamos que fue necesario integrar dos veces por partes para obtener argumentos rigurosos que permitieran llegar a las conclusiones (esperadas).

Sería interesante ahora trasladar este proceso de *integración por partes* desde un punto de vista teórico de sistemas. Efectivamente, posteriormente se mostrará, en la Subsección 5.5.2, que esta interpretación en términos de cambios de base equivalentes es la guía que ayuda en la elección de la ley de control proporcional que resuelve el Problema ARISV.

5.3.1 Descripciones importantes de los sistemas

Se debe enfatizar que la primera integración por partes es la solución temporal (es decir, en el tiempo) de la descripción de Fliess en el espacio de estados [25, Fliess 1990]:

$$\begin{aligned} \dot{w} &= [-1/\varepsilon]w + [1/\varepsilon](-\varepsilon\dot{f}) \\ y &= [-1/\varepsilon]w. \end{aligned}$$

Esta descripción se puede obtener de (5.1) con el simple cambio de variable $w = x - f$.

Ahora, note que la segunda integración por partes corresponde a la solución temporal de la descripción de Fliess:

$$\begin{aligned} \dot{z} &= [-1/\varepsilon]z + [1/\varepsilon](\varepsilon^2\ddot{f}) \\ y &= [-1/\varepsilon]z + [1]\dot{f}. \end{aligned} \tag{5.2}$$

Esta descripción también se puede obtener de (5.1) con el cambio de variable $z = w - (-\varepsilon\dot{f}) = x - f + \varepsilon\dot{f}$.

Las descripciones implícitas de (5.1) y (5.2) tienen la siguiente forma:

i) Haciendo $\xi_1 = x$ y $\xi_2 = f$ en (5.1) se obtiene:

$$\begin{aligned} \left[\begin{array}{cc} \boxed{1 \quad 0} \\ 0 \quad 0 \end{array}\right]\dot{\xi} &= \left[\begin{array}{cc} -1/\varepsilon & 0 \\ \multicolumn{2}{c}{\boxed{0 \qquad 1}} \end{array}\right]\xi + \left[\begin{array}{c} 1/\varepsilon \\ -1 \end{array}\right] f \\ y &= \left[\begin{array}{cc} -1/\varepsilon & 1/\varepsilon \end{array}\right]\xi \end{aligned} \tag{5.3}$$

ii) Haciendo $\zeta_1 = z$, $\zeta_2 = f$, $\zeta_3 = -\varepsilon\dot{\zeta}_2$ y $\zeta_4 = \dot{\zeta}_3$ en (5.2) se obtiene:

$$\begin{aligned} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]\dot{\zeta} &= \left[\begin{array}{cccc} -1/\varepsilon & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \end{array}\right]\zeta + \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ -1 \end{array}\right] f \\ y &= \left[\begin{array}{cccc} -1/\varepsilon & 0 & -1/\varepsilon & 0 \end{array}\right] \end{aligned} \tag{5.4}$$

5.3.2 Propiedad Interna

Note que el sistema (5.3) es internamente propio. En efecto, esto se obtiene del hecho que (ver la Proposición 4 del Capitulo 2):

$$\mathcal{X} = \ker \left[\begin{array}{cc} 1 & 0 \end{array}\right] \oplus \ker \left[\begin{array}{cc} 0 & 1 \end{array}\right].$$

También, note que el sistema (5.4) no es internamente propio debido a que (ver la Proposición 4 del Capitulo 2):

$$\ker \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \cap \ker \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \end{array}\right] \neq \{0\}.$$

5.3.3 Propiedad Externa

Puesto que el sistema (5.3) es internamente propio, también es externamente propio. Con respecto al sistema (5.4) primero se necesita obtener el sistema cocientado por $\mathcal{R}^*_{ao}$ en el co-dominio (ver Corolario 1 del Capitulo 2):[2]

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & -\varepsilon \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \end{bmatrix} \dot{\bar{\zeta}} = \begin{bmatrix} \boxed{\begin{matrix} \text{-}1/\varepsilon & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix} \bar{\zeta} + \begin{bmatrix} \boxed{\begin{matrix} 1/\varepsilon \\ -1 \end{matrix}} \\ \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix} f$$

$$y = \begin{bmatrix} \boxed{\begin{matrix} -1/\varepsilon & 1/\varepsilon \end{matrix}} & 0 & 0 \end{bmatrix}.$$

Aplicando el algoritmo (1.15) a este sistema se obtiene $\mathcal{R}^*_{ao} = \{e_3, e_4\}$. De esta manera, dentro de los cuadros se encuentra el sistema inducido mencionado en el Corolario 1, el cual no es otro que el sistema (5.3). Entonces, (5.4) es externamente propio y externamente equivalente a (5.3).

De esta discusión se concluye que realizar dos integraciones por partes es equivalente a aplicar, en la descripción de Fliess en el espacio de estado, el cambio de variable:[3] $z = x - f + \varepsilon\dot{f}$. El sistema obtenido con este cambio de variable únicamente agrega variables descriptoras diferencialmente redundantes y permanece externamente propio y externamente equivalente al sistema original. Como se verá muy pronto, el subespacio diferencialmente redundante agregado nos permite llevar el sistema a una forma estructural agradable.

5.4 Descomposiciones Geométricas Básicas

El objetivo de esta sección es obtener una serie de descomposiciones de los espacios del sistema, los cuales nos permitirán proponer una aproximación propia de la ley de control proporcional derivativa, $u(t) = F_p x(t) + F_d \dot{x}(t)$, presentada en el Capítulo 2.

Para propósitos de aproximación, primero se modificará la condición geométrica (2.18) de la siguiente manera:

$$\dim(\mathcal{V}^* \cap E^{-1}\mathcal{B}) \geq \dim(\mathcal{K}_E) + \dim(\frac{\mathcal{K}_E + \mathcal{V}^*}{\mathcal{V}^*}). \tag{5.5}$$

Por simplicidad podemos asumir, sin pérdida de generalidad, que una retroalimentación proporcional preliminar ha sido aplicada tal que:

$$A\mathcal{V}^* \cap \mathcal{B} = \{0\} \text{ y } A\mathcal{V}^* \subset E\mathcal{V}^*. \tag{5.6}$$

De manera similar a la mostrada en [17, Bonilla y Malabre 2002], se realiza una descomposición de $\mathcal{K}_E$, $E^{-1}\mathcal{B}$, $\mathcal{V}^*$ y el espacio $\mathcal{X}$ como sigue ($\mathcal{X}_0$, $\mathcal{X}_{\mathcal{V}^*}$, $\mathcal{X}_3$ y $\mathcal{X}_{\mathcal{K}_E}$ son subespacios complementarios

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[2] Pre-multiplique (5.4a) por $\begin{bmatrix} 1 & -1/\varepsilon & 1 & -1/\varepsilon \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ y haga $\zeta = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bar{\zeta}$.

[3] En el caso de n integraciones por partes el cambio de variable es: $z_n = x + \Sigma_{i=1}^{n}(-1)^i \varepsilon^{i-1} f^{(i-1)}$

cualesquiera):

$$\begin{cases} \mathcal{K}_E = (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \\ E^{-1}\mathcal{B} = ((\mathcal{V}^* \cap E^{-1}\mathcal{B}) + \mathcal{K}_E) \oplus \mathcal{X}_3 \\ \mathcal{V}^* = \mathcal{X}_{\mathcal{V}^*} \oplus (\mathcal{V}^* \cap E^{-1}\mathcal{B}) \\ \mathcal{X} = (\mathcal{V}^* + E^{-1}\mathcal{B}) \oplus \mathcal{X}_0. \end{cases} \tag{5.7}$$

En vista de (5.5), existen $\mathcal{X}_1$ y $\mathcal{X}_2 \subset E^{-1}\mathcal{B}$ tal que:

$$\begin{gathered} \mathcal{V}^* \cap E^{-1}\mathcal{B} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \\ \text{con } \mathcal{X}_2 \approx \mathcal{X}_{\mathcal{K}_E} \text{ y } \dim \mathcal{X}_1 \geq \dim \mathcal{X}_{\mathcal{K}_E}. \end{gathered} \tag{5.8}$$

De (5.7) y (5.8) se obtiene:

$$\begin{cases} \mathcal{X} = \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \\ \mathcal{V}^* = \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \\ E^{-1}\mathcal{B} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3. \end{cases} \tag{5.9}$$

De esta manera, $\mathcal{B}$, $E\mathcal{V}^*$ y $\mathcal{E}$ se descomponen de la siguiente forma (recuerde 2.9a):

$$\begin{aligned} \mathcal{B} &= E\mathcal{X}_1 \oplus E\mathcal{X}_2 \oplus E\mathcal{X}_3 \\ E\mathcal{V}^* &= E\mathcal{X}_{\mathcal{V}^*} \oplus E\mathcal{X}_1 \oplus E\mathcal{X}_2 \\ \mathcal{E} &= E\mathcal{X}_{\mathcal{V}^*} \oplus E\mathcal{X}_1 \oplus E\mathcal{X}_2 \oplus E\mathcal{X}_3 \oplus E\mathcal{X}_0. \end{aligned} \tag{5.10}$$

También, $\mathcal{U}$ se puede descomponer de la siguiente manera (recuerde que $\mathcal{K}_B = \{0\}$):

$$\mathcal{U} = B^{-1}E\mathcal{X}_1 \oplus B^{-1}E\mathcal{X}_2 \oplus B^{-1}E\mathcal{X}_3 \tag{5.11}$$

Basándonos en las descomposiciones anteriores se definen las siguientes proyecciones naturales ($i \in \{\mathcal{V}^*, 1, 2, \mathcal{K}_E\}$):[4]

$$\begin{cases} Q_{\mathcal{X}_i} : \mathcal{X} \to \mathcal{X}_i; \qquad Q_{VE} : \mathcal{X} \to \mathcal{V}^* \cap \mathcal{K}_E \\ Q_{E30} : \mathcal{X} \to \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0; \qquad P_{\mathcal{X}_{\mathcal{V}^*}} : \mathcal{E} \to E\mathcal{X}_{\mathcal{V}^*} \\ P_1 : \mathcal{E} \to E\mathcal{X}_1; \qquad P_{230} : \mathcal{E} \to E\mathcal{X}_2 \oplus E\mathcal{X}_3 \oplus E\mathcal{X}_0 \\ R_1 : \mathcal{U} \to B^{-1}E\mathcal{X}_1; \qquad R_2 : \mathcal{U} \to B^{-1}E\mathcal{X}_2 \\ R_3 : \mathcal{U} \to B^{-1}E\mathcal{X}_3 \end{cases} \tag{5.12}$$

y los siguientes mapas de inserción:

$$\begin{cases} V_{\mathcal{X}_i} : \mathcal{X}_i \to \mathcal{X}; \qquad V_{VE} : \mathcal{V}^* \cap \mathcal{K}_E \to \mathcal{X} \\ V_{E30} : \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \to \mathcal{X}; \qquad W_1 : B^{-1}E\mathcal{X}_1 \to \mathcal{U} \\ W_2 : B^{-1}E\mathcal{X}_2 \to \mathcal{U}; \qquad W_3 : B^{-1}E\mathcal{X}_3 \to \mathcal{U}. \end{cases} \tag{5.13}$$

Gracias a las proyecciones (5.12) y a las inserciones (5.13) se obtiene una descripción implícita más precisa de (1.2) en el siguiente lema. Este lema será el punto de partida para obtener la solución propuesta del Problema 5 planteado en la Sección 5.2.

[4] Las proyecciones naturales son proyectadas a lo largo de los subespacios complementarios definidos en (5.9a), (5.10c) y (5.11).

Lema 8 *Si (5.5) y (5.6) se satisfacen, entonces el sistema (1.2) se puede expresar de la siguiente manera:*

$$\begin{cases} T_V Q_{\mathcal{X}_{\mathcal{V}^*}} \dot{x} = P_{\mathcal{X}_{\mathcal{V}^*}} A x \\ T_1 Q_{\mathcal{X}_1} \dot{x} = L_1 R_1 u \\ (K_1 Q_{\mathcal{X}_2} + N_1 Q_{E30}) \dot{x} = A_1 Q_{E30} x + (L_2 R_2 + L_3 R_3) u \\ y = \bar{C} Q_{E30} x \end{cases} \tag{5.14}$$

donde $T_V : \mathcal{X}_{\mathcal{V}} \to E\mathcal{X}_{\mathcal{V}^*}$ *(*$T_V = P_{\mathcal{X}_{\mathcal{V}^*}} E V_{\mathcal{X}_{\mathcal{V}^*}}$*),* $T_1 : \mathcal{X}_1 \to E\mathcal{X}_1$ *(*$T_1 = P_1 E V_{X_1}$*) y* $L_1 : B^{-1}E\mathcal{X}_1 \to E\mathcal{X}_1$ *(*$L_1 = P_1 B W_1$*) son isomorfismos; los mapas* $L_2 = P_{230} B W_2$ *y* $L_3 = P_{230} B W_3$ *son mónicos y tal que* $\operatorname{Im} L_2 = E\mathcal{X}_2$ *y* $\operatorname{Im} L_3 = E\mathcal{X}_3$*; los mapas* A_1 *y* $\bar{C}$ *se definen como* $A_1 = P_{E30} A V_{230}$ *y* $\bar{C} = C V_{E30}$*. Los mapas* K_1 *y* N_1 *se definen de la siguiente manera:*

$$K_1 = P_{E30} E V_{\mathcal{X}_2}; \ \textit{satisfaciendo} \ \operatorname{Im} K_1 = E\mathcal{X}_2 \quad y \quad \mathcal{K}_{K_1} = \{0\} \tag{5.15}$$

$$N_1 = P_{E30} E V_{230}; \ \textit{satisfaciendo} \ \operatorname{Im} N_1 = E(\mathcal{X}_3 \oplus \mathcal{X}_0), \quad y \quad \mathcal{K}_{N_1} = \mathcal{X}_{\mathcal{K}_E} \tag{5.16}$$

También se necesita el siguiente lema.

Lema 9 *Definamos las siguientes proyecciones naturales:*

$$\bar{Q}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \to \mathcal{X}_{\mathcal{K}_E} \ \textit{a lo largo de} \ \mathcal{X}_3 \oplus \mathcal{X}_0. \tag{5.17}$$

Entonces el mapa $T_{230} : (N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \bar{Q}_{\mathcal{X}_{\mathcal{K}_E}})$ *es un isomorfismo. Además* $Q_{\mathcal{X}_{\mathcal{K}_E}} = \bar{Q}_{\mathcal{X}_{\mathcal{K}_E}} Q_{E30}$

5.5 Solución al Problema ARISV

En esta sección se presenta la contribución principal de este Capítulo. En el Teorema 11 se dan las condiciones bajo las cuales la aproximación, resuelve, en un sentido aproximado, el Problema ARISV.

Para evitar tecnicismos innecesarios, en esta Sección asumiremos que:[5]

$$\mathcal{X}_2 \approx \mathcal{X}_{\mathcal{K}_E} \approx \mathcal{X}_1 \tag{5.18}$$

5.5.1 Retroalimentación Derivativa

La retroalimentación derivativa propuesta en [17, Bonilla y Malabre 2002] fue (recuerde que $K_1 = P_{E20} E V_{\mathcal{X}_2}$):

$$L_2 R_2 u^* = -K_1(-Q_{\mathcal{X}_2} + T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) \dot{x} + L_2 R_2 \bar{u} \tag{5.19}$$

[5] Si la $\dim \mathcal{X}_1 > \dim \mathcal{X}_{\mathcal{K}_E}$,entonces sólo se tiene que trabajar con una proyección adecuada sobre un subespacio $\mathcal{X}_1'$ de $\mathcal{X}_1$ tal que $\mathcal{X}_1' \approx \mathcal{X}_{\mathcal{K}_E}$.

donde $T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \to \mathcal{X}_2$ es un isomorfismo (recuerde (5.7a) y (5.8)). Aplicando (5.19) y la retroalimentación $L_1R_1u^* = -(1/\varepsilon)T_1Q_{\mathcal{X}_1}x$ a (5.14), se obtiene (recuerde que $Q_{\mathcal{X}_{\mathcal{K}_E}} = \bar{Q}_{\mathcal{X}_{\mathcal{K}_E}}Q_{E30}$ y que el mapa $T_{230} = (N_1 + K_1T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}}\bar{Q}_{\mathcal{X}_{\mathcal{K}_E}})$ es un isomorfismo, Lema 9):

$$\begin{aligned}
\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & \mathrm{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{\mathrm{I}} \end{bmatrix} \dot{x} &= \begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ 0 & (1/\varepsilon)\mathrm{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{\bar{A}} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \boxed{\bar{B}} \end{bmatrix} \bar{u}_2 \\
y^* &= \begin{bmatrix} 0 & 0 & 0 & 0 & \boxed{\bar{\mathrm{C}}} \end{bmatrix}
\end{aligned} \tag{5.20}$$

donde: $\begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \end{bmatrix} = T^{(-1)}_{\mathcal{V}}P_{\mathcal{X}_{\mathcal{V}^*}}A$, $\bar{A} = T^{(-1)}_{230}A_1$, $\bar{B} = T^{(-1)}_{230}L_2$, $\bar{u}_2 = R_2\bar{u}$. Se usa y^* en lugar de y para distinguirlo del caso de retroalimentación proporcional. El subsistema $\Sigma^s(I, \bar{A}, \bar{B}, \bar{C})$ encerrado en los cuadros es el sistema cociente en espacio de estado mencionado en el Teorema 6. Asumiendo controlabilidad en la parte dinámica común de la descripción implícita global (2.8), se sigue que el par $(\bar{A}, \bar{B})$ es controlable; de esta manera supondremos que el mapa $\bar{A}$ se ha hecho Hurwitz por una retroalimentación proporcional de estado previa.

Note que estas propiedades estructurales y los resultados son independientes de la estructura interna activa de cada $\mathcal{K}_{D_i}$ particular.

5.5.2 Retroalimentación Proporcional

Basándonos en la retroalimentación derivativa (5.19) se propone la retroalimentación proporcional (recuerde que $T_1 = P_1EV_{\mathcal{X}_1}$):

$$L_1R_1u = -(1/\varepsilon)T_1(Q_{\mathcal{X}_1} - T^{\mathcal{X}_1}_{\mathcal{X}_2}Q_{\mathcal{X}_2} + T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}}Q_{\mathcal{X}_{\mathcal{K}_E}})x \tag{5.21}$$

$$L_2R_2u = K_1g + L_2R_2\bar{u} \tag{5.22}$$

$$g = -(1/\varepsilon)(T^{\mathcal{X}_2}_{\mathcal{X}_1}Q_{\mathcal{X}_1} - Q_{\mathcal{X}_2} + T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}}Q_{\mathcal{X}_{\mathcal{K}_E}})x \tag{5.23}$$

donde $T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \to \mathcal{X}_1$, $T^{\mathcal{X}_1}_{\mathcal{X}_2} : \mathcal{X}_2 \to \mathcal{X}_1$, $T^{\mathcal{X}_2}_{\mathcal{X}_1} : \mathcal{X}_1 \to \mathcal{X}_2$ y $T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \to \mathcal{X}_2$ son isomorfismos (recuerde (5.18)). Aplicando la retroalimentación (5.21)-(5.23) al sistema (5.14) se obtiene:

$$\begin{cases} T_VQ_{\mathcal{X}_{\mathcal{V}^*}}\dot{x} = P_{\mathcal{X}_{\mathcal{V}^*}}Ax \\ T_1Q_{\mathcal{X}_1}\dot{x} = -\frac{1}{\varepsilon}T_1Q_{\mathcal{X}_1}x + \frac{1}{\varepsilon}T_1(T^{\mathcal{X}_1}_{\mathcal{X}_2}Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}}Q_{\mathcal{X}_{\mathcal{K}_E}})x \\ (K_1Q_{\mathcal{X}_2} + N_1Q_{E30})\dot{x} = A_1Q_{E30}x + K_1g + L_2R_2\bar{u} \\ 0 = \frac{1}{\varepsilon}T^{\mathcal{X}_2}_{\mathcal{X}_1}Q_{\mathcal{X}_1}x - \frac{1}{\varepsilon}(Q_{\mathcal{X}_2} - T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}}Q_{\mathcal{X}_{\mathcal{K}_E}})x + g. \end{cases} \tag{5.24}$$

Basándonos en la Subsección 5.3 (integración por partes), se hace el siguiente cambio de variable:

$$\begin{aligned}
Q_iz &= Q_ix,, \quad \forall i \neq \mathcal{X}_1 \quad \text{y} \\
Q_{\mathcal{X}_1}z &= \frac{1}{\varepsilon}Q_{\mathcal{X}_i}x - \frac{1}{\varepsilon}(T^{\mathcal{X}_1}_{\mathcal{X}_2}Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}}Q_{\mathcal{X}_{\mathcal{K}_E}})x + (T^{\mathcal{X}_1}_{\mathcal{X}_2}Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}}Q_{\mathcal{X}_{\mathcal{K}_E}})\dot{x}.
\end{aligned} \tag{5.25}$$

Entonces el sistema (5.24) toma la siguiente forma (recuerde que $T_{\mathcal{X}_1}^{\mathcal{X}_2} T_{\mathcal{X}_2}^{\mathcal{X}_1} = I$ y $T_{\mathcal{X}_1}^{\mathcal{X}_2} T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_1} = T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_2}$):

$$\begin{cases} T_V Q_{\mathcal{X}_{\mathcal{V}^*}} \dot{z} = P_{\mathcal{X}_{\mathcal{V}^*}} A z \\ T_1 Q_{\mathcal{X}_1} \dot{z} = -(1/\varepsilon) T_1 Q_{\mathcal{X}_1} z + T_1 (T_{\mathcal{X}_2}^{\mathcal{X}_1} Q_{\mathcal{X}_2} - T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_1} Q_{\mathcal{X}_{\mathcal{K}_E}}) \ddot{z} \\ (N_1 + K_1 T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_2} \bar{Q}_{\mathcal{X}_{\mathcal{K}_E}}) Q_{E30} \dot{z} = A_1 Q_{E30} z - K_1 T_{\mathcal{X}_1}^{\mathcal{X}_2} Q_{\mathcal{X}_1} z + L_2 R_2 \bar{u} \\ 0 = T_{\mathcal{X}_1}^{\mathcal{X}_2} Q_{\mathcal{X}_1} z - (Q_{\mathcal{X}_2} - T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_2} Q_{\mathcal{X}_{\mathcal{K}_E}}) \dot{z} + g \end{cases} . \tag{5.26}$$

Ahora definimos $h := (T_{\mathcal{X}_2}^{\mathcal{X}_1} Q_{\mathcal{X}_2} - T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_1} \bar{Q}_{\mathcal{X}_{\mathcal{K}_E}} Q_{E30}) \dot{z}$ y construimos el espacio $\mathcal{Z} = \mathcal{X} \oplus \{g\} \oplus \{h\}$. Entonces (5.26) toma la siguiente forma:

$$\begin{aligned} \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{I} & 0 & 0 & 0 & 0 & -\mathrm{I} \\ 0 & 0 & 0 & 0 & \boxed{I} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_7^{(-1)} & 0 & -Y_1 & 0 & 0 \end{bmatrix} \dot{z} &= \begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & 0 & 0 \\ 0 & (-1/\varepsilon)\mathrm{I} & 0 & 0 & 0 & 0 & 0 \\ 0 & -X_6 & 0 & 0 & \boxed{\bar{A}} & 0 & 0 \\ 0 & X_7 & 0 & 0 & 0 & I & -X_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \boxed{\bar{B}} \\ 0 \\ 0 \end{bmatrix} \bar{u}_2 \\ y &= \begin{bmatrix} 0 & 0 & 0 & 0 & \boxed{\bar{C}} & 0 & 0 \end{bmatrix} \end{aligned} \tag{5.27}$$

donde $\begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \end{bmatrix} = T_V^{(-1)} P_{\mathcal{X}_{\mathcal{V}^*}}$, $X_6 = T_{230}^{(-1)} K_1 T_{\mathcal{X}_1}^{\mathcal{X}_2}$, $X_7 = T_{\mathcal{X}_1}^{\mathcal{X}_2}$, $\bar{A} = T_{230}^{(-1)} A_1$, $\bar{B} = T_{230}^{(-1)} L_2$, $Y_1 = T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_1} \bar{Q}_{\mathcal{X}_{\mathcal{K}_E}}$, y $\bar{u}_2 = R_2 \bar{u}$. Note que el subsistema $\Sigma^s(I, \bar{A}, \bar{B}, \bar{C})$ encerrado en los cuadros es el mismo que se obtuvo en (5.20), pero ahora es perturbado por el subsistema exponencialmente estable en estado estacionario de ganancia ε, $\Sigma^f(I, -(1/\varepsilon)I, -I, -X_6)$ encerrado por las líneas punteadas. Aunque la ganancia en estado estacionario puede ser muy pequeña (pero nunca cero), es necesario que la variación de estructura interna, la cual se hace casi no-observable sea exponencialmente estable. De esta manera, sólo se pueden aceptar variaciones de estructura interna, $\mathcal{K}_{D_i}$, que pertenecen al siguiente conjunto (recuerde que $\bar{A}$ es Hurwitz y $\varepsilon > 0$):

$$\Gamma_F(D_i) = \left\{ \mathcal{K}_{D_i} | \det \begin{bmatrix} E - (A + BF) \\ -D_i \end{bmatrix} \text{ es Hurwitz} \right\}. \tag{5.28}$$

Si este conjunto es vacio entonces no existen soluciones al Problema ARISV.

5.5.3 Semi-rechazo de la variación de estructura interna

Teorema 11 *Suponga que se cumplen las condiciones del Teorema 6 del Capitulo 2. Si además se satisface (2.9) pero con (5.5) en lugar de la segunda expresión en (2.9), entonces existe una retroalimentación proporcional que resuelve el problema ARISV para todo* $\mathcal{K}_{D_i} \in \Gamma_F(D_i)$.

5.6 Ejemplo Ilustrativo

Ahora consideremos nuevamente el ejemplo (1.4). Para este sistema se tiene:

$$\mathcal{V}^* = \{e_1, e_2\}, \quad \mathcal{K}_E = \{e_3\}, \quad E^{-1}\mathcal{B} = \{e_2, e_3\}$$

por lo tanto:

$$\mathcal{X}_{\mathcal{V}^*} = \{e_1\}, \quad \mathcal{X}_2 = \{e_2\}, \quad \mathcal{X}_{\mathcal{K}_E} = \{e_3\}, \quad \mathcal{X}_1 = \mathcal{V}^* \cap \mathcal{K}_E = \mathcal{X}_3 = \mathcal{X}_0 = \{0\}$$

donde se observa que la condición geométrica (5.5) no se satisface. Para acercarnos más a dicha condición, se agrega un integrador externo al sistema (1.4):[6]

$$\begin{aligned} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \dot{x} &= \left[\begin{array}{cccc} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \hline \alpha & 0 & \beta & 1 \end{array}\right] x + \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] u \\ y &= \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right] x. \end{aligned} \tag{5.29}$$

Para este sistema se tiene:

$$\mathcal{V}^* = \{e_1, e_2, e_3\}, \quad \mathcal{K}_E = \{e_4\}, \quad E^{-1}\mathcal{B} = \{e_2, e_3, e_4\}$$

de esta manera:

$$\begin{gathered} \mathcal{X}_{\mathcal{V}^*} = \{e_1\}, \quad \mathcal{X}_1 = \{e_2\}, \quad \mathcal{X}_2 = \{e_3\}, \quad \mathcal{X}_{\mathcal{K}_E} = \{e_4\} \\ \mathcal{X}_1 = \mathcal{V}^* \cap \mathcal{K}_E = \mathcal{X}_3 = \mathcal{X}_0 = \{0\}, \end{gathered}$$

y ahora se satisface la condición geométrica (5.5). De (5.29) se obtiene:

$$\begin{gathered} L_1 = L_2 = T_V = T_1 = K_1 = T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} = T^{\mathcal{X}_1}_{\mathcal{X}_2} = T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} = T^{\mathcal{X}_2}_{\mathcal{X}_1} = \bar{Q}_{\mathcal{X}_{\mathcal{K}_E}} = 1 \\ N_1 = 0, \quad A_1 = -1, \quad R_1 = \left[\begin{array}{cc} 1 & 0 \end{array}\right], \quad R_2 = \left[\begin{array}{cc} 0 & 1 \end{array}\right] \\ Q_{\mathcal{X}_{\mathcal{V}^*}} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right], \quad Q_{\mathcal{X}_1} = \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \end{array}\right], \quad Q_{\mathcal{X}_2} = \left[\begin{array}{cccc} 0 & 0 & 1 & 0 \end{array}\right] \\ Q_{\mathcal{X}_{\mathcal{K}_E}} = Q_{E30} = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right], \quad P_{\mathcal{X}_{\mathcal{V}^*}} = \left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right]. \end{gathered}$$

5.6.1 Retroalimentación Proporcional Derivativa

De (5.19) se obtiene la retroalimentación derivativa:

$$L_2 R_2 u^* = -\left[\begin{array}{cccc} 0 & 0 & -1 & 1 \end{array}\right] \dot{x} + \left[\begin{array}{cc} 0 & 1 \end{array}\right] \bar{u};$$

la retroalimentación proporcional está dada por la expresión:

$$L_1 R_1 u^* = -\frac{1}{\varepsilon} \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \end{array}\right].$$

El sistema en lazo cerrado toma la forma:

$$\begin{aligned} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{0} \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \dot{x} &= \left[\begin{array}{cccc} 0 & 0 & 1 & -1 \\ 0 & -1/\varepsilon & 0 & 0 \\ 0 & 0 & 0 & \boxed{-1} \\ \hline \alpha & 0 & \beta & 1 \end{array}\right] x + \left[\begin{array}{c} 0 \\ 0 \\ \boxed{1} \\ \hline 0 \end{array}\right] \bar{u}_2 \\ y &= \left[\begin{array}{cccc} 0 & 0 & 0 & \boxed{1} \end{array}\right] x. \end{aligned} \tag{5.30}$$

[6] **(i)** El segundo renglón y la segunda columna corresponden al integrador externo agregado; **(ii)** se ha aplicado previamente una retroalimentación proporcional para obtener (5.6); **(iii)** en la parte inferior se ha agregado la ecuación algebraica $0 = D_i x$

El polinomio característico del sistema (5.30) en lazo cerrado está dado por:

$$\det\left[\lambda\left(\mathbb{E}-\mathbb{B}F_d\right)-\left(\mathbb{A}_i+\mathbb{B}F_p\right)\right]=(\lambda+1)\left(\lambda+1/\varepsilon\right)\left(\beta\lambda+\alpha\right).$$

por lo tanto:

$$\Gamma_{(F_p,F_d)}(\bar{D})=\{(\alpha,\beta)\,|\alpha\cdot\beta>0 \text{ o } (\beta=0\ \&\ \alpha\neq 0)\} \tag{5.31}$$

con $\bar{D}=\begin{bmatrix}\alpha & 0 & \beta & 1\end{bmatrix}$.

5.6.2 Retroalimentación Proporcional

De (5.21), (5.22) y (5.23) se obtiene:

$$L_2R_2u=g+\begin{bmatrix}0 & 1\end{bmatrix}\bar{u},\quad g=-\frac{1}{\varepsilon}\begin{bmatrix}0 & 1 & -1 & 1\end{bmatrix}x,\quad L_1R_1u=-\frac{1}{\varepsilon}\begin{bmatrix}0 & 1 & -1 & 1\end{bmatrix}x.$$

Ahora el sistema en lazo toma la forma:

$$\begin{aligned}\left[\begin{array}{cccccc}1 & 0 & 0 & 0 & 0 & 0\\ 0 & \boxed{1} & 0 & 0 & 0 & \boxed{-1}\\ 0 & 0 & 0 & \boxed{1} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0\end{array}\right]\dot{z} &= \left[\begin{array}{cccccc}0 & 0 & 1 & -1 & 0 & 0\\ 0 & \boxed{-(1/\varepsilon)} & 0 & 0 & 0 & 0\\ 0 & \boxed{-1} & 0 & \boxed{-1} & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0 & 0 & 1\\ \hline \alpha & 0 & \beta & 1 & 0 & 0\end{array}\right]z+\left[\begin{array}{c}0\\ 0\\ \boxed{1}\\ 0\\ 0\\ \hline 0\end{array}\right]\bar{u}_2 \\ y &= \begin{bmatrix}0 & 0 & 0 & \boxed{1} & 0 & 0\end{bmatrix}z.\end{aligned} \tag{5.32}$$

El polinomio característico del sistema (5.32) esta dado por la expresión:

$$\det\left[\lambda\left(\mathbb{E}-\left(\mathbb{A}_i+\mathbb{B}F_p\right)\right)\right]=(\lambda+1)\left(\lambda+1/\varepsilon\right)\left(\beta\lambda+\alpha\right)-(\beta+1)\,\lambda^3.$$

Por lo tanto:

$$\Gamma_F(\tilde{D})=\left\{(\alpha,\beta)\,|\alpha<\min\left\{-\left(\frac{1}{\varepsilon}+1\right)\beta,-\left(\frac{\beta}{1+\varepsilon}\right),0\right\}\right\}. \tag{5.33}$$

con $\tilde{D}=\begin{bmatrix}\alpha & 0 & \beta & 1 & 0 & 0\end{bmatrix}$.

Comentario 4 *De este ejemplo ilustrativo se puede mencionar lo siguiente:*

1. *Con la retroalimentación proporcional y derivativa se obtiene una descripción en espacio de estado controlable desacoplado de la variación de estructura interna (ver en (5.30) el sub-sistema encerrado en los cuadros).*

2. *Si además se desea estabilidad interna, la variación de estructura (en el caso de la retroalimentación proporcional derivativa) debe presentarse en la región (5.31). Note que los cuatro casos considerados en el Capítulo 1,* $(\alpha,\beta)\in\{(-1,-1),(-1,\ 0),(-1,-5),(1,1)\}$, *están dentro de* $\Gamma_{(F_p,F_d)}(D_i)$.

3. *Para la retroalimentación proporcional, el Problema ARISV tiene una solución si la variación de estructura interna se presenta en la región (5.33). Para este caso, solamente se pueden considerar las estructuras* $(\alpha, \beta) \in \{(-1,-1),(-1,0),(-1,-5)\}$*; el caso* $(\alpha, \beta) = (1,1)$ *no se puede presentar.*

4. *Puesto que el sub-sistema encerrado en los cuadros en (5.30) y (5.32) es el mismo, las propiedades estructurales son, también, las mismas.*

5. *La región de posibles variaciones que preservan la estabilidad interna se reduce cuando se realiza una aproximación de la retroalimentación proporcional derivativa, es decir, para el caso proporcional derivativo esta región está formada por el primer y tercer ortante, mientras que para el caso proporcional la región se reduce al tercer ortante.*

6. *Si se satisface la condición geométrica (2.18b) pero no la condición (5.5), solamente se necesitan agregar algunos integradores hasta satisfacer (5.5).*

Capítulo 6

Optimización de Sistemas Escalera

En este capítulo se obtiene una ley de control *sub-óptima*, u, del sistema global (1.3), presentado en el Capítulo 1, como una combinación lineal de los controles óptimos, u_i^* $(i = 1, ..., n)$, de cada sistema. Esta ley de control es menos óptima cuando el sistema i está activo que su control óptimo asociado, pero se conduce a un compromiso global satisfactorio. Este esquema es conveniente cuando no se sabe con suficiente precisión que sistema está activo (artículo en proceso de elaboración).

6.1 Résumé en français

Dans ce chapitre on décrit une loi de commande sous-optimale du système global montré dans le Chapitre 1, et obtenue comme une combinaison linéaire des commandes optimales de chaque système. Cette loi de commande est moins *bonne* quand le *i-ème* système est actif que lorsque l'on applique sa commande optimale associée ; mais elle conduit à un compromis global satisfaisant. Ce schéma est utile quand on ne sait pas précisément quel système est actif (un article est en cours de rédaction sur ce point).

Dans la Section 6.2 on montre un cas particulier pour deux systèmes ($n = 2$). On y considère d'abord le système global, qui est nécessaire pour obtenir la commande sous-optimale.

Dans la Section 6.3, on décrit la procédure générale pour obtenir une loi de commande sous-optimale comme une combinaison linéaire des lois de commandes optimales de chaque système particulier, et également, on obtient une borne pour l'erreur. Dans la sous-section 6.3.1, on obtient la loi de commande sous-optimale pour la paire de systèmes considérée dans la section 6.2, de même que la borne pour l'erreur, et finalement on montre les résultats de la simulation.

6.2 Introducción

En esta Sección se muestra un caso particular para obtener una ley de control óptima cuando los parámetros (α, β) del sistema global solamente toman los valores $(\alpha, \beta) = (-1, -1)$ y $(\alpha, \beta) = (-1, 0)$, es decir, cuando se presentan los dos primeros casos mencionados en el Capitulo 1. Para referirnos al caso $(\alpha, \beta) = (-1, -1)$ lo haremos llamándolo sistema 1 y para el segundo caso, sistema 2. Sus leyes de control óptimas asociadas se representarán como u_1^* y u_2^*, respectivamente.

Como se mencionó anteriormente, la ley de control sub-óptimo única será una combinación lineal de los controles optimos de cada sistema. Considere nuevamente el sistema global:

$$\begin{aligned}\left[\begin{array}{ccc}1&0&0\\0&1&0\\\hline 0&0&0\end{array}\right]\dot{x} &= \left[\begin{array}{ccc}0&1&-1\\1&0&-1\\\hline \alpha&\beta&1\end{array}\right]x+\left[\begin{array}{c}0\\1\\\hline 0\end{array}\right]u \\ y &= \left[\begin{array}{ccc}0&0&1\end{array}\right]x.\end{aligned} \tag{6.1}$$

Como se mencionó en el Capítulo 2, se puede realizar un cambio de base para obtener:

$$\begin{aligned}\left[\begin{array}{cc|c}1&0&0\\0&1&0\\\hline 0&0&0\end{array}\right]\dot{\xi} &= \left[\begin{array}{cc|c}-1&0&0\\\rho_1&\rho_2&-1\\\hline 0&0&1\end{array}\right]\xi+\left[\begin{array}{c}1\\0\\\hline 0\end{array}\right]u \\ y &= \left[\begin{array}{cc|c}-\beta&-\rho_3&1\end{array}\right]\xi \\ \rho_1 &= 1+\beta,\quad \rho_2=1+\alpha+\beta,\quad \rho_3=\alpha+\beta.\end{aligned} \tag{6.2}$$

En la Tabla 6.1 muestra los valores de ρ_1, ρ_2 y ρ_3 asi como la función de transferencia para diferentes valores de α y β.

α	β	ρ_1	ρ_2	ρ_3	Función de Transferencia
-1	-1	0	-1	-2	$(p+1)y=u$
-1	0	1	0	-1	$p(p+1)y=u$
-1	-5	-4	-5	-6	$(p+5)(p+1)y=(5p+1)u$
1	1	2	3	2	$(p-3)y=-u$

Tabla 6.1: Función de transferencia y valores de ρ_1, ρ_2 y ρ_3

El sistema reducido de (6.2) es:

$$\begin{aligned}\begin{bmatrix}1&0\\0&1\end{bmatrix}\dot{\xi} &= \begin{bmatrix}-1&0\\\rho_1&\rho_2\end{bmatrix}\xi+\begin{bmatrix}1\\0\end{bmatrix}u \\ y &= \begin{bmatrix}-\beta&-\rho_3\end{bmatrix}\xi \\ \rho_1 &= 1+\beta,\quad \rho_2=1+\alpha+\beta,\quad \rho_3=\alpha+\beta.\end{aligned} \tag{6.3}$$

La solución de la ecuación de estado (6.3) es:

- Para $\rho_2=-1$:

$$\xi=\begin{bmatrix}e^{-t}&0\\\rho_1te^{-t}&e^{-t}\end{bmatrix}\xi_o+\int_0^T\begin{bmatrix}e^{-(t+\tau)}\\\rho_1(t-\tau)e^{-(t-\tau)}\end{bmatrix}u(\tau)d\tau$$

$$\begin{aligned}y(t) &= -[(\beta+\rho_3\rho_1t)\xi_{10}+\rho_3\xi_{20}]e^{-t}+\int_0^t[-(\beta+\rho_3\rho_1(t-\tau))e^{-(t-\tau)}]u(\tau)d\tau \\ \dot{y}(t) &= -[(-\beta-\rho_3\rho_1t+\rho_3\rho_1)\xi_{10}-\rho_3]e^{-t}-\beta u(t)+\int_0^t[(\beta+\rho_3\rho_1(t-\tau)-\rho_3\rho_1)e^{-(t-\tau)}]u(\tau)d\tau\end{aligned}$$

- Para $\rho_2 \neq -1$:

$$\xi = \begin{bmatrix} e^{-t} & 0 \\ \frac{\rho_1}{1+\rho_2}(-e^{-t} + e^{\rho_2 t}) & e^{\rho_2 t} \end{bmatrix} \xi_o + \int_0^T \begin{bmatrix} e^{-(t+\tau)} \\ \frac{\rho_1}{1+\rho_2}(-e^{-(t-\tau)} + e^{\rho_2(t-\tau)}) \end{bmatrix} u(\tau) d\tau$$

$$\begin{aligned} y(t) &= -[((\beta - \frac{\rho_3\rho_1}{1+\rho_2})e^{-t} + \frac{\rho_3\rho_1}{1+\rho_2}e^{\rho_2 t})\xi_{10} + \rho_3 e^{\rho_2 t}\xi_{20}] + \\ &\quad + \int_0^t [-(\beta - \frac{\rho_3\rho_1}{1+\rho_2})e^{-(t-\tau)} + \frac{\rho_3\rho_1}{1+\rho_2}e^{\rho_2(t-\tau)}]u(\tau)d\tau \\ \dot{y}(t) &= -[((-\beta + \frac{\rho_3\rho_1}{1+\rho_2})e^{-t} + \frac{\rho_3\rho_2\rho_1}{1+\rho_2}e^{\rho_2 t})\xi_{10} + \rho_3\rho_2 e^{\rho_2 t}\xi_{20}] - \beta u(t) + \\ &\quad + \int_0^t [(\beta - \frac{\rho_3\rho_1}{1+\rho_2})e^{-(t-\tau)} - \frac{\rho_3\rho_2\rho_1}{1+\rho_2}e^{\rho_2(t-\tau)}]u(\tau)d\tau \end{aligned}$$

Problema 6 *Sea $u \in L_2[0,T]$. Se desea minimizar $\|u(t)\|$ sujeto a:*

$$(y|u) = k_1 \quad y \quad (\dot{y}|u) = k_2$$

con condiciones iniciales y finales $y(0) = \dot{y}(0) = u(0) = 0$, $y(T) = r$; $\dot{y}(T) = 0$; $u(T) = 0$, respectivamente, donde:

$$\left.\begin{array}{l} k_1 = ((\beta + \rho_3\rho_1 T)\xi_{10} + \rho_3\xi_{20})e^{-T} \\ y = -(\beta + \rho_3\rho_1(T-t))e^{-(T-t)} \\ k_2 = ((-\beta - \rho_3\rho_1 T + \rho_3\rho_1)\xi_{10} - \rho_3\xi_{20})e^{-T} + \beta \\ \dot{y} = (\beta + \rho_3\rho_1(T-t) - \rho_3\rho_1)e^{-(T-t)} \end{array}\right\} \text{ para } \rho_2 = -1 \tag{6.4}$$

$$\left.\begin{array}{l} k_1 = ((\beta - \frac{\rho_3\rho_1}{1+\rho_2})e^{-T} + \frac{\rho_3\rho_1}{1+\rho_2}e^{\rho_2 T})\xi_{10} + \rho_3 e^{\rho_2 T}\xi_{20} \\ y = -(\beta - \frac{\rho_3\rho_1}{1+\rho_2})e^{-(T-t)} - \frac{\rho_3\rho_1}{1+\rho_2}e^{\rho_2(T-t)} \\ k_2 = ((-\beta + \frac{\rho_3\rho_1}{1+\rho_2})e^{-T} + \frac{\rho_3\rho_2\rho_1}{1+\rho_2}e^{\rho_2 T})\xi_{10} + \rho_3\rho_2 e^{\rho_2 T}\xi_{20} + \beta \\ \dot{y} = (\beta - \frac{\rho_3\rho_1}{1+\rho_2})e^{-(T-t)} - \frac{\rho_3\rho_2\rho_1}{1+\rho_2}e^{\rho_2(T-t)} \end{array}\right\} \text{ para } \rho_2 \neq -1 \tag{6.5}$$

$$(f|g) = \int_0^T f(t)g(t)dt, \quad ||f|| = \sqrt{(f|f)}$$

En la Tabla 6.2 se muestran los valores de k_1, k_2 para diferentes valores de α y β.

Solución: La solución general de este problema es (ver por ejemplo [37, Luenberger 1969]):

$$u^* = k_1 y_1 + k_2 y_2$$

$$\left\{\begin{array}{l} (y_1|y_1)k_1 + (y_1|y_2)k_2 = r \\ (y_2|y_1)k_1 + (y_2|y_2)k_2 = 0 \end{array}\right.$$

donde los valores de y_1, y_2 y la forma de los controles óptimos se muestran en la Tabla 6.3

α	β	k_1	k_2
-1	-1	$-\left(\xi_{10}+2\xi_{20}\right)e^{-T}$	$\left(\xi_{10}+2\xi_{20}\right)e^{-T}-1$
-1	0	$\xi_{10}e^{-T}-(\xi_{10}+\xi_{20})$	$-\xi_{10}e^{-T}$
-1	-5	$\xi_{10}e^{-T}-6(\xi_{10}+\xi_{20})e^{-5T}$	$-\xi_{10}e^{-T}+30(\xi_{10}+\xi_{20})e^{-5T}-5$
1	1	$(\xi_{10}+2\xi_{20})e^{3T}$	$3(\xi_{10}+2\xi_{20})e^{3T}+1$

Tabla 6.2: Valores de k_1 y k_2

α	β	ρ_1	ρ_2	ρ_3	y_1	y_2	$u^*(0)$	$u^*(t)\ t\in(0,T)$	$u^*(T)$
-1	-1	0	-1	-2	$e^{-(T-t)}$	$-e^{-(T-t)}$	0	be^t	r
-1	0	1	0	-1	$1-e^{-(T-t)}$	$e^{-(T-t)}$	0	$a+be^t$	0
-1	-5	-4	-5	-6	$6e^{-5(T-t)}-e^{-(T-t)}$	$e^{-(T-t)}-30e^{-5(T-t)}$	0	be^t+ce^{5t}	0
1	1	2	3	2	$-e^{3(T-t)}$	$-3e^{3(T-t)}$	0	de^{-3t}	0

Tabla 6.3: Forma de y_1, y_2 y los controles

6.2.1 Ley de Control Óptima del Sistema 1

Procedamos a calcular el valor de b para el caso $(\alpha,\beta)=(-1,-1)$:

$$\begin{aligned} y(t) &= \int_0^t e^{-(t-\tau)}u(\tau)d\tau \\ \dot{y}(t) &= u(t)-\int_0^t e^{-(t-\tau)}u(\tau)d\tau = u(t)-y(t) \end{aligned}$$

$$(y|be^t)=r\rightarrow b=r/(y|e^t)$$

donde:

$$(y|e^t)=e^{-T}\int_0^T e^{2t}dt=e^{-T}\left(\frac{1}{2}e^{2T}-\frac{1}{2}\right)=\sinh T$$

Por lo tanto:

$$u_1^*(t)=\begin{cases} 0 & \text{si} \quad t=0 \\ \frac{r}{\sinh T}e^t & \text{si} \quad 0\le t\le T \\ r & \text{si} \quad t=T \end{cases}.$$

Además, $\|u_1^*\|=\frac{e^{T/2}}{\sqrt{\sinh T}}r$. Para $T=10$, $r=10$ se obtiene:

$$u_1^*(t)=\begin{cases} 0 & \text{si} \quad t=0 \\ \frac{10}{\sinh 10}e^t & \text{si} \quad 0<t\le 10 \\ 10 & \text{si} \quad t=10 \end{cases}$$

y su norma toma el valor $\|u_1^*\|=14.142$

En la Figura 6.1 se muestran los resultados de simulación. En ella se pueden observar el comportamiento del control óptimo, la salida, su derivada y los estados.

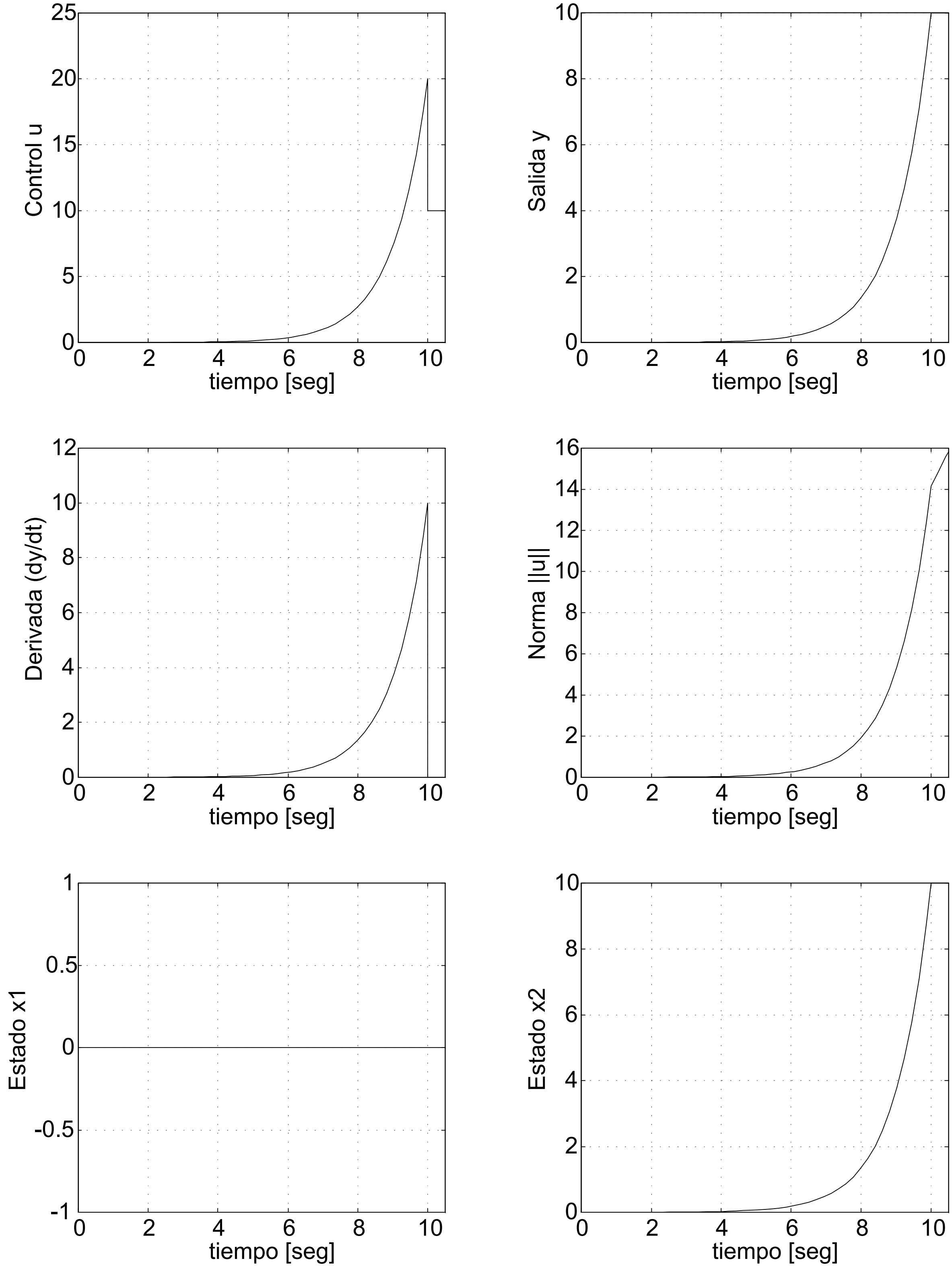

Figura 6.1: Resultados de la simulación del sistema 1.

6.2.2 Ley de Control Óptima del Sistema 2

Procedamos a calcular los valores de a y b para el caso $(\alpha, \beta) = (-1, 0)$. Para ello, considere el siguiente sistema de ecuaciones:

$$\left.\begin{array}{l}(y|a+be^t) = r \\ (\dot{y}|a+be^t) = 0\end{array}\right\} \rightarrow \left\{\begin{array}{l}a(y|1) + b(y|e^t) = r \\ a(\dot{y}|1) + b(\dot{y}|e^t) = 0\end{array}\right.$$

o en forma matricial:

$$\begin{bmatrix}(y|1) & (y|e^t) \\ (\dot{y}|1) & (\dot{y}|e^t)\end{bmatrix}\begin{bmatrix}a \\ b\end{bmatrix} = \begin{bmatrix}r \\ 0\end{bmatrix}$$

donde:

$$\begin{array}{rcl}(y|1) & = & T - (1-e^{-T}); \quad (y|e^t) = \cosh T - 1 \\ (\dot{y}|1) & = & 1 - e^{-T}; \quad (\dot{y}|e^t) = \sinh T\end{array}$$

Entonces:

$$a = \frac{(y_2|e^t)}{\Delta} r; \quad b = -\frac{(y_2|1)}{\Delta} r$$

donde:

$$\Delta = (y|1)(\dot{y}|e^t) - (\dot{y}|1)(y|e^t) = T\sinh T + 2(1-\cosh T)$$

por lo tanto, el control óptimo para el sistema 2 está dado por la expresión:

$$u_2^*(t) = \left\{\begin{array}{ll}0, & t=0 \\ \frac{r}{\Delta}(\sinh T - (1-e^{-T})e^t), & 0 < t \leq T\end{array}\right.$$

y su norma:

$$\|u_2^*\| = e^{T/2}\sqrt{\frac{\sinh T}{Te^T \sinh T - (e^T-1)^2}} r.$$

Para $T = 10$, $r = 10$ se obtiene:

$$u_2^*(t) = \left\{\begin{array}{ll}0, & t=0 \\ \frac{10}{10\sinh 10 + 2 - 2\cosh 10}\left(\sinh 10 - \left(1-e^{-10}\right)e^t\right), & 0 < t \leq 10\end{array}\right. .$$

En la Figura 6.2 se muestran los resultados de simulación. En ella se pueden observar el comportamiento del control óptimo, la salida, su derivada y los estados.

La norma toma el valor $\|u_2^*\| = 3.5355$

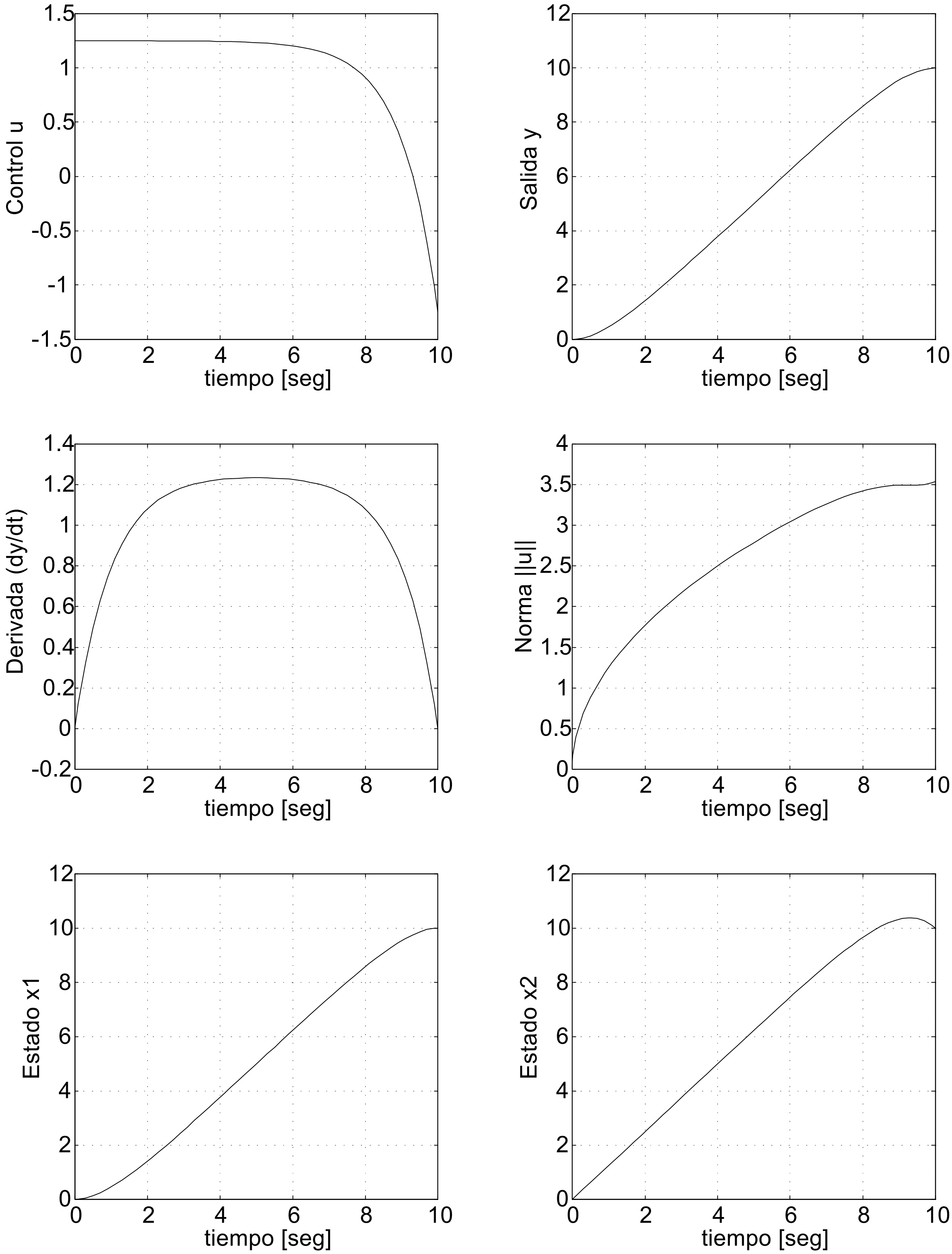

Figura 6.2: Resultados de la simulación del sistema 2.

6.3 Ley de Control Sub-óptima

En esta sección se desea resolver el siguiente problema.

Problema 7 *Considere n sistemas descritos por las funcionales:*

$$y_i^* = (z_i|u_i^*), i = 1, ..., n \tag{6.6}$$

para los cuales se han obtenido sus leyes de control óptimo $u_1^*, ..., u_n^*$. Obtener la mejor combinación lineal de controles:

$$u = \sum_{k=1}^{n} \gamma_k u_k^* \tag{6.7}$$

donde:

$$\sum_{k=1}^{n} \gamma_k = 1 \tag{6.8}$$

tal que se minimice la desviación final de la evolución del sistema cuando se aplica la ley de control (6.7), i.e., minimizar:

$$e^2 = \sum_{i=1}^{n} (y_i^* - y_i)^2 \tag{6.9}$$

donde:

$$y_i = (z_i|u), \quad \textit{con } i = 1, ..., n \tag{6.10}$$

Solución: La ley de control lineal (6.7) sujeto a (6.8) toma la forma:

$$u = \sum_{k=1}^{n-1} \gamma_k u_k^* + \left(1 - \sum_{k=1}^{n-1} \gamma_k\right) u_n^*, \tag{6.11}$$

por lo que las funcionales (6.10) pueden re-escribirse como:

$$y_i = \sum_{k=1}^{n-1} (z_i|u_k^* - u_n^*) \gamma_k + (z_i|u_n^*).$$

Definiendo los siguientes errores parciales:

$$\begin{aligned} e_i &= f_{i,i} - \sum_{k=1}^{n-1} \gamma_k f_{i,k}, \quad \text{con } i = 1, ..., n-1 \\ e_n &= -\sum_{k=1}^{n-1} \gamma_k f_{n,k} \end{aligned}$$

donde:

$$f_{i,k} := (z_i|u_k^* - u_n^*). \tag{6.12}$$

La funcional (6.9) toma la siguiente forma:

$$e^2 = \sum_{i=1}^{n-1} f_{i,i}^2 - 2\sum_{i=1}^{n-1} f_{i,i}\underline{\gamma}^T \underline{f}_i + \sum_{i=1}^{n} \underline{\gamma}^T \underline{f}_i \underline{f}_i^T \underline{\gamma} \tag{6.13}$$

donde:

$$\underline{\gamma}^T := \left[\gamma_1, ..., \gamma_{n-1}\right], \quad \underline{f}_i^T := [f_{i,1}, ..., f_{i,n-1}] \tag{6.14}$$

Sean $F = \sum_{i=1}^{n} \underline{f}_i \underline{f}_i^T$ y $\underline{g}_i^T = [f_{i,1} \cdots f_{n,i}]$ con $i = 1, ..., n$.

Lema 10 *Si el conjunto* $\left\{\underline{g}_1, ..., \underline{g}_{n-1}\right\}$ *es linealmente independiente, entonces* F *es invertible.*

Prueba: *Suponga* F *singular, entonces existe* $x \in \mathbb{R}^{n-1}$*, con* $x \neq 0$ *tal que* $Fx = 0$*, i.e.,*

$$\sum_{i=n}^{n} \underline{f}_i \underline{f}_i^T x = 0 \Rightarrow \sum_{i=1}^{n} (\underline{f}_i^T x)^2 = 0 \Rightarrow \underline{f}_i^T x = 0; \quad i = 1, ..., n$$

esto conduce a $\left[\underline{g}_1 | \cdots | \underline{g}_{n-1}\right] x = 0$ *lo que implica que* $x = 0$ *(contradicción).* ∎

Definiendo las siguientes matrices:

$$\underline{a} = F^{-1} \sum_{i=1}^{n-1} f_{i,i} \underline{f}_i, \quad \underline{b}^T = [f_{1,1} \cdots f_{n-1,n-1}]$$

se tiene que la funcional (6.13) se re-escribe de la siguiente forma:

$$e^2 = (\underline{\gamma} - \underline{a})^T F (\underline{\gamma} - \underline{a}) + \underline{b}^T \underline{b} - \underline{a}^T F \underline{a}. \tag{6.15}$$

Derivando (6.15) con respecto a $\underline{\gamma}$ e igualando a cero se obtiene:

$$\frac{\partial e^2}{\partial \underline{\gamma}} = 2F(\underline{\gamma} - \underline{a}) = 0,$$

i.e.,

$$\boxed{\underline{\gamma}^* = \underline{a}} \tag{6.16}$$

y además:

$$\frac{\partial^2 e^2}{\partial \underline{\gamma}^2} = 2F > 0,$$

siendo el valor mínimo de la funcional (6.15):

$$e^{*2} = \underline{b}^T b - \underline{\gamma}^{*T} F \underline{\gamma}^* \tag{6.17}$$

Lema 11 *La funcional minimizada (6.17) está acotada como sigue:*

$$0 \leq e^{*2} \leq \underline{b}^T \underline{b}(1 - \lambda_m(H^T F^{-1} H)). \tag{6.18}$$

Prueba: Sea $H := \left[\underline{f}_1 | \cdots | \underline{f}_{n-1}\right]$, entonces:

$$\sum_{i=1}^{n-1} f_{i,i} \underline{f}_i = H\underline{b}$$

$$\underline{a} = F^{-1} H b = \underline{\gamma}^*,$$

i.e.,

$$e^{*2} = \underline{b}^T b - (F^{-1} H \underline{b})^T F (F^{-1} H \underline{b}) = \underline{b}^T \left(I - H^T F^{-1} H\right) \underline{b}.$$

De la desigualdad de Raleigh:

$$x^T x \lambda_{\min}(H^T F^{-1} H) \leq x^T H^T F^{-1} H x \leq x^T x \lambda_{\max}(H^T F^{-1} H)$$

$$-x^T x \lambda_M(H^T F^{-1} H) \leq -x^T H^T F^{-1} H x \leq -x^T x \lambda_m(H^T F^{-1} H)$$

Sumando $x^T x$:

$$x^T x(1 - \lambda_M(H^T F^{-1} H)) \leq x^T (I - H^T F^{-1} H) x \leq x^T x(1 - \lambda_m(H^T F^{-1} H))$$

entonces se obtiene (6.18). ■

6.3.1 Cálculo de la Ley de Control Sub-óptima u^* en $[0, T_o]$

En esta subsección se obtendrá una ley de control sub-óptima del par de sistemas introducidos en la Sección 6.2. Observe que para este caso se tiene $n = 2$, por lo que el control (6.7) toma la siguiente forma:

$$u = \gamma_1 u_1^* + (1 - \gamma_1) u_2^*.$$

De (6.12) se tiene:

$$\begin{aligned} f_{1,1} &= (z_1 | u_1^* - u_2^*) = \int_0^T e^{-(T-\tau)} (u_1^* - u_2^*) d\tau \\ f_{2,1} &= (z_2 | u_1^* - u_2^*) = \int_0^T (1 - e^{-(T-\tau)}) (u_1^* - u_2^*) d\tau \end{aligned}$$

entonces, de (6.14):

$$\underline{f}_1 = f_{1,1}; \quad \underline{f}_2 = f_{2,1}$$

por lo tanto:

$$F = f_{1,1}^2 + f_{2,1}^2$$

y finalmente de (6.16) se tiene:

$$\gamma_1 = \underline{a} = \frac{f_{1,1}^2}{\left(f_{1,1}^2 + f_{2,1}^2\right)} = \frac{\left(\int_0^{T_0} e^{-(T_0-t)}(u_1^* - u_2^*)dt\right)^2}{\left(\int_0^{T_0} e^{-(T_0-t)}(u_1^* - u_2^*)dt\right)^2 + \left(\int_0^{T_0}(1 - e^{-(T_0-t)})(u_1^* - u_2^*)dt\right)^2}$$

Para $T = r = 10$ y $T_0 = 1$ se tiene que (recuerde que u_1^* y u_2^* se obtuvieron en la Sección 6.2):

$$\begin{aligned} u_1^* &= \frac{10}{\sinh 10} e^t \\ u_2^* &= 1.25 - 1.1349 \times 10^{-4} e^t \\ \gamma &= 0.7469 \end{aligned}$$

Ahora procedamos a calcular la cota del error. Para este casos se tiene $H = \underline{f}_1 = f_{1,1} = \underline{b}$, entonces $H^T F^{-1} H = \frac{f_{1,1}^2}{\left(\underline{f}_{1,1}^2 + \underline{f}_{2,1}^2\right)} = \gamma = 0.7469$ y por lo tanto $\lambda_m(H^T F^{-1} H) = 0.7469$. De (6.18) se tiene:

$$0 \leq e^{*2} \leq \left(\int_0^{T_0} e^{-(T_0-t)}(u_1^* - u_2^*)dt\right)^2 (1 - 0.7469) = 0.1575$$

6.3.2 Ejemplos y Simulaciones

Los resultados de la simulación para el caso particular (sistemas 1 y 2) mostrado en la Sección 6.2 con $T_0 = 1$ *seg* y $\gamma = 0.7469$ se muestran en las siguientes Figuras.

1. La Figura 6.3 muestra, en la columna de la izquierda, la respuesta del sistema 1 y cuando se le aplica su control óptimo u_1^* y en la columna de la derecha la respuesta cuando se aplica el control sub-óptimo u. La Figura 6.4 muestra, en la columna de la izquierda, la respuesta del sistema 2 cuando se le aplica su control óptimo u_2^* y en la columna de la derecha la respuesta cuando se aplica el control sub-óptimo u.

2. La Figura 6.5 muestra la evolución del error para los sistemas 1 y 2. La Figura 6.6 muestra la evolución del error del sistema global y la cota superior.

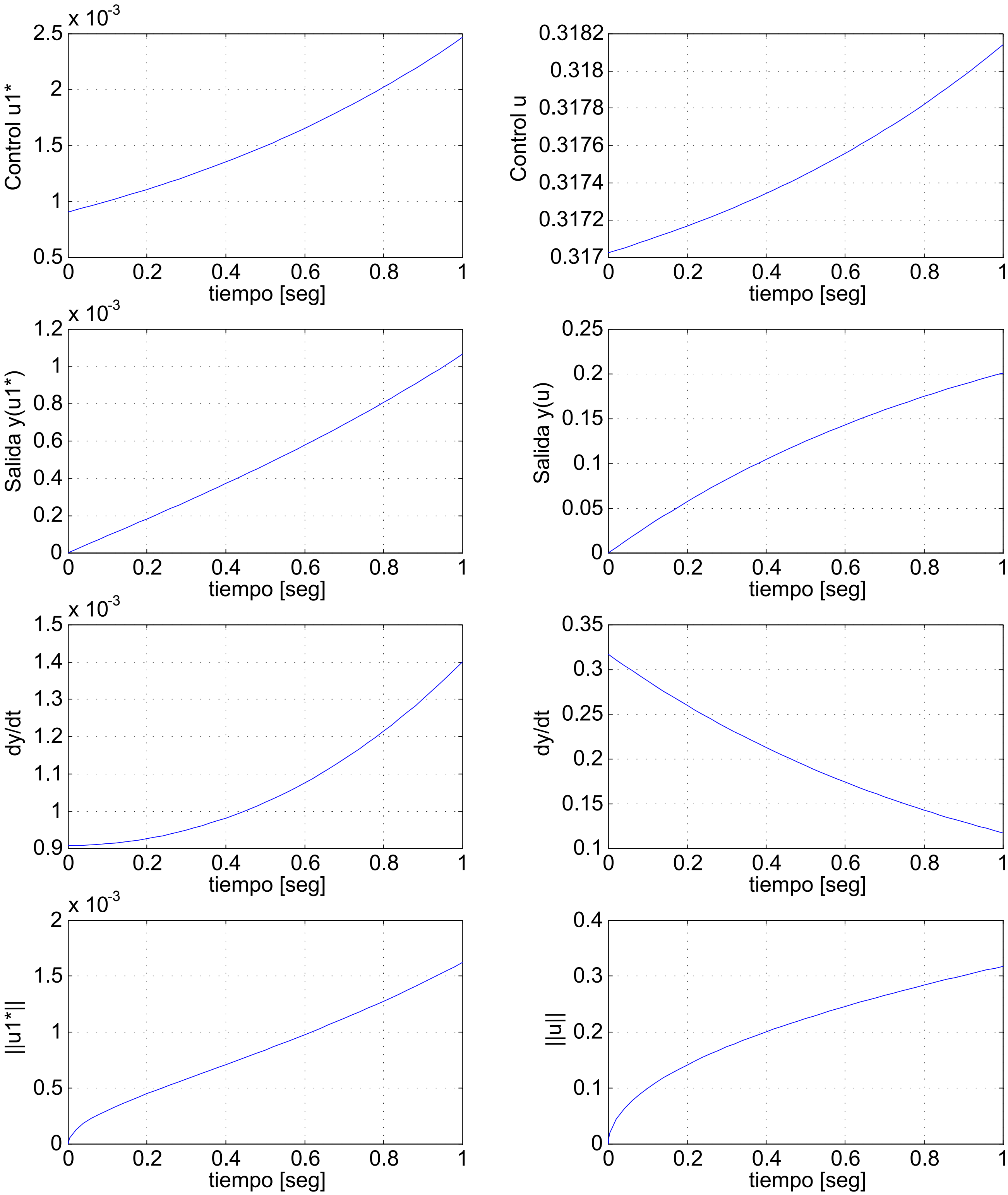

Figura 6.3: Respuesta del Sistema 1 ante su control óptimo u_1^* y el control subóptimo u^*.

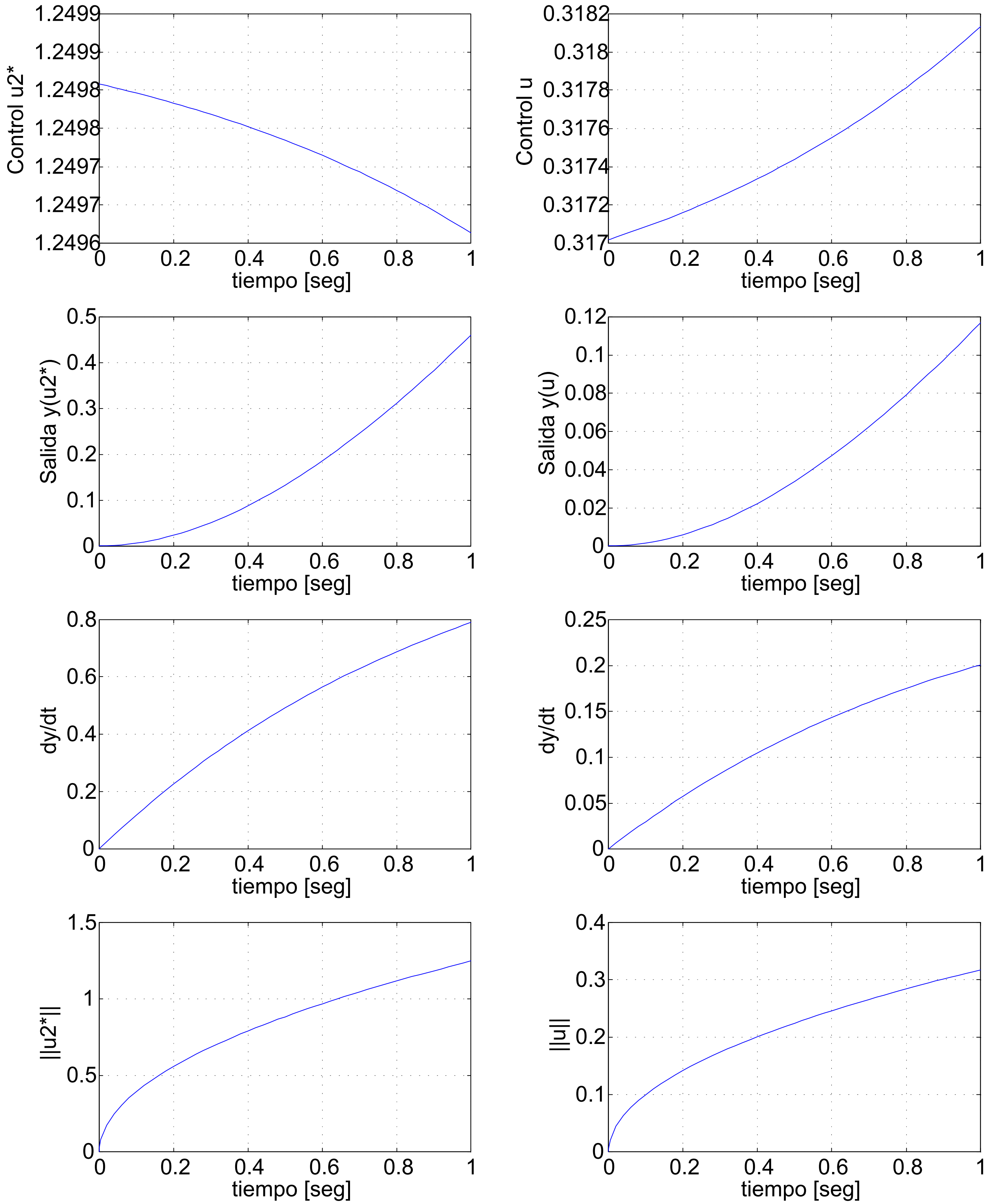

Figura 6.4: Respuesta del Sistema 2 ante su control óptimo u_2^* y el control subóptimo u^*.

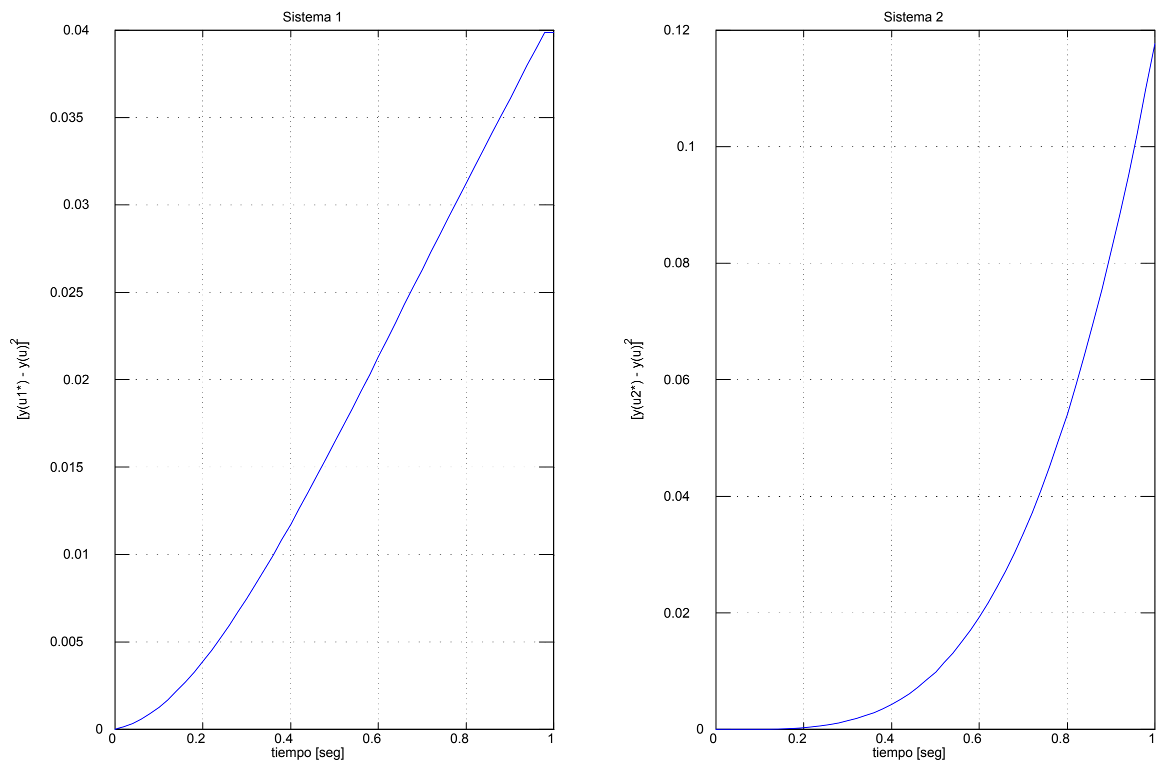

Figura 6.5: Evolución del error para los sistemas 1 y 2.

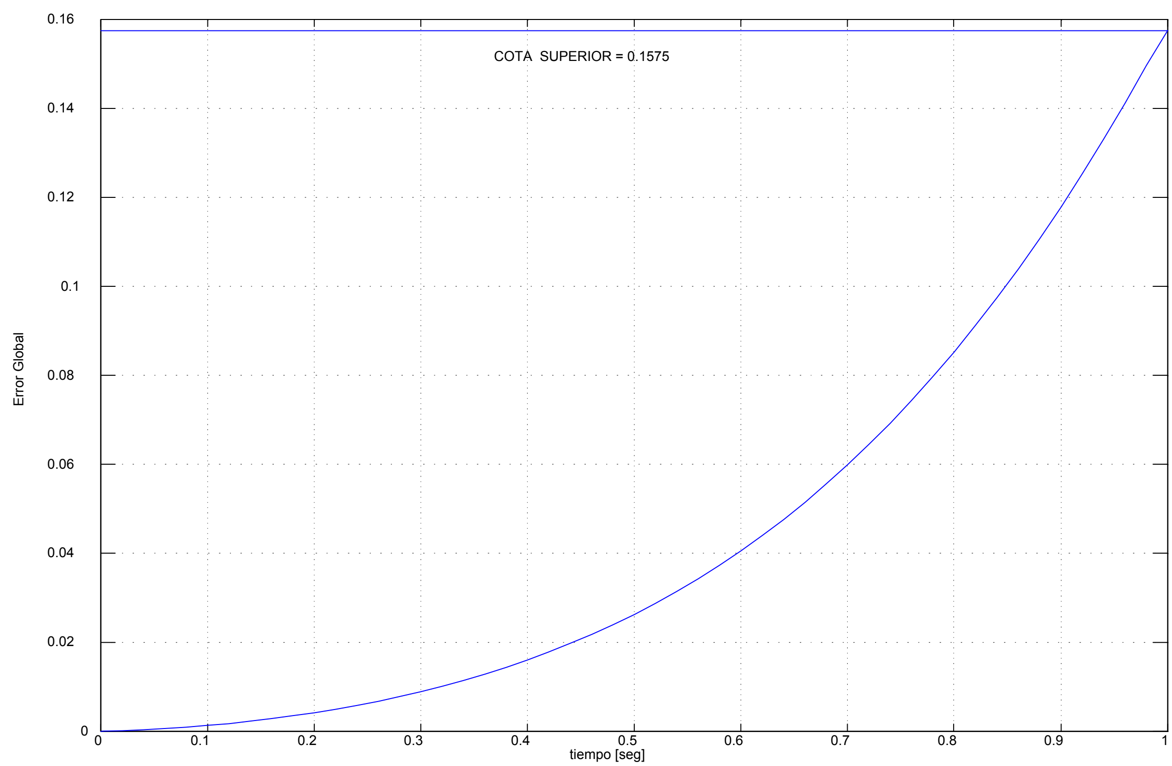

Figura 6.6: Evolución del error global y la cota

Resultados y Perspectivas

Como se mencionó en los capítulos anteriores, los sistemas implícitos rectangulares descritos por modelos matemáticos del tipo:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t); \\ y &= Cx(t) \end{aligned}$$

se pueden utilizar para modelar una clase muy amplia de sistemas lineales. Se presentaron las condiciones necesarias y suficientes (expresados en términos de un modelo implícito global) para controlarlos de tal manera que presenten un comportamiento único, sin importar las variaciones de estructura interna.

La implementación de leyes de control proporcional derivativa presuponen que las variables internas x y su derivada $\dot{x}$ están disponibles, lo que no es generalmente el caso. Para solucionar este problema se puede utilizar un *detector de estructura* cuya finalidad es determinar cual es la estructura *activa* entre las que han sido descritas en el modelo. Una de las aportaciones del presente trabajo fue mejorar los resultados obtenidos por [5, Bonilla et al. 2000] y demostrar que el *esquema del detector de estructura adaptable* converge en un tiempo finito. Para ello se obtiene una cota para el tiempo de convergencia del algoritmo (articulo en proceso de elaboración).

Otra aportación muy importante fue la presentación de un procedimiento sistemático sencillo para aproximar de manera exponencial leyes de control (o, más generalmente, de filtros) impropias por sistemas propios (ver [20, Bonilla et al. 2003]).

Otra contribución importante es la obtención de ley de control proporcional (de alta ganancia). El diseño es similar al procedimiento utilizado para obtener leyes de control proporcional y derivativa. Un punto muy interesante es la *interpretación teórica de sistemas* del proceso clásico de la *integración por partes* que equivale a un cambio de base particular ([21, Bonilla et al. 2003]).

A partir del sistema global (1.3) se ha obtenido una ley de control *sub-óptima* u, como una combinación lineal de los controles óptimos, u_i^*, de cada sistema, que es, obviamente, menos óptima que cuando el *i-ésimo* sistema activo se le aplica su control óptimo asociado, pero que conduce a un compromiso global satisfactorio. Este esquema es conveniente cuando no se sabe con precisión que sistema está activo, es decir, durante el proceso de detección de estructura mediante el detector adaptable propuesto en el Capítulo 3. Una vez detectado el *i-ésimo* sistema activo se procede a aplicarle su control óptimo asociado (artículo en proceso de elaboración).

Una perspectiva de este trabajo es estudiar el siguiente caso. Primero obtener un comportamiento único, mediante las técnicas mostradas en el Capítulo 2 y posteriormente aplicar (al sistema sin variación interna obtenido) metodologías clásicas del regulador óptimo.

Résumé en français

Comme on l'a déjà mentionné, les Systèmes Implicites Rectangulaires décrits par des modèles du type $E\dot{x}(t) = Ax(t) + Bu(t)$, $y = Cx(t)$ peuvent être utilisés pour modéliser une classe très large de systèmes linéaires. On a donné des conditions nécessaires et suffisantes, (exprimées sur un système implicite global), qui garantissent l'existence d'une commande assurant un comportement unique à la sortie, quelles que soient les variations de structure interne.

La mise en oeuvre de lois de commande Proportionnelle et Dérivée présuppose que les variables internes et ses dérivées sont disponibles, ce qui n'est généralement pas le cas, et qui oblige à les observer. Pour contourner cette difficulté on propose un détecteur de structure, dont la finalité est de déterminer quelle est la structure interne active parmi celles qui ont été décrites dans le modèle. Une contribution de ce travail a été améliorer un précédent résultat de Bonilla et al ([5]). On a ainsi montré que ce détecteur de structure adaptatif converge en un temps fini et on obtient une borne pour le temps de convergence de l'algorithme (un article est en cours de rédaction).

Une deuxième contribution très important a été donner une procédure systématique pour approximer de manière exponentielle des lois de commande (ou, plus généralement, de filtres) impropres par des systèmes propres à grand gain.

Une troisième contribution a été l'obtention d'une loi de commande purement proportionnelle (à grand gain). La synthèse est différente, mais très proche de celle du procédé utilisé pour obtenir la loi de commande Proportionnelle et Dérivée. Un point que nous trouvons particulièrement intéressant est l'interprétation, du point de vue de la théorie des systèmes, du procédé classique de l'intégration par parties, et qui est traduit par un changement de base (généralisé) particulier.

Pour le cas d'une famille de n systèmes linéaires avec commutations internes, on a proposé une loi de commande sous-optimale obtenue comme une combinaison linéaire des commandes optimales de chaque système. Cette loi de commande est moins *bonne* quand le *i-ème* système est actif que lorsque l'on applique sa commande optimale associée ; mais elle conduit à un compromis global satisfaisant. Ce schéma est utile quand on ne sait pas précisément quel système est actif (un article est en cours de rédaction sur ce point); ceci est le cas lorsque l'on détecte la structure interne en utilisant l'algorithme de détection adaptatif proposé dans le Chapitre 3. Une fois détecté le *i-ème* système actif, on applique sa loi de commande optimale associée.

Une suite naturelle pour ce travail de thèse est d'étudier et de comparer la demarche consistant à rendre tout d'abord le comportement unique (en utilisant les techniques du Chapitre 2) puis à appliquer ensuite (au system sans variation interne ainsi obtenue) la méthodologie classique du régulateur optimal.

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Apéndice A

Proper Exponential Approximation of Non Proper Compensator: MIMO Case (42nd IEEE-CDC2003, Maui, Hawaii, USA)

Proper Exponential Approximation of Non Proper Compensators: The MIMO Case

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Abstract— In some control problems, it may be convenient, at least from the analysis point of view, to use non proper compensators. However, as far as their implementation is concerned, proper approximations have to be designed. In the present paper, we first show how exponential approximations can be rather easily designed (Lemma 1). Then, we characterize, in geometric terms, the external properness of an implicit description (Theorem 1 and Corollary 1). Finally, the combination of those two results solves the problem of proper exponential approximation and generalizes to the MIMO case a previous result from Bonilla and Lozano (Theorem 2).

Notation

Script capitals $\mathcal{V}$, $\mathcal{W}$, ..., denote linear spaces with elements v, w, ...; the dimension of a space $\mathcal{V}$ is denoted $\dim(\mathcal{V})$; when $\mathcal{V} \subset \mathcal{W}$, $\frac{\mathcal{W}}{\mathcal{V}}$ or $\mathcal{W}/\mathcal{V}$ stands for the quotient space $\mathcal{W}$ modulo $\mathcal{V}$; the direct sum of independent spaces is written as $\oplus$. Given a map $X : \mathcal{V} \to \mathcal{W}$, Im $X = X\mathcal{V}$ denotes its image, and $\mathcal{K}_X$ or sometimes Ker X denotes its kernel. We write $X^{-1}\mathcal{T}$ for the inverse image of $\mathcal{T}$ by X. $\{x, y, z\}$ stands for the subspace spanded by x, y and z. e_i stands for the vector with a 1 in its $i-th$ component and 0 in its other components. $\mathcal{L}^{-1}\{\cdot\}$ denotes the inverse Laplace Transform; s is the complex Laplace variable; p is the derivative operator d/dt. $\underline{\chi}_k^i$ denotes a kx1 vector whose i-th component is 1 and the others are zero. $\underline{1}_k$ denotes a kx1 vector whose all components are 1. I_k denotes a kxk identity matrix, or simply I when the size does not have to be explicitly indicated. $U\{v^T\}$ denotes an upper triangular Toeplitz matrix with first row vector v^T. $L\{v\}$ denotes a lower triangular Toeplitz matrix with first column vector v. $D\{X_1, ..., X_k\}$ denotes a block diagonal matrix whose diagonal block are the matrices $X_1, ..., X_k$. Examples: $\underline{\chi}_2^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{1}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $U\{[\ a \quad b\ ]\} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, $L\left\{\begin{bmatrix} a \\ b \end{bmatrix}\right\} = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$, $D\{X_1, X_2\} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$.

I. Introduction

In many control problems (e.g.decoupling, disturbance decoupling,...), one often has to consider non proper compensators. This is because, either proper exact solutions do not exist, or obtaining them (often based on inversion techniques) is much easier; see for example [5], [6] and [7]. Then, for their effective implementation, proper approximations must be designed which do not "much" alter the control objective. We propose here a systematic procedure for proper exponential approximation. More precisely, we give a solution to the following problem:

Problem 1: Given the non-proper compensator, $\Sigma^c : \mathcal{U} \to \mathcal{Y}$, with realization:

$$N\dot{\omega}(t) = \omega(t) + \Gamma u(t) \quad ; \quad y^*(t) = \Delta\omega(t) \tag{1}$$

where $N \colon \mathcal{W} \to \mathcal{W}$, $\Gamma : \mathcal{U} \to \mathcal{W}$ and $\Delta : \mathcal{W} \to \mathcal{Y}$ are linear operators, N is a nilpotent operator and $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{W}$ are the input, the output and the descriptor variable spaces, respectively. Find a strictly proper filter, $\Sigma^f : \mathcal{Y} \to \mathcal{Y}$, with realization:

$$\dot{z}(t) = A(\varepsilon) z(t) + B(\varepsilon) y^*(t) \quad ; \quad y(t) = Cz(t) \tag{2}$$

such that:

1) $\lim_{\varepsilon \to 0} \|y(t) - y^*(t)\| \leq K\mathrm{e}^{-\beta t}$, with $K, \beta > 0$ and Σ^f is internally stable for all $\varepsilon > 0$
2) The transfer function matrix of the overall system, $\Sigma^f \circ \Sigma^c$, is proper.

We are going to assume that the non proper compensator (1) is completely observable, and then its Kronecker canonical form has only row minimal indices blocks (see [8], [10] and [9]). And thus, when system (1) is carried to its Kronecker canonical form, we get

$$\begin{array}{c} N = D\{N_1, \ldots, N_n\}, \ \Delta = D\{\Delta_1^T, \ldots, \Delta_n^T\} \\ N_i = L\left\{\underline{\chi}^2_{(k_i+1)}\right\}, \ \Delta_i = \underline{\chi}^{(k_i+1)}_{(k_i+1)}, \text{ with } i = 1, \ldots, n \end{array} \tag{3}$$

Note that $k_i \geq 0$, $i = 1, \ldots, n$, denote the orders of the poles at infinity of compensator (1).

In Section II we propose an internally stable strictly proper realization, Σ^f, which external behavior exponentially approaches that of the non proper compensator, Σ^c (solution of part 1 of Problem 1). In Section III we characterize the external properness, through some nice geometric results (Theorem 1 and its Corollary 1). In Section IV we show that our construction of Σ^f makes the transfer function matrix, $\left(\Sigma^f \circ \Sigma^c\right)(s)$, externally proper (solution of part 2 of Problem 1). This leads to the structural Theorem 2, which generalizes to the MIMO case some previous SISO result from Bonilla and Lozano [4]. In Section V we detail an illustrative example and we conclude in Section VI.

II. Exponential Approximation

Lemma 1: Let us consider the system, $\Sigma^f : \mathcal{Y} \to \mathcal{Y}$,

$$\begin{array}{l} \dot{\bar{x}}(t) = A_\beta \bar{x}(t) - \varepsilon^{k+1} y(t) \\ \varepsilon \dot{\hat{x}}(t) = A_o \hat{x}(t) + B_o \left(\bar{x}(t) + y^*(t)\right) \ ; \quad y(t) = C_o \hat{x}(t) \end{array} \tag{4}$$

where $\hat{x} \in \widehat{\mathcal{X}}$; $\bar{x}, y, y^* \in \mathcal{Y}$, and $\varepsilon > 0$ such that:

1. A_β and A_o are Hurwitz,
2. $\mathcal{L}^{-1}\left\{\left(\mathrm{sI} - \frac{1}{\varepsilon}A_o\right)^{-1}\right\} = \overline{A}_o(t,\varepsilon)\mathrm{e}^{-t/\varepsilon}$,
3. The elements of $\overline{A}_o(t,\varepsilon)$ are polynomials in the variable t/ε, with degrees less than or equal to κ,
4. $\int_0^\infty C_o \overline{A}_o(\lambda)\mathrm{e}^{-\lambda} B_o \mathrm{d}\lambda = \mathrm{I}$, $\overline{A}_o(\lambda) = \overline{A}_o(\varepsilon\lambda, \varepsilon)$,
5. $\det\left(\varepsilon^{\kappa-1} C_o (\mathrm{sI} - \frac{1}{\varepsilon}A_o)^{-1} B_o (\mathrm{sI} - A_\beta)^{-1}\right) = \frac{\bar{h}(\varepsilon)\bar{f}(\mathrm{s})}{\bar{g}(\mathrm{s})}$,

where $\bar{h} \in \mathbb{R}[\varepsilon]$ & $\bar{f}$, $\bar{g} \in \mathbb{R}[\mathrm{s}]$, with $\bar{g}$ Hurwitz.

Then:

$$\lim_{\varepsilon \to 0} \left(y(t) - y^*(t)\right) = \mathrm{e}^{A_\beta t} \bar{x}(0) \, ; \ t > 0 \tag{5}$$

$$\det \begin{bmatrix} (\mathrm{sI} - A_\beta) & \varepsilon^{k+1} C_o \\ -\frac{1}{\varepsilon} B_o & (\mathrm{sI} - \frac{1}{\varepsilon} A_o) \end{bmatrix} \text{ is Hurwitz} \tag{6}$$

Before proving this Lemma let us show that the following choice of A_o, B_o and C_o satisfies the requirements:

$$\begin{array}{c} A_o = D\{A_1, \ldots, A_n\}, \ B_o = D\{b_1, \ldots, b_n\} \\ C_o = D\{c_1^T, \ldots, c_n^T\} \end{array} \tag{7}$$

$$\begin{array}{c} A_i = -\mathrm{I}_{k_i} + U\{(\underline{\chi}_{k_i}^2)^T\}, \ b_i = \underline{\chi}_{k_i}^{k_i}, \ c_i = \underline{\chi}_{k_i}^1, \\ \text{with } i = 1, ..., n. \text{ and } \kappa = \max\{k_1, \ldots, k_n\} \end{array} \tag{8}$$

Indeed, $\mathcal{L}^{-1}\{(\varepsilon \mathrm{sI} - A_i)^{-1}\} = \frac{1}{\varepsilon} U\left\{\begin{bmatrix} 1 & \frac{t/\varepsilon}{1!} & \cdots \frac{(t/\varepsilon)^{k_i - 1}}{(k_i - 1)!}\end{bmatrix}\right\}\mathrm{e}^{-t/\varepsilon}$, and then $\int_0^\infty c_i^T \overline{A}_i(\varepsilon\lambda, \varepsilon)\mathrm{e}^{-\lambda} b_i \mathrm{d}\lambda = c_i^T U\{\underline{1}_{k_i}^T\} b_i = 1$. Moreover the Markov's parameters of each subsystem $\left\{A_i, b_i, c_i^T\right\}$ satisfy:

$$\begin{array}{l} h_{i,j+1} = c_i^T A_i^j b_i = 0, \text{ for } j = 0, 1, \ldots, k_i - 2, \\ i = 1, \ldots, n \ \text{ and } \ h_{i,k_i} = 1, \text{ for } i = 1, \ldots, n \end{array} \tag{9}$$

Furthemore: $\det\left[\mathrm{I} + \varepsilon^\kappa C_o (\mathrm{sI} - \frac{1}{\varepsilon} A_o)^{-1} B_o (\mathrm{sI} - A_\beta)^{-1}\right] = \Pi_{i=1}^n \left(1 + \varepsilon \frac{\varepsilon^{\kappa - k_i}}{(\mathrm{s} + 1/\varepsilon)^{k_i}(\mathrm{s} + \beta)}\right)$.

Proof of Lemma 1

$$\begin{array}{l} \det \begin{bmatrix} (\mathrm{sI} - A_\beta) & \varepsilon^{k+1} C_o \\ -\frac{1}{\varepsilon} B_o & (\mathrm{sI} - \frac{1}{\varepsilon} A_o) \end{bmatrix} = \\ = \det[\mathrm{sI} - A_\beta] \det[\mathrm{sI} - \frac{1}{\varepsilon} A_o] \det[\mathrm{I} + \varepsilon^\kappa C_o (\mathrm{sI} - \frac{1}{\varepsilon} A_o)^{-1} \\ \qquad B_o (\mathrm{sI} - A_\beta)^{-1}] \\ = \det[\mathrm{sI} - A_\beta] \det[\mathrm{sI} - \frac{1}{\varepsilon} A_o] \left(1 + \varepsilon \frac{f(\mathrm{s}, \varepsilon)}{g(\mathrm{s})}\right) \end{array}$$

where $g(\mathrm{s})$ divides $\bar{g}(\mathrm{s})$ and $f(\mathrm{s}, \varepsilon)$ divides $\bar{h}(\varepsilon)\bar{f}(\mathrm{s}, \varepsilon)$. Then by Routh-Hurwitz there exists ε^* such that the right hand side is Hurwitz for all $\varepsilon \in (0, \varepsilon^*]$.

The solution of (4) is:

$$\begin{array}{l} \bar{x}(t) = \mathrm{e}^{A_\beta t} \bar{x}(0) - \varepsilon^{k+1} \int_0^t \mathrm{e}^{A_\beta (t - \sigma)} y(\sigma) \mathrm{d}\sigma; \\ y(t) = C_o \overline{A}_o(t, \varepsilon) \mathrm{e}^{-t/\varepsilon} \hat{x}(0) + C_o \int_0^t \frac{1}{\varepsilon} \overline{A}_o(t - \tau, \varepsilon) \\ \qquad \mathrm{e}^{-(t-\tau)/\varepsilon} B_o \left(\bar{x}(\tau) + y^*(\tau)\right) \mathrm{d}\tau \end{array}$$

Doing the change of variable, $\tau = t - \varepsilon\lambda$, we get: $y(t) = C_o \overline{A}_o(t, \varepsilon)\mathrm{e}^{-t/\varepsilon}\bar{x}(0) + C_o \int_0^{t/\varepsilon} \overline{A}_o(\lambda)\mathrm{e}^{-\lambda} B_o \mathrm{e}^{A_\beta (t - \varepsilon\lambda)} \bar{x}(0)\mathrm{d}\lambda - \varepsilon^{k+1} C_o \int_0^{t/\varepsilon} \overline{A}_o(\lambda)\mathrm{e}^{-\lambda} B_o \int_0^{t - \varepsilon\lambda} \mathrm{e}^{A_\beta (t - \varepsilon\lambda - \sigma)} y(\sigma)\mathrm{d}\sigma\mathrm{d}\lambda + C_o \int_0^{t/\varepsilon} \overline{A}_o(\lambda)\mathrm{e}^{-\lambda} B_o y^*(t - \varepsilon\lambda)\mathrm{d}\lambda$. Then $\lim_{\varepsilon \to 0}\left(y(t) - y^*(t)\right) = \mathrm{e}^{A_\beta t}\bar{x}(0)$. ■

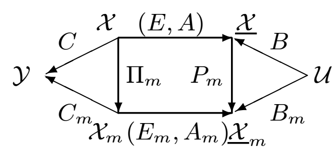
III. External Properness

Definition 1: The implicit system

$$E\dot{x}(t) = Ax(t) + Bu(t) \ ; \quad y(t) = Cx(t) \tag{10}$$

where $E : \mathcal{X} \to \underline{\mathcal{X}}$, $A : \mathcal{X} \to \underline{\mathcal{X}}$, $B : \mathcal{U} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ are linear operators, is externally proper iff its externally minimal part is internally proper.

It is shown in [3] that the system (E, A, B, C) is externally equivalent to the externally minimal system (E_m, A_m, B_m, C_m), shown in Fig. 1.



$$\mathcal{X}_m = \frac{\mathcal{V}_\mathcal{X}^*}{\mathcal{V}_o^* + \mathcal{V}_\mathcal{X}^* \cap \mathcal{R}_{a0}^*} \ ; \ \underline{\mathcal{X}}_m = \frac{E\mathcal{V}_\mathcal{X}^* + \mathrm{Im}\ B}{E\mathcal{V}_o^* + A\left(\mathcal{V}_\mathcal{X}^* \cap \mathcal{R}_{a0}^*\right)}$$

Fig. 1. External minimal realization of a given (E, A, B, C)) realization. Π_m and P_m are canonical projections.

The subspace $\mathcal{V}_\mathcal{X}^*$ characterizes (together with $E\mathcal{V}_\mathcal{X}^* + \mathrm{Im}\ B$) the set of all possible trajectories which are not identically zero for any input u, which is the limit of:

$$\mathcal{V}_\mathcal{X}^o = \mathcal{X} \ ; \ \ \mathcal{V}_\mathcal{X}^{\mu+1} = A^{-1}\left(E\mathcal{V}_\mathcal{X}^\mu + \mathrm{Im}\ B\right).$$

The subspace $\mathcal{V}_o^*$ characterizes (together with $E\mathcal{V}_o^*$) the set of all exponential trajectories which are unobservable at the output y. This subspace is the limit of:

$$\mathcal{V}_o^o = \mathcal{X} \ ; \quad \mathcal{V}_o^{\mu+1} = \mathcal{K}_C \cap A^{-1} E \mathcal{V}_o^\mu.$$

The subspace $\mathcal{R}_{a0}^*$ characterizes (together with $A\mathcal{R}_{a0}^*$) the set of all trajectories due to pure differential actions with no influence on the input-output trajectories. This subspace is the limit of:

$$\mathcal{R}_{ao}^o = \mathcal{K}_C \cap \mathcal{K}_E \ ; \quad \mathcal{R}_{ao}^{\mu+1} = \mathcal{K}_C \cap E^{-1} A \mathcal{R}_{ao}^\mu. \tag{11}$$

Theorem 1: If (10) is observable and has no trajectories identically null, no matter the input action, namely $\mathcal{V}_o^* = \{0\}$ and $\mathcal{V}_\mathcal{X}^* = \mathcal{X}$, (10) is externally proper iff

$$\mathcal{V}_{\mathcal{X}_o}^* + \mathcal{S}_{\mathcal{X}_o}^* = \mathcal{X} \ ; \quad \mathcal{V}_{\mathcal{X}_o}^* \cap \mathcal{S}_{\mathcal{X}_o}^* \subset \mathcal{R}_{a0}^* \tag{12}$$

$$\dim \left(\frac{\mathcal{V}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^* + \overline{\mathcal{T}}_1^2}{\mathcal{V}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^* + \overline{\mathcal{T}}_1^1} \right) = 0 \tag{13}$$

where $\mathcal{V}_{\mathcal{X}_o}^*$ and $\mathcal{S}_{\mathcal{X}_o}^*$ are respectively the limits of:

$$\begin{array}{l} \mathcal{V}_{\mathcal{X}_o}^o = \mathcal{X} \ , \quad \mathcal{V}_{\mathcal{X}_o}^{\mu+1} = A^{-1} E \mathcal{V}_{\mathcal{X}_o}^\mu \\ \mathcal{S}_{\mathcal{X}_o}^o = \mathcal{K}_E \ , \quad \mathcal{S}_{\mathcal{X}_o}^{\mu+1} = E^{-1} A \mathcal{S}_{\mathcal{X}_o}^\mu \end{array} \tag{14}$$

and $\overline{\mathcal{T}}_1^\mu$ and $\overline{\mathcal{T}}_2^\mu$ are extracted from the two algorithms:

$$\overline{\mathcal{T}}_1^o = \mathcal{R}_{a0}^* \ , \quad \overline{\mathcal{T}}_1^{\mu+1} = E^{-1}A\left(\overline{\mathcal{T}}_1^\mu + \mathcal{R}_{a0}^*\right)$$
$$\overline{\mathcal{T}}_2^o = \mathcal{X} \ , \quad \overline{\mathcal{T}}_2^{\mu+1} = A^{-1}E\overline{\mathcal{T}}_2^\mu + \mathcal{R}_{a0}^*$$

$\mathcal{V}_{\mathcal{X}_o}^*$ characterizes (together with $E\mathcal{V}_{\mathcal{X}_o}^*$) the exponential trajectories and $\mathcal{S}_{\mathcal{X}_o}^*$ characterizes (together with $A\mathcal{S}_{\mathcal{X}_o}^*$) the set of all trajectories due to pure differential actions (see [11] and [1]); let us note that $\mathcal{S}_{\mathcal{X}_o}^* \supset \mathcal{R}_{a0}^*$.

Proof of Theorem 1 The proof is done in 4 steps:

1. Let us first recall that a pencil $[\lambda\mathbb{F} - \mathbb{G}]$ is internally proper iff (see [8], [1], [2]): it is regular, i.e. $\det[\lambda\mathbb{F} - \mathbb{G}] \neq 0$, and it has no infinite zeros of order greater than one, i.e. there exist no derivators.

From [9] we have that the regularity is equivalent to:

$$\mathcal{X} = \mathcal{A}_1^* \oplus \mathcal{A}_2^*$$

Also, the absence of infinite zeros of order greater than one is equivalent to:

$$\dim\left(\frac{\mathcal{A}_2^* + \mathcal{A}_1^2}{\mathcal{A}_2^* + \mathcal{A}_1^1}\right) = 0$$

where $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ are respectively the limits of:

$$\mathcal{A}_1^o = \{0\} \ , \quad \mathcal{A}_1^{\mu+1} = \mathbb{F}^{-1}\mathbb{G}\mathcal{A}_1^\mu$$
$$\mathcal{A}_2^o = \mathcal{X} \ , \quad \mathcal{A}_2^{\mu+1} = \mathbb{G}^{-1}\mathbb{F}\mathcal{A}_2^\mu$$

2. Let us next show that the system (E_m, A_m, B_m, C_m) is internally proper iff:

$$\mathcal{X} = \mathcal{T}_1^* + \mathcal{T}_2^* \ ; \quad \mathcal{T}_1^* \cap \mathcal{T}_2^* = \text{Ker }\Pi_m$$

$$\dim\left(\frac{\mathcal{T}_2^* + \mathcal{T}_1^2}{\mathcal{T}_2^* + \mathcal{T}_1^1}\right) = \dim\left(\frac{(\mathcal{T}_2^* + \mathcal{T}_1^2) \cap \text{Ker }\Pi_m}{(\mathcal{T}_2^* + \mathcal{T}_1^1) \cap \text{Ker }\Pi_m}\right)$$

where $\mathcal{T}_1^o = \text{Ker }\Pi_m$, $\mathcal{T}_1^{\mu+1} = E^{-1}(A\mathcal{T}_1^\mu + \text{Ker }P_m)$, and $\mathcal{T}_2^o = \mathcal{X}$, $\mathcal{T}_2^{\mu+1} = A^{-1}(E\mathcal{T}_2^\mu + \text{Ker }P_m)$.

Indeed, from the first item (E_m, A_m, B_m, C_m) is regular iff $\mathcal{X}_m = \mathcal{A}_{1m}^* \oplus \mathcal{A}_{2m}^*$, where $\mathcal{A}_{1m}^*$ and $\mathcal{A}_{2m}^*$ are respectively the limits of:

$$\mathcal{A}_{1,m}^o = \{0\} \ , \quad \mathcal{A}_{1m}^{\mu+1} = E_m^{-1}A_m\mathcal{A}_{1m}^\mu$$
$$\mathcal{A}_{2m}^o = \mathcal{X}_m \ , \quad \mathcal{A}_{2m}^{\mu+1} = A_m^{-1}E_m\mathcal{A}_{2m}^\mu$$

Now from the commutative diagram of Fig. 1 we get: $\Pi_m^{-1}\mathcal{A}_{1m}^{\mu+1} = (E_m\Pi_m)^{-1}A_m\mathcal{A}_{1m}^\mu = E^{-1}P_m^{-1}A_m\Pi_m\Pi_m^{-1}\mathcal{A}_{1m}^\mu = E^{-1}P_m^{-1}P_mA\Pi_m^{-1}\mathcal{A}_{1m}^\mu = E^{-1}(A\Pi_m^{-1}\mathcal{A}_{1m}^\mu + \text{Ker }P_m)$, namely: $\mathcal{T}_1^\mu = \Pi_m^{-1}\mathcal{A}_{1m}^\mu$. In a similar way $\mathcal{T}_2^\mu = \Pi_m^{-1}\mathcal{A}_{2m}^\mu$. And thus,

$$\mathcal{X}_m = \mathcal{A}_{1m}^* \oplus \mathcal{A}_{2m}^* \ \text{ iff } \ \mathcal{X} = \mathcal{T}_1^* + \mathcal{T}_2^* \ \text{ and } \ \mathcal{T}_1^* \cap \mathcal{T}_2^* = \text{Ker }\Pi_m$$

On the other hand

$$\dim\left(\frac{\mathcal{A}_{2m}^*+\mathcal{A}_{1m}^2}{\mathcal{A}_{2m}^*+\mathcal{A}_{1m}^1}\right) = \dim\left(\frac{\Pi_m\left(\mathcal{T}_2^*+\mathcal{T}_1^2\right)}{\Pi_m\left(\mathcal{T}_2^*+\mathcal{T}_1^1\right)}\right)$$
$$= \dim\left(\frac{\mathcal{T}_2^*+\mathcal{T}_1^2}{\mathcal{T}_2^*+\mathcal{T}_1^1}\right) - \dim\left(\frac{(\mathcal{T}_2^*+\mathcal{T}_1^2)\cap\text{Ker }\Pi_m}{(\mathcal{T}_2^*+\mathcal{T}_1^1)\cap\text{Ker }\Pi_m}\right)$$

And thus

$$\dim\left(\frac{\mathcal{A}_{2m}^*+\mathcal{A}_{1m}^2}{\mathcal{A}_{2m}^*+\mathcal{A}_{1m}^1}\right) = 0 \ \text{ iff}$$
$$\dim\left(\frac{\mathcal{T}_2^*+\mathcal{T}_1^2}{\mathcal{T}_2^*+\mathcal{T}_1^1}\right) = \dim\left(\frac{(\mathcal{T}_2^*+\mathcal{T}_1^2)\cap\text{Ker }\Pi_m}{(\mathcal{T}_2^*+\mathcal{T}_1^1)\cap\text{Ker }\Pi_m}\right)$$

3. Let us now show that: If $\mathcal{V}_o^* = \{0\}$ and $\mathcal{V}_{\mathcal{X}}^* = \mathcal{X}$ then $\mathcal{T}_1^\mu = \overline{\mathcal{T}}_1^\mu$ and $\mathcal{T}_2^\mu = \overline{\mathcal{T}}_2^\mu$; moreover, $\overline{\mathcal{T}}_1^* = \mathcal{S}_{\mathcal{X}_o}^*$ and $\overline{\mathcal{T}}_2^* = \mathcal{V}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^*$.

Indeed, from the commutative diagram of Fig. 1 we get for this case $\text{Ker }\Pi_m = \mathcal{R}_{a0}^*$ and $\text{Ker }P_m = A\mathcal{R}_{a0}^*$, and thus $\mathcal{T}_1^\mu = \overline{\mathcal{T}}_1^\mu$ and $\mathcal{T}_2^\mu = \overline{\mathcal{T}}_2^\mu$.

Note that $\overline{\mathcal{T}}_1^1 = E^{-1}A\mathcal{R}_{a0}^* \supset \mathcal{K}_E = \mathcal{S}_{\mathcal{X}_o}^o + \mathcal{R}_{a0}^*$. Let us then assume that $\overline{\mathcal{T}}_1^\mu \supset \mathcal{S}_{\mathcal{X}_o}^{\mu-1} + \mathcal{R}_{a0}^*$, which implies $\overline{\mathcal{T}}_1^{\mu+1} \supset \mathcal{S}_{\mathcal{X}_o}^\mu + E^{-1}A\mathcal{R}_{a0}^* \supset \mathcal{S}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^*$.

On the other hand, $\overline{\mathcal{T}}_1^o = \mathcal{R}_{a0}^* \subset \mathcal{S}_{\mathcal{X}_o}^*$, assuming then $\overline{\mathcal{T}}_1^\mu \subset \mathcal{S}_{\mathcal{X}_o}^*$ we get $\overline{\mathcal{T}}_1^{\mu+1} \subset \mathcal{S}_{\mathcal{X}_o}^*$. Therefore:

$$\mathcal{S}_{\mathcal{X}_o}^* = \mathcal{S}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^* \subset \overline{\mathcal{T}}_1^* \subset \mathcal{S}_{\mathcal{X}_o}^*$$

Now in view that $\overline{\mathcal{T}}_2^o = \mathcal{X} = \mathcal{V}_{\mathcal{X}_o}^o = \mathcal{V}_{\mathcal{X}_o}^o + \mathcal{R}_{a0}^*$, let us assume that $\overline{\mathcal{T}}_2^\mu = \mathcal{V}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^*$. This assumption implies (recall that $E\mathcal{R}_{a0}^* \subset A\mathcal{R}_{a0}^*$):

$$\begin{aligned}\overline{\mathcal{T}}_2^{\mu+1} &= A^{-1}\left(E\mathcal{V}_{\mathcal{X}_o}^\mu + E\mathcal{R}_{a0}^*\right) + \mathcal{R}_{a0}^* \\ &= A^{-1}\left(E\mathcal{V}_{\mathcal{X}_o}^\mu + E\mathcal{R}_{a0}^* + A\mathcal{R}_{a0}^*\right) \\ &= A^{-1}\left(E\mathcal{V}_{\mathcal{X}_o}^\mu + A\mathcal{R}_{a0}^*\right) \\ &= A^{-1}E\mathcal{V}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^* = \mathcal{V}_{\mathcal{X}_o}^{\mu+1} + \mathcal{R}_{a0}^*\end{aligned}$$

namely $\overline{\mathcal{T}}_2^\mu = \mathcal{V}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^*$, which implies $\overline{\mathcal{T}}_2^* = \mathcal{V}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^*$.

4. Finally, let us note that:

$$\overline{\mathcal{T}}_1^* \cap \overline{\mathcal{T}}_2^* = \mathcal{S}_{\mathcal{X}_o}^* \cap \left(\mathcal{V}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^*\right) = \mathcal{S}_{\mathcal{X}_o}^* \cap \mathcal{V}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^*$$

which proves (12). Also:

$$\dim\left(\frac{(\overline{\mathcal{T}}_2^*+\overline{\mathcal{T}}_1^2)\cap\text{Ker }\Pi_m}{(\overline{\mathcal{T}}_2^*+\overline{\mathcal{T}}_1^1)\cap\text{Ker }\Pi_m}\right) =$$
$$= \dim\left(\frac{(\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^2)\cap\mathcal{R}_{a0}^*}{(\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^1)\cap\mathcal{R}_{a0}^*}\right) = \dim\left(\frac{\mathcal{R}_{a0}^*}{\mathcal{R}_{a0}^*}\right) = 0$$

which proves (13). ∎

Now if the implicit description (10) has only exponential and derivative modes (neither minimal row indices nor minimal column indices) then (see [8], [1] and [9]):

$$\mathcal{X} = \mathcal{V}_{\mathcal{X}_o}^* \oplus \mathcal{S}_{\mathcal{X}_o}^*$$

In this case, (12) is automatically satisfied. Since $\overline{\mathcal{T}}_1^\mu \subset \overline{\mathcal{T}}_1^* = \mathcal{S}_{\mathcal{X}_o}^*$, implies that $\mathcal{V}_{\mathcal{X}_o}^* \cap \mathcal{T}_1^\mu = \mathcal{V}_{\mathcal{X}_o}^* \cap \mathcal{S}_{\mathcal{X}_o}^* \cap \overline{\mathcal{T}}_1^\mu = \{0\}$. And since $\overline{\mathcal{T}}_1^{\mu+1} \supset \mathcal{S}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^* \supset \mathcal{R}_{a0}^*$, we get:

$$\dim\left(\frac{\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^2}{\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^1}\right) =$$
$$= \dim\left(\frac{\mathcal{V}_{\mathcal{X}_o}^*+\overline{\mathcal{T}}_1^2}{\mathcal{V}_{\mathcal{X}_o}^*+\overline{\mathcal{T}}_1^1}\right) = \dim\left(\frac{\mathcal{V}_{\mathcal{X}_o}^*\oplus\overline{\mathcal{T}}_1^2}{\mathcal{V}_{\mathcal{X}_o}^*\oplus\overline{\mathcal{T}}_1^1}\right) = \dim\left(\frac{\overline{\mathcal{T}}_1^2}{\overline{\mathcal{T}}_1^1}\right)$$

Therefore $\dim\left(\frac{\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^2}{\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^1}\right) = 0$ iff $\overline{\mathcal{T}}_1^1 = \overline{\mathcal{T}}_2^2 = \overline{\mathcal{T}}_1^*$, namely: $E^{-1}A\mathcal{R}_{a0}^* = \mathcal{S}_{\mathcal{X}_o}^*$.

We have proved in this way the following Corollary:

Corollary 1: If the implicit system (10) is exponentially observable, has no trajectories identically null, no matter the input action, and has only integral and derivative actions, namely:

$$\mathcal{V}_o^* = \{0\}\ ,\quad \mathcal{V}_{\mathcal{X}}^* = \mathcal{X}\ ,\quad \text{and}\quad \mathcal{X} = \mathcal{V}_{\mathcal{X}_o}^* \oplus \mathcal{S}_{\mathcal{X}_o}^*$$

then (10) is externally proper iff

$$E^{-1}A\mathcal{R}_{a0}^* = \mathcal{S}_{\mathcal{X}_o}^* \tag{15}$$

IV. Proper Exponential Approximation

Embedding the non proper compensator (1) and the strictly proper filter (4) into an implicit description, we get that the overall system, $\left(\Sigma^f \circ \Sigma^c\right)$, is:

$$E\dot{x}(t) = Ax(t) + Bu(t)\ ;\quad y(t) = Cx(t) \tag{16}$$

where:

$$\begin{array}{l} E = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & N \end{bmatrix},\ A = \begin{bmatrix} A_p & B_p \\ 0 & \mathrm{I} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}, \\ C = \begin{bmatrix} C_p & 0 \end{bmatrix} \\ \text{with: } A_p = \begin{bmatrix} A_\beta & -\varepsilon^{k+1}C_o \\ \frac{1}{\varepsilon}B_o & \frac{1}{\varepsilon}A_o \end{bmatrix},\ B_p = \begin{bmatrix} 0 \\ \frac{1}{\varepsilon}B_o\Delta \end{bmatrix}, \\ C_p = \begin{bmatrix} 0 & C_o \end{bmatrix},\quad \text{and}\quad x^T = \begin{bmatrix} \overline{x}^T & \widehat{x}^T & \omega^T \end{bmatrix} \end{array} \tag{17}$$

Taking into account the particular forms of Σ^c and Σ^f, we get from (3), (7) and (17):

$$\begin{array}{l} B_p = [B_{p_1}|B_{p_2}|\cdots|B_{p_n}]\ ;\quad B_{p_i} = [0|\cdots|0|b_{p_i}] \\ b_{p_i}^T = \left[0\middle|\,0\middle|\cdots\middle|0\middle|\,\frac{1}{\varepsilon}b_i^T\,\middle|\,0\middle|\cdots\middle|0\right],\ i = 1, ..., n. \end{array} \tag{18}$$

Lemma 2: Let us define the two following changes of basis: $R = \begin{bmatrix} \mathrm{I} & R_p \\ 0 & \mathrm{I} \end{bmatrix}$, $L = \begin{bmatrix} \mathrm{I} & L_p \\ 0 & \mathrm{I} \end{bmatrix}$, where

$$\begin{array}{l} R_p = \begin{bmatrix} R_{p_1}| & R_{p_2}| & \cdots & |R_{p_n} \end{bmatrix} \\ R_{p_i} = \begin{bmatrix} A_p^{k_i-1}b_{p_i}\Big| & \cdots & \Big|A_pb_{p_i} & \Big|b_{p_i} & \Big|0 \end{bmatrix} \\ L_p = -\left(A_pR_p + B_p\right). \end{array} \tag{19}$$

Then

$$R_p + L_pN = 0 \tag{20}$$

$$\begin{array}{l} C_pA_p^jb_{p_i} = 0,\ \text{for } j = 0, 1, \ldots, k_i - 2,\ \&\ i = 1, \ldots, n \\ C_pA_p^{k_i-1}b_{p_i} = \frac{1}{\varepsilon^{k_i}}\underline{\chi}_n^i,\ \text{for } i = 1, \ldots, n \end{array} \tag{21}$$

Proof of Lemma 2 From (19) and (3) we get:

$$\begin{array}{l} -L_pN = \left(A_pR_p + B_p\right)N \\ = \left[\left[\ A_p^{k_1}b_{p_1}\Big|\ \cdots\ \Big|A_pb_{p_1}\ \Big|b_{p_1}\ \right]N_1\Big|\right. \\ \quad \cdots\ \left.\Big|\left[\ A_p^{k_n}b_{p_n}\Big|\ \cdots\ \Big|A_pb_{p_n}\ \Big|b_{p_n}\ \right]N_n\ \right] \\ = \left[\left[\ A_p^{k_1-1}b_{p_1}\Big|\ \cdots\ \Big|b_{p_1}\ \Big|0\ \right]\Big|\right. \\ \quad \cdots\ \left.\Big|\left[\ A_p^{k_n-1}b_{p_n}\Big|\ \cdots\ \Big|b_{p_n}\ \Big|0\ \right]\ \right] = R_p. \end{array}$$

From (18), (17), (7) and (9) we get (21). ■

Now in view of Lemma 2, we have:

$$\begin{array}{l} LER = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & N \end{bmatrix}\ ,\quad LAR = \begin{bmatrix} A_p & 0 \\ 0 & \mathrm{I} \end{bmatrix}, \\ CR = \begin{bmatrix} C_p & C_n \end{bmatrix} \end{array} \tag{22}$$

where:

$$C_n = D\left\{\nu_1^T, ..., \nu_n^T\right\}\ ;\quad \nu_i = \frac{1}{\varepsilon^{k_i}}\underline{\chi}_{(k_i+1)}^1\ ,\quad i = 1, ..., n. \tag{23}$$

And thus, by simple computation we can check from (22) and (23) that

$$E^{-1}A\mathcal{R}_{a0}^* = \mathcal{S}_{\mathcal{X}_o}^*\ ,$$

that is to say, the overall system (16)-(17) is externally proper and satisfies Lemma 1.

From (3), we can easily see that the integer n corresponds to the number of chains of derivators in the non proper compensator Σ^c in (1), each chain of length k_i. On the other hand, the zero Markov parameters which appear in (21) express the fact that the orders of the zeros at infinity of the strictly proper filter Σ^f are greater than or equal to k_i. This corresponds to the following:

Theorem 2: Let the proper filter Σ^f, be designed as in Lemma 1 in order to approximate in an exponential way the non proper compensator Σ^f. Then the overall system $\left(\Sigma^f \circ \Sigma^c\right)$ is externally proper iff the orders of the zeros at infinity of Σ^f are respectively greater than or equal to k_i.

This result generalizes to the MIMO case the filter introduced in [4] for SISO systems (see Fact 2.1).

Remark 1: As mentioned in the Introduction, one important application of our results rests upon the practical synthesis of non proper compensators, like P.D. feedback (see for instance [5], [6] and [7]). Note that the same procedure can be used identically to implement other kinds of non proper systems, for instance state derivative observers, i.e. systems which give an exponential estimate of $\dot{x}(t)$ and $x(t)$ (see mainly (5) with $y^*(t) = \begin{bmatrix} \dot{x}^T(t) & x^T(t) \end{bmatrix}^T$ and Theorem 2).

V. Illustrative Example

In order to clarify the principal ideas of this paper, let us consider the following non proper system with one pole at infinity of order two:

$$\begin{array}{rcl} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\dot{\xi} & = & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\xi + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}u \\ y^* & = & \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\xi \end{array} \tag{24}$$

In the following two Subsections we are going to consider the approximation proposed in Lemma 1. In the first approximation we take a zero at infinity of order one and in the second one we take order two.

A. First approximation

Let us consider the system of Fig. 2, which descriptor form is ($\overline{x} = \begin{bmatrix} x & z & \xi^T \end{bmatrix}^T$):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \dot{\overline{x}} = \begin{bmatrix} -\beta & -\varepsilon^2 & 0 & 0 & 0 \\ \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} & 0 & 0 & \frac{1}{\varepsilon} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overline{x} + \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \overline{x} \tag{25}$$

Comparing system (25) with (16)–(18), we get: $A_p = \begin{bmatrix} -\beta & -\varepsilon^2 \\ 1/\varepsilon & -1/\varepsilon \end{bmatrix}$, $b_{p1} = \begin{bmatrix} 0 \\ 1/\varepsilon \end{bmatrix}$, and $B_p = B_{p1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\varepsilon \end{bmatrix}$. And then, $R_p = \begin{bmatrix} -\varepsilon & 0 & 0 \\ -1/\varepsilon^2 & 1/\varepsilon & 0 \end{bmatrix}$ and $L_p = \begin{bmatrix} -\varepsilon\beta - 1 & \varepsilon & 0 \\ 1 - 1/\varepsilon^3 & 1/\varepsilon^2 & -1/\varepsilon \end{bmatrix}$, namely: $R = \begin{bmatrix} I_2 & R_p \\ 0 & I_3 \end{bmatrix}$ and $L = \begin{bmatrix} I_2 & L_p \\ 0 & I_3 \end{bmatrix}$. Premultiplying (25) by L and doing $\overline{x} = R\widetilde{x}$, we get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \dot{\widetilde{x}} = \begin{bmatrix} -\beta & -\varepsilon^2 & 0 & 0 & 0 \\ \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \widetilde{x} + \begin{bmatrix} (\varepsilon\beta + 1) & (1/\varepsilon^3 - 1) & -1 & 0 & 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & -\frac{1}{\varepsilon^2} & \frac{1}{\varepsilon} & 0 \end{bmatrix} \widetilde{x} \tag{26}$$

Doing the correspondence of (26) with (22), we can easily identify matrices N, A_p, C_p and C_n. Applying now algorithms (11) and (14) to (26), we get:

$$\mathcal{R}^*_{a0} = \{e_5\}\,;\; E^{-1}A\mathcal{R}^*_{a0} = \{e_4, e_5\}\,;\; \mathcal{S}^*_{\mathcal{X}_o} = \{e_3, e_4, e_5\}$$

We then realize that: $\mathcal{S}^*_{\mathcal{X}_o} \neq E^{-1}A\mathcal{R}^*_{a0}$. And thus, Corollary 1 tells us that (25) is not externally proper.

B. Second approximation

Let us consider the system of Fig. 3, which descriptor form is ($\overline{x} = \begin{bmatrix} x & z_1 & z_2 & \xi^T \end{bmatrix}^T$):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \dot{\overline{x}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} -\beta & -\varepsilon^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon & 0 & 0 & 1/\varepsilon^2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overline{x}$$
$$y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \overline{x} \tag{27}$$

Comparing system (27) with (16)–(18), we get: $A_p = \begin{bmatrix} -\beta & -\varepsilon^2 & 0 \\ 0 & 0 & 1 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon \end{bmatrix}$, $b_{p1} = \begin{bmatrix} 0 \\ 0 \\ 1/\varepsilon^2 \end{bmatrix}$, and $B_p = B_{p1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\varepsilon^2 \end{bmatrix}$. And then, $R_p = \begin{bmatrix} 0 & 0 & 0 \\ 1/\varepsilon^2 & 0 & 0 \\ -2/\varepsilon^3 & 1/\varepsilon^2 & 0 \end{bmatrix}$ and $L_p = \begin{bmatrix} 1 & 0 & 0 \\ 2/\varepsilon^3 & -1/\varepsilon^2 & 0 \\ -3/\varepsilon^4 & 2/\varepsilon^3 & -1/\varepsilon^2 \end{bmatrix}$, namely: $R = \begin{bmatrix} I_3 & R_p \\ 0 & I_3 \end{bmatrix}$ and $L = \begin{bmatrix} I_3 & L_p \\ 0 & I_3 \end{bmatrix}$. Premultiplying (27) by L and doing $\overline{x} = R\widetilde{x}$, we get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \dot{\widetilde{x}} = \begin{bmatrix} -1 \\ -2/\varepsilon^3 \\ 3/\varepsilon^4 \\ -1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} -\beta & -\varepsilon^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \widetilde{x}$$
$$y = \begin{bmatrix} 0 & 1 & 0 & 1/\varepsilon^2 & 0 & 0 \end{bmatrix} \widetilde{x} \tag{28}$$

Doing the correspondence of (28) with (22), we can easily identify matrices N, A_p, C_p and C_n. Applying now algorithms (11) and (14) to (28), we get:

$$\mathcal{R}^*_{a0} = \{e_5, e_6\}\,;\; E^{-1}A\mathcal{R}^*_{a0} = \{e_4, e_5, e_6\}\,;$$
$$\mathcal{S}^*_{\mathcal{X}_o} = \{e_4, e_5, e_6\}$$

We then realize that: $\mathcal{S}^*_{\mathcal{X}_o} = E^{-1}A\mathcal{R}^*_{a0}$. And thus, Corollary 1 tell us that system (27) is externally proper. Indeed, it is externally equivalent to:

$$\begin{aligned} \dot{\widehat{x}} &= \begin{bmatrix} -\beta & -\varepsilon^2 & 0 \\ 0 & 0 & 1 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon \end{bmatrix} \widehat{x} + \begin{bmatrix} -1 \\ -2/\varepsilon^3 \\ 3/\varepsilon^4 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \widehat{x} + (1/\varepsilon^2) u \end{aligned} \tag{29}$$

VI. Conclusion

In this paper we have solved the problem of approximating non proper MIMO systems by stable externally proper systems.

The main geometric result (Theorem 1 and Corollary 1) characterizes the external properness of an implicit description.

This, combined with our proposed approximation (Lemma 1) solves the problem and generalizes to the MIMO case (Theorem 2) a previous result from Bonilla and Lozano [4].

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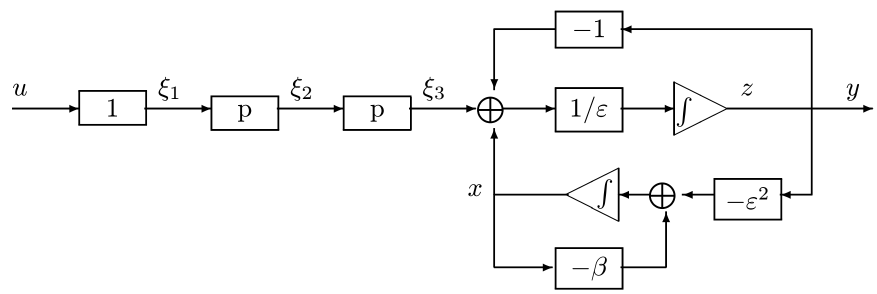

Fig. 2. First approximation.

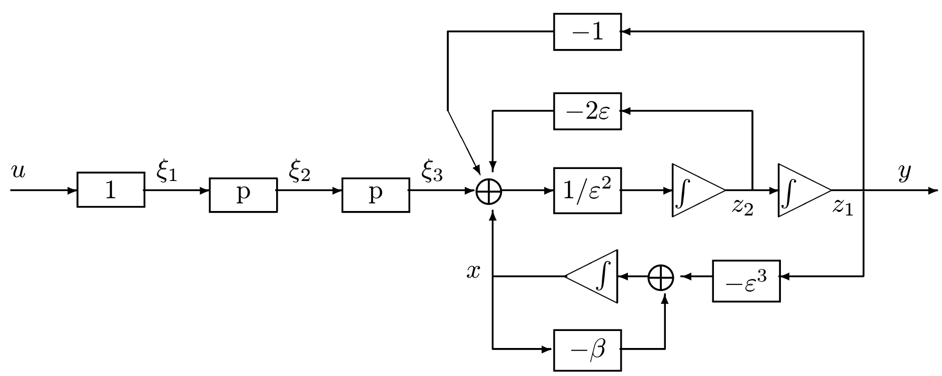

Fig. 3. Second approximation.

Apéndice B

Almost Rejection of Internal Structural Variations in Linear Systems (42nd IEEE-CDC2003, Maui, Hawaii, USA)

Almost Rejection of Internal Structural Variations in Linear Systems

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Abstract— This paper deals with linear systems having internal structural variations. Under some light assumptions, the control of such systems is indeed possible, thanks to the very nice setting of implicit models and Proportional and Derivative (PD) feedbacks. However, the effective design of such PD feedbacks usually requires suitable approximations for the derivatives, hence Proportional approximations of the PD feedback "exact" solution. The aim of the present contribution is to directly takle an "almost" version of the problem by pure (high gain) Proportional feedback. The design is different, but very close to the one related to PD feedbacks, mainly with respect to the associated geometric splittings. As an interesting by-product, a system theoretical interpretation of the classical process of "integration by parts" is given and shown to be equivalent to some particular changes of bases.

Notation

Script capitals $\mathcal{V}, \mathcal{W}, \ldots$, denote linear spaces with elements $v, w, \ldots$; the dimension of a space $\mathcal{V}$ is denoted $\dim(\mathcal{V})$; $\mathcal{V} \approx \mathcal{W}$ stands for $\dim(\mathcal{V}) = \dim(\mathcal{W})$; when $\mathcal{V} \subset \mathcal{W}$, $\frac{\mathcal{W}}{\mathcal{V}}$ or $\mathcal{W}/\mathcal{V}$ stands for the quotient space $\mathcal{W}$ modulo $\mathcal{V}$; the direct sum of independent spaces is written as $\oplus$. Given a map $X : \mathcal{V} \to \mathcal{W}$, Im $X = X\mathcal{V}$ denotes its image, and $\mathcal{K}_X$ or sometimes Ker X denotes its kernel. For the special maps $E : \mathcal{X} \to \underline{\mathcal{X}}$ and $B : \mathcal{U} \to \underline{\mathcal{X}}$, their images are denoted by $\mathcal{E}$ and $\mathcal{B}$, respectively. We write $X^{(-1)}$ for the inverse map of X (when it exists) in order to avoid confusions with $X^{-1}\mathcal{T}$, the inverse image of $\mathcal{T}$ by X. $\{x, y, z\}$ stands for the subspace spanded by x, y and z. e_i stands for the vector with a 1 in its $i-th$ component and 0 otherwise.

I. Introduction

Consider the implicit description, $\Sigma\ (E, A, B, C)$:

$$E\dot{x} = Ax + Bu \quad ; \quad y = Cx \tag{1}$$

where $E : \mathcal{X} \to \underline{\mathcal{X}}$, $A : \mathcal{X} \to \underline{\mathcal{X}}$, $B : \mathcal{U} \to \underline{\mathcal{X}}$ and $C : \mathcal{X} \to \mathcal{Y}$ are linear maps of appropriate dimensions, with $\mathcal{K}_B = \{0\}$ and Im $C = \mathcal{Y}$. x, u and y are the descriptor variable, the input and the output. When $\dim(\underline{\mathcal{X}}) < \dim(\mathcal{X})$, it is possible to describe linear systems with an internal variable structure (see [4]). For example:

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{x} &= \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \end{aligned} \tag{2}$$

with the additional constraint: $[\alpha\ \beta\ 1]x = 0$. If $(\alpha,\ \beta) = (-1,\ -1)$, then $\dot{y} + y = u$. If $(\alpha,\ \beta) = (-1,\ 0)$, then $\ddot{y} + \dot{y} = u$. If $(\alpha,\ \beta) = (-1,\ -5)$, then $\ddot{y} + 6\dot{y} + 5y = 5\dot{u} + u$. Finally, if $(\alpha,\ \beta) = (1,\ 1)$, then $\dot{y} - 3y = -u$.

Problem 1: (Bonilla and Malabre [4] (Part I)) Let us consider a set of strictly proper linear systems embedded in a set of implicit global descriptions $\Sigma_i^g(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$:

$$\mathbb{E}\dot{x} = \mathbb{A}x + \mathbb{B}u \quad ; \quad y = Cx \tag{3}$$

with $\mathbb{E} = \begin{bmatrix} E^T & 0 \end{bmatrix}^T$, $\mathbb{A}_i = \begin{bmatrix} A^T & D_i^T \end{bmatrix}^T$, $\mathbb{B} = \begin{bmatrix} B^T & 0 \end{bmatrix}^T$ $(i = 1, \ldots, n)$; E and D_i are epic.
– Under which conditions can this set of linear systems be controlled by a fixed P.D. state feedback, $u = F_p x + F_d \dot{x}$, assigning a fixed external closed-loop behaviour, and which synthesis is based on the common internal structure, described by $E\dot{x} = Ax + Bu$?
Bonilla and Malabre [4] have found what it is the common internal structure of the set of linear systems (3) which enables to solve Problem 1. They have given a procedure to synthesize P.D. feedbacks for rendering unobservable the variation of structure and assigning at will the closed-loop output dynamics. Such a synthesis procedure relied on the following Theorems (see Section II for definition and related properties of $\mathcal{V}^*$):

Theorem 2: (Bonilla and Malabre [4] (Part II)) If

$$\text{Im } A + \mathcal{B} \subset \mathcal{E} \quad \text{and} \quad \dim\left(\mathcal{V}^* \cap E^{-1}\mathcal{B}\right) \geq \dim\left(\mathcal{K}_E\right) \tag{4}$$

there then exists a P.D. feedback, $u = F_p^* x + F_d^* \dot{x}$, with $(F_p^*, F_d^*) \in \mathrm{F}(\mathcal{V}^*)$ (see Section II), such that:

$$\text{Im } (E - BF_d) = \mathcal{E} \quad \text{and} \quad \text{Ker } (E - BF_d) \subset \mathcal{V}^* \tag{5}$$

Theorem 3: (Bonilla and Malabre [4] (Part II)) Let $(F_p^*, F_d^*) \in \mathrm{F}(\mathcal{V}^*)$, as in Theorem 2. When (3) is fed back with: $u = F_p^* x + F_d^* \dot{x} + v$, it is externally equivalent to: $\dot{\hat{x}} = E_*^{(-1)} A_* \hat{x} + E_*^{(-1)} B_* v$ and $y = C_* \hat{x}$. The isomorphism E_* and the maps A_* and B_* are induced from the closed loop system by the cannonical projections $\Pi : \mathcal{E} \to \mathcal{E}/E_{F^*}\mathcal{V}^*$ and $\Phi : \mathcal{X} \to \mathcal{X}/\mathcal{V}^*$.

Theorem 4: (Bonilla and Malabre [4] (Part II)) Given any pair (F_p^*, F_d^*), as in Theorem 3. If (1), related to (3), is controllable[1] then $\dot{\hat{x}} = E_*^{(-1)} A_* \hat{x} + E_*^{(-1)} B_* v$ and $y = C_* \hat{x}$ is also controllable (in the classical sense).

[1]This controllability depends on the exogenous input, u, and also on the degree of freedom, characterized by $\mathcal{K}_{D_i}$, which acts as endogenous inputs.

The solution found in [4] is based on the use of (P,D) feedbacks which are friends of $\mathcal{V}^*$. From the geometric condition (4.b) of Theorem 2, we can realize that the derivative part of the control law, F_d, is crucial for solving Problem 1, namely a static feedback is generally not sufficient. A natural question from a practical point of view is -what can we do if we are restricted to use Proportional feedbacks?; more precisely -how to approximate the P.D. feedback obtained from Theorem 2 in order to be close to the nice structural properties given in Theorems 3 and 4?. This question is stated in the following Almost Rejection of the Internal Structural Variations Problem (ARISV-Problem):

Problem 5 (ARISV-Problem): Given the implicit global descriptions (3), let $u^* = F_p^* x + F_d^* \dot{x} + \bar{u}$ be a control law solving Problem 1 and let y^* be the output obtained with this P.D. feedback. For a given $\delta \in \mathbb{R}^+$ find a Proportional state feedback, $u = F_p x + \bar{u}$, such that its closed loop output, y, satisfies ($t^*(\delta)$ is some fixed time depending on the given δ):

$$|y - y^*| \leq \delta \qquad \forall \;\; t \;\geq\; t^*(\delta)$$

In this paper we solve this ARISV-Problem. For this we introduce in Section II some needed background material. In Section III, we find some basic geometric decompositions and in Section IV the problem is solved.

II. Background

A. Subspaces and Related Properties

Related with (1) are the well known subspaces:

The supremal (A,E,B) invariant subspace contained in $\mathcal{K}_C$, $\mathcal{V}_\Sigma^* := \sup\{\mathcal{V} \subset \mathcal{K}_C \mid A\mathcal{V} \subset E\mathcal{V} + \text{Im } B\}$, limit of: $\mathcal{V}^0 = \mathcal{X}$ & $\mathcal{V}^{\mu+1} = \mathcal{K}_C \cap A^{-1}(E\mathcal{V}^\mu + \text{Im } B)$, $\mu \geq 0$.
Let $\mathrm{F}(\mathcal{V}_\Sigma^*)$ denote the set of all (F_p, F_d) such that $(A + BF_p)\mathcal{V}_\Sigma^* \subset (E - BF_d)\mathcal{V}_\Sigma^*$. Such (F_p, F_d) is called a friend pair of $\mathcal{V}_\Sigma^*$. The following result is well known:

Fact 6: (i) For any (F_p, F_d), for the closed loop system, $\Sigma_F(E - BF_d, A + BF_p, B, C)$ there holds: $\mathcal{V}_\Sigma^* = \mathcal{V}_{\Sigma_F}^*$; then, we just write $\mathcal{V}^*$ to identify $\mathcal{V}_\Sigma^*$ or $\mathcal{V}_{\Sigma_F}^*$ (ii) For any F_d, there exists F_p^* s.t. $(F_p^*, F_d) \in \mathrm{F}(\mathcal{V}_\Sigma^*)$.
The supremal almost (A,E) controllability subspace contained in $\mathcal{K}_C$, $\mathcal{R}_{a0}^* := \inf\{\mathcal{R} \subset \mathcal{K}_C \mid \mathcal{R} = \mathcal{K}_C \cap E^{-1}(A\mathcal{R})\}$, is the limit of the algorithm:

$$\mathcal{R}_{a0}^0 = \mathcal{K}_C \cap \mathcal{K}_E \;\; ; \;\; \mathcal{R}_{a0}^{\mu+1} = \mathcal{K}_C \cap E^{-1}(A\mathcal{R}_{a0}^\mu) \text{ for } \mu \geq 0 \tag{6}$$

$\mathcal{R}_{a0}^*$ characterizes (together with $A\mathcal{R}_{a0}^*$) the set of all trajectories due to differential actions with no influence on the input-output trajectories. This set is called differentially redundant descriptor variables.

Proposition 7: (Bonilla and Malabre [3]) Given an implicit description, $\Sigma(E,A,B,C)$, $\mathcal{R}_{a0}^*$ characterizes the differentially redundant descriptor variables. $\widehat{\Sigma}(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$: $\widehat{E}\dot{\hat{x}} = \widehat{A}\hat{x} + \widehat{B}u$ and $y = \widehat{C}\hat{x}$ with $\widehat{E} : \mathcal{X}/\mathcal{R}_{a0}^* \to \underline{\mathcal{X}}/A\mathcal{R}_{a0}^*$, $\widehat{A} : \mathcal{X}/\mathcal{R}_{a0}^* \to \underline{\mathcal{X}}/A\mathcal{R}_{a0}^*$, $\widehat{B} : \mathcal{U} \to \underline{\mathcal{X}}/A\mathcal{R}_{a0}^*$, and $\widehat{C} : \mathcal{X}/\mathcal{R}_{a0}^* \to \mathcal{Y}$, has no differentially redundant descriptor variables. $\Sigma(E,A,B,C)$ and $\widehat{\Sigma}(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$ are externally equivalent.

B. Properness

Definition 8: (Bernhard [2], Armentano [1]) The system $\mathbb{F}\dot{x} = \mathbb{G}x + v$ is internally proper iff $[\lambda\mathbb{F} - \mathbb{G}]$ is regular (square and $\det(\lambda\mathbb{F} - \mathbb{G}) \neq 0$) and it has no infinite zero of order greater than one (no derivators).

Proposition 9: (Bonilla and Malabre [4] (Part I)) The global descriptions (3) are internally proper iff:

$$\text{Ker } D_i \oplus \text{Ker } E = \mathcal{X}$$

Definition 10: An implicit description, $\Sigma(E,A,B,C)$ is called externally proper if it is externally equivalent to some internally proper system.

Corollary 11: An implicit description, $\Sigma(E,A,B,C)$ is externally proper if its induced system, $\widehat{\Sigma}(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$ (defined in Proposition 7), is internally proper.

C. Integration by Parts

In this Subsection we find the equivalence, in the framework of system theory, of one powerful tool of functional analysis: the integration by parts. For this let us consider the following proper system:

$$\dot{x} = [-1/\varepsilon]\, x + [1/\varepsilon]\, f \;\; ; \;\; y = [-1/\varepsilon]\, x + [1/\varepsilon]\, f \tag{7}$$

where y is the output and f is an input at least twice differentiable and such that $f, \dot{f}, \ddot{f} \in L_\infty$, with $x(0) = x_0$, $f(0) = f_0$ and $\dot{f}(0) = \dot{f}_0$; ε is a positive parameter. Let us analyze the external behaviour when ε tends to zero. For this, let us obtain the solution of (7):

$$\begin{array}{c} x(t) = \mathrm{e}^{-t/\varepsilon} x_0 + \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon} f(\tau) d\tau \;\; , \;\; \text{and} \\ y(t) = -\frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon} x_0 + \frac{1}{\varepsilon} f(t) - \frac{1}{\varepsilon^2}\int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon} f(\tau) d\tau \end{array}$$

Then $|y(t)| \leq \frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon}|x_0| + \frac{1}{\varepsilon}|f(t)| + \frac{1}{\varepsilon}\|f\|_\infty$. As we can not conclude anything when $\varepsilon \to 0$, let us integrate by parts:

$$\begin{array}{c} x(t) - f(t) = \mathrm{e}^{-t/\varepsilon}(x_0 - f_0) - \int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon} \dot{f}(\tau) d\tau \;\; , \\ \text{and} \;\; y(t) = -\frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon}(x_0 - f_0) + \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon} \dot{f}(\tau) d\tau \end{array}$$

Then $|y(t)| \leq \frac{1}{\varepsilon}\mathrm{e}^{-\frac{t}{\varepsilon}}|x_0 - f_0| + \|\dot{f}\|_\infty$, and we can only conclude that y is bounded when $\varepsilon \to 0$ and for $t > 0$. Let us integrate by parts one more time:

$$\begin{array}{rl} x(t) - f(t) + \varepsilon \dot{f}(t) & = \mathrm{e}^{-t/\varepsilon}(x_0 - f_0 + \varepsilon \dot{f}_0) \\ & \quad + \varepsilon \int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon} \ddot{f}(\tau) d\tau \;\; , \;\; \text{and} \\ y(t) - \dot{f}(t) & = \;\; -\frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon}(x_0 - f_0 + \varepsilon \dot{f}_0) \\ & \quad - \int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon} \ddot{f}(\tau) d\tau \end{array}$$

Then: $|y(t) - \dot{f}(t)| \leq \frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon}|x_0 - f_0 + \varepsilon \dot{f}_0| + \varepsilon\|\ddot{f}\|_\infty$. Therefore, $y(t) \overset{\varepsilon \to 0}{\longrightarrow} \dot{f}(t) \;\; \forall \; t > 0$.

From this simple analysis, we realize that it was necessary to integrate by parts twice in order to get rigorous arguments for (expected) conclusions. We would like now to translate this process of "integration by parts" into a system theoretical point of view. Indeed, we shall later show, in Section III(B), that this interpretation in terms of equivalent changes of bases is the guide which fully enlights the choice of the particular proportional state feedback law solving the Almost RISV problem.

1) Useful System Descriptions: Let us first note that the first integration by parts is the time solution of the Fliess state space description [5]: $\dot{w} = [-1/\varepsilon]\, w + [1/\varepsilon]\,(-\varepsilon \dot{f})$ and $y = [-1/\varepsilon]\, w$. This description can be obtained from (7) with the simple change of variable $w = x - f$. Let us next note that the second integration by parts corresponds to the time solution of the Fliess state space description:

$$\dot{z} = [-1/\varepsilon]\, z + [1/\varepsilon]\,(\varepsilon^2 \ddot{f}) \quad ; \quad y = [-1/\varepsilon]\, z + [1]\, \dot{f} \tag{8}$$

This description can be obtained from (7) with the change of variable $z = w - (-\varepsilon \dot{f}) = x - f + \varepsilon \dot{f}$.
The implicit descriptions (7) and (8) are:

(i) Doing $\xi_1 = x$ and $\xi_2 = f$ in (7), we get:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{\xi} = \begin{bmatrix} -1/\varepsilon & 0 \\ 0 & 1 \end{bmatrix} \xi + \begin{bmatrix} 1/\varepsilon \\ -1 \end{bmatrix} f, \qquad y = \begin{bmatrix} -1/\varepsilon & 1/\varepsilon \end{bmatrix} \xi \tag{9}$$

(ii) Doing $\zeta_1 = z$, $\zeta_2 = f$, $\zeta_3 = -\varepsilon \dot{\zeta}_2$ and $\zeta_4 = \dot{\zeta}_3$ in (8), we get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\zeta} = \begin{bmatrix} -1/\varepsilon & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \zeta + \begin{bmatrix} 0 & 0 & 0 & -1 \end{bmatrix}^T f, \qquad y = \begin{bmatrix} -1/\varepsilon & 0 & -1/\varepsilon & 0 \end{bmatrix} \zeta \tag{10}$$

2) Internal Properness: Let us first note that system (9) is internally proper. Indeed, this follows from the fact that (c.f. Proposition 9): $\mathcal{X} = \operatorname{Ker} \begin{bmatrix} 1 & 0 \end{bmatrix} \oplus \operatorname{Ker} \begin{bmatrix} 0 & 1 \end{bmatrix}$. Let us next note that system (10) is not internally proper. Indeed, this follows from the fact that (c.f. Proposition 9): $\operatorname{Ker} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cap \operatorname{Ker} \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \neq \{0\}$.

3) External Properness: Since system (9) is internally proper, it is also externally proper. With respect to system (10), we first need to obtain the system quotiented by $\mathcal{R}^*_{a0}$ in the domain and by $A\mathcal{R}^*_{a0}$ in the co-domain (c.f. Corollary 11):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\epsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \dot{\bar{\zeta}} = \begin{bmatrix} -1/\varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bar{\zeta} + \begin{bmatrix} 1/\varepsilon & -1 & 0 & 0 \end{bmatrix}^T f, \qquad y = \begin{bmatrix} -1/\varepsilon & 1/\varepsilon & 0 & 0 \end{bmatrix} \bar{\zeta}$$

$\mathcal{X}/\mathcal{R}^*_{a0}$; $\frac{\mathcal{X}}{A\mathcal{R}^*_{a0}}$

Applying the algorithm (6) to this system, we realize that $\mathcal{R}^*_{a0} = \{e_3,\ e_4\}$. And thus, inside the solid line boxes we find the induced system claimed in Corollary 11, which is nothing else than system (9). Then, (10) is externally proper and externally equivalent to (9).

From this discussion, we conclude that performing two integrations by parts is equivalent to applying, in the Fliess state space description, the change of variable:[2] $z = x - f + \varepsilon \dot{f}$. The system obtained with this change of variable only adds differential redundant descriptor variables and remains externally proper and externally equivalent to the original system. As we will see later on, the added differential redundant subspace enables us to bring the system into a nice structural form.

III. Basic Geometric Decompositions

Let us modify the geometric condition (4.b) as follows:

$$\dim\left(\mathcal{V}^* \cap E^{-1}\mathcal{B}\right) \geq \dim\left(\mathcal{K}_E\right) + \dim\left((\mathcal{K}_E + \mathcal{V}^*)/\mathcal{V}^*\right) \tag{11}$$

In order to simplify, we can also assume, without any loss of generality, that a preliminary proportional feedback has been applied such that:

$$A\mathcal{V}^* \cap \mathcal{B} = \{0\} \quad \text{and} \quad A\mathcal{V}^* \subset E\mathcal{V}^* \tag{12}$$

In a similar way as in [4] (Part II), let us decompose $\mathcal{K}_E$, $E^{-1}\mathcal{B}$, $\mathcal{V}^*$, and the space $\mathcal{X}$ as follows ($\mathcal{X}_0$, $\mathcal{X}_{\mathcal{V}^*}$, $\mathcal{X}_3$, and $\mathcal{X}_{\mathcal{K}_E}$ are any complementary subspaces):

$$\begin{cases} \mathcal{K}_E = (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \\ E^{-1}\mathcal{B} = ((\mathcal{V}^* \cap E^{-1}\mathcal{B}) + \mathcal{K}_E) \oplus \mathcal{X}_3 \\ \mathcal{V}^* = \mathcal{X}_{\mathcal{V}^*} \oplus (\mathcal{V}^* \cap E^{-1}\mathcal{B}) \quad ; \quad \mathcal{X} = (\mathcal{V}^* + E^{-1}\mathcal{B}) \oplus \mathcal{X}_0 \end{cases} \tag{13}$$

In view of (11), there exist $\mathcal{X}_1,\ \mathcal{X}_2 \subset E^{-1}\mathcal{B}$, such that:

$$\begin{array}{c} \mathcal{V}^* \cap E^{-1}\mathcal{B} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E), \quad \text{with:} \\ \mathcal{X}_2 \approx \mathcal{X}_{\mathcal{K}_E} \quad \text{and} \quad \dim \mathcal{X}_1 \geq \dim \mathcal{X}_{\mathcal{K}_E} \end{array} \tag{14}$$

From (13) and (14) we get:

$$\begin{cases} \mathcal{X} = \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \\ \mathcal{V}^* = \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \\ E^{-1}\mathcal{B} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \end{cases} \tag{15}$$

Thus, $\mathcal{B}$, $E\mathcal{V}^*$, and $\mathcal{E}$ are decomposed as (recall (4.a)):

$$\begin{array}{c} \mathcal{B} = E\mathcal{X}_1 \oplus E\mathcal{X}_2 \oplus E\mathcal{X}_3 \ ; \ E\mathcal{V}^* = E\mathcal{X}_{\mathcal{V}^*} \oplus E\mathcal{X}_1 \oplus E\mathcal{X}_2 \\ \mathcal{E} = E\mathcal{X}_{\mathcal{V}^*} \oplus E\mathcal{X}_1 \oplus E\mathcal{X}_2 \oplus E\mathcal{X}_3 \oplus E\mathcal{X}_0 \end{array} \tag{16}$$

Also $\mathcal{U}$ can be decomposed as (recall that $\mathcal{K}_B = \{0\}$):

$$\mathcal{U} = B^{-1}E\mathcal{X}_1 \oplus B^{-1}E\mathcal{X}_2 \oplus B^{-1}E\mathcal{X}_3 \tag{17}$$

[2] In the case of n integration by parts the change of variable is: $z_n = x + \sum_{i=1}^{n} (-1)^i \varepsilon^{i-1} f^{(i-1)}$

Based on the above decompositions, let us define the following natural projections ($i \in \{\mathcal{V}^*, 1, 2, \mathcal{K}_E\}$):[3]

$$\left\{ \begin{array}{c} Q_{\mathcal{X}_i} : \mathcal{X} \to \mathcal{X}_i \ ; \quad Q_{VE} : \mathcal{X} \to \mathcal{V}^* \cap \mathcal{K}_E \\ Q_{E30} : \mathcal{X} \to \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \ ; \quad P_{\mathcal{X}_{\mathcal{V}^*}} : \mathcal{E} \to E\mathcal{X}_{\mathcal{V}^*} \\ P_1 : \mathcal{E} \to E\mathcal{X}_1 \ ; \quad P_{230} : \mathcal{E} \to E\mathcal{X}_2 \oplus E\mathcal{X}_3 \oplus E\mathcal{X}_0 \\ R_1 : \mathcal{U} \to B^{-1}E\mathcal{X}_1 \ ; \quad R_2 : \mathcal{U} \to B^{-1}E\mathcal{X}_2 \\ R_3 : \mathcal{U} \to B^{-1}E\mathcal{X}_3 \end{array} \right. \tag{18}$$

and the following insertion maps:

$$\left\{ \begin{array}{c} V_{\mathcal{X}_i} : \mathcal{X}_i \to \mathcal{X} \ ; \quad V_{VE} : \mathcal{V}^* \cap \mathcal{K}_E \to \mathcal{X} \\ V_{E30} : \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \to \mathcal{X} \ ; \quad W_1 : \to B^{-1}E\mathcal{X}_1 \to \mathcal{U} \\ W_2 : \to B^{-1}E\mathcal{X}_2 \to \mathcal{U} \ ; \quad W_3 : \to B^{-1}E\mathcal{X}_3 \to \mathcal{U} \end{array} \right. \tag{19}$$

Thanks to the projections (18) and the insertions (19) so defined, we get a more precise implicit description of (1) in the following Lemma, proved in the Appendix:

Lemma 12: If (11) and (12) hold, then system (1) can be expressed as:

$$\begin{array}{l} T_V Q_{\mathcal{X}_{\mathcal{V}^*}} \dot{x} = P_{\mathcal{X}_{\mathcal{V}^*}} A x \\ T_1 Q_{\mathcal{X}_1} \dot{x} = L_1 R_1 u \\ (K_1 Q_{\mathcal{X}_2} + N_1 Q_{E30}) \dot{x} = A_1 Q_{E30} x + (L_2 R_2 + L_3 R_3) u \\ y = \overline{C} Q_{E30} x \end{array} \tag{20}$$

where $T_V : \mathcal{X}_{\mathcal{V}^*} \leftrightarrow E\mathcal{X}_{\mathcal{V}^*}$ ($T_V = P_{\mathcal{X}_{\mathcal{V}^*}} E V_{\mathcal{X}_{\mathcal{V}^*}}$), $T_1 : \mathcal{X}_1 \leftrightarrow E\mathcal{X}_1$ ($T_1 = P_1 E V_{\mathcal{X}_1}$) and $L_1 : B^{-1}E\mathcal{X}_1 \leftrightarrow E\mathcal{X}_1$ ($L_1 = P_1 B W_1$) are isomorphisms. $L_2 = P_{230} B W_2$ and $L_3 = P_{230} B W_3$ are monic and such that Im $L_2 = E\mathcal{X}_2$ and Im $L_3 = E\mathcal{X}_3$. K_1 and N_1 are defined as follows:

$$K_1 = P_{E30} E V_{\mathcal{X}_2}; \ \text{Im } K_1 = E\mathcal{X}_2, \ \mathcal{K}_{K_1} = \{0\} \tag{21}$$

$$N_1 = P_{E30} E V_{230}; \ \text{Im } N_1 = E(\mathcal{X}_3 \oplus \mathcal{X}_0), \ \mathcal{K}_{N_1} = \mathcal{X}_{\mathcal{K}_E} \tag{22}$$

A_1 and $\overline{C}$ are: $A_1 = P_{E30} A V_{230}$ and $\overline{C} = C V_{E30}$.
We need the following Lemma, proved in the Appendix:

Lemma 13: Let us define the natural projection:

$$\overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} : \ \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \to \mathcal{X}_{\mathcal{K}_E} \quad \text{along} \quad \mathcal{X}_3 \oplus \mathcal{X}_0 \tag{23}$$

Then the map $T_{230} = (N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}})$ is an isomorphism. Moreover $Q_{\mathcal{X}_{\mathcal{K}_E}} = \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} Q_{E30}$.

IV. Solution to the ARISV-Problem

We propose in this Section the main contribution of the paper. Namely, we give in Theorem 14 the conditions under which the causal approximation of a PD state feedback solves, in an approximated sense, the RISV problem. To avoid unnecessary heavy technicalities, in this Section we are going to assume that (c.f. (14)):[4]

$$\mathcal{X}_2 \approx \mathcal{X}_{\mathcal{K}_E} \approx \mathcal{X}_1 \tag{24}$$

[3]The natural projections are projected along the complementary subspaces defined in (15.a), (16.c) and (17).

[4]If $\dim \mathcal{X}_1 > \dim \mathcal{X}_{\mathcal{K}_E}$ we just have to work with an adequate projection on a subspace $\mathcal{X}_1'$ of $\mathcal{X}_1$ such that $\mathcal{X}_1' \approx \mathcal{X}_{\mathcal{K}_E}$.

A. Derivative Feedback

The derivative feedback proposed in [4] (Part II) for proving Theorem 2 was (recall that $K_1 = P_{E20} E V_{\mathcal{X}_2}$):

$$L_2 R_2 u^* = -K_1 \left(-Q_{\mathcal{X}_2} + T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} \right) \dot{x} + L_2 R_2 \bar{u} \tag{25}$$

where $T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \leftrightarrow \mathcal{X}_2$ is an isomorphism (recall (13.a) and (14)). Applying (25) and the feedback $L_1 R_1 u^* = -(1/\varepsilon) T_1 Q_{\mathcal{X}_1} x$ to (20), we get (recall that $Q_{\mathcal{X}_{\mathcal{K}_E}} = \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} Q_{E30}$ and that the map $T_{230} = (N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}})$ is an isomorphism; c.f Lemma 13):

$$\begin{array}{rcl} \mathrm{I}(Q_{\mathcal{X}_{\mathcal{V}^*}} \dot{x}) & = & T_V^{(-1)} P_{\mathcal{X}_{\mathcal{V}^*}} A x \\ \boxed{\mathrm{I}}(Q_{\mathcal{X}_1} \dot{x}) & = & \boxed{-(1/\varepsilon)\mathrm{I}}(Q_{\mathcal{X}_1} x) \\ \boxed{\mathrm{I}}(Q_{E30} \dot{x}) & = & \boxed{\bar{A}}(Q_{E30} x) + \boxed{\bar{B}}\bar{u}_2 \\ y^* & = & \boxed{\bar{C}}(Q_{E30} x) \end{array} \tag{26}$$

where: $\bar{A} = T_{230}^{(-1)} A_1$, $\bar{B} = T_{230}^{(-1)} L_2$, and $\bar{u}_2 = R_2 \bar{u}$, and $x = (Q_{\mathcal{X}_{\mathcal{V}^*}} x) + (Q_{\mathcal{X}_1} x) + (Q_{\mathcal{X}_2} x) + (Q_{VE} x) + (Q_{E30} x)$. We write y^* instead of y in order to distinguish it from the proportional feedback case. The subsystem $\Sigma^s(\mathrm{I}, \ \bar{A}, \ \bar{B}, \ \bar{C})$ enclosed by the solid line boxes is the state space quotient system mentioned in Theorem 3. Assuming controllability of the common dynamic part (1), of the implicit global descriptions (3), it follows from Theorem 4 that the pair $(\bar{A}, \ \bar{B})$ is controllable; thus, we are going to assume that the map $\bar{A}$ has been made Hurwitz by a previous proportional state feedback. Let us note that these structural properties and results are independent on the active internal structure of each particular $\mathcal{K}_{D_i}$.

B. Proportional Feedback

Based on the derivative feedback (25), let us propose the proportional feedback (recall that $T_1 = P_1 E V_{\mathcal{X}_1}$):

$$L_1 R_1 u = -(1/\varepsilon) T_1 (Q_{\mathcal{X}_1} - T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} + T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) x \tag{27}$$

$$L_2 R_2 u = K_1 g + L_2 R_2 \bar{u} \tag{28}$$

$$g = -(1/\varepsilon)(T^{\mathcal{X}_2}_{\mathcal{X}_1} Q_{\mathcal{X}_1} - Q_{\mathcal{X}_2} + T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) x \tag{29}$$

where: $T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \leftrightarrow \mathcal{X}_1$, $T^{\mathcal{X}_1}_{\mathcal{X}_2} : \mathcal{X}_2 \leftrightarrow \mathcal{X}_1$, $T^{\mathcal{X}_2}_{\mathcal{X}_1} : \mathcal{X}_1 \leftrightarrow \mathcal{X}_2$, and $T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \leftrightarrow \mathcal{X}_2$ are isomorphisms (recall (24)). Applying the feedback (27)–(29) to (20), we get:

$$\begin{array}{l} T_V Q_{\mathcal{X}_{\mathcal{V}^*}} \dot{x} = P_{\mathcal{X}_{\mathcal{V}^*}} A x \\ T_1 Q_{\mathcal{X}_1} \dot{x} = -\frac{1}{\varepsilon} T_1 Q_{\mathcal{X}_1} x + \frac{1}{\varepsilon} T_1 \left(T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} \right) x \\ (K_1 Q_{\mathcal{X}_2} + N_1 Q_{E30}) \dot{x} = A_1 Q_{E30} x + K_1 g + L_2 R_2 \bar{u} \\ 0 = \frac{1}{\varepsilon} T^{\mathcal{X}_2}_{\mathcal{X}_1} Q_{\mathcal{X}_1} x - \left(Q_{\mathcal{X}_2} - T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} \right) x + g \end{array} \tag{30}$$

Based on the Integration by Parts Section, let us do the following change of variable:

$$\begin{array}{l} Q_i z = Q_i x \quad , \quad \forall \ i \neq x_1 \quad \text{and} \\ Q_{\mathcal{X}_1} z = Q_{\mathcal{X}_1} x - \frac{1}{\varepsilon} (T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) x \\ \qquad\qquad + (T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) \dot{x} \end{array} \tag{31}$$

Then, system (30) takes the following form (recall that $T_{\mathcal{X}_1}^{\mathcal{X}_2} T_{\mathcal{X}_2}^{\mathcal{X}_1} = \mathrm{I}$ and $T_{\mathcal{X}_1}^{\mathcal{X}_2} T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_1} = T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_2}$):

$$\begin{array}{l} T_V Q_{\mathcal{X}_{\mathcal{V}^*}} \dot{z} = P_{\mathcal{X}_{\mathcal{V}^*}} A z \\ T_1 Q_{\mathcal{X}_1} \dot{z} = -(1/\varepsilon) T_1 Q_{\mathcal{X}_1} z + T_1 (T_{\mathcal{X}_2}^{\mathcal{X}_1} Q_{\mathcal{X}_2} - T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_1} Q_{\mathcal{X}_{\mathcal{K}_E}}) \ddot{z} \\ (N_1 + K_1 T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_2} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}) Q_{E30} \dot{z} = A_1 Q_{E30} z - K_1 T_{\mathcal{X}_1}^{\mathcal{X}_2} Q_{\mathcal{X}_1} z \\ \qquad\qquad\qquad\qquad\qquad + L_2 R_2 \bar{u} \\ 0 = T_{\mathcal{X}_1}^{\mathcal{X}_2} Q_{\mathcal{X}_1} z - (Q_{\mathcal{X}_2} - T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_2} Q_{\mathcal{X}_{\mathcal{K}_E}}) \dot{z} + g \end{array} \tag{32}$$

Let us define $h := (T_{\mathcal{X}_2}^{\mathcal{X}_1} Q_{\mathcal{X}_2} - T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_1} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} Q_{E30}) \dot{z}$ and let us build the space $\mathcal{Z} = \mathcal{X} \oplus \{g\} \oplus \{h\}$. Then (32) takes the following form:

$$\begin{array}{l} \mathrm{I}(Q_{\mathcal{X}_{\mathcal{V}^*}} \dot{z}) = T_V^{(-1)} P_{\mathcal{X}_{\mathcal{V}^*}} A z \\ \boxed{\mathrm{I}}(Q_{\mathcal{X}_1} \dot{z}) - \boxed{\mathrm{I}}(\dot{h}) = -\boxed{(1/\varepsilon)\mathrm{I}}(Q_{\mathcal{X}_1} z) \\ \boxed{\mathrm{I}}(Q_{E30} \dot{z}) = -\boxed{X_6}(Q_{\mathcal{X}_1} z) + \boxed{\bar{A}}(Q_{E30} z) + \boxed{\bar{B}}(\bar{u}_2) \\ 0 = X_7 (Q_{\mathcal{X}_1} z) + \mathrm{I}(g) - X_7(h) \\ X_7^{(-1)}(Q_{\mathcal{X}_2} \dot{z}) - Y_1 (Q_{E30} \dot{z}) = \mathrm{I}(h) \\ y = \boxed{\bar{C}} z \end{array} \tag{33}$$

where: $X_6 = T_{230}^{(-1)} K_1 T_{\mathcal{X}_1}^{\mathcal{X}_2}$, $X_7 = T_{\mathcal{X}_1}^{\mathcal{X}_2}$, $Y_1 = T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_1} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}$, $\bar{u}_2 = R_2 \bar{u}$, and $z = (Q_{\mathcal{X}_{\mathcal{V}^*}} x) + (Q_{\mathcal{X}_1} x) + (Q_{\mathcal{X}_2} x) + (Q_{VE} x) + (Q_{E30} x) + (g) + (h)$. Let us note that the subsystem $\Sigma^s(\mathrm{I},\ \bar{A},\ \bar{B},\ \bar{C})$ enclosed by the solid line boxes is the same as the one obtained in (26); but now it is perturbed by the fast exponentially stable subsystem of steady state gain ε, $\Sigma^f(\mathrm{I},\ -(1/\varepsilon)\mathrm{I},\ -\mathrm{I},\ -X_6)$, enclosed by the dash line boxes. Although the steady state gain can be very small (but never zero), it is necessary that the internal structure variation which is rendered almost unobservable be exponentially stable. So we can only accept internal structure variations, namely $\mathcal{K}_{D_i}$, belonging to the following set (recall that $\bar{A}$ is Hurwitz and $\varepsilon > 0$):

$$\Gamma_F(D_i) = \left\{ \mathcal{K}_{D_i} \;\middle|\; \det \begin{bmatrix} E - (A + BF) \\ -D_i \end{bmatrix} \text{ is Hurwitz} \right\} \tag{34}$$

Theorem 14: Under the same conditions as in Theorems 2, 3 and 4, but with (11) in place of the second in (4), there exists a proportional feedback solving the ARISV-Problem for all $\mathcal{K}_{D_i} \in \Gamma_F(D_i)$.

Proof:

From (26) and (33) we get ($Q_{E30} z_0$, $Q_{\mathcal{X}_1} z_0$ and $Q_{E30} x_0$ are initial conditions):

$$\begin{array}{l} y - y^* = \bar{C} \mathrm{e}^{\bar{A}t} Q_{E30} (z_0 - x_0) - \bar{C} \int_0^t \mathrm{e}^{\bar{A}(t-\tau)} X_6 \cdot \\ \cdot (\mathrm{e}^{-\tau/\varepsilon} Q_{\mathcal{X}_1} z_0 + \int_0^\tau \mathrm{e}^{-(\tau-\sigma)/\varepsilon} \dot{h}(\sigma) d\sigma) d\tau \end{array}$$

Assuming $\mathcal{K}_{D_i} \in \Gamma_F(D_i)$, the closed loop systems (26) and (33) are exponentially stable, which implies: $\dot{h} \in L_\infty$ and the existence of $c, a \in \mathbb{R}^+$ such that $\left| \mathrm{e}^{\bar{A}t} \right| \leq c \mathrm{e}^{-at}$. There then exist $k_1, k_2, k_3 \in \mathbb{R}^+$ such that:

$$|y - y^*| \leq k_1 \mathrm{e}^{-at} + k_2 \|\dot{h}\|_\infty \varepsilon + k_3 \varepsilon \mathrm{e}^{-t/\varepsilon}$$

Therefore, given a $\delta \in \mathbb{R}^+$ there exist $t^*, \varepsilon^* \in \mathbb{R}^+$ s.t.

$$|y - y^*| \leq \delta \qquad \forall\ \ t \geq t^* \quad \& \quad \varepsilon \in (0,\ \varepsilon^*)$$

Let us come back to the example (2). For this system, we have: $\mathcal{V}^* = \{e_1,\ e_2\}$, $\mathcal{K}_E = \{e_3\}$ and $E^{-1}\mathcal{B} = \{e_2,\ e_3\}$; and thus: $\mathcal{X}_{\mathcal{V}^*} = \{e_1\}$, $\mathcal{X}_2 = \{e_2\}$, $\mathcal{X}_{\mathcal{K}_E} = \{e_3\}$, $\mathcal{X}_1 = \mathcal{V}^* \cap \mathcal{K}_E = \mathcal{X}_3 = \mathcal{X}_0 = \{0\}$. And thus the geometric condition (11) is not satisfied. In order to get closer to such a condition, let us add an external integrator to system (2), namely:[5]

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \dot{x} = \left[\begin{array}{cccc} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \hline \alpha & 0 & \beta & 1 \end{array}\right] x + \left[\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] u$$
$$y = \left[\begin{array}{cccc} 0 & 0 & 0 & 1 \end{array}\right] x \tag{35}$$

For this system, we have: $\mathcal{V}^* = \{e_1,\ e_2,\ e_3\}$, $\mathcal{K}_E = \{e_4\}$ and $E^{-1}\mathcal{B} = \{e_2,\ e_3, e_4\}$; and thus: $\mathcal{X}_{\mathcal{V}^*} = \{e_1\}$, $\mathcal{X}_1 = \{e_2\}$, $\mathcal{X}_2 = \{e_3\}$, $\mathcal{X}_{\mathcal{K}_E} = \{e_4\}$, $\mathcal{X}_1 = \mathcal{V}^* \cap \mathcal{K}_E = \mathcal{X}_3 = \mathcal{X}_0 = \{0\}$. And now the geometric condition (11) is satisfied. Also from (35) we get: $L_1 = L_2 = T_V = T_1 = K_1 = T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_2} = T_{\mathcal{X}_2}^{\mathcal{X}_1} = T_{\mathcal{X}_{\mathcal{K}_E}}^{\mathcal{X}_1} = T_{\mathcal{X}_1}^{\mathcal{X}_2} = \bar{Q}_{\mathcal{X}_{\mathcal{K}_E}} = 1$, $N_1 = 0$, $A_1 = -1$ $R_1 = [\ 1 \quad 0\ ]$, $R_2 = [\ 0 \quad 1\ ]$, $Q_{\mathcal{X}_{\mathcal{V}^*}} = [\ 1 \quad 0 \quad 0 \quad 0\ ]$, $Q_{\mathcal{X}_1} = [\ 0 \quad 1 \quad 0 \quad 0\ ]$, $Q_{\mathcal{X}_2} = [\ 0 \quad 0 \quad 1 \quad 0\ ]$, $Q_{\mathcal{X}_{\mathcal{K}_E}} = Q_{E30} = [\ 0 \quad 0 \quad 0 \quad 1\ ]$, $P_{\mathcal{X}_{\mathcal{V}^*}} = [\ 1 \quad 0 \quad 0\ ]$.

P.D. Feedback: From (25) we get the derivative feedback: $L_2 R_2 u^* = -[\ 0 \quad 0 \quad -1 \quad 1\ ]\dot{x} + [\ 0 \quad 1\ ]\bar{u}$; the proportional feedback is: $L_1 R_1 u^* = -(1/\varepsilon)[\ 0 \quad 1 \quad 0 \quad 0\ ]x$. The closed loop system is:

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \dot{x} = \left[\begin{array}{cccc} 0 & 0 & 1 & -1 \\ 0 & -\frac{1}{\varepsilon} & 0 & 0 \\ 0 & 0 & 0 & \boxed{-1} \\ \hline \alpha & 0 & \beta & 1 \end{array}\right] x + \left[\begin{array}{c} 0 \\ 0 \\ \boxed{1} \\ \hline 0 \end{array}\right] \bar{u}_2$$
$$y^* = \left[\begin{array}{cccc} 0 & 0 & 0 & \boxed{1} \end{array}\right] x \tag{36}$$

The characteristic polynomial of the closed loop system (36) is: $\det[\lambda((\mathbb{E} - \mathbb{B}F_d) - (\mathbb{A}_i + \mathbb{B}F_p)] = (\lambda + 1)(\lambda + 1/\varepsilon)(\beta\lambda + \alpha)$. And then ($\overline{D} = [\ \alpha \quad 0 \quad \beta \quad 1\ ]$):

$$\Gamma_{(F_p, F_d)}(\overline{D}) = \{(\alpha,\ \beta) \ \mid \alpha \cdot \beta > 0 \text{ or } (\beta = 0 \ \&\ \alpha \neq 0)\} \tag{37}$$

P. Feedback: From (27), (28) and (29) we get: $L_2 R_2 u = g + [0\ 1]\bar{u}$, $g = -\frac{1}{\varepsilon}[0\ 1\ -1\ 1]x$, and $L_1 R_1 u =$

[5] (i) The second row and second column correspond to the added external integrator, (ii) we have applied a previous proportional feedback to get (12) and (iii) in the bottom we have added the algebraic equation $0 = D_i x$ (c.f. (3)).

$-\frac{1}{\varepsilon}[0\ 1\ \ -1\ 1]x$. The closed loop system is (c.f. (33)):

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & \boxed{-1} \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]\dot{z} =$$

$$= \left[\begin{array}{cccccc} 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & \boxed{-(1/\varepsilon)} & 0 & 0 & 0 & 0 \\ 0 & \boxed{-1} & 0 & \boxed{-1} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \alpha & 0 & \beta & 1 & 0 & 0 \end{array}\right] z + \left[\begin{array}{c} 0 \\ 0 \\ \boxed{1} \\ 0 \\ 0 \\ \hline 0 \end{array}\right] \bar{u}_2$$

$$y = \left[\begin{array}{cccccc} 0 & 0 & 0 & \boxed{1} & 0 & 0 \end{array}\right] z \tag{38}$$

The characteristic polynomial of the closed loop system (38) is: $\det[\lambda(\mathbb{E} - (\mathbb{A}_i + \mathbb{B}F_p)] = (\lambda+1)(\lambda+1/\varepsilon)(\beta\lambda+\alpha) - (\beta+1)\lambda^3$. And then ($\widetilde{D} = \left[\begin{array}{cccccc} \alpha & 0 & \beta & 1 & 0 & 0 \end{array}\right]$):

$$\Gamma_F(\widetilde{D}) = \left\{ (\alpha, \beta) \,\middle|\, \alpha < \min\left\{ -\left(\tfrac{1}{\varepsilon}+1\right)\beta, -\left(\tfrac{\beta}{1+\varepsilon}\right), 0 \right\} \right\} \tag{39}$$

The region of the possible variations preserving the internal stability is reduced when the P.D. feedback is approximated. For the P.D. case this region is the first and third orthant; for the P. case the region is the third orthant; the case $(\alpha, \beta) = (1, 1)$ cannot be kept.

Appendix

A. Proof of Lemma 12

This proof is done in 8 steps: 1) From (15) and (17), any $x \in \mathcal{X}$ and any $u \in \mathcal{U}$ can be expressed as: $x = (V_{\mathcal{X}_{\mathcal{V}^*}} Q_{\mathcal{X}_{\mathcal{V}^*}} + V_{\mathcal{X}_1} Q_{\mathcal{X}_1} + V_{\mathcal{X}_2} Q_{\mathcal{X}_2} + V_{VE} Q_{VE} + V_{E30} Q_{E30})x$ and $u = (W_1 R_1 + W_2 R_2 + W_3 R_3)u$. 2) From (16.c) and (13.a) we get: $P_{\mathcal{X}_{\mathcal{V}^*}} E V_i = 0$, for $i \in \{ \mathcal{X}_1, \mathcal{X}_2, VE, E30 \}$; $P_1 E V_i = 0$, for $i \in \{ \mathcal{X}_{\mathcal{V}^*}, \mathcal{X}_2, VE, E30 \}$; $P_{230} E V_i = 0$, for $i \in \{ \mathcal{X}_{\mathcal{V}^*}, \mathcal{X}_1, VE \}$. 3) From (16.c), (12) and (16.b) we get: $P_1 A = 0$ and $P_{230} A V_i = 0$ for $i \in \{ \mathcal{X}_{\mathcal{V}^*}, \mathcal{X}_1, \mathcal{X}_2, VE \}$. 4) From (16.c), (16.a) and (17.c) we get: $P_{\mathcal{X}_{\mathcal{V}^*}} B = 0$; $P_1 B W_i = 0$, for $i \in \{ 2, 3 \}$; $P_{230} B W_1 = 0$. 5) From (15.a) and (15.b) we get: $C V_i = 0$, for $i \in \{ \mathcal{X}_{\mathcal{V}^*}, \mathcal{X}_1, \mathcal{X}_2, VE \}$. 6) From (18), (16.c), (16.a), (17), (13.a) and (15.a), we get: Im $K_1 = P_{230} E \mathcal{X}_2 = E \mathcal{X}_2$ and $\mathcal{K}_{K_1} = \mathcal{X}_2 \cap E^{-1} \mathcal{K}_{P_{230}} = \mathcal{X}_2 \cap E^{-1} E(\mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1) = \mathcal{X}_2 \cap (\mathcal{K}_E \oplus \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1) = \{0\}$. 7) From (19), (18), (16.c), (13.a) and (15.a), we get: Im $N_1 = P_{230} E(\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0) = E(\mathcal{X}_3 \oplus \mathcal{X}_0)$ and $\mathcal{K}_{N_1} = (\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0) \cap E^{-1} \mathcal{K}_{P_{230}} = (\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0) \cap E^{-1} E(\mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1) = (\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0) \cap (\mathcal{K}_E \oplus \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1) = \mathcal{X}_{\mathcal{K}_E}$. 8) From (19), (17) and (18), we get: Im $L_1 = P_1 B B^{-1} E \mathcal{X}_1 = P_1 E \mathcal{X}_1 = E \mathcal{X}_1$ and $\mathcal{K}_{L_1} = B^{-1} E \mathcal{X}_1 \cap B^{-1} \mathcal{K}_{P_1} = B^{-1}(E \mathcal{X}_1 \cap \mathcal{K}_{P_1}) = \mathcal{K}_B = \{0\}$. And for $i \in \{2, 3\}$: Im $L_i = P_{230} B B^{-1} E \mathcal{X}_i = P_{230} E \mathcal{X}_i = E \mathcal{X}_i$ and $\mathcal{K}_{L_i} = B^{-1} E \mathcal{X}_i \cap B^{-1} \mathcal{K}_{P_{230}} = B^{-1}(E \mathcal{X}_i \cap \mathcal{K}_{P_{230}}) = \mathcal{K}_B = \{0\}$.

B. Proof of Lemma 13

This proof is done in 4 steps: 1) From (23), (21), (22) and (18), we get: $\left(N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) : \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \to E(\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_0)$. 2) The domain, $\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0$, and the co-domain, $E(\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_0)$, are isomorphic. Indeed, from (24), (13.a) and (15.a), we get: $\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \approx \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \approx E(\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_0)$. 3) The map $\left(N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right)$ is monic. This item is proved in 3 steps: (i) Let us first note that (21), (22) and (23) imply that: Im $N_1 = E(\mathcal{X}_3 \oplus \mathcal{X}_0)$ and Im $\left(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) = K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \mathcal{X}_{\mathcal{K}_E} = K_1 \mathcal{X}_2 =$ Im $K_1 = E\mathcal{X}_2$, and then:

$$E(\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_0) = \text{Im } N_1 \oplus \text{Im } \left(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) \tag{40}$$

(ii) Let us next note that (22), (21) and (23) imply that: $\mathcal{K}_{N_1} = \mathcal{X}_{\mathcal{K}_E}$ and Ker $\left(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) = \overline{Q}^{-1}_{\mathcal{X}_{\mathcal{K}_E}} T^{\mathcal{X}_{\mathcal{K}_E}}_{\mathcal{X}_2} \mathcal{K}_{K_1} =$ Ker $\overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} = \mathcal{X}_3 \oplus \mathcal{X}_0$, and then:

$$\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 = \mathcal{K}_{N_1} \oplus \text{Ker } (K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}) \tag{41}$$

(iii) Let us now take a $x \in$ Ker $(N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}})$, i.e. $(N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}})x = 0$. In view of (41) there exist unique $x_1 \in \mathcal{K}_{N_1}$ and $x_2 \in$ Ker $\left(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right)$ such that $x = x_1 + x_2$. Then: $N_1 x_2 = -K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} x_1$. From this last equality and from (40) we have: $N_1 x_2 \in$ Im $N_1 \cap$ Im $(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}) = \{0\}$, which together with (41) imply $x_2 \in \mathcal{K}_{N_1} \cap$ Ker $\left(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) = \{0\}$. As $x_2 = \{0\}$, we get from (41): $x_1 \in$ Ker $(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}) \cap \mathcal{K}_{N_1} = \{0\}$. Therefore: Ker $\left(N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) = \{0\}$. 4) T_{230} is an isomorphism because it is a square monic map.

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Apéndice C

Structural Proper Exponential Approximation of Non Proper Systems: The MIMO Case (Submitted to the IEEE-TAC)

Structural Proper Exponential Approximation of Non Proper Systems: The MIMO Case

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Abstract

In some control and observations problems, it may be convenient, at least from the analysis point of view, to use non proper systems. However, as far as their implementation is concerned, proper approximations have to be designed. In the present paper, we first show how exponential approximations can be rather easily designed (Lemma 1). Then, we characterize, in geometric terms, the external properness of an implicit description (Theorem 3 and Corollary 1). Finally, the combination of those two results solves the problem of proper exponential approximation and generalizes to the MIMO case (Theorem 4) a previous result from Bonilla and Lozano.

Index Terms

PD feedbacks, proper approximations, implicit systems, linear systems

Notation

Script capitals $\mathcal{V}$, $\mathcal{W}$, …, denote linear spaces with elements v, w, …; the dimension of a space $\mathcal{V}$ is denoted dim($\mathcal{V}$); when $\mathcal{V} \subset \mathcal{W}$, $\frac{\mathcal{W}}{\mathcal{V}}$ or $\mathcal{W}/\mathcal{V}$ stands for the quotient space $\mathcal{W}$ modulo $\mathcal{V}$; the direct sum of independent spaces is written as $\oplus$. Given a linear map $X : \mathcal{V} \to \mathcal{W}$, Im $X = X\mathcal{V}$ denotes its image, and $\mathcal{K}_X$ or sometimes Ker X denotes its kernel; we write $X^{-1}\mathcal{T}$ for the inverse image of the subspace $\mathcal{T}$ by the linear map X, and we write $X^{(-1)}$ to stand its inverse map (in the case of having a bijection); we write Mat X for its matrix in some particular bases (in the domain and in the co-domain). Mat $X \approx$ Mat Y, means that there exist a pair of elementary operations matrices T_1 and T_2 such that Mat $X = T_1(\text{Mat } Y)T_2$. $\{x, y, z\}$ stands for the subspace spanned by the vectors x, y and z. e_i stands for the vector with a 1 in its $i-th$ component and 0 in its other components. $\mathcal{L}^{-1}\{\cdot\}$ denotes the inverse Laplace Transform; s is the complex Laplace variable; p is the derivative operator d/dt; $\mathbb{R}$[x] stands for the set of monic polynomials in the indeterminate x and with coefficients in $\mathbb{R}$, the real numbers set. $\mathbb{R}^+$ is the set of positive real numbers. M[x] stands for the polynomial matrix set in the indeterminate x and with elements in $\mathbb{R}$. $\|\cdot\|$ stands for the Euclidean norm (in case of vectors) and also the induced Euclidean norm (in case of linear maps). $\underline{\chi}_k^i$ denotes a $k \times 1$ vector whose i-th component is 1 and the others are zero. I_k denotes a $k \times k$ identity matrix, or simply I when the size does not have to be explicitly indicated. $\mathrm{E}_{n,m}^{i,j}$ denotes a $n \times m$ matrix whose (i, j)-th component is 1 and the others are zero, or simply $\mathrm{E}_n^{i,j}$ when $n = m$. 0 denotes a zero matrix (of appropriate dimensions). $U\{v^T\}$ denotes an upper triangular Toeplitz matrix with first row vector v^T. $L\{v\}$ denotes a lower triangular Toeplitz matrix with first column vector v. N_n denotes the nilpotent matrix $(n \geq 2)$ $L\left\{\underline{\chi}_n^2\right\}$. $D\{X_1, ..., X_k\}$ denotes a block diagonal matrix whose diagonal blocks are the matrices $X_1, ..., X_k$. Examples: $\underline{\chi}_2^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathrm{E}_{2,3}^{1,2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $U\left\{\begin{bmatrix} a & b \end{bmatrix}\right\} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, $L\left\{\begin{bmatrix} a \\ b \end{bmatrix}\right\} = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$.

I. Introduction

The use of derivative actions was very familiar to practical control pioneers (see for example Coinsidine [10]) who proposed useful approximations of such derivative actions (see for example Jackson [13]). With the introduction of the so called generalized systems (also named descriptor, singular and implicit systems) by Rosenbrock [17], it was possible to study derivative actions from a more structural point of view. Also some control problems (e.g.decoupling, disturbance decoupling,...) were considered in the framework of the implicit systems using non proper compensators. This is because, either proper exact solutions do not exist, or obtaining them (often based on inversion techniques) is much easier; see for example [18], [7], [8] and [9]. Then, for their effective implementation, proper approximations must be designed which do not "much" alter the control objective.

In [5] was studied a generalization of the approximation of Jackson [13] and was also described a procedure to perturb the whole system (the proper system plus the non proper compensator) by a static feedback (a sufficient geometric condition was given); in both cases the approximation depends upon a positive real number ε (the quality of the approximation is inversely proportional to ε). For the generalization of Jackson's procedure proposed in [5], it was shown that it does not always function correctly because of stability problems. On the contrary, for the feedback perturbation, stability is not a problem. For both cases, there are transitories function of the parameter ε, namely terms of the type $k\mathrm{e}^{-\varepsilon t}$. These terms can trouble the digital simulations, or the practical synthesis, when the positive parameter ε is too small. In order to overcome this inconvenience in [6] was proposed an approximation for SISO systems which nicely separates the quality of the approximation (given by the inverse of the positive parameter ε) from the convergence ratio (given by a positive parameter β in terms of the type $k\mathrm{e}^{-\beta t}$). The extension of the approximation proposed in [6] to the MIMO case is not quite direct since it is necessary to realize a structural study, which is the aim of this paper. We propose here a systematic procedure for proper exponential approximation which gives a solution to the following problem:

Problem 1: Given the non-proper compensator, $\Sigma^c : \mathcal{U} \to \mathcal{Y}$, with realization:

$$N\dot{\omega}(t) = \omega(t) + \Gamma u(t) \quad ; \quad y^*(t) = \Delta\omega(t) \tag{1}$$

where N: $\mathcal{W} \to \mathcal{W}$, $\Gamma : \mathcal{U} \to \mathcal{W}$ and $\Delta : \mathcal{W} \to \mathcal{Y}$ are linear operators, N is a nilpotent operator, Γ is a map such that the matrix $[N \ \Gamma]$ is epic, and $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{W}$ are the input, the output and the descriptor variable spaces, respectively. Find a strictly proper filter, $\Sigma^f : \mathcal{Y} \to \mathcal{Y}$, with realization:

$$\dot{\zeta}(t) = A(\varepsilon)\,\zeta(t) + B(\varepsilon)\,y^*(t) \quad ; \quad y(t) = C\zeta(t) \tag{2}$$

such that:

1) $\lim_{\varepsilon \to 0} \|y(t) - y^*(t)\| \leq K\mathrm{e}^{-\beta t}$, with $K, \beta > 0$ and Σ^f is internally stable for all $\varepsilon > 0$
2) The transfer function matrix of the overall system, $\Sigma^f \circ \Sigma^c$, is proper.

In other words, we are looking for a proper filter, Σ^f, which makes proper the overall system, $\Sigma^f \circ \Sigma^c$, and which output, $y(t)$, exponentially tends to the non proper behaviour of Σ^c. The interest is to finally synthesize the overall proper system $\Sigma^f \circ \Sigma^c$ as a proper approximation of Σ^c.

We are going to assume that the non proper compensator (1) is completely observable, and then its Kronecker canonical form has only row minimal indices blocks (see [12], [16] and [15]). And thus, when system (1) is carried to its Kronecker canonical form, we get

$$N = D\{N_1, \ldots, N_n\}, \ \Delta = D\{\Delta_1^T, \ldots, \Delta_n^T\}; \quad \text{where: } N_i = L\left\{\underline{\chi}^2_{(k_i+1)}\right\}, \ \Delta_i = \underline{\chi}^{(k_i+1)}_{(k_i+1)} \tag{3}$$

Note that $k_i > 0$, $i = 1, \ldots, n$, denote the orders of the poles at infinity of compensator (1).

In Section II we propose an internally stable strictly proper realization, Σ^f, which external behavior exponentially approaches that of the non proper compensator, Σ^c (solution of part 1 of Problem 1). In Section III we characterize the external properness, through some nice geometric results presented in Theorem 3 and in its Corollary 1; necessary and sufficient conditions are given. In Section IV we show that our construction of Σ^f makes the transfer function matrix, $\left(\Sigma^f \circ \Sigma^c\right)(s)$, externally proper (solution of part 2 of Problem 1). This leads to the structural Theorem 4, which generalizes to the MIMO case some previous SISO result from Bonilla and Lozano [6]. In Section V we detail an illustrative example and we conclude in Section VI. All the proofs are in the Appendix.

II. Exponential Approximation

In the next Lemma we propose an approximation which solves Problem 1.

Lemma 1: Let us consider the filter, $\Sigma^f : \mathcal{Y} \to \mathcal{Y}$,

$$\dot{\bar{x}}(t) = A_\beta \bar{x}(t) - \varepsilon^{\kappa+1} y(t) \ ; \quad \varepsilon \dot{\hat{x}}(t) = A_o \hat{x}(t) + B_o\left(\bar{x}(t) + y^*(t)\right) \ ; \quad y(t) = C_o \hat{x}(t) \tag{4}$$

where $\hat{x} \in \widehat{\mathcal{X}}$; $\bar{x}, y, y^* \in \mathcal{Y}$, and $\varepsilon > 0$ such that:

- H1. A_β and A_o are Hurwitz,
- H2. $\mathcal{L}^{-1}\left\{(\mathrm{sI} - (1/\varepsilon)A_o)^{-1}\right\} = \overline{A}_o(t, \varepsilon)\mathrm{e}^{-t/\varepsilon}$,
- H3. $\overline{A}_o(t, \varepsilon) \in M[t/\varepsilon]$, with degrees less than or equal to a positive integer κ,
- H4. $\int_0^\infty C_o \overline{A}_o(\lambda)\mathrm{e}^{-\lambda} B_o \mathrm{d}\lambda = \mathrm{I}$, $\overline{A}_o(\lambda) = \overline{A}_o(\varepsilon\lambda, \varepsilon)$,
- H5. $\det\left[\mathrm{I} + \varepsilon^\kappa C_o(\mathrm{sI} - (1/\varepsilon)A_o)^{-1}B_o(\mathrm{sI} - A_\beta)^{-1}\right] = \Pi_{i=1}^n \left(1 + \varepsilon\frac{h_i(\varepsilon)f_i(\mathrm{s})}{g_i(\mathrm{s},\varepsilon)}\right)$, where ($\varepsilon$ is a given positive real number) $f_i(\mathrm{s})$, $g_i(\mathrm{s}, \varepsilon) \in \mathbb{R}[\mathrm{s}]$, the $g_i(\mathrm{s}, \varepsilon)$ are Hurwitz polynomials and $h_i(\varepsilon) \in \mathbb{R}[\varepsilon]$.
- H6. $y^*(t)$ is bounded and Lipchitz continuous.

Then $\det \begin{bmatrix} (\mathrm{sI} - A_\beta) & \varepsilon^{k+1}C_o \\ -(1/\varepsilon)B_o & (\mathrm{sI} - (1/\varepsilon)A_o) \end{bmatrix}$ is Hurwitz and $\lim\limits_{\varepsilon \to 0}(y(t) - y^*(t)) = \mathrm{e}^{A_\beta t}\bar{x}(0)\,;\ t > 0$.

Let us show that the following choice of A_o, B_o and C_o satisfies the requirements:

$$A_o = D\{A_1, \ldots, A_n\}, \ B_o = D\{b_1, \ldots, b_n\}, \ C_o = D\{c_1^T, \ldots, c_n^T\}, \ A_\beta = -\beta \mathrm{I}_n \tag{5}$$

$$A_i = -\mathrm{I}_{k_i} + U\{(\underline{\chi}^2_{k_i})^T\}, \ b_i = \underline{\chi}^{k_i}_{k_i}, \ c_i = \underline{\chi}^1_{k_i}, \quad \text{with } i = 1, ..., n \ \& \ \kappa = \max\{k_1, \ldots, k_n\} \tag{6}$$

Indeed, $\mathcal{L}^{-1}\{(\mathrm{sI}-(1/\varepsilon)A_i)^{-1}\} = U\left\{\begin{bmatrix}1 & \frac{t/\varepsilon}{1!} & \cdots & \frac{(t/\varepsilon)^{k_i-1}}{(k_i-1)!}\end{bmatrix}\right\}\mathrm{e}^{-t/\varepsilon}$, and then $\int_0^\infty c_i^T \overline{A}_i(\varepsilon\lambda,\varepsilon)\mathrm{e}^{-\lambda} b_i \mathrm{d}\lambda = c_i^T U\{\underline{1}_{k_i}^T\}\ b_i = 1$. Moreover the Markov's parameters of each subsystem $\left\{A_i, b_i, c_i^T\right\}$ satisfy for the $k_i > 1$ (for the $k_i = 1$, we simply have $h_{i,k_i} = c_i^T b_i = 1$):

$$h_{i,j+1} = c_i^T A_i^j b_i = 0 \text{ for } j = 0, 1, \ldots, k_i - 2;\ i = 1, \ldots, n \ \&\ h_{i,k_i} = 1 \text{ for } i = 1, \ldots, n \tag{7}$$

Furthemore: $\det\left[\mathrm{I} + \varepsilon^\kappa C_o(\mathrm{sI} - \frac{1}{\varepsilon}A_o)^{-1}B_o(\mathrm{sI} - A_\beta)^{-1}\right] = \Pi_{i=1}^n\left(1 + \varepsilon\frac{\varepsilon^{\kappa-k_i}}{(\mathrm{s}+1/\varepsilon)^{k_i}(\mathrm{s}+\beta)}\right)$.

III. External Properness

We are interested in this Section in finding geometric conditions which guarantee the properness of the external behaviour, given by the set of input-output trajectories, of the concatenation of the nonproper compensator (1) with the strictly proper filter (4). In order to tackle this problem we have to define the meaning of external properness. For this, we use the three well known notions of external equivalence, external minimality, and internal properness (recalled hereafter):

Definition 1: (Willems [19]) Two models are called externally equivalent iff the corresponding sets of all possible trajectories for the external variables (external behaviors) are the same.

Definition 2: (Kuijper [14] and Bonilla and Malabre [3]) A given (E, A, B, C) implicit description, $E\dot{x}(t) = Ax(t) + Bu(t)$ and $y(t) = Cx(t)$, where $E, A \in \mathbb{R}^{\underline{n}\times n}$, $B \in \mathbb{R}^{\underline{n}\times m}$, and $C \in \mathbb{R}^{p\times n}$, with $\underline{n}$ and n not necessarily equal, is minimal among all externally equivalent descriptions of the same type iff: 1) the corresponding descriptor equation has the least possible number of rows, and 2) the descriptor variable has the least possible number of components.

Definition 3: (Bernhard [2], Armentano [1]) The system $\mathbb{F}\dot{x} = \mathbb{G}x + v$ is internally proper iff the pencil $[\lambda\mathbb{F} - \mathbb{G}]$ is regular (square and with full rank) and it has no infinite zero of order greater than one (there are no derivators).

In the light of these notions, we associate external properness with the properness of the external behaviour of the overall system; more precisely:

Definition 4: The implicit system

$$E\dot{x}(t) = Ax(t) + Bu(t) \quad ; \quad y(t) = Cx(t) \tag{8}$$

where $E : \mathcal{X} \to \underline{\mathcal{X}}$, $A : \mathcal{X} \to \underline{\mathcal{X}}$, $B : \mathcal{U} \to \mathcal{X}$ and $C : \mathcal{X} \to \mathcal{Y}$ are linear operators, is externally proper iff its externally minimal part is internally proper.

Indeed, it is shown in [3] that the implicit system (E, A, B, C) is externally equivalent to a minimal description (E_m, A_m, B_m, C_m), called the externally minimal part of the system. Also in [14] was given necesary and sufficient conditions for external minimality. Let us recall these results:

Theorem 1: (Kuijper [14]) A given matrix description (E, A, B, C) is minimal (among all externally equivalent descriptions of the type (8) iff: (*i*) the matrix $[E\ B]$ is epic, (*ii*) the matrix $[E^T\ C^T]^T$ is monic, and (*iii*) the matrix $\begin{bmatrix}\lambda E - A \\ C\end{bmatrix}$ has full column rank for all complex number λ.

Theorem 2: (Bonilla and Malabre [3]) A Given implicit description (E, A, B, C) is externally equivalent to the minimal implicit description (E_m, A_m, B_m, C_m), whose maps are uniquely defined as follows:

$$\begin{gathered} E_m\Pi_m = P_mE \quad ; \quad A_m\Pi_m = P_mA \quad ; \quad B_m = P_mB \quad ; \quad C_m\Pi_m = C \\ \Pi_m : \mathcal{X} \to \mathcal{V}^*_{\mathcal{X}}/(\mathcal{V}^*_o + \mathcal{V}^*_{\mathcal{X}} \cap \mathcal{R}^*_{a0}) : \text{ canonical projection} \\ P_m : \underline{\mathcal{X}} \to (E\mathcal{V}^*_{\mathcal{X}} + \text{Im } B)/(E\mathcal{V}^*_o + A(\mathcal{V}^*_{\mathcal{X}} \cap \mathcal{R}^*_{a0})) : \text{ canonical projection} \end{gathered} \tag{9}$$

$\mathcal{V}^*_{\mathcal{X}}$ characterizes (together with $E\mathcal{V}^*_{\mathcal{X}} + \text{Im } B$) the set of all possible trajectories which are not identically zero for any input u, and is the limit of:

$$\mathcal{V}^o_{\mathcal{X}} = \mathcal{X} \ , \quad \mathcal{V}^{\mu+1}_{\mathcal{X}} = A^{-1}\left(E\mathcal{V}^{\mu}_{\mathcal{X}} + \text{Im } B\right).$$

$\mathcal{V}^*_o$ characterizes (together with $E\mathcal{V}^*_o$) the set of all exponential trajectories which are unobservable at the output y. This subspace is the limit of:

$$\mathcal{V}^o_o = \mathcal{X} \ , \quad \mathcal{V}^{\mu+1}_o = \mathcal{K}_C \cap A^{-1}E\mathcal{V}^{\mu}_o.$$

$\mathcal{R}^*_{a0}$ characterizes (together with $A\mathcal{R}^*_{a0}$) the set of all trajectories due to pure differential actions with no influence on the input-output trajectories (we call them differential redundant). It is the limit of:

$$\mathcal{R}^o_{ao} = \mathcal{K}_C \cap \mathcal{K}_E \ , \quad \mathcal{R}^{\mu+1}_{ao} = \mathcal{K}_C \cap E^{-1}A\mathcal{R}^{\mu}_{ao}. \tag{10}$$

Let $\mathcal{V}^*_{\mathcal{X}_o}$ and $\mathcal{S}^*_{\mathcal{X}_o}$ be respectively the limits of:

$$\mathcal{V}^o_{\mathcal{X}_o} = \mathcal{X} \ , \quad \mathcal{V}^{\mu+1}_{\mathcal{X}_o} = A^{-1}E\mathcal{V}^{\mu}_{\mathcal{X}_o} \quad \text{and} \quad \mathcal{S}^o_{\mathcal{X}_o} = \mathcal{K}_E \ , \quad \mathcal{S}^{\mu+1}_{\mathcal{X}_o} = E^{-1}A\mathcal{S}^{\mu}_{\mathcal{X}_o} \tag{11}$$

$\mathcal{V}^*_{\mathcal{X}_o}$ characterizes (together with $E\mathcal{V}^*_{\mathcal{X}_o}$) the exponential trajectories and $\mathcal{S}^*_{\mathcal{X}_o}$ characterizes (together with $A\mathcal{S}^*_{\mathcal{X}_o}$) the set of all trajectories due to pure differential actions (see [1]).

The idea is then to make differential redundant the set of all trajectories due to pure differential actions (note that $\mathcal{S}^*_{\mathcal{X}_o} \supset \mathcal{R}^*_{a0}$). In the next two results, we give geometric characterizations of external properness:

Theorem 3: If (8) is observable and has no trajectories identically null, no matter the input action, namely $\mathcal{V}^*_o = \{0\}$ and $\mathcal{V}^*_{\mathcal{X}} = \mathcal{X}$, (8) is externally proper iff

$$\mathcal{V}^*_{\mathcal{X}_o} + \mathcal{S}^*_{\mathcal{X}_o} = \mathcal{X} \ , \quad \mathcal{V}^*_{\mathcal{X}_o} \cap \mathcal{S}^*_{\mathcal{X}_o} \subset \mathcal{R}^*_{a0} \quad \text{and} \quad \dim\left(\mathcal{V}^*_{\mathcal{X}_o} + \mathcal{R}^*_{a0} + \overline{\mathcal{T}}^2_1\right)/\left(\mathcal{V}^*_{\mathcal{X}_o} + \mathcal{R}^*_{a0} + \overline{\mathcal{T}}^1_1\right) = 0 \tag{12}$$

where $\overline{\mathcal{T}}^{\mu}_1$ and $\overline{\mathcal{T}}^{\mu}_2$ are extracted from the two algorithms:

$$\overline{\mathcal{T}}^o_1 = \mathcal{R}^*_{a0} \ , \quad \overline{\mathcal{T}}^{\mu+1}_1 = E^{-1}A\left(\overline{\mathcal{T}}^{\mu}_1 + \mathcal{R}^*_{a0}\right) \quad \text{and} \quad \overline{\mathcal{T}}^o_2 = \mathcal{X} \ , \quad \overline{\mathcal{T}}^{\mu+1}_2 = A^{-1}(E\overline{\mathcal{T}}^{\mu}_2 + \mathcal{R}^*_{a0})$$

Let us note that the three conditions (i)-(iii) of Theorem 1 are related with the subspaces of Theorem 2 as follows (see [4]): (i) iff $E\mathcal{V}^*_{\mathcal{X}} + \text{Im } B = \underline{\mathcal{X}}$, (ii) iff $\mathcal{R}^*_{a0} = \{0\}$, and (iii) iff $\mathcal{V}^*_o = \{0\}$[1]. And thus, we are just focusing on the differential redundancy, which is the one which enables us by cascading the filter (4) to hide on the output the non proper modes of system (1), without loosing external equivalence.

[1]In the case that there is no algebraic restrictions on the input space $\mathcal{U}$ (equivalently $E^{-1}\text{Im } B \subset \mathcal{V}^*_{\mathcal{X}}$), (i) is equivalent to $\mathcal{V}^*_{\mathcal{X}} = \mathcal{X}$ (the systems satisfying this geometric condition are called strict in [11]).

If in addition, the dynamic part of the system is only composed by integrators and derivators[2], we get:

Corollary 1: If the implicit system (8) is exponentially observable, has no trajectories identically null, no matter the input action, and has only integral and derivative actions, namely, $\mathcal{V}_o^* = \{0\}$, $\mathcal{V}_{\mathcal{X}}^* = \mathcal{X}$ and $\mathcal{X} = \mathcal{V}_{\mathcal{X}_o}^* \oplus \mathcal{S}_{\mathcal{X}_o}^*$, then (8) is externally proper iff

$$E^{-1} A \mathcal{R}_{a0}^* = \mathcal{S}_{\mathcal{X}_o}^* \tag{13}$$

IV. Proper Exponential Approximation

Embedding the non proper compensator (1) and the strictly proper filter (4) into an implicit description, we get that the overall system, $\left(\Sigma^f \circ \Sigma^c\right)$, is ($x = [\overline{x}^T \ \widehat{x}^T \ \omega^T]^T$):

$$E\dot{x}(t) = Ax(t) + Bu(t) \ ; \quad y(t) = Cx(t) \tag{14}$$

$$E = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & N \end{bmatrix}, \quad A = \begin{bmatrix} A_p & B_p \\ 0 & \mathrm{I} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}, \quad C = \begin{bmatrix} C_p & 0 \end{bmatrix} \tag{15}$$

$$A_p = \begin{bmatrix} A_\beta & -\varepsilon^{k+1} C_o \\ (1/\varepsilon) B_o & (1/\varepsilon) A_o \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ (1/\varepsilon B_o)\Delta \end{bmatrix}, \quad C_p = \begin{bmatrix} 0 & C_o \end{bmatrix} \tag{16}$$

Taking into account the particular forms of Σ^c and Σ^f, we get from (3), (5) and (15):

$$B_p = [B_{p_1} | B_{p_2} | \cdots | B_{p_n}] \ , \ B_{p_i} = [0 | \cdots | 0 | b_{p_i}] \ ; \ b_{p_i}^T = \left[0 \middle| \ 0 \middle| \cdots \middle| 0 \middle| \ \frac{1}{\varepsilon} b_i^T \middle| \ 0 \middle| \cdots \middle| 0 \right], \ i = 1, ..., n. \tag{17}$$

Lemma 2: Let us define the two matrices: $R = \begin{bmatrix} \mathrm{I} & R_p \\ 0 & \mathrm{I} \end{bmatrix}$ and $L = \begin{bmatrix} \mathrm{I} & L_p \\ 0 & \mathrm{I} \end{bmatrix}$, where

$$R_p = \begin{bmatrix} R_{p_1} | & R_{p_2} | & \cdots & | R_{p_n} \end{bmatrix} \ ; \ R_{p_i} = \begin{bmatrix} A_p^{k_i - 1} b_{p_i} | & \cdots & | A_p b_{p_i} & | b_{p_i} & | 0 \end{bmatrix} \ ; \ L_p = -\left(A_p R_p + B_p\right). \tag{18}$$

Then

$$R_p + L_p N = 0 \tag{19}$$

$$\begin{aligned} &\text{For the } k_i > 1: \quad C_p A_p^j b_{p_i} = 0, \text{ for } j = 0, 1, \ldots, k_i - 2, \ \& \ i = 1, \ldots, n \ ; \ C_p A_p^{k_i - 1} b_{p_i} = (1/\varepsilon^{k_i}) \underline{\chi}_n^i, \text{ for } i = 1, \ldots, n \\ &\text{For the } k_i = 1: \quad C_p b_{p_i} = (1/\varepsilon^{k_i}) \underline{\chi}_n^i, \text{ for } i = 1, \ldots, n \end{aligned} \tag{20}$$

Now in view of Lemma 2, we have:

$$LER = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & N \end{bmatrix}, \quad LAR = \begin{bmatrix} A_p & 0 \\ 0 & \mathrm{I} \end{bmatrix}, \quad LB = \begin{bmatrix} L_p \Gamma \\ \Gamma \end{bmatrix}, \quad CR = \begin{bmatrix} C_p & C_n \end{bmatrix} \tag{21}$$

$$C_n = D\left\{\nu_1^T, ..., \nu_n^T\right\} \ ; \quad \nu_i = \frac{1}{\varepsilon^{k_i}} \underline{\chi}_{(k_i+1)}^1, \quad i = 1, ..., n. \tag{22}$$

[2]Its associated pencil $[\lambda E - A]$ is regular, i.e. square and $\det[\lambda E - A] \neq 0$, viz $\mathcal{X} = \mathcal{V}_{\mathcal{X}_o}^* \oplus \mathcal{S}_{\mathcal{X}_o}^*$ (see [12])

Let us note that the implicit system (LER, LAR, LB, CR) satisfies the assumptions of Corollary 1. Indeed (recall (21), (16), (5), (6) and , (3)):

1) Mat $[LER\ LB] = \text{Mat}\begin{bmatrix} \text{I} & 0 & L_p\Gamma \\ 0 & N & \Gamma \end{bmatrix}$, then ($[N\ \Gamma]$ was assumed epic): $\mathcal{V}^*_{\mathcal{X}} = \mathcal{X}$.

2) $\text{Mat}\left[\begin{array}{c} \lambda LER - LAR \\ \hline CR \end{array}\right] = \text{Mat}\left[\begin{array}{cc|c} (\lambda\text{I} - A_\beta) & \varepsilon^{\kappa+1}C_o & 0 \\ -\frac{1}{\varepsilon}B_o & (\lambda\text{I} - \frac{1}{\varepsilon}A_o) & 0 \\ 0 & 0 & (\lambda N - \text{I}) \\ \hline 0 & C_o & C_n \end{array}\right] \approx \text{Mat}\begin{bmatrix} 0 & 0 \\ D\{M_1, \ldots, M_n\} & 0 \\ 0 & \text{I}_{n(\kappa+1)} \end{bmatrix}$,

where $M_i = \text{Mat}\begin{bmatrix} \underline{\chi}^{k_i}_{k_i} & U\{(\underline{\chi}^2_{k_i})^T\} \\ 0 & (\underline{\chi}^1_{k_i})^T \end{bmatrix}$, with $i = 1, \ldots n$. Then $\mathcal{V}^*_o = \{0\}$.

3) $\det[\lambda LER - LAR] = \det(\lambda\text{I} - A_\beta)\det(\lambda\text{I} - (1/\varepsilon)A_o)\det[\text{I} + \varepsilon^\kappa C_o\,(\lambda\text{I} - (1/\varepsilon)A_o)^{(-1)}B_o(\lambda\text{I} - A_\beta)^{(-1)}] = (\lambda+\beta)^n$ $(\lambda + (1/\varepsilon))^n\ \Pi^n_{i=1}\left(1 + \varepsilon\frac{\varepsilon^{\kappa-k_i}}{(\text{s}+1/\varepsilon)^{k_i}(\text{s}+\beta)}\right) \neq 0$, then: $\mathcal{X} = \mathcal{V}^*_{\mathcal{X}_o} \oplus \mathcal{S}^*_{\mathcal{X}_o}$.

And thus, by simple computation, we can check from (21) and (22) that $E^{-1}A\mathcal{R}^*_{a0} = \mathcal{S}^*_{\mathcal{X}_o}$, that is to say, the overall system (14)-(15) is externally proper and satisfies Lemma 1.

From (3), we can easily see that the integer n corresponds to the number of chains of derivators in the non proper compensator Σ^c in (1), each chain of length k_i. On the other hand, the zero Markov parameters which appear in (20) express the fact that the orders of the zeros at infinity of the strictly proper filter Σ^f are greater than or equal to k_i. This corresponds to the following:

Theorem 4: Let the proper filter Σ^f, be designed as in Lemma 1 in order to approximate in an exponential way the non proper compensator Σ^f. Then the overall system $\left(\Sigma^f \circ \Sigma^c\right)$ is externally proper iff the orders of the zeros at infinity of Σ^f are respectively greater than or equal to k_i.

As mentioned in the Introduction, one important application of our results rests upon the practical synthesis of non proper compensators, like P.D. feedback (see for instance [18], [7], [8] and [9]). Note that the same procedure can be used identically to implement other kinds of non proper systems, for instance state derivative observers, i.e. systems which give an exponential estimate of $\dot{x}$ and x (see Lemma 1 with $y^* = \begin{bmatrix} \dot{x}^T & x^T \end{bmatrix}^T$ and Theorem 4).

V. Illustrative Example

In order to clarify the principal ideas of this paper, let us consider the following system:

$$\left[\begin{array}{c|c|c} \text{N}_3 & 0 & 0 \\ \hline 0 & Z_1 & 0 \\ \hline 0 & 0 & \text{N}_2 \end{array}\right]\dot{\xi} = \left[\begin{array}{c|c|c} \text{I}_3 & 0 & 0 \\ \hline X_1 & X_2 & 0 \\ \hline 0 & 0 & \text{I}_2 \end{array}\right]\xi + \left[\begin{array}{c} Y_1 \\ \hline Y_2 \\ \hline Y_3 \end{array}\right]u \quad ; \quad z = \left[\begin{array}{ccc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]\xi \tag{23}$$

where $Z_1 = \text{E}^{2,2}_2$, $X_1 = \text{E}^{2,2}_{2,3} - \text{E}^{1,2}_{2,3}$, $X_2 = \text{E}^{1,1}_2 + \text{E}^{1,2}_2 - \text{E}^{2,2}_2$, $Y_1 = -\text{E}^{1,2}_3 - \text{E}^{2,1}_3$, $Y_2 = 2\text{E}^{1,1}_{2,3} - 2\text{E}^{2,1}_{2,3}$, and $Y_3 = -\text{E}^{1,1}_{2,3} - \text{E}^{1,2}_{2,3} - \text{E}^{1,3}_{2,3}$; which input-output description is described by the following differential equation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & (\text{p}+1) & 0 \\ 0 & 0 & (\text{p}+1) \end{bmatrix} z = \begin{bmatrix} \text{p} & \text{p}^2 & 0 \\ -\text{p}(\text{p}+2) & -\text{p}^3 & 0 \\ \text{p}^2 & \text{p}^2 & (\text{p}^2-1) \end{bmatrix} u \tag{24}$$

Let us carry system (23) into the Kronecker canonical form. For this, let us define $\bar{z} = T_{L_o} z$ and $\bar{\xi} = T_R^{(-1)}\xi$ and let us multiply on the left the descriptor equation by T_L, where $T_{L_o} = \mathrm{E}_3^{1,1} + \mathrm{E}_3^{3,1} + \mathrm{E}_3^{3,2} + \mathrm{E}_3^{2,3}$, $T_L = \left[\begin{array}{c|c|c} \mathrm{I}_3 & 0 & 0 \\ \hline 0 & 0 & T_1 \\ \hline T_2 & T_3 & 0 \end{array}\right]$, and $T_R = \left[\begin{array}{c|c|c} \mathrm{I}_3 & 0 & 0 \\ \hline T_4 & 0 & X_2 \\ \hline 0 & T_5 & 0 \end{array}\right]$, with $T_1 = \mathrm{E}_2^{1,1} - \mathrm{E}_2^{2,1} + \mathrm{E}_2^{2,2}$, $T_2 = \mathrm{E}_{3,2}^{2,2} - \mathrm{E}_{3,2}^{1,1} - \mathrm{E}_{3,2}^{2,1}$, $T_3 = \mathrm{E}_2^{1,1} - \mathrm{E}_2^{2,2}$, $T_4 = \mathrm{E}_{3,2}^{1,2} + \mathrm{E}_{3,2}^{2,1}$, $T_5 = \mathrm{E}_2^{1,1} + \mathrm{E}_2^{2,1} + \mathrm{E}_2^{2,2}$; namely:

$$\begin{bmatrix} \mathrm{N}_3 & & & \\ & \mathrm{N}_2 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} \dot{\bar{\xi}} = \begin{bmatrix} \mathrm{I}_3 & & & \\ & \mathrm{I}_2 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \bar{\xi} + \left[\begin{array}{c} Y_1 \\ \hline Y_4 \\ \hline Y_5 \\ \hline Y_6 \end{array}\right] u \;\; ; \;\; \bar{z} = \left[\begin{array}{ccc|cc|c|c} 0 & 0 & 1 & & & & \\ & & & 0 & 1 & & 1 \\ 0 & 1 & 0 & & & 1 & 1 \end{array}\right] \bar{\xi} \tag{25}$$

where $\bar{\xi} = [\; \bar{\xi}_{i,1}^T \;\; \bar{\xi}_{i,2}^T \;\; \bar{\xi}_{p,1}^T \;\; \bar{\xi}_{p,2}^T \;]^T$ and $Y_4 = Y_3 + \mathrm{E}_{2,3}^{2,1} + \mathrm{E}_{2,3}^{2,2} + \mathrm{E}_{2,3}^{2,3}$, $Y_5 = 2\mathrm{E}_{1,3}^{1,1} + \mathrm{E}_{1,3}^{1,2}$, and $Y_6 = \mathrm{E}_{1,3}^{1,1} + \mathrm{E}_{1,3}^{1,2}$. In order to obtain the particular form proposed in (3), let us decompose (25) as follows (recall that $z = T_{L_o}^{(-1)}\bar{z}$):

$$\begin{bmatrix} \mathrm{N}_3 & & \\ & \mathrm{N}_2 & \\ & & \mathrm{N}_2 \end{bmatrix} \dot{w} = \begin{bmatrix} \mathrm{I}_3 & & \\ & \mathrm{I}_2 & \\ & & \mathrm{I}_2 \end{bmatrix} w + \underbrace{\left[\begin{array}{c} Y_1 \\ \hline Y_4 \\ \hline Y_7 \end{array}\right]}_{\Gamma} u \;\; ; \;\; y^* = \left[\begin{array}{ccc|cc|cc} 0 & 0 & 1 & & & & \\ & & & 0 & 1 & & \\ & & & & & 0 & 1 \end{array}\right] w \tag{26}$$

$$\dot{\bar{\xi}}_{p,2} = -\bar{\xi}_{p,2} + \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} u \;\; ; \;\; z = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} y^* + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \bar{\xi}_{p,2} + \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} u \tag{27}$$

where $Y_7 = -\mathrm{E}_{2,3}^{1,2} - \mathrm{E}_{2,3}^{2,1}$. Note that the input-output description of (26) is:

$$y^* = \begin{bmatrix} \mathrm{p} & \mathrm{p}^2 & 0 \\ (\mathrm{p}-1) & (\mathrm{p}-1) & (\mathrm{p}-1) \\ 1 & \mathrm{p} & 0 \end{bmatrix} u \tag{28}$$

Let us now proceed to approximate the non-proper system (26).

1) Comparing (26) with (1) and (3), we get: $n = 3$, $k_1 = 2$, and $k_2 = k_3 = 1$; thus $\kappa = 2$. Matrix Γ is indicated in (26) itself.

2) Using the proposition (5) and (6), we get the following overall system (c.f. (14)-(16)):

$$\left[\begin{array}{cc|ccc} \mathrm{I}_3 & & & & \\ & \mathrm{I}_4 & & & \\ \hline & & \mathrm{N}_3 & & \\ & & & \mathrm{N}_2 & \\ & & & & \mathrm{N}_2 \end{array}\right] \dot{x} = \left[\begin{array}{cc|ccc} -\beta \mathrm{I}_3 & -\varepsilon^3 X_3 & 0 & 0 & 0 \\ (1/\varepsilon)X_4 & -(1/\varepsilon)X_5 & (1/\varepsilon)\mathrm{E}_{4,3}^{2,3} & (1/\varepsilon)\mathrm{E}_{4,2}^{3,2} & (1/\varepsilon)\mathrm{E}_{4,2}^{4,2} \\ \hline & & \mathrm{I}_3 & & \\ & & & \mathrm{I}_2 & \\ & & & & \mathrm{I}_2 \end{array}\right] x + \left[\begin{array}{c} \\ \hline Y_1 \\ \hline Y_4 \\ \hline Y_7 \end{array}\right] u \tag{29}$$

$$y = \left[\begin{array}{c|c|ccc} \quad & X_3 & & & \end{array}\right] x$$

where: $X_3 = D\{(\underline{\chi}_2^1)^T, 1, 1\}$, $X_4 = D\{\underline{\chi}_2^2, 1, 1\}$, and $X_5 = D\{\mathrm{I}_2 - \mathrm{E}_2^{1,2}, 1, 1\}$.

3) The matrices R_p and L_p defined in Lemma 2 are (see (18)): $R_p = \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline R_1 & R_2 & R_3 \end{array}\right]$ and $L_p = \left[\begin{array}{c|c|c} L_1 & L_2 & L_3 \\ \hline L_4 & L_5 & L_6 \end{array}\right]$; where $R_1 = (1/\varepsilon^2)(\mathrm{E}_{4,3}^{1,1} - \mathrm{E}_{4,3}^{2,1}) + (1/\varepsilon)\mathrm{E}_{4,3}^{2,2}$, $R_2 = (1/\varepsilon)\mathrm{E}_{4,2}^{3,1}$, $R_3 = (1/\varepsilon)\mathrm{E}_{4,2}^{4,1}$, $L_1 = \varepsilon \mathrm{E}_3^{1,1}$,

$L_2 = \varepsilon^2 \mathrm{E}_{3,2}^{2,1}$, $L_3 = \varepsilon^2 \mathrm{E}_{3,2}^{3,1}$, $L_4 = (1/\varepsilon^3)(2\mathrm{E}_{4,3}^{1,1} - \mathrm{E}_{4,3}^{2,1}) + (1/\varepsilon^2)(\mathrm{E}_{4,3}^{2,2} - \mathrm{E}_{4,3}^{1,2}) - (1/\varepsilon)\mathrm{E}_{4,3}^{2,3}$, $L_5 = (1/\varepsilon^2)\mathrm{E}_{4,2}^{3,1} - (1/\varepsilon)\mathrm{E}_{4,2}^{3,2}$, and $L_6 = (1/\varepsilon^2)\mathrm{E}_{4,2}^{4,1} - (1/\varepsilon)\mathrm{E}_{4,2}^{4,2}$. Note that (19) and (20) are satisfied.

4) Defining $\bar{x} = \begin{bmatrix} \mathrm{I} & R_p \\ 0 & \mathrm{I} \end{bmatrix}^{(-1)} x$ and premultiplying (29) by $\begin{bmatrix} \mathrm{I} & L_p \\ 0 & \mathrm{I} \end{bmatrix}$, we get (c.f. (21)):

$$\left[\begin{array}{cc|ccc} \mathrm{I}_3 & & & & \\ & \mathrm{I}_4 & & & \\ \hline & & \mathrm{N}_3 & & \\ & & & \mathrm{N}_2 & \\ & & & & \mathrm{N}_2 \end{array}\right] \dot{\bar{x}} = \left[\begin{array}{cc|ccc} -\beta\mathrm{I}_3 & -\varepsilon^3 X_3 & & & \\ (1/\varepsilon)X_4 & -(1/\varepsilon)X_5 & & & \\ \hline & & \mathrm{I}_3 & & \\ & & & \mathrm{I}_2 & \\ & & & & \mathrm{I}_2 \end{array}\right] \bar{x} + \left[\begin{array}{c} -\varepsilon Y_8 \\ \hline (1/\varepsilon)Y_9 \\ \hline Y_1 \\ \hline Y_4 \\ \hline Y_7 \end{array}\right] u \tag{30}$$
$$y = \left[\begin{array}{c|c|c} \quad & X_3 & X_6 \end{array}\right] \bar{x}$$

where: $Y_8 = \begin{bmatrix} 0 & 1 & 0 \\ \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & 0 \end{bmatrix}$, $Y_9 = \begin{bmatrix} 1/\varepsilon & -2/\varepsilon^2 & 0 \\ -1/\varepsilon & 1/\varepsilon^2 & 0 \\ -(1/\varepsilon+1) & -(1/\varepsilon+1) & -(1/\varepsilon+1) \\ 1 & -1/\varepsilon & 0 \end{bmatrix}$, and $X_6 = D\{(1/\varepsilon^2)(\underline{\chi_3^1})^T, (1/\varepsilon)(\underline{\chi_2^1})^T, (1/\varepsilon)(\underline{\chi_2^1})^T\}$.

5) Applying the algorithms (10) and (11) to (30), we get: $\mathcal{R}_{a0}^* = \{e_9,\ e_{10},\ e_{12},\ e_{14}\}$ and $E^{-1}A\mathcal{R}_{a0}^* = \{e_8,\ e_9,\ e_{10},\ e_{11},\ e_{12},\ e_{14}\} = \mathcal{S}_{\mathcal{X},0}^*$. And thus, since the assumptions of Corollary 1 are satisfied, system (30) is externally proper; and it is externally equivalent to:

$$\dot{\hat{x}} = \left[\begin{array}{c|c} -\beta\mathrm{I}_3 & -\varepsilon^3 X_3 \\ \hline (1/\varepsilon)X_4 & -(1/\varepsilon)X_5 \end{array}\right] \hat{x} + \left[\begin{array}{c} -\varepsilon Y_8 \\ \hline (1/\varepsilon)Y_9 \end{array}\right] u \quad ; \quad y = \left[\begin{array}{c|c} 0 & X_3 \end{array}\right] \hat{x} + \left[(1/\varepsilon^2)Y_8 \right] u \tag{31}$$

Note that the input-output description of (31) is:

$$\underbrace{\begin{bmatrix} (\varepsilon\mathrm{p}+1)^2(\mathrm{p}+\beta)+\varepsilon^3 & 0 & 0 \\ 0 & (\varepsilon\mathrm{p}+1)(\mathrm{p}+\beta)+\varepsilon^3 & 0 \\ 0 & 0 & (\varepsilon\mathrm{p}+1)(\mathrm{p}+\beta)+\varepsilon^3 \end{bmatrix}}_{F(\mathrm{p})} y = (\mathrm{p}+\beta) \begin{bmatrix} \mathrm{p} & \mathrm{p}^2 & 0 \\ (\mathrm{p}-1) & (\mathrm{p}-1) & (\mathrm{p}-1) \\ 1 & \mathrm{p} & 0 \end{bmatrix} u \tag{32}$$

Thus from (32) and (28), we get: $F(\mathrm{p})y(t) = (\mathrm{p}+\beta)y^*(t)$. There then exists a positive real number, ε^*, such that $\det F(\mathrm{p})$ is Hurwitz for all $\varepsilon \in (0,\ \varepsilon^*)$. Moreover, we realize that there are dominant poles in $-\beta$ and that the other poles are very close to $-1/\varepsilon$ (for ε very small). Furthermore, $y(t) \approx y^*(t) + \mathrm{e}^{-\beta t}(y(0) - y^*(0))$ (for ε very small). The filter looked for is given by (31), (27.a) and (cf (27.b)):

$$z = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \bar{\xi}_{p,2} + \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} u \tag{33}$$

VI. Conclusion

In this paper we have solved the problem of approximating non proper MIMO systems by stable externally proper systems. The proposed approximation (Lemma 1) is an extension to the MIMO case of the proposition given in [6] for SISO systems. This extension preserves the original nice features, namely separates the quality of the approximation, given by the inverse of the positive parameter ε, from the convergence ratio, given by a positive parameter β. The main result is Theorem 3 and its Corollary 1, where necessary and sufficient geometric conditions are given to guarantee external properness of a given implicit description. In Theorem 4 we combine Lemma 1 and Corollary 1 in order to find a structural solution for Problem 1.

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Appendix

A. Proof of Lemma 1

$\det\begin{bmatrix} (\mathrm{sI}-A_\beta) & \varepsilon^{k+1}C_o \\ -(1/\varepsilon)B_o & (\mathrm{sI}-(1/\varepsilon)A_o) \end{bmatrix} = \det[\mathrm{sI}-A_\beta]\det[\mathrm{sI}-(1/\varepsilon)A_o]\det[\mathrm{I}+\varepsilon^\kappa C_o(\mathrm{sI}-(1/\varepsilon)A_o)^{-1}B_o(\mathrm{sI}-A_\beta)^{-1}]$ $= \det[\mathrm{sI}-A_\beta]\det[\mathrm{sI}-(1/\varepsilon)A_o]\Pi_{i=1}^n\left(1+\varepsilon\frac{h_i(\varepsilon)f_i(\mathrm{s})}{g_i(\mathrm{s},\varepsilon)}\right)$. Then by Routh-Hurwitz there exists a sufficiently small positive real number, ε^*, such that the right hand side is Hurwitz for all $\varepsilon\in(0,\varepsilon^*]$.

The solution of (4) is (with the initial conditions: $\bar{x}(0)=\bar{x}_0$ and $\hat{x}(0)=\hat{x}_0$): $\bar{x}(t)=\mathrm{e}^{A_\beta t}\bar{x}_0 - \varepsilon^{k+1}\int_0^t \mathrm{e}^{A_\beta(t-\sigma)}y(\sigma)\mathrm{d}\sigma$ and $y(t)=C_o\overline{A}_o(t,\varepsilon)\mathrm{e}^{-t/\varepsilon}\hat{x}_0 + C_o\int_0^t \frac{1}{\varepsilon}\overline{A}_o(t-\tau,\varepsilon)\mathrm{e}^{-(t-\tau)/\varepsilon}B_o\left(\bar{x}(\tau)+y^*(\tau)\right)\mathrm{d}\tau$. Doing the change of variable, $\tau=t-\varepsilon\lambda$, we get:

$$\begin{aligned} y(t)-\left(\mathrm{e}^{A_\beta t}\bar{x}_0+y^*(t)\right) &= C_o\overline{A}_o(t,\varepsilon)\mathrm{e}^{-t/\varepsilon}\hat{x}_0+\int_0^{t/\varepsilon}C_o\overline{A}_o(\lambda)\mathrm{e}^{-\lambda}B_o\left(\mathrm{e}^{A_\beta(t-\varepsilon\lambda)}-\mathrm{e}^{A_\beta t}\right)\bar{x}_0\mathrm{d}\lambda \\ &+\int_0^{t/\varepsilon}C_o\overline{A}_o(\lambda)\mathrm{e}^{-\lambda}B_o\left(y^*(t-\varepsilon\lambda)-y^*(t)\right)\mathrm{d}\lambda-\left(\mathrm{I}-\int_0^{t/\varepsilon}C_o\overline{A}_o(\lambda)\mathrm{e}^{-\lambda}B_o\mathrm{d}\lambda\right)\left(\mathrm{e}^{A_\beta t}\bar{x}_0+y^*(t)\right) \\ &-\varepsilon^{k+1}\int_0^{t/\varepsilon}C_o\overline{A}_o(\lambda)\mathrm{e}^{-\lambda}B_o\left(\int_0^{t-\varepsilon\lambda}\mathrm{e}^{A_\beta(t-\varepsilon\lambda-\sigma)}y(\sigma)\mathrm{d}\sigma\right)\mathrm{d}\lambda \end{aligned} \tag{34}$$

Let us do the following observations:

(O1) The boundedness of y^* and the exponential stability of (4) imply the boundedness of $\bar{x}$, $\hat{x}$ and y. Then $\exists\ K_y,\ K_{y^*}\in\mathbb{R}^+$, such that: $\|y^*(t)\|\le K_{y^*}$ and $\|y(t)\|\le K_y$.

(O2) A_β Hurwitz implies: $\exists\ K_\beta,\ \beta\in\mathbb{R}^+$, s.t. $\|\mathrm{e}^{A_\beta t}\|\le K_\beta\mathrm{e}^{-\beta t}$.

(O3) $\mathrm{e}^{A_\beta t}$ continuous implies: $\|\mathrm{e}^{A_\beta(t-\varepsilon\lambda)}-\mathrm{e}^{A_\beta t}\|\le\varepsilon\lambda\max_{\{0\le\tau\le t/\varepsilon\}}\|A_\beta\mathrm{e}^{A_\beta\tau}\|\le\varepsilon\lambda\|A_\beta\|K_\beta$.

(O4) y^* Lipschitz implies: $\exists\ \ell\in\mathbb{R}^+$, s.t. $\|y^*(t_1)-y^*(t_2)\|\le\ell|t_1-t_2|\quad\forall\quad t_1,t_2\in\mathbb{R}^+\cup\{0\}$.

(O5) $\overline{A}_o(t,\varepsilon)\in\mathrm{M}[t/\varepsilon]$ implies: $\exists\ \pi\in\mathbb{R}[t/\varepsilon]$, $\eta\in\mathbb{R}[\lambda]$, s.t. $\|C_o\overline{A}_o(t,\varepsilon)\hat{x}_0\|=\pi(t/\varepsilon)$, $\|C_o\overline{A}_o(\lambda)B_o\|=\eta(\lambda)$.

Applying norms in (34) and taking into account (O1)–(O5), we get:

$$\begin{aligned} &\|y(t)-\left(\mathrm{e}^{A_\beta t}\bar{x}_0+y^*(t)\right)\|\le\pi(t/\varepsilon)\mathrm{e}^{-t/\varepsilon}+\varepsilon\left(\|A_\beta\|K_\beta\|\bar{x}_0\|+\ell\right)\int_0^{t/\varepsilon}\eta(\lambda)\lambda\mathrm{e}^{-\lambda}\mathrm{d}\lambda \\ &+\left\|\mathrm{I}-\int_0^{t/\varepsilon}C_o\overline{A}_o(\lambda)\mathrm{e}^{-\lambda}B_o\mathrm{d}\lambda\right\|\left(K_\beta\mathrm{e}^{-\beta t}\bar{x}_0+K_{y^*}\right)+\varepsilon^{k+1}K_y\left(\frac{K_\beta}{\beta}\right)\int_0^{t/\varepsilon}\eta(\lambda)\mathrm{e}^{-\lambda}\mathrm{d}\lambda \end{aligned}$$

Then (recall H4): $\lim_{\varepsilon\to0}(y(t)-y^*(t))=\mathrm{e}^{A_\beta t}\bar{x}_0$.

B. Proof of Theorem 3

The proof is done in 4 steps:

1. Let us first recall (see Definition 3) that a pencil $[\lambda\mathbb{F}-\mathbb{G}]$ is internally proper iff (see [12], [1], [2]): it is regular, i.e. $\det[\lambda\mathbb{F}-\mathbb{G}]\ne0$, and it has no infinite zeros of order greater than one, i.e. there exist no derivators.

From [15], regularity is equivalent to $\mathcal{X}=\mathcal{A}_1^*\oplus\mathcal{A}_2^*$. Also, the absence of infinite zeros of order greater than one is equivalent to $\dim\left((\mathcal{A}_2^*+\mathcal{A}_1^2)/(\mathcal{A}_2^*+\mathcal{A}_1^1)\right)=0$, where $\mathcal{A}_1^*$ and $\mathcal{A}_2^*$ are respectively the limits of $\mathcal{A}_1^o=\{0\}$, $\mathcal{A}_1^{\mu+1}=\mathbb{F}^{-1}\mathbb{G}\mathcal{A}_1^\mu$, and, $\mathcal{A}_2^o=\mathcal{X}$, $\mathcal{A}_2^{\mu+1}=\mathbb{G}^{-1}\mathbb{F}\mathcal{A}_2^\mu$.

2. Let us next show that the system (E_m,A_m,B_m,C_m) is internally proper iff

$$\mathcal{X}=\mathcal{T}_1^*+\mathcal{T}_2^*,\ \mathcal{T}_1^*\cap\mathcal{T}_2^*=\mathrm{Ker}\ \Pi_m\ \text{and}\ \dim\left(\frac{\mathcal{T}_2^*+\mathcal{T}_1^2}{\mathcal{T}_2^*+\mathcal{T}_1^1}\right)=\dim\left(\frac{(\mathcal{T}_2^*+\mathcal{T}_1^2)\cap\mathrm{Ker}\ \Pi_m}{(\mathcal{T}_2^*+\mathcal{T}_1^1)\cap\mathrm{Ker}\ \Pi_m}\right)$$

where $\mathcal{T}_1^o = \text{Ker } \Pi_m$, $\mathcal{T}_1^{\mu+1} = E^{-1}(A\mathcal{T}_1^\mu + \text{Ker } P_m)$, and $\mathcal{T}_2^o = \mathcal{X}$, $\mathcal{T}_2^{\mu+1} = A^{-1}(E\mathcal{T}_2^\mu + \text{Ker } P_m)$.
Indeed, from the first item (E_m, A_m, B_m, C_m) is regular iff $\mathcal{X}_m = \mathcal{A}_{1m}^* \oplus \mathcal{A}_{2m}^*$, where $\mathcal{A}_{1m}^*$ and $\mathcal{A}_{2m}^*$ are respectively the limits of $\mathcal{A}_{1,m}^o = \{0\}$, $\mathcal{A}_{1m}^{\mu+1} = E_m^{-1} A_m \mathcal{A}_{1m}^\mu$ and $\mathcal{A}_{2m}^o = \mathcal{X}_m$, $\mathcal{A}_{2m}^{\mu+1} = A_m^{-1} E_m \mathcal{A}_{2m}^\mu$. Now from (9) we get: $\Pi_m^{-1}\mathcal{A}_{1m}^{\mu+1} = (E_m\Pi_m)^{-1}A_m\mathcal{A}_{1m}^\mu = E^{-1}P_m^{-1}A_m\Pi_m\Pi_m^{-1}\mathcal{A}_{1m}^\mu = E^{-1}P_m^{-1}P_mA\Pi_m^{-1}\ \mathcal{A}_{1m}^\mu = E^{-1}(A\Pi_m^{-1}\mathcal{A}_{1m}^\mu + \text{Ker } P_m)$, namely: $\mathcal{T}_1^\mu = \Pi_m^{-1}\mathcal{A}_{1m}^\mu$. In a similar way $\mathcal{T}_2^\mu = \Pi_m^{-1}\mathcal{A}_{2m}^\mu$. And thus,

$$\mathcal{X}_m = \mathcal{A}_{1m}^* \oplus \mathcal{A}_{2m}^* \quad \text{iff} \quad \mathcal{X} = \mathcal{T}_1^* + \mathcal{T}_2^* \quad \text{and} \quad \mathcal{T}_1^* \cap \mathcal{T}_2^* = \text{Ker } \Pi_m$$

On the other hand: $\dim\left(\frac{\mathcal{A}_{2m}^*+\mathcal{A}_{1m}^2}{\mathcal{A}_{2m}^*+\mathcal{A}_{1m}^1}\right) = \dim\left(\frac{\Pi_m(\mathcal{T}_2^*+\mathcal{T}_1^2)}{\Pi_m(\mathcal{T}_2^*+\mathcal{T}_1^1)}\right) = \dim\left(\frac{\mathcal{T}_2^*+\mathcal{T}_1^2}{\mathcal{T}_2^*+\mathcal{T}_1^1}\right) - \dim\left(\frac{(\mathcal{T}_2^*+\mathcal{T}_1^2)\cap\text{Ker } \Pi_m}{(\mathcal{T}_2^*+\mathcal{T}_1^1)\cap\text{Ker } \Pi_m}\right)$. And thus

$$\dim\left(\frac{\mathcal{A}_{2m}^*+\mathcal{A}_{1m}^2}{\mathcal{A}_{2m}^*+\mathcal{A}_{1m}^1}\right) = 0 \quad \text{iff} \quad \dim\left(\frac{\mathcal{T}_2^*+\mathcal{T}_1^2}{\mathcal{T}_2^*+\mathcal{T}_1^1}\right) = \dim\left(\frac{(\mathcal{T}_2^*+\mathcal{T}_1^2)\cap\text{Ker } \Pi_m}{(\mathcal{T}_2^*+\mathcal{T}_1^1)\cap\text{Ker } \Pi_m}\right)$$

3. Let us now show that: If $\mathcal{V}_o^* = \{0\}$ and $\mathcal{V}_{\mathcal{X}}^* = \mathcal{X}$ then $\mathcal{T}_1^\mu = \overline{\mathcal{T}}_1^\mu$ and $\mathcal{T}_2^\mu = \overline{\mathcal{T}}_2^\mu$; moreover, $\overline{\mathcal{T}}_1^* = \mathcal{S}_{\mathcal{X}_o}^*$ and $\overline{\mathcal{T}}_2^* = \mathcal{V}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^*$.
Indeed, from (9) we get for this case Ker $\Pi_m = \mathcal{R}_{a0}^*$ and Ker $P_m = A\mathcal{R}_{a0}^*$, and thus $\mathcal{T}_1^\mu = \overline{\mathcal{T}}_1^\mu$ and $\mathcal{T}_2^\mu = \overline{\mathcal{T}}_2^\mu$.

Note that $\overline{\mathcal{T}}_1^1 = E^{-1}A\mathcal{R}_{a0}^* \supset \mathcal{K}_E = \mathcal{S}_{\mathcal{X}_o}^o + \mathcal{R}_{a0}^*$. Let us then assume that $\overline{\mathcal{T}}_1^\mu \supset \mathcal{S}_{\mathcal{X}_o}^{\mu-1} + \mathcal{R}_{a0}^*$, which implies $\overline{\mathcal{T}}_1^{\mu+1} \supset \mathcal{S}_{\mathcal{X}_o}^\mu + E^{-1}A\mathcal{R}_{a0}^* \supset \mathcal{S}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^*$.
On the other hand, $\overline{\mathcal{T}}_1^o = \mathcal{R}_{a0}^* \subset \mathcal{S}_{\mathcal{X}_o}^*$, assuming then $\overline{\mathcal{T}}_1^\mu \subset \mathcal{S}_{\mathcal{X}_o}^*$ we get $\overline{\mathcal{T}}_1^{\mu+1} \subset \mathcal{S}_{\mathcal{X}_o}^*$. Therefore: $\mathcal{S}_{\mathcal{X}_o}^* = \mathcal{S}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^* \subset \overline{\mathcal{T}}_1^* \subset \mathcal{S}_{\mathcal{X}_o}^*$.

Now in view that $\overline{\mathcal{T}}_2^o = \mathcal{X} = \mathcal{V}_{\mathcal{X}_o}^o = \mathcal{V}_{\mathcal{X}_o}^o + \mathcal{R}_{a0}^*$, let us assume that $\overline{\mathcal{T}}_2^\mu = \mathcal{V}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^*$. This assumption implies (recall that $E\mathcal{R}_{a0}^* \subset A\mathcal{R}_{a0}^*$): $\overline{\mathcal{T}}_2^{\mu+1} = A^{-1}(E\mathcal{V}_{\mathcal{X}_o}^\mu + E\mathcal{R}_{a0}^*) + \mathcal{R}_{a0}^* = A^{-1}(E\mathcal{V}_{\mathcal{X}_o}^\mu + E\mathcal{R}_{a0}^* + A\mathcal{R}_{a0}^*) = A^{-1}(E\mathcal{V}_{\mathcal{X}_o}^\mu + A\mathcal{R}_{a0}^*) = A^{-1}E\mathcal{V}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^* = \mathcal{V}_{\mathcal{X}_o}^{\mu+1} + \mathcal{R}_{a0}^*$, namely $\overline{\mathcal{T}}_2^\mu = \mathcal{V}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^*$, which implies $\overline{\mathcal{T}}_2^* = \mathcal{V}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^*$.

4. Finally, let us note that: $\overline{\mathcal{T}}_1^* \cap \overline{\mathcal{T}}_2^* = \mathcal{S}_{\mathcal{X}_o}^* \cap (\mathcal{V}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^*) = \mathcal{S}_{\mathcal{X}_o}^* \cap \mathcal{V}_{\mathcal{X}_o}^* + \mathcal{R}_{a0}^*$, which proves (12.a) and (12.b). Also: $\dim\left(\frac{(\overline{\mathcal{T}}_2^*+\overline{\mathcal{T}}_1^2)\cap\text{Ker } \Pi_m}{(\overline{\mathcal{T}}_2^*+\overline{\mathcal{T}}_1^1)\cap\text{Ker } \Pi_m}\right) = \dim\left(\frac{(\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^2)\cap\mathcal{R}_{a0}^*}{(\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^1)\cap\mathcal{R}_{a0}^*}\right) = \dim\left(\frac{\mathcal{R}_{a0}^*}{\mathcal{R}_{a0}^*}\right) = 0$, which proves (12.c).

C. Proof of Corollary 1

If the implicit description (8) has only exponential and derivative modes (neither minimal row indices nor minimal column indices) then (see [12], [1] and [15]): $\mathcal{X} = \mathcal{V}_{\mathcal{X}_o}^* \oplus \mathcal{S}_{\mathcal{X}_o}^*$. In this case, (12.a) and (12.b) is automatically satisfied. Since $\overline{\mathcal{T}}_1^\mu \subset \overline{\mathcal{T}}_1^* = \mathcal{S}_{\mathcal{X}_o}^*$, implies that $\mathcal{V}_{\mathcal{X}_o}^* \cap \mathcal{T}_1^\mu = \mathcal{V}_{\mathcal{X}_o}^* \cap \mathcal{S}_{\mathcal{X}_o}^* \cap \overline{\mathcal{T}}_1^\mu = \{0\}$. And since $\overline{\mathcal{T}}_1^{\mu+1} \supset \mathcal{S}_{\mathcal{X}_o}^\mu + \mathcal{R}_{a0}^* \supset \mathcal{R}_{a0}^*$, we get: $\dim\left(\frac{\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^2}{\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^1}\right) = \dim\left(\frac{\mathcal{V}_{\mathcal{X}_o}^*+\overline{\mathcal{T}}_1^2}{\mathcal{V}_{\mathcal{X}_o}^*+\overline{\mathcal{T}}_1^1}\right) = \dim\left(\frac{\mathcal{V}_{\mathcal{X}_o}^*\oplus\overline{\mathcal{T}}_1^2}{\mathcal{V}_{\mathcal{X}_o}^*\oplus\overline{\mathcal{T}}_1^1}\right) = \dim\left(\frac{\overline{\mathcal{T}}_1^2}{\overline{\mathcal{T}}_1^1}\right)$.

Therefore $\dim\left(\frac{\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^2}{\mathcal{V}_{\mathcal{X}_o}^*+\mathcal{R}_{a0}^*+\overline{\mathcal{T}}_1^1}\right) = 0$ iff $\overline{\mathcal{T}}_1^1 = \overline{\mathcal{T}}_2^2 = \overline{\mathcal{T}}_1^*$, namely: $E^{-1}A\mathcal{R}_{a0}^* = \mathcal{S}_{\mathcal{X}_o}^*$.

D. Proof of Lemma 2

From (18) and (3) we get: $-L_pN = (A_pR_p + B_p)\,N = \Big[\,\big[\; A_p^{k_1}b_{p_1} \big| \;\cdots\; \big| A_pb_{p_1} \big| b_{p_1} \;\big]\, N_1 \Big| \cdots$
$\cdots \Big| \big[\; A_p^{k_n}b_{p_n} \big| \;\cdots\; \big| A_pb_{p_n} \big| b_{p_n} \;\big]\, N_n \;\Big] = \Big[\,\big[\; A_p^{k_1-1}b_{p_1} \big| \;\cdots\; \big| b_{p_1} \big| 0 \;\big] \Big| \cdots$
$\cdots \Big| \big[\; A_p^{k_n-1}b_{p_n} \big| \;\cdots\; \big| b_{p_n} \big| 0 \;\big] \;\Big] = R_p$. From (17), (15), (5) and (7) we get (20).

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I. Global Changes

We have taken into account all the comments of the four Reviewers, and the Associate Editor. The objectives and the contribution have been explained in more details, the position with respect to previous results precised, and the illustration of the proposed method better shown on a MIMO example.

In Section II we answer to the different points raised by Reviewers. And in order to simplify the revision task, we have added, in Section III, the Illustrative Example shown in the fifth Section of the revised paper, but in a more explicit way.

II. Reviewers Comments

A. Reviewer number 1

1) Questions:

Comments to Authors

(Paper No.: IEEE AC TN03-01-11 by

J. P. Martinez, M. Bonilla E. and M. Malabre)

1) General comments This paper studies three related problems. a) proper approximation of non-proper systems, b) characterization of external properness of implicit systems, and c) the combination of 1) and 2). See Problem 1, pl-2. Solutions to these problems are presented.

(a) Requirement 1) of Problem 1 is apparently equivalent to a stable observer for $\Delta\omega$. Considering the fact that there are many results on observer design for implicit systems, the paper fails to link these results and does not show the advantages of the solution given in Lemma 1.

(b) The concept of external properness is defined (Def. 1 p3) in terms of internal properness of the externally minimal part. It is assumed implicitly in the proof of Theorem 1 [2), p5] that this internal properness dependent on a particular (or any such?) pair of E_m and A_m only. While the conditions (12) and (13) of Theorem 1 depend on the pair E and A, as well as on C but not on B (why not?). In addition, the relationship between Def. 1 and the usual understanding of properness from transfer function needs to be explained and clarified. The assumptions in Theorem 1 are not explanatory and lessen the contribution of the paper, as Theorem 1 is the main result of the paper.

(c) The basic fact that Σ, and Σ^f are connected in series make the result of Theorem 2 [p8] simple and natural, if not trivial. Yet this observation is not utilized to facilitate [the understanding of] the result.

(d) In addition, the special selection of system (1) and its Kronecker canonical form (3) for the proper approximation problem, and especially the assumption in Theorem 1 [p4] that system (10) is observable and has no trajectories identically null lessen the contribution of the paper, since the properness is usually not expected to be associated with observability and/or controllability.

(e) Finally, the title indicates the paper is about MIMO case. But the results on SISO are not reviewed. There are no discussion about the respective results.

2) Technical correctness and presentation

The results and derivations are technically correct, and the proof are well organized, which makes the logic quite clear. However there is a lack of explanation of motivations (for the problems considered, and the geometric approach used) and a lack of presentation of relevant results in the literatures (e.g., these on observer design or proper approximation). The English needs to be polished, for example, the word "which" in p2, 1. 12, p9, 1. 12, and p10, 1.7 should read "whose". The presentation of Part B of section V needs to be simplified to avoid repetition of Part A. Also "non proper" or "non-proper"?

3) Detailed suggestions and questions

(a) Notation: The basic notations of the geometric approach (e.g., the inverse image functions E^{-1} and A^{-1} need to be mentioned or referred.

(b) Introduction: The review of literatures on proper approximation and relevant results should be provided. One such example is the result in [4] for SISO systems, another example is the paper by M. Bonilla E., M. Malabre and M. Fonseca, "On the approximation of non proper control laws," Int. J. Control, vol. 68, pp. 775-796, 1997. The requirement in 1) of Problem 1 that

Σ^f is internally stable for all $\varepsilon > 0$ is not necessary and is indeed not achieved by the filter that Lemma offers.

(c) Section II: It seems that the filter given in Lemma 1 is of kind of high-gain observer, is it a special design of observer for the particular system (1)? Also add a little more arguments that lead to $lim_{\varepsilon \to 0}(y(t) - y*(t)) = e^{A_\beta t}\bar{x}(0)$ [p3, 1. -8].

(d) Section Ill: Why the external properness is defined in term of the internal properness of a (equivalent to any?) minimal part? Note that it is well-accepted that the properness can be defined by the transfer function. The assumptions of Theorem 1 should be discussed.

p4. 1. -7: What kind of observability?

p5. 1. 2: A pair of brackets is missing in the last equation.

p5. 1, -9: Γ_1^* and $\Gamma_2^{'*}$ are missing.

p6. 1. 9, 11, and 13: How can these assumptions be made? are they implied by the assumptions of Theorem 1?

(e) Section IV: p8, 1.13: The assumptions of Corollary 1 need to be verified.

(f) Section V: p9 1. 6-7: change ξ into ω.

(g) Section VI: The conclusion should be modified with the assumptions stated.

(h) References: [7] is published. vol. 46, no.1, pp. 60-65.

2) Answers to Reviewer 1:

1) General comments

(a) No, Condition 1) of Problem 1 is not related with an observer. The output $y^* = \Delta\omega$ is only the output (of the controller) and not an internal variable that would have to be observed. Condition 1) means that Σ^f is a stable filter, which modification on the output of Σ^c is almost null (in an exponential way). Condition 2) is the most important for our objective: the aim is to filter Σ^c without modifying it too much, but mainly insuring the properness of the whole system $\Sigma^f \circ \Sigma^c$.

(b, c, d) In order to clarify more, we have introduced in Section III a detailed description of the needed concepts.

In this Section we are working with concepts related to the external behaviour of the systems and thus the external properness is independent of the particular realization chosen.

Indeed, in the paper

(*) M. Kuijper : "Descriptor representations without direct feed through term", Automatica, vol. 28, 3, pp. 633-637, 1992

it is stated the existence of relations between all the minimal realizations: they only differ by changes of basis in the domain and co-domain; with respect to [3], it is equivalent to choosing bases spanning explicitly $\mathcal{X}_m$ and $\underline{\mathcal{X}}_m$. More precisely, (E, A, B, C) and (E', A', B', C') are two (externally) minimal realizations iff there exists two invertible matrices M and N, such that: $B' = MB$, $M(sE - A) = (sE' - A')N$ and $C'N = C$, which implies (in the case of regular systems): $C(sE - A)^{(-1)}B = C'(sE' - A')^{(-1)}B'$; Therefore both realizations have the same properness property.

With respect to the fact that the map B does not play any role in the conditions (12), let us note that we are working with external minimality and not with internal minimality. Indeed, internal minimality (introduced by Kalman) is used in the transfer function framework and in the case of classical state space case it is equivalent to both observability and controllability. While external minimality (introduced by Willems) is used when working in the time domain (which is our case), since there are cases where the non controllable modes are necessary to describe their associated behaviours, In fact, in the paper

(**) M. Fliess : "A simple definition of hidden modes, poles and zeros", Kybernetika, vol. 27 (3), pp. 186-189, 1991

it was pointed out the necessity of external minimality when trying to describe systems like $\dot{y} = \dot{u}$, in which case only the observability characterizes the minimality of any state description of such a system.

For the generalized systems, Kuijper has shown that there are three conditions for having external minimality, namely: (i) $[E\ B]$ epic, (ii) $[E^T,\ C^T]^T$ monic and (iii) observability of the exponential modes. Later in [3] it was shown that the minimal part of the system is found in the subspaces $\mathcal{X}_m$ and $\underline{\mathcal{X}}_m$. In our Results of Section III, Theorem 3 and Corollary 1, we are asking for satisfying $\mathcal{V}_o^* = \{0\}$ and $\mathcal{V}_{\mathcal{X}}^* = \mathcal{X}$, which are equivalent to satisfy condition (i) and (iii) given by Kuijper. The other condition of Kuijper is related to the notion that we call differential redundancy, and this is precisely the part that we use to solve the problem.

In the new version of the paper, we widely speak about all these aspects in Section III.

(e) When approximating non proper SISO systems, there is normally a positive parameter, say ε, which has to tend to zero in order to improve the approximation; but this parameter introduces terms of the type $ke^{-\varepsilon t}$ which can trouble digital simulations. In order to solve this problem, for SISO systems, in [6] was introduced an exponential mode which aim is to separate quality of approximation (given by the positive parameter ε) and rate of convergence (given by a positive parameter β in terms of the type $ke^{-\beta t}$). The extension of this proposition to the MIMO case is not straightforward and to get it, it is necessary to realize a carefully structural study, which is the principal aim of the paper.

This point is explained in the Introduction of the new version.

2) Technical correctness and presentation : they are completely taken into account in the new version.
3) Detailed suggestions and questions

(a) We have done it in the new version.

(b) We have added the reference [5] and we do a complete comparison of its results.

The requirement of internally stability of Σ^f asked in the point 1) of Problem 1 is crucial for proving Lemma 1. In fact, we have made the proof more explicit with respect to that point and we have added the hypothesis H5 to enhance this.

(c) We have already replied on this in 2.1.2.1(a) and in 2.1.2.3(b).

(d) We have already detailed this in 2.1.2.1(b, c, d). As we are working in the time domain only exists one kind of observability, the so called exponential observability. People working in a distributional framework (distributions and generalized derivatives; as Cobb do) make a distinction between finite (un)observability and infinite or impulsive (un)observability; in our time domain framework (the derivatives being in the usual (not distributional) sense) the impulsive unobservability is identified with a differential redundancy (see for example [3]).

B. Reviewer number 2

1) Questions:

COMMENTS TO AUTHOR(S):

The question treated in this paper may be of interest when control policies using differentiators are adopted to cope with disturbance decoupling problems where the standard, well-known, geometric conditions are not satisfied. In this area, the contribution by J. C. Willems, Systems and Control Letters, vol. 1, n. 4, 1982, should be acknowledged along with [5,6,7]. Since the paper main motivation is the need of finding proper systems to replace (at the implementation stage) the non proper systems solving the problem from a merely theoretic point of view, the discussion could be improved by a neat analysis of how much the control objective are compromised when the suggested approximations are introduced. The authors should also clarify how different choices of the various parameters imply different degree of accuracy. No mentions are made to other literature dealing with this same problem in the MIMO case: is it totally lacking? There is something wrong in reference [7]!

2) Answers to Reviewer 2: We have added the references of Willems and we have completed the reference [9].

As we have pointed in 2.1.2.1(e) the aim of the positive parameter ε is to give certain quality of the approximation and the aim of the positive parameter β is to give the rate of convergence; this is summarized by the equation: $\lim_{\varepsilon\to 0}(y(t)-y^{*}(t)) = \mathrm{e}^{A_\beta t}\bar{x}(0)\,,\ t>0.$

Indeed, in our last comment of the new MIMO example of Section V, we point out that the differential equation of the error signals is $F(\mathrm{p})y = (\mathrm{p} + \beta)y^*$, where $F(\mathrm{p})$ is under braced in equation (32); from these equations we realize that we have dominant poles in $-\beta$ and the other ones are very close to $-1/\varepsilon$ (for ε very small), and thus $y \approx y^* + ke^{-\beta t}$.

C. Reviewer number 3

1) Questions:

COMMENTS TO AUTHOR(S):

This paper is very difficult to understand. The authors really need to work on refining their results, stating them concisely and putting them into proper perspective relative to the work of other researchers. For example, I notice that virtually every reference is to a paper published by the same group of authors.

Problem 1 is not sufficiently motivated. What is the significance of conditions 1) and 2) on p. 2? What is the role played by the parameter ε? Similar remarks refer to the long list of conditions in Lemma 1.

It's not at all clear that anything worthwhile is accomplished by Theorem 1. The condition of external properness is replaced by a mass of subspace iterations. What intuitive significance can be obtained from this result?

The authors like to toss around phrases that they have not carefully defined, such as "externally minimal part", "internally proper", "derivators", "exponential and derivative modes", "minimal row (column) indices", "exponentially observable", "integral and derivative actions", and "state derivative observers". Most of these terms probably have reasonable definitions, but they do need to be defined in the paper.

By the time I reached the main result, Theorem 2, I was hopelessly confused, so I cannot pass judgment on the merit of the authors conclusions.

2) Answers to Reviewer 3: See answers to Reviewers 1 and 2.

D. Reviewer number 4

1) Questions:

Comments to Authors:

This paper addresses the synthesis of a strictly proper, internally stable filter for MIMO improper systems. Although the topic is an interesting one, there are a number of problems with this paper:

1) The structure of the filter poses serious concern. With ϵ tending to zero, the filter reverts to one with poles at infinity. The magnitudes of the filter gains are also unacceptably high. In order words, since ε is directly tied to the filter bandwidth, it has to be small to produce a good approximation yet on the other hand, it also leads to a practically unsound design.
2) The problem of improperness is local, i.e. even though $\Sigma^f \circ \Sigma$ is proper, its realizability is not assured if it still contains improper elements. For example, the original derivative chain (1) still appears in Figures 2 and 3. It would be more meaningful if the authors carry this step further and come up with a proper realization for $(\Sigma'^f \circ \Sigma)$ directly.
3) Band-limiting a derivative chain should be done in the context of the closed loop system rather on the controller alone. It is not clear how the filter impacts on the closed loop system (stability, robustness, performance, etc.).
4) There are no clear error bounds on the choice of ε, β. The asymptotic results are generally not helpful for synthesis/design purposes.
5) Finally, sections of the paper, especially the abstract and the conclusions, are not well written. There are also a number of typos/grammatical errors that should be weeded out.

2) Answers to Reviewer 4:

1) We have tried to answer this question by choosing in Section V a better illustrative MIMO example, which takes into account a large set of possibilities (see equation (25)) and points out the separate roles of ε and β, see equations (28) and (32) (see also answer 2.2.2).
2) $\Sigma^f \circ \Sigma^c$ being proper, its realization is up to the user. Of course the improper elements coming initially from Σ^c are no longer present.
   We have changed to a better illustrative MIMO example.
3) See answer 2.2.2.
4) See answer 2.2.2.

5) We have done it.

III. Illustrative Example

In order to clarify the principal ideas of this paper, let us consider the following system:

$$\left[\begin{array}{ccc|cc|cc} 0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ \hline 0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ \hline 0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0 \end{array}\right]\dot{\xi} = \left[\begin{array}{ccc|cc|cc} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ \hline 0&-1&0&1&1&0&0\\ 0&1&0&0&-1&0&0\\ \hline 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{array}\right]\xi + \left[\begin{array}{ccc} 0&-1&0\\ -1&0&0\\ 0&0&0\\ \hline 2&0&0\\ -2&0&0\\ \hline -1&-1&-1\\ 0&0&0 \end{array}\right]u$$

$$z = \left[\begin{array}{ccc|cc|cc} 0&0&1&0&0&0&0\\ 0&0&-1&1&0&0&0\\ 1&0&0&0&-1&-1&1 \end{array}\right]\xi \tag{23}$$

which input-output description is described by the following differential equation:

$$\begin{bmatrix} 1&0&0\\ 0&(\mathrm{p}+1)&0\\ 0&0&(\mathrm{p}+1) \end{bmatrix} z = \begin{bmatrix} \mathrm{p}&\mathrm{p}^2&0\\ -\mathrm{p}(\mathrm{p}+2)&-\mathrm{p}^3&0\\ \mathrm{p}^2&\mathrm{p}^2&(\mathrm{p}^2-1) \end{bmatrix} u \tag{24}$$

Let us carry system (23) into the Kronecker canonical form. For this, let us define $\bar{z} = T_{L_o} z$ and $\bar{\xi} = T_R^{(-1)}\xi$ and let us multiply on the left the descriptor equation by T_L, where $T_{L_o} = \begin{bmatrix} 1&0&0\\ 0&0&1\\ 1&1&0 \end{bmatrix}$, $T_L = \left[\begin{array}{ccc|cc|cc} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ \hline 0&0&0&0&0&1&0\\ 0&0&0&0&0&-1&1\\ \hline -1&0&0&1&0&0&0\\ -1&1&0&0&-1&0&0 \end{array}\right]$, and $T_R = \left[\begin{array}{ccc|cc|cc} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ \hline 0&1&0&0&0&1&1\\ 1&0&0&0&0&0&-1\\ \hline 0&0&0&1&0&0&0\\ 0&0&0&1&1&0&0 \end{array}\right]$, namely:

$$\left[\begin{array}{ccc|cc|c|c} 0&0&0&&&&\\ 1&0&0&&&&\\ 0&1&0&&&&\\ \hline &&&0&0&&\\ &&&1&0&&\\ \hline &&&&&0&\\ \hline &&&&&&1 \end{array}\right]\dot{\bar{\xi}} = \left[\begin{array}{ccc|cc|c|c} 1&0&0&&&&\\ 0&1&0&&&&\\ 0&0&1&&&&\\ \hline &&&1&0&&\\ &&&0&1&&\\ \hline &&&&&1&\\ \hline &&&&&&-1 \end{array}\right]\bar{\xi} + \left[\begin{array}{ccc} 0&-1&0\\ -1&0&0\\ 0&0&0\\ \hline -1&-1&-1\\ 1&1&1\\ \hline 2&1&0\\ \hline 1&1&0 \end{array}\right]u$$

$$\bar{z} = \left[\begin{array}{ccc|cc|c|c} 0&0&1&&&&\\ \hline &&&0&1&&1\\ \hline 0&1&0&&&1&1 \end{array}\right]\bar{\xi} \tag{25}$$

where $\bar{\xi} = [\ \bar{\xi}_{i,1}^T \quad \bar{\xi}_{i,2}^T \quad \bar{\xi}_{p,1}^T \quad \bar{\xi}_{p,2}^T\ ]^T$. In order to obtain the particular form proposed in (??), let us decompose (25) as follows (recall that $z = T_{L_o}^{(-1)} \bar{z}$):

$$
\left[\begin{array}{ccc|cc|cc}
0 & 0 & 0 & & & & \\
1 & 0 & 0 & & & & \\
0 & 1 & 0 & & & & \\
\hline
 & & & 0 & 0 & & \\
 & & & 1 & 0 & & \\
\hline
 & & & & & 0 & 0 \\
 & & & & & 1 & 0
\end{array}\right] \dot{w} = \left[\begin{array}{ccc|cc|cc}
1 & 0 & 0 & & & & \\
0 & 1 & 0 & & & & \\
0 & 0 & 1 & & & & \\
\hline
 & & & 1 & 0 & & \\
 & & & 0 & 1 & & \\
\hline
 & & & & & 1 & 0 \\
 & & & & & 0 & 1
\end{array}\right] w + \underbrace{\left[\begin{array}{ccc}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\
\hline
-1 & -1 & -1 \\
1 & 1 & 1 \\
\hline
0 & -1 & 0 \\
-1 & 0 & 0
\end{array}\right]}_{\Gamma} u \tag{26}
$$

$$
y^* = \left[\begin{array}{ccc|cc|cc}
0 & 0 & 1 & & & & \\
\hline
 & & & 0 & 1 & & \\
\hline
 & & & & & 0 & 1
\end{array}\right] w
$$

$$
\dot{\bar{\xi}}_{p,2} = -\bar{\xi}_{p,2} + \left[\begin{array}{ccc} 1 & 1 & 0 \end{array}\right] u \quad ; \quad z = \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] y^* + \left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right] \bar{\xi}_{p,2} + \left[\begin{array}{ccc} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] u \tag{27}
$$

Note that the input-output description of (26) is:

$$
y^* = \left[\begin{array}{ccc} \mathrm{p} & \mathrm{p}^2 & 0 \\ (\mathrm{p}-1) & (\mathrm{p}-1) & (\mathrm{p}-1) \\ 1 & \mathrm{p} & 0 \end{array}\right] u \tag{28}
$$

Let us now proceed to approximate the non-proper system (26).

1) Comparing (26) with (1) and (3), we get: $n = 3$, $k_1 = 2$, and $k_2 = k_3 = 1$; thus $\kappa = 2$. Matrix Γ is indicated in (26) itself.

2) Using the proposition (5) and (6), we get the following overall system (c.f. (14)-(16)):

$$
\left[\begin{array}{ccc|cccc|ccc|cc|cc}
1&0&0&&&&&&&&&&& \\
0&1&0&&&&&&&&&&& \\
0&0&1&&&&&&&&&&& \\
\hline
&&&1&0&0&0&&&&&&& \\
&&&0&1&0&0&&&&&&& \\
&&&0&0&1&0&&&&&&& \\
&&&0&0&0&1&&&&&&& \\
\hline
&&&&&&&0&0&0&&&& \\
&&&&&&&1&0&0&&&& \\
&&&&&&&0&1&0&&&& \\
\hline
&&&&&&&&&&0&0&& \\
&&&&&&&&&&1&0&& \\
\hline
&&&&&&&&&&&&0&0 \\
&&&&&&&&&&&&1&0
\end{array}\right]\dot{x} =
\left[\begin{array}{ccc}
&& \\ && \\ && \\ && \\ && \\ && \\ && \\
\hline
0&-1&0 \\
-1&0&0 \\
0&0&0 \\
\hline
-1&-1&-1 \\
1&1&1 \\
\hline
0&-1&0 \\
-1&0&0
\end{array}\right]u
$$

$$
+\left[\begin{array}{ccc|cccc|ccc|cc|cc}
-\beta&0&0&-\varepsilon^3&0&&&0&&&0&&0& \\
0&-\beta&0&&&-\varepsilon^3&&0&&&0&&0& \\
0&0&-\beta&&&&-\varepsilon^3&0&&&0&&0& \\
\hline
0&&&-1/\varepsilon&1/\varepsilon&&&0&&&0&&0& \\
1/\varepsilon&&&0&-1/\varepsilon&&&1/\varepsilon&&&0&&0& \\
&1/\varepsilon&&&&-1/\varepsilon&&0&&&1/\varepsilon&&0& \\
&&1/\varepsilon&&&&-1/\varepsilon&0&&&0&&1/\varepsilon& \\
\hline
&&&&&&&1&0&0&&&& \\
&&&&&&&0&1&0&&&& \\
&&&&&&&0&0&1&&&& \\
\hline
&&&&&&&&&&1&0&& \\
&&&&&&&&&&0&1&& \\
\hline
&&&&&&&&&&&&1&0 \\
&&&&&&&&&&&&0&1
\end{array}\right]x \tag{29}
$$

$$
y = \left[\begin{array}{ccc|cccc|ccccccc}
&&&1&0&&&&&&&&& \\
&&&&&1&&&&&&&& \\
&&&&&&1&&&&&&&
\end{array}\right]x
$$

3) The matrices R_p and L_p defined in Lemma 2 are (see (18)):

$$
R_p = \left[\begin{array}{ccc|cc|cc}
0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0 \\
\hline
1/\varepsilon^2&0&0&0&0&0&0 \\
-1/\varepsilon^2&1/\varepsilon&0&0&0&0&0 \\
0&0&0&1/\varepsilon&0&0&0 \\
0&0&0&0&0&1/\varepsilon&0
\end{array}\right]
\text{ and } L_p = \left[\begin{array}{ccc|cc|cc}
\varepsilon&0&0&0&0&0&0 \\
0&0&0&\varepsilon^2&0&0&0 \\
0&0&0&0&0&\varepsilon^2&0 \\
\hline
2/\varepsilon^3&-1/\varepsilon^2&0&0&0&0&0 \\
-1/\varepsilon^3&1/\varepsilon^2&-1/\varepsilon&0&0&0&0 \\
0&0&0&1/\varepsilon^2&-1/\varepsilon&0&0 \\
0&0&0&0&0&1/\varepsilon^2&-1/\varepsilon
\end{array}\right];
$$

note that (19) and (20) are satisfied.

4) Defining $\bar{x} = \begin{bmatrix} \mathrm{I} & R_p \\ 0 & \mathrm{I} \end{bmatrix}^{(-1)} x$ and premultiplying (29) by $\begin{bmatrix} \mathrm{I} & L_p \\ 0 & \mathrm{I} \end{bmatrix}$, we get (c.f. (21)):

$$
\left[\begin{array}{ccc|cccc|ccc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right] \dot{\bar{x}} =
\left[\begin{array}{ccc}
0 & -\varepsilon & 0 \\
-\varepsilon^2 & -\varepsilon^2 & -\varepsilon^2 \\
0 & -\varepsilon^2 & 0 \\
\hline
1/\varepsilon^2 & -2/\varepsilon^3 & 0 \\
-1/\varepsilon^2 & 1/\varepsilon^3 & 0 \\
-(1/\varepsilon^2+1/\varepsilon) & -(1/\varepsilon^2+1/\varepsilon) & -(1/\varepsilon^2+1/\varepsilon) \\
1/\varepsilon & -1/\varepsilon^2 & 0 \\
\hline
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0 \\
\hline
-1 & -1 & -1 \\
1 & 1 & 1 \\
\hline
0 & -1 & 0 \\
-1 & 0 & 0
\end{array}\right] u
$$

$$
+ \left[\begin{array}{ccc|cccc|ccc|cc|cc}
-\beta & 0 & 0 & -\varepsilon^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\beta & 0 & 0 & 0 & -\varepsilon^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\beta & 0 & 0 & 0 & -\varepsilon^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & -1/\varepsilon & 1/\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/\varepsilon & 0 & 0 & 0 & -1/\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/\varepsilon & 0 & 0 & 0 & -1/\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/\varepsilon & 0 & 0 & 0 & -1/\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \bar{x}
$$

$$
y = \left[\begin{array}{ccc|cccc|ccc|cc|cc}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/\varepsilon^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/\varepsilon & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1/\varepsilon & 0
\end{array}\right] \bar{x}
\tag{30}
$$

5) Applying the algorithms (10) and (11) to (30), we get: $\mathcal{R}^*_{a0} = \{e_9,\ e_{10},\ e_{12},\ e_{14}\}$ and $E^{-1}A\mathcal{R}^*_{a0} = \{e_8,\ e_9,\ e_{10},\ e_{11},\ e_{12},\ e_{14}\} = \mathcal{S}^*_{\mathcal{X},0}$. And thus, since the assumptions of Corollary 1 are satisfied, system (30) is externally proper; and it is externally equivalent to:

$$
\begin{aligned}
\dot{\hat{x}} &= \left[\begin{array}{ccc|cccc}
-\beta & 0 & 0 & -\varepsilon^3 & 0 & 0 & 0 \\
0 & -\beta & 0 & 0 & 0 & -\varepsilon^3 & 0 \\
0 & 0 & -\beta & 0 & 0 & 0 & -\varepsilon^3 \\
\hline
0 & 0 & 0 & -1/\varepsilon & 1/\varepsilon & 0 & 0 \\
1/\varepsilon & 0 & 0 & 0 & -1/\varepsilon & 0 & 0 \\
0 & 1/\varepsilon & 0 & 0 & 0 & -1/\varepsilon & 0 \\
0 & 0 & 1/\varepsilon & 0 & 0 & 0 & -1/\varepsilon
\end{array}\right]\hat{x} + \left[\begin{array}{ccc}
0 & -\varepsilon & 0 \\
-\varepsilon^2 & -\varepsilon^2 & -\varepsilon^2 \\
0 & -\varepsilon^2 & 0 \\
\hline
1/\varepsilon^2 & -2/\varepsilon^3 & 0 \\
-1/\varepsilon^2 & 1/\varepsilon^3 & 0 \\
-(1/\varepsilon^2+1/\varepsilon) & -(1/\varepsilon^2+1/\varepsilon) & -(1/\varepsilon^2+1/\varepsilon) \\
1/\varepsilon & -1/\varepsilon^2 & 0
\end{array}\right]u \\
y &= \left[\begin{array}{ccc|cccc}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]\hat{x} + \left[\begin{array}{ccc}
0 & 1/\varepsilon^2 & 0 \\
1/\varepsilon & 1/\varepsilon & 1/\varepsilon \\
0 & 1/\varepsilon & 0
\end{array}\right]u
\end{aligned}
\tag{31}
$$

Note that the input-output description of (31) is:

$$
\underbrace{\left[\begin{array}{ccc}
(\varepsilon \mathrm{p}+1)^2(\mathrm{p}+\beta)+\varepsilon^3 & 0 & 0 \\
0 & (\varepsilon \mathrm{p}+1)(\mathrm{p}+\beta)+\varepsilon^3 & 0 \\
0 & 0 & (\varepsilon \mathrm{p}+1)(\mathrm{p}+\beta)+\varepsilon^3
\end{array}\right]}_{F(\mathrm{p})} y = (\mathrm{p}+\beta)\left[\begin{array}{ccc}
\mathrm{p} & \mathrm{p}^2 & 0 \\
(\mathrm{p}-1) & (\mathrm{p}-1) & (\mathrm{p}-1) \\
1 & \mathrm{p} & 0
\end{array}\right]u
\tag{32}
$$

Thus from (32) and (28), we get: $F(\mathrm{p})y(t) = (\mathrm{p}+\beta)y^*(t)$. There then exists a positive real number, ε^*, such that $\det F(\mathrm{p})$ is Hurwitz for all $\varepsilon \in (0,\ \varepsilon^*)$. Moreover, we realize that there are dominant poles in $-\beta$ and that the other poles are very close to $-1/\varepsilon$ (for ε very small). Furthermore, $y(t) \approx y^*(t) + \mathrm{e}^{-\beta t}(y(0) - y^*(0))$ (for ε very small). The filter looked for is given by (31), (27.a) and (cf (27.b)):

$$
z = \left[\begin{array}{ccc}
1 & 0 & 0 \\
-1 & 0 & 1 \\
0 & 1 & 0
\end{array}\right]y + \left[\begin{array}{c}
0 \\ 1 \\ 1
\end{array}\right]\bar{\xi}_{p,2} + \left[\begin{array}{ccc}
0 & 0 & 0 \\
-2 & -1 & 0 \\
0 & 0 & 0
\end{array}\right]u
\tag{33}
$$

Apéndice D

Almost Rejection of Internal Structural Variations in Linear Systems (Submitted to Automatica)

Almost Rejection of Internal Structural Variations in Linear Systems

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Abstract

This paper deals with linear systems having internal structural variations. Under some weak assumptions, the control of such systems is indeed possible, thanks to the very nice setting of implicit models and Proportional and Derivative (PD) feedbacks. However, the effective design of such PD feedbacks usually requires suitable approximations for the derivatives, hence Proportional approximations of the PD feedback "exact" solution. The aim of the present contribution is to directly tackle an "almost" version of the problem by pure (high gain) Proportional feedback. The design is different, but very close to the one related to PD feedbacks, mainly with respect to the associated geometric splittings. As an interesting by-product, a system theoretical interpretation of the classical process of "integration by parts" is given and shown to be equivalent to some particular changes of bases.

Key words: Linear systems, implicit systems, variable structure systems, PD feedbacks, proper approximations.

Notation

Script capitals $\mathcal{V}$, $\mathcal{W}$, ..., denote linear spaces with elements v, w, ...; the dimension of a space $\mathcal{V}$ is denoted $\dim(\mathcal{V})$; $\mathcal{V} \approx \mathcal{W}$ stands for $\dim(\mathcal{V}) = \dim(\mathcal{W})$; when $\mathcal{V} \subset \mathcal{W}$, $\frac{\mathcal{W}}{\mathcal{V}}$ or $\mathcal{W}/\mathcal{V}$ stands for the quotient space $\mathcal{W}$ modulo $\mathcal{V}$; the direct sum of independent spaces is written as $\oplus$. Given a linear map $X : \mathcal{V} \to \mathcal{W}$, Im $X = X\mathcal{V}$ denotes its image, and $\mathcal{K}_X$ or sometimes Ker X denotes its kernel. For the two special maps $E : \mathcal{X} \to \underline{\mathcal{X}}$ and $B : \mathcal{U} \to \underline{\mathcal{X}}$, their images are denoted by $\mathcal{E}$ and $\mathcal{B}$, respectively. We write $X^{(-1)}$ for the inverse map of X (when it exists) in order to avoid confusions with $X^{-1}\mathcal{T}$, the inverse image of the subspace $\mathcal{T}$ by the linear map X. $\{x, y, z\}$ stands for the subspace spanded by x, y and z. e_i stands for the vector with a 1 in its $i - th$ component and 0 otherwise.

1 Introduction

Consider the implicit description, $\Sigma\ (E, A, B, C)$: [3]

$$E\dot{x} = Ax + Bu \ \ ; \ \ y = Cx \tag{1}$$

where $E : \mathcal{X} \to \underline{\mathcal{X}}$, $A : \mathcal{X} \to \underline{\mathcal{X}}$, $B : \mathcal{U} \to \underline{\mathcal{X}}$ and $C : \mathcal{X} \to \mathcal{Y}$ are linear maps of appropriate dimensions, with $\mathcal{K}_B = \{0\}$ and Im $C = \mathcal{Y}$. x, u and y are the descriptor variable, the input and the output. In (Bonilla and Malabre

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[3] For the sake of shortness, and except when needed, we write x, $\dot{x}$, u, y, ... in place of $x(t)$, $\dot{x}(t)$, $u(t)$, $y(t)$, ...

1991), it was shown that when $\dim(\underline{\mathcal{X}}) < \dim(\mathcal{X})$, it is possible to describe linear systems with an internal variable structure (see Bonilla and Malabre (2003)). For example the implicit flat description:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \ ; \ y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x \tag{2}$$

with the additional constraint: $[\alpha\ \beta\ 1]x = 0$.

If $(\alpha,\ \beta) = (-1,\ -1)$, namely $x_3 = x_1 + x_2$, then the input-output description $\dot{y} + y = u$. If $(\alpha,\ \beta) = (-1,\ 0)$, namely $x_3 = x_1$, then the input-output description is $\ddot{y} + \dot{y} = u$. If $(\alpha,\ \beta) = (-1,\ -5)$, namely $x_3 = x_1 + 5x_2$, then the input-output description is $\ddot{y} + 6\dot{y} + 5y = 5\dot{u} + u$. Finally, if $(\alpha,\ \beta) = (1,\ 1)$, namely $x_3 = -(x_1 + x_2)$, then the input-output description is $\dot{y} - 3y = -u$.

Bonilla and Malabre (2003) have considered the following problem:

Problem 1 *(Bonilla and Malabre 2003) Let us consider a set of strictly proper linear systems embedded in the following set of implicit global descriptions $\Sigma_i^g(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$:*[4]

$$\mathbb{E}\dot{x} = \mathbb{A}x + \mathbb{B}u \ ; \ y = Cx \tag{3}$$

with $\mathbb{E} = \begin{bmatrix} E^T & 0 \end{bmatrix}^T$, $\mathbb{A}_i = \begin{bmatrix} A^T & D_i^T \end{bmatrix}^T$, $\mathbb{B} = \begin{bmatrix} B^T & 0 \end{bmatrix}^T$ $(i = 1, \ldots, n)$; E and D_i are epic.
– Under which conditions can this set of linear systems be controlled by a fixed P.D. state feedback, $u = F_p x + F_d \dot{x}$, assigning a fixed external closed-loop behaviour, and which synthesis is based on the common internal structure, described by $E\dot{x} = Ax + Bu$?

Bonilla and Malabre (2003) have found what it is the common internal structure of the set of linear systems (3) which enables to solve Problem 1. They have given a procedure to synthesize P.D. feedbacks for rendering unobservable the variation of structure and assigning at will the closed-loop output dynamics. Such a synthesis procedure relied on the following Theorems (see the background Section 2 for the definition of $\mathcal{V}^*$ and some of its related properties, as well as the definitions of external equivalence and external minimality):

Theorem 2 *(Bonilla and Malabre 2003) If the following two geometric conditions hold*

$$\mathrm{Im}\ A + \mathcal{B} \subset \mathcal{E} \quad \text{and} \quad \dim\left(\mathcal{V}^* \cap E^{-1}\mathcal{B}\right) \geq \dim\left(\mathcal{K}_E\right) \tag{4}$$

there then exists a P.D. feedback, $u = F_p^* x + F_d^* \dot{x}$, with $(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*)$ (see Section II), such that:

$$\mathrm{Im}\ (E - BF_d) = \mathcal{E} \quad \text{and} \quad \mathrm{Ker}\ (E - BF_d) \subset \mathcal{V}^* \tag{5}$$

Theorem 3 *(Bonilla and Malabre 2003) Let $(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*)$, as in Theorem 2. When (3) is fed back with: $u = F_p^* x + F_d^* \dot{x} + v$, it is externally equivalent to: $\dot{\hat{x}} = E_*^{(-1)} A_* \hat{x} + E_*^{(-1)} B_* v$ and $y = C_* \hat{x}$. The isomorphism E_* and the maps A_* and B_* are induced from the closed loop system by the cannonical projections $\Pi : \mathcal{E} \to \mathcal{E}/E_{F^*}\mathcal{V}^*$ and $\Phi : \mathcal{X} \to \mathcal{X}/\mathcal{V}^*$ and $\hat{x} = \Phi x$.*

Theorem 4 *(Bonilla and Malabre 2003) Given any pair (F_p^*, F_d^*), as in Theorem 3. If the implicit system (1), related to the implicit global descriptions (3), is controllable*[5] *then the quotient system $\dot{\hat{x}} = E_*^{(-1)} A_* \hat{x} + E_*^{(-1)} B_* v$ and $y = C_* \hat{x}$ is also controllable (in the classical sense).*

The solution found in (Bonilla and Malabre 2003) is based on the use of (P, D) feedbacks which are *friends* of $\mathcal{V}^*$. From the geometric condition (4.b) of Theorem 2, we can realize that the derivative part of the control law, F_d,

[4] The dynamic part of the $\Sigma_i^g(\mathbb{E}, \mathbb{A}, \mathbb{B}, C)$ which is active for any choice of D_i will be denoted as $\Sigma(E, A, B, C)$ as in (1).
[5] This controllability depends on the exogenous input, u, and also on the degree of freedom, characterized by $\mathcal{K}_{D_i}$, which acts as endogenous inputs.

is crucial for solving Problem 1, namely a static feedback is generally not sufficient. A natural question from a practical point of view is -*what can we do if we are restricted to use Proportional feedbacks?*; more precisely -*how to approximate the P.D. feedback obtained from Theorem 2 in order to be close to the nice structural properties given in Theorems 3 and 4?*. This question is stated in the following *Almost Rejection of the Internal Structural Variations Problem (ARISV-Problem)*:

Problem 5 (*ARISV-Problem*) *Given the implicit global descriptions (3), let $u^* = F_p^* x + F_d^* \dot{x} + \bar{u}$ be a control law solving Problem 1 and let y^* be the output obtained with this P.D. feedback. For a given $\delta \in \mathbb{R}^+$ find a Proportional state feedback, $u = F_p x + \bar{u}$, such that its closed loop output, y, satisfies ($t^*(\delta)$ is some fixed time depending on the given δ):*

$$|y - y^*| \leq \delta \qquad \forall \;\; t \;\geq\; t^*(\delta)$$

In this paper we solve this *ARISV-Problem*. For this we introduce in Section 2 some needed background material. In Section 3, we find some basic geometric decompositions and in Section 4 the problem is solved. In Section 5 we give an illustrative example and in Section 6 some concluding remarks are given.

2 Background

2.1 Structural Definitions

Definition 6 *(Willems 1983) Two models are called externally equivalent iff the corresponding sets of all possible trajectories for the external variables (external behaviors) are the same.*

Definition 7 *(Kuijper (1992) and Bonilla and Malabre (1995)) A given implicit description, $E\dot{x} = Ax + Bu$ and $y = Cx$, where $E, A \in \mathbb{R}^{\underline{n} \times n}$, $B \in \mathbb{R}^{\underline{n} \times m}$, and $C \in \mathbb{R}^{p \times n}$, with $\underline{n}$ and n not necessarily equal, is minimal among all externally equivalent descriptions of the same type iff:* 1) *the corresponding descriptor equation has the least possible number of rows, and* 2) *the descriptor variable has the least possible number of components.*

Definition 8 *(Bernhard (1982) and Armentano (1986)) $\mathbb{F}\dot{x} = \mathbb{G}x + v$ is internally proper iff the pencil $[\lambda\mathbb{F} - \mathbb{G}]$ is regular (square and $\det(\lambda\mathbb{F} - \mathbb{G}) \neq 0$) and it has no infinite zero of order greater than one (there are no derivators).*

Kuijper (1992) gave necesary and sufficient conditions for external minimality and Bonilla and Malabre (1995) showed that a given implicit description (E, A, B, C) is externally equivalent to the minimal implicit description (E_m, A_m, B_m, C_m), called the externally minimal part of the system whose maps are uniquely defined as follows:

$$E_m \Pi_m = P_m E \;\;;\;\; A_m \Pi_m = P_m A \;\;;\;\; B_m = P_m B \;\;;\;\; C_m \Pi_m = C$$

where $\Pi_m : \mathcal{X} \to \mathcal{V}_{\mathcal{X}}^* / (\mathcal{V}_o^\star + \mathcal{V}_{\mathcal{X}}^* \cap \mathcal{R}_{a0}^*)$ and $P_m : \underline{\mathcal{X}} \to (E\mathcal{V}_{\mathcal{X}}^* + \text{Im } B)/(E\mathcal{V}_o^\star + A(\mathcal{V}_{\mathcal{X}}^* \cap \mathcal{R}_{a0}^*))$ are canonical projections (see next subsection for subspace definitions).

Definition 9 *(Pacheco et al 2003) The implicit system, $E\dot{x} = Ax + Bu$ and $y = Cx$, is externally proper iff its externally minimal part is internally proper.*

2.2 Subspaces and Related Properties

Related with any implicit system (1) are the well known following subspaces: (see Özçaldiran (1986), Malabre (1987) and Bonilla and Malabre (1995)):

The supremal (A, E, B) invariant subspace contained in $\mathcal{K}_C$, $\mathcal{V}_\Sigma^* := \sup \{\mathcal{V} \subset \mathcal{K}_C \mid A\mathcal{V} \subset E\mathcal{V} + \text{Im } B\}$, limit of: $\mathcal{V}^0 = \mathcal{X}$ and $\mathcal{V}^{\mu+1} = \mathcal{K}_C \cap A^{-1}(E\mathcal{V}^\mu + \text{Im } B)$ for $\mu \geq 0$.
Let $\mathbf{F}(\mathcal{V}_\Sigma^*)$ denote the set of all (F_p, F_d) such that $(A + BF_p)\mathcal{V}_\Sigma^* \subset (E - BF_d)\mathcal{V}_\Sigma^*$. Such (F_p, F_d) is called a friend pair of $\mathcal{V}_\Sigma^*$. The following result is well known:

Fact 10 (i) *For any (F_p, F_d), for the closed loop system, $\Sigma_F(E - BF_d, A + BF_p, B, C)$ there holds: $\mathcal{V}_\Sigma^* = \mathcal{V}_{\Sigma_F}^*$; then, we just write $\mathcal{V}^*$ to identify $\mathcal{V}_\Sigma^*$ or $\mathcal{V}_{\Sigma_F}^*$* (ii) *For any F_d, there exists F_p^* s.t. $(F_p^*, F_d) \in \mathbf{F}(\mathcal{V}_\Sigma^*)$.*

The supremal (A, E, B) invariant subspace contained in $\mathcal{X}$, $\mathcal{V}^*_{\mathcal{X}} := \sup\{\mathcal{V} \subset \mathcal{X} \mid A\mathcal{V} \subset E\mathcal{V} + \text{Im } B\}$, is the limit of the non-increasing algorithm: $\mathcal{V}^o_{\mathcal{X}} = \mathcal{X}$ and $\mathcal{V}^{\mu+1}_{\mathcal{X}} = A^{-1}(E\mathcal{V}^{\mu}_{\mathcal{X}} + \text{Im } B)$ for $\mu \geq 0$. $\mathcal{V}^*_{\mathcal{X}}$ characterizes (together with $E\mathcal{V}^*_{\mathcal{X}} + \text{Im } B$) the set of all possible trajectories which are not identically zero for any input u,

The supremal (A, E) invariant subspace contained in Ker C, $\mathcal{V}^*_o := \sup\{\mathcal{V} \subset \text{Ker } C \mid A\mathcal{V} \subset E\mathcal{V}\}$, is the limit of the non-increasing algorithm: $\mathcal{V}^o_o = \mathcal{X}$ and $\mathcal{V}^{\mu+1}_o = \mathcal{K}_C \cap A^{-1}E\mathcal{V}^{\mu}_o$ for $\mu \geq 0$. $\mathcal{V}^*_o$ characterizes (together with $E\mathcal{V}^*_o$) the set of all exponential trajectories which are unobservable at the output y.

The supremal almost (A, E) controllability subspace contained in $\mathcal{K}_C$, $\mathcal{R}^*_{a0} := \inf\{\mathcal{R} \subset \mathcal{K}_C \mid \mathcal{R} = \mathcal{K}_C \cap E^{-1}(A\mathcal{R})\}$, is the limit of the non-decreasing algorithm:

$$\mathcal{R}^0_{a0} = \mathcal{K}_C \cap \mathcal{K}_E \ ; \ \ \mathcal{R}^{\mu+1}_{a0} = \mathcal{K}_C \cap E^{-1}(A\mathcal{R}^{\mu}_{a0}) \text{ for } \mu \geq 0 \tag{6}$$

$\mathcal{R}^*_{a0}$ characterizes (together with $A\mathcal{R}^*_{a0}$) the set of all trajectories due to differential actions with no influence on the input-output trajectories. $\mathcal{R}^*_{a0}$ is called the set of differentially redundant descriptor variables.

Proposition 11 *(Bonilla and Malabre 1995) Given an implicit description,* $\Sigma(E, A, B, C)$, $\mathcal{R}^*_{a0}$ *characterizes the differentially redundant descriptor variables. Moreover the induced system,* $\widehat{\Sigma}(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$*:* $\widehat{E}\dot{\hat{x}} = \widehat{A}\hat{x} + \widehat{B}u$ *and* $y = \widehat{C}\hat{x}$ *with* $\widehat{E} : \mathcal{X}/\mathcal{R}^*_{a0} \to \underline{\mathcal{X}}/A\mathcal{R}^*_{a0}$, $\widehat{A} : \mathcal{X}/\mathcal{R}^*_{a0} \to \underline{\mathcal{X}}/A\mathcal{R}^*_{a0}$, $\widehat{B} : \mathcal{U} \to \underline{\mathcal{X}}/A\mathcal{R}^*_{a0}$*, and* $\widehat{C} : \mathcal{X}/\mathcal{R}^*_{a0} \to \mathcal{Y}$*, has no differentially redundant descriptor variables. Furthermore the systems* $\Sigma(E, A, B, C)$ *and* $\widehat{\Sigma}(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$ *are externally equivalent.*

2.3 Properness

Proposition 12 *(Bonilla and Malabre 2003) The global descriptions (3) are internally proper iff:* $\mathcal{K}_{D_i} \oplus \mathcal{K}_E = \mathcal{X}$.

In view that, from the three subspaces, $\mathcal{V}^*_{\mathcal{X}}$, $\mathcal{V}^*_o$ and $\mathcal{R}^*_{a0}$, characterizing the externally minimal part of (1), only $\mathcal{R}^*_{a0}$ is related with non proper modes then external properness (see Definition 9) is characterized as follows:

Corollary 13 *An implicit description,* $\Sigma(E, A, B, C)$ *is externally proper if its induced system,* $\widehat{\Sigma}(\widehat{E}, \widehat{A}, \widehat{B}, \widehat{C})$ *(defined in Proposition 11), is internally proper.*

2.4 Integration by Parts

In this Subsection we find the equivalence, in the framework of system theory, of one powerful tool of functional analysis: the integration by parts. For this let us consider the following proper system:

$$\dot{x} = [-1/\varepsilon]\, x + [1/\varepsilon]\, f \quad ; \quad y = [-1/\varepsilon]\, x + [1/\varepsilon]\, f \tag{7}$$

where y is the output and f is an input at least twice differentiable and such that $f, \dot{f}, \ddot{f} \in L_\infty$, with $x(0) = x_0$, $f(0) = f_0$ and $\dot{f}(0) = \dot{f}_0$; ε is a positive parameter. We are interested in analyzing the external behaviour when the positive parameter ε tends to zero. Namely, we want to have a rigorous and simple setting from which to argue the "obvious" property that $y(t)$ tends to $\dot{f}(t)$ when ε tends to zero. For this, let us obtain the solution of (7):[6]

$$x(t) = \mathrm{e}^{-t/\varepsilon}x_0 + \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon} f(\tau)d\tau \ , \quad \text{and} \quad y(t) = -\frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon}x_0 + \frac{1}{\varepsilon}f(t) - \frac{1}{\varepsilon^2}\int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon} f(\tau)d\tau$$

From this time-domain solution, we get $|y(t)| \leq \frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon}|x_0| + \frac{1}{\varepsilon}|f(t)| + \frac{1}{\varepsilon}\|f\|_\infty$. As we can not conclude anything when $\varepsilon \to 0$, let us integrate by parts:

$$x(t) - f(t) = \mathrm{e}^{-t/\varepsilon}(x_0 - f_0) - \int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon}\dot{f}(\tau)d\tau \ , \quad \text{and} \quad y(t) = -\frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon}(x_0 - f_0) + \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon}\dot{f}(\tau)d\tau$$

[6] For better clarity, we write here the time dependency of the variables.

From this integration by parts, we get: $|y(t)| \leq \frac{1}{\varepsilon}\mathrm{e}^{-\frac{t}{\varepsilon}}|x_0 - f_0| + \|\dot{f}\|_\infty$, and we can only conclude that y is bounded when $\varepsilon \to 0$ and for $t > 0$. Let us integrate by parts one more time:

$$x(t)-f(t)+\varepsilon\dot{f}(t) = \mathrm{e}^{-t/\varepsilon}(x_0-f_0+\varepsilon\dot{f}_0)+\varepsilon\int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon}\ddot{f}(\tau)d\tau \;\&\; y(t)-\dot{f}(t) = -\frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon}(x_0-f_0+\varepsilon\dot{f}_0)-\int_0^t \mathrm{e}^{-(t-\tau)/\varepsilon}\ddot{f}(\tau)d\tau$$

From this integration by parts, we get: $|y(t) - \dot{f}(t)| \leq \frac{1}{\varepsilon}\mathrm{e}^{-t/\varepsilon}|x_0 - f_0 + \varepsilon\dot{f}_0| + \varepsilon\|\ddot{f}\|_\infty$. Therefore, $y(t) \overset{\varepsilon\to 0}{\longrightarrow} \dot{f}(t)$ $\forall\, t > 0$.

From this simple analysis, we realize that it was necessary to integrate by parts twice in order to get rigorous arguments for (expected) conclusions. We would like now to translate this process of “integration by parts” into a system theoretical point of view. Indeed, we shall later show, in Section 4.2, that this interpretation in terms of equivalent changes of bases is the guide which fully enlights the choice of the particular proportional state feedback law solving the ARISV problem.

2.4.1 Useful System Descriptions

Let us first note that the expression of the first integration by parts is the time solution of the *Fliess state space description* (Fliess 1990): $\dot{w} = [-1/\varepsilon]\, w + [1/\varepsilon]\,(-\varepsilon\dot{f})$ and $y = [-1/\varepsilon]\, w$. This description can be obtained from (7) with the simple change of variable $w = x - f$. Let us next note that the expression of the second integration by parts corresponds to the time solution of the *Fliess state space description*:

$$\dot{z} = [-1/\varepsilon]\, z + [1/\varepsilon]\,(\varepsilon^2\ddot{f}) \quad ; \quad y = [-1/\varepsilon]\, z + [1]\,\dot{f} \tag{8}$$

This description can be obtained from (7) with the change of variable $z = w - (-\varepsilon\dot{f}) = x - f + \varepsilon\dot{f}$.
The implicit descriptions of systems (7) and (8) are:

(i) Doing $\xi_1 = x$ and $\xi_2 = f$ in (7), we get:

$$\begin{bmatrix} \boxed{1\;\;0} \\ 0\;\;0 \end{bmatrix}\dot{\xi} = \begin{bmatrix} -1/\varepsilon & 0 \\ \boxed{0 \;\; 1} \end{bmatrix}\xi + \begin{bmatrix} 1/\varepsilon \\ -1 \end{bmatrix} f \quad ; \quad y = \begin{bmatrix} -1/\varepsilon & 1/\varepsilon \end{bmatrix}\xi \tag{9}$$

(ii) Doing $\zeta_1 = z$, $\zeta_2 = f$, $\zeta_3 = -\varepsilon\dot{\zeta}_2$ and $\zeta_4 = \dot{\zeta}_3$ in (8), we get:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\dot{\zeta} = \begin{bmatrix} -1/\varepsilon & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}\zeta + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} f \quad ; \quad y = \begin{bmatrix} -1/\varepsilon & 0 & -1/\varepsilon & 0 \end{bmatrix}\zeta \tag{10}$$

2.4.2 Internal Properness

Let us first note that system (9) is internally proper. Indeed, this follows from the fact that (c.f. Proposition 12): $\mathcal{X} = \mathrm{Ker}\,[1\;0] \oplus \mathrm{Ker}\,[0\;1]$. Let us next note that system (10) is not internally proper. Indeed, this follows from the fact that (c.f. Proposition 12): $\mathrm{Ker}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cap \mathrm{Ker}\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \neq \{0\}$.

2.4.3 External Properness

Since system (9) is internally proper, it is also externally proper. For system (10), in order to check the external properness, we first need to obtain the system quotiented by $\mathcal{R}^*_{a0}$ in the domain and by $A\mathcal{R}^*_{a0}$ in the co-domain (c.f.

Corollary 13). Applying the matricial procedure of (Bonilla and Malabre 1997) to system (10), we get:[7]

$$\overset{\mathcal{X}/\mathcal{R}^*_{a0}}{\longleftrightarrow} \quad \frac{\mathcal{X}}{A\mathcal{R}^*_{a0}} \updownarrow \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & -\epsilon \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \end{bmatrix} \dot{\bar{\zeta}} = \begin{bmatrix} \boxed{\begin{matrix} -1/\varepsilon & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{bmatrix} \bar{\zeta} + \begin{bmatrix} \boxed{\begin{matrix} 1/\varepsilon \\ -1 \end{matrix}} \\ \begin{matrix} 0 \\ 0 \end{matrix} \end{bmatrix} f \quad ; \quad y = \begin{bmatrix} \boxed{\begin{matrix} -1/\varepsilon & 1/\varepsilon \end{matrix}} & 0 & 0 \end{bmatrix} \bar{\zeta}$$

Applying the algorithm (6) to this system, we realize that $\mathcal{R}^*_{a0} = \{e_3,\ e_4\}$. And thus, inside the solid line boxes we find the induced system claimed in Corollary 13, which is nothing else than system (9). Then, (10) is externally proper and externally equivalent to (9).

From this discussion, we conclude that performing two integrations by parts is equivalent to applying, in the *Fliess state space description*, the change of variable:[8] $z = x - f + \varepsilon \dot{f}$. The system obtained with this change of variable only adds differential redundant descriptor variables and remains externally proper and externally equivalent to the original system. As we will see later on, the added differential redundant subspace enables us to bring the system into a nice structural form.

3 Basic Geometric Decompositions

In order to solve the *ARISV-Problem* we modify the geometric condition (4.b) as follows:

$$\dim\left(\mathcal{V}^* \cap E^{-1}\mathcal{B}\right) \geq \dim\left(\mathcal{K}_E\right) + \dim\left((\mathcal{K}_E + \mathcal{V}^*)/\mathcal{V}^*\right) \tag{11}$$

In order to simplify, we can also assume, without any loss of generality, that a preliminary proportional feedback has been applied such that:

$$A\mathcal{V}^* \cap \mathcal{B} = \{0\} \quad \text{and} \quad A\mathcal{V}^* \subset E\mathcal{V}^* \tag{12}$$

In a similar way as in (Bonilla and Malabre 2003), let us decompose $\mathcal{K}_E$, $E^{-1}\mathcal{B}$, $\mathcal{V}^*$, and the space $\mathcal{X}$ as follows ($\mathcal{X}_0$, $\mathcal{X}_{\mathcal{V}^*}$, $\mathcal{X}_3$, and $\mathcal{X}_{\mathcal{K}_E}$ are any complementary subspaces):

$$\begin{cases} \mathcal{K}_E = (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \quad ; \quad E^{-1}\mathcal{B} = ((\mathcal{V}^* \cap E^{-1}\mathcal{B}) + \mathcal{K}_E) \oplus \mathcal{X}_3 \\ \qquad \mathcal{V}^* = \mathcal{X}_{\mathcal{V}^*} \oplus (\mathcal{V}^* \cap E^{-1}\mathcal{B}) \quad ; \quad \mathcal{X} = (\mathcal{V}^* + E^{-1}\mathcal{B}) \oplus \mathcal{X}_0 \end{cases} \tag{13}$$

In view of (11), there then exist $\mathcal{X}_1$, $\mathcal{X}_2 \subset E^{-1}\mathcal{B}$, subspaces of $E^{-1}\mathcal{B}$, $\mathcal{X}_1$ and $\mathcal{X}_2$, such that:

$$\mathcal{V}^* \cap E^{-1}\mathcal{B} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E), \quad \text{with: } \mathcal{X}_2 \approx \mathcal{X}_{\mathcal{K}_E} \quad \text{and} \quad \dim \mathcal{X}_1 \geq \dim \mathcal{X}_{\mathcal{K}_E} \tag{14}$$

From (13) and (14), $\mathcal{X}$, $E^{-1}\mathcal{B}$ and $\mathcal{V}^*$ can be decomposed as follows:

$$\begin{cases} \mathcal{X} = \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \quad ; \quad \mathcal{V}^* = \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \\ \qquad E^{-1}\mathcal{B} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \end{cases} \tag{15}$$

Thus, $\mathcal{B}$, $E\mathcal{V}^*$, and $\mathcal{E}$ are decomposed as (recall (4.a)):

$$\mathcal{B} = E\mathcal{X}_1 \oplus E\mathcal{X}_2 \oplus E\mathcal{X}_3 \ ; \ E\mathcal{V}^* = E\mathcal{X}_{\mathcal{V}^*} \oplus E\mathcal{X}_1 \oplus E\mathcal{X}_2 \quad ; \quad \mathcal{E} = E\mathcal{X}_{\mathcal{V}^*} \oplus E\mathcal{X}_1 \oplus E\mathcal{X}_2 \oplus E\mathcal{X}_3 \oplus E\mathcal{X}_0 \tag{16}$$

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[7] Just pre-multiply (10.a) by $\begin{bmatrix} 1 & -1/\varepsilon & 1 & -1/\varepsilon \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ and do: $\zeta = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \bar{\zeta}$.

[8] In the case of n integration by parts the change of variable is: $z_n = x + \sum_{i=1}^{n} (-1)^i \varepsilon^{i-1} f^{(i-1)}$

Also $\mathcal{U}$ can be decomposed as (recall that $\mathcal{K}_B = \{0\}$):

$$\mathcal{U} = B^{-1}E\mathcal{X}_1 \oplus B^{-1}E\mathcal{X}_2 \oplus B^{-1}E\mathcal{X}_3 \tag{17}$$

Based on the above geometric decompositions, let us define the following natural projections ($i \in \{\mathcal{V}^*, 1, 2, \mathcal{K}_E\}$): [9]

$$\begin{cases} Q_{\mathcal{X}_i} : \mathcal{X} \to \mathcal{X}_i \ ; \quad Q_{VE} : \mathcal{X} \to \mathcal{V}^* \cap \mathcal{K}_E \ ; \quad Q_{E30} : \mathcal{X} \to \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \ ; \quad P_{\mathcal{X}_{\mathcal{V}^*}} : \mathcal{E} \to E\mathcal{X}_{\mathcal{V}^*} \ ; \quad P_1 : \mathcal{E} \to E\mathcal{X}_1 \\ P_{230} : \mathcal{E} \to E\mathcal{X}_2 \oplus E\mathcal{X}_3 \oplus E\mathcal{X}_0 \ ; \quad R_1 : \mathcal{U} \to B^{-1}E\mathcal{X}_1 \ ; \quad R_2 : \mathcal{U} \to B^{-1}E\mathcal{X}_2 \ ; \quad R_3 : \mathcal{U} \to B^{-1}E\mathcal{X}_3 \end{cases} \tag{18}$$

and the following insertion maps:

$$\begin{cases} V_{\mathcal{X}_i} : \mathcal{X}_i \to \mathcal{X} \ ; \quad V_{VE} : \mathcal{V}^* \cap \mathcal{K}_E \to \mathcal{X} \ ; \quad V_{E30} : \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \to \mathcal{X} \ ; \quad W_1 : \to B^{-1}E\mathcal{X}_1 \to \mathcal{U} \\ W_2 : \to B^{-1}E\mathcal{X}_2 \to \mathcal{U} \ ; \quad W_3 : \to B^{-1}E\mathcal{X}_3 \to \mathcal{U} \end{cases} \tag{19}$$

Thanks to the projections (18) and the insertions (19) so defined, we get a more precise implicit description of (1) in the following Lemma, proved in the Appendix:

Lemma 14 *If (11) and (12) hold, then system (1) can be expressed as:*

$$\begin{bmatrix} T_V Q_{\mathcal{X}_{\mathcal{V}^*}} \\ T_1 Q_{\mathcal{X}_1} \\ K_1 Q_{\mathcal{X}_2} + N_1 Q_{E30} \end{bmatrix} \dot{x} = \begin{bmatrix} P_{\mathcal{X}_{\mathcal{V}^*}} A \\ 0 \\ A_1 Q_{E30} \end{bmatrix} x + \begin{bmatrix} 0 \\ L_1 R_1 \\ L_2 R_2 + L_3 R_3 \end{bmatrix} u \ ; \quad y = \begin{bmatrix} \overline{C} Q_{E30} \end{bmatrix} x \tag{20}$$

where the maps $T_V : \mathcal{X}_{\mathcal{V}^*} \leftrightarrow E\mathcal{X}_{\mathcal{V}^*}$ ($T_V = P_{\mathcal{X}_{\mathcal{V}^*}} E V_{\mathcal{X}_{\mathcal{V}^*}}$), $T_1 : \mathcal{X}_1 \leftrightarrow E\mathcal{X}_1$ ($T_1 = P_1 E V_{\mathcal{X}_1}$) and $L_1 : B^{-1}E\mathcal{X}_1 \leftrightarrow E\mathcal{X}_1$ ($L_1 = P_1 B W_1$) are isomorphisms. The maps $L_2 = P_{230} B W_2$ and $L_3 = P_{230} B W_3$ are monic and such that $\mathrm{Im}\, L_2 = E\mathcal{X}_2$ *and* $\mathrm{Im}\, L_3 = E\mathcal{X}_3$. *The maps K_1 and N_1 are defined as follows:*

$$K_1 = P_{E30} E V_{\mathcal{X}_2} \ , \quad \textit{satisfying} \ : \ \mathrm{Im}\, K_1 = E\mathcal{X}_2 \ , \quad \textit{and} \quad \mathcal{K}_{K_1} = \{0\} \tag{21}$$

$$N_1 = P_{E30} E V_{230} \ , \quad \textit{satisfying} \ : \ \mathrm{Im}\, N_1 = E(\mathcal{X}_3 \oplus \mathcal{X}_0) \ , \quad \textit{and} \quad \mathcal{K}_{N_1} = \mathcal{X}_{\mathcal{K}_E} \tag{22}$$

The maps A_1 and $\overline{C}$ are defined as: $A_1 = P_{E30} A V_{230}$ and $\overline{C} = C V_{E30}$.

This Lemma will be our starting point for finding the proposed solution. We also need the following Lemma, proved in the Appendix:

Lemma 15 *Let us define the natural projection:*

$$\overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} : \ \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \to \mathcal{X}_{\mathcal{K}_E} \quad \textit{along} \quad \mathcal{X}_3 \oplus \mathcal{X}_0 \tag{23}$$

Then the map $T_{230} = (N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}})$ is an isomorphism. Moreover $Q_{\mathcal{X}_{\mathcal{K}_E}} = \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} Q_{E30}$.

4 Solution to the *ARISV-Problem*

We propose in this Section the main contribution of the paper. Namely, we give in Theorem 16 the conditions under which the causal approximation of a PD state feedback solves, in an approximated sense, the RISV problem. To avoid unnecessary heavy technicalities, in this Section we are going to assume that (c.f. (14)): [10]

$$\mathcal{X}_2 \approx \mathcal{X}_{\mathcal{K}_E} \approx \mathcal{X}_1 \tag{24}$$

[9] The natural projections are projected along the complementary subspaces defined in (15.a), (16.c) and (17).

[10] If $\dim \mathcal{X}_1 > \dim \mathcal{X}_{\mathcal{K}_E}$ we just have to work with an adequate projection on a subspace $\mathcal{X}_1^{'}$ of $\mathcal{X}_1$ such that $\mathcal{X}_1^{'} \approx \mathcal{X}_{\mathcal{K}_E}$.

4.1 Derivative Feedback

The derivative feedback proposed in (Bonilla and Malabre 2003) for proving Theorem 2 was (recall that $K_1 = P_{E20} E V_{\mathcal{X}_2}$):

$$L_2 R_2 u^* = -K_1 \left(-Q_{\mathcal{X}_2} + T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} \right) \dot{x} + L_2 R_2 \bar{u} \tag{25}$$

where $T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \leftrightarrow \mathcal{X}_2$ is an isomorphism (recall (13.a) and (14)). Applying (25) and the feedback $L_1 R_1 u^* = -(1/\varepsilon) T_1 Q_{\mathcal{X}_1} x$ to (20), we get:[11]

$$\begin{bmatrix} \mathrm{I} & 0 & 0 & 0 & 0 \\ 0 & \boxed{\mathrm{I}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{\mathrm{I}} \end{bmatrix} \dot{x} = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ 0 & \boxed{-(1/\varepsilon)\mathrm{I}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{\bar{A}} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \boxed{\bar{B}} \end{bmatrix} \bar{u}_2 \quad ; \quad y^* = \begin{bmatrix} 0 & 0 & 0 & \boxed{\bar{C}} \end{bmatrix} x \tag{26}$$

where: $\begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \end{bmatrix} = T_V^{(-1)} P_{\mathcal{X}_{\mathcal{V}^*}} A$, $X_6 = T_{230}^{(-1)} K_1 T^{\mathcal{X}_2}_{\mathcal{X}_1}$, $\bar{A} = T_{230}^{(-1)} A_1$, $\bar{B} = T_{230}^{(-1)} L_2$, and $\bar{u}_2 = R_2 \bar{u}$ and $x = (Q_{\mathcal{X}_{\mathcal{V}^*}} x) + (Q_{\mathcal{X}_1} x) + (Q_{\mathcal{X}_2} x) + (Q_{VE} x) + (Q_{E30} x)$. We write y^* instead of y in order to distinguish it from the proportional feedback case. The subsystem $\Sigma^s(\mathrm{I},\ \bar{A},\ \bar{B},\ \bar{C})$ enclosed by the solid line boxes is the state space quotient system mentioned in Theorem 3. Assuming controllability of the common dynamic part (1), of the implicit global descriptions (3), it follows from Theorem 4 that the pair $(\bar{A},\ \bar{B})$ is controllable; thus, we are going to assume that the map $\bar{A}$ has been made Hurwitz by a previous proportional state feedback. Let us note that these structural properties and results are independent on the active internal structure of each particular $\mathcal{K}_{D_i}$.

4.2 Proportional Feedback

Based on the derivative feedback (25), let us propose the following proportional feedback (recall that $T_1 = P_1 E V_{\mathcal{X}_1}$):

$$L_1 R_1 u = -(1/\varepsilon) T_1 (Q_{\mathcal{X}_1} - T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} + T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) x \tag{27}$$

$$L_2 R_2 u = K_1 g + L_2 R_2 \bar{u} \tag{28}$$

$$g = -(1/\varepsilon)(T^{\mathcal{X}_2}_{\mathcal{X}_1} Q_{\mathcal{X}_1} - Q_{\mathcal{X}_2} + T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) x \tag{29}$$

where: $T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \leftrightarrow \mathcal{X}_1$, $T^{\mathcal{X}_1}_{\mathcal{X}_2} : \mathcal{X}_2 \leftrightarrow \mathcal{X}_1$, $T^{\mathcal{X}_2}_{\mathcal{X}_1} : \mathcal{X}_1 \leftrightarrow \mathcal{X}_2$, and $T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X}_{\mathcal{K}_E} \leftrightarrow \mathcal{X}_2$ are isomorphisms (recall (24)). Applying the feedback (27)–(29) to (20), we get:

$$\begin{aligned} & T_V Q_{\mathcal{X}_{\mathcal{V}^*}} \dot{x} = P_{\mathcal{X}_{\mathcal{V}^*}} A x \\ & T_1 Q_{\mathcal{X}_1} \dot{x} = -\tfrac{1}{\varepsilon} T_1 Q_{\mathcal{X}_1} x + \tfrac{1}{\varepsilon} T_1 \left(T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} \right) x \\ & (K_1 Q_{\mathcal{X}_2} + N_1 Q_{E30}) \dot{x} = A_1 Q_{E30} x + K_1 g + L_2 R_2 \bar{u} \\ & 0 = \tfrac{1}{\varepsilon} T^{\mathcal{X}_2}_{\mathcal{X}_1} Q_{\mathcal{X}_1} x - \left(Q_{\mathcal{X}_2} - T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} \right) x + g \end{aligned} \tag{30}$$

Based on the Integration by Parts Section, let us do the following change of variable:

$$\begin{aligned} & Q_i z = Q_i x \quad , \quad \forall\ i \neq \mathcal{X}_1 \quad \text{and} \\ & Q_{\mathcal{X}_1} z = Q_{\mathcal{X}_1} x - \tfrac{1}{\varepsilon} (T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) x + (T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) \dot{x} \end{aligned} \tag{31}$$

[11] Recall that $Q_{\mathcal{X}_{\mathcal{K}_E}} = \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} Q_{E30}$ and that the map $T_{230} = (N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}})$ is an isomorphism; c.f Lemma 15

Then, system (30) takes the following form (recall that $T^{\mathcal{X}_2}_{\mathcal{X}_1} T^{\mathcal{X}_1}_{\mathcal{X}_2} = \mathrm{I}$ and $T^{\mathcal{X}_2}_{\mathcal{X}_1} T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} = T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}}$):

$$\begin{aligned}
&T_V Q_{\mathcal{X}_{\mathcal{V}^*}} \dot{z} = P_{\mathcal{X}_{\mathcal{V}^*}} A z \\
&T_1 Q_{\mathcal{X}_1} \dot{z} = -(1/\varepsilon) T_1 Q_{\mathcal{X}_1} z + T_1 (T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) \ddot{z} \\
&(N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}) Q_{E30} \dot{z} = A_1 Q_{E30} z - K_1 T^{\mathcal{X}_2}_{\mathcal{X}_1} Q_{\mathcal{X}_1} z + L_2 R_2 \bar{u} \\
&0 = T^{\mathcal{X}_2}_{\mathcal{X}_1} Q_{\mathcal{X}_1} z - (Q_{\mathcal{X}_2} - T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) \dot{z} + g
\end{aligned} \tag{32}$$

Let us define $h := (T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} Q_{E30}) \dot{z}$ and let us build the space $\mathcal{Z} = \mathcal{X} \oplus \{g\} \oplus \{h\}$. Then (32) takes the following form:

$$\begin{bmatrix} \mathrm{I} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{\mathrm{I}} & 0 & 0 & 0 & 0 & \boxed{-\mathrm{I}} \\ 0 & 0 & 0 & 0 & \boxed{\mathrm{I}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_7^{(-1)} & 0 & -Y_1 & 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & 0 & 0 \\ 0 & \boxed{-(1/\varepsilon)\mathrm{I}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{-X_6} & 0 & 0 & \boxed{\bar{A}} & 0 & 0 \\ 0 & X_7 & 0 & 0 & 0 & \mathrm{I} & -X_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{I} \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \boxed{\bar{B}} \\ 0 \\ 0 \end{bmatrix} \bar{u}_2$$
$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & \boxed{\bar{C}} & 0 & 0 \end{bmatrix} z \tag{33}$$

where: $\begin{bmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \end{bmatrix} = T_V^{(-1)} P_{\mathcal{X}_{\mathcal{V}^*}} A$, $X_6 = T_{230}^{(-1)} K_1 T^{\mathcal{X}_2}_{\mathcal{X}_1}$, $X_7 = T^{\mathcal{X}_2}_{\mathcal{X}_1}$, $Y_1 = T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}$, and $\bar{u}_2 = R_2 \bar{u}$ and $z = (Q_{\mathcal{X}_{\mathcal{V}^*}} x) + (Q_{\mathcal{X}_1} x) + (Q_{\mathcal{X}_2} x) + (Q_{VE} x) + (Q_{E30} x) + (g) + (h)$. Let us note that the subsystem $\Sigma^s(\mathrm{I},\ \bar{A},\ \bar{B},\ \bar{C})$ enclosed by the solid line boxes is the same as the one obtained in (26); but now it is perturbed by the fast exponentially stable subsystem of steady state gain ε, $\Sigma^f(\mathrm{I},\ -(1/\varepsilon)\mathrm{I},\ -\mathrm{I},\ -X_6)$, enclosed by the dash line boxes. Although the steady state gain can be very small (but never zero), it is necessary that the internal structure variation which is rendered almost unobservable be exponentially stable. So we can only accept internal structure variations, namely $\mathcal{K}_{D_i}$, belonging to the following set (recall that $\bar{A}$ is already Hurwitz and $\varepsilon > 0$):

$$\Gamma_F(D_i) = \left\{ \mathcal{K}_{D_i} \;\middle|\; \det \begin{bmatrix} E - (A + BF) \\ -D_i \end{bmatrix} \text{ is Hurwitz} \right\} \tag{34}$$

If this set is empty there then exists no solution to the *ARISV-Problem*.

Theorem 16 *Under the same conditions as in Theorems 2, 3 and 4, but with (11) in place of the second in (4), there exists a proportional feedback solving the* ARISV-Problem *for all* $\mathcal{K}_{D_i} \in \Gamma_F(D_i)$.

Proof:

From (26) and (33) we get ($Q_{E30} z_0$, $Q_{\mathcal{X}_1} z_0$ and $Q_{E30} x_0$ are initial conditions):

$$y - y^* = \bar{C} \mathrm{e}^{\bar{A}t} Q_{E30}(z_0 - x_0) - \bar{C} \int_0^t \mathrm{e}^{\bar{A}(t-\tau)} X_6 \left(\mathrm{e}^{-\tau/\varepsilon} Q_{\mathcal{X}_1} z_0 + \int_0^\tau \mathrm{e}^{-(\tau-\sigma)/\varepsilon} \dot{h}(\sigma) d\sigma \right) d\tau$$

Assuming $\mathcal{K}_{D_i} \in \Gamma_F(D_i)$, the closed loop systems (26) and (33) are exponentially stable, which implies: $\dot{h} \in L_\infty$ and the existence of $c, a \in \mathbb{R}^+$ such that $\left| \mathrm{e}^{\bar{A}t} \right| \leq c \mathrm{e}^{-at}$. There then exist $k_1, k_2, k_3 \in \mathbb{R}^+$ such that:

$$|y - y^*| \leq k_1 \mathrm{e}^{-at} + k_2 \|\dot{h}\|_\infty \varepsilon + k_3 \varepsilon \mathrm{e}^{-t/\varepsilon}$$

Therefore, given a $\delta \in \mathbb{R}^+$ there exist $t^*, \varepsilon^* \in \mathbb{R}^+$ s.t.

$$|y - y^*| \leq \delta \qquad \forall\ t \geq t^* \quad \& \quad \varepsilon \in (0,\ \varepsilon^*)$$

5 Illustrative Example

Let us come back to the example (2). For this system, we have: $\mathcal{V}^* = \{e_1,\ e_2\}$, $\mathcal{K}_E = \{e_3\}$ and $E^{-1}\mathcal{B} = \{e_2,\ e_3\}$; and thus: $\mathcal{X}_{\mathcal{V}^*} = \{e_1\}$, $\mathcal{X}_2 = \{e_2\}$, $\mathcal{X}_{\mathcal{K}_E} = \{e_3\}$, $\mathcal{X}_1 = \mathcal{V}^* \cap \mathcal{K}_E = \mathcal{X}_3 = \mathcal{X}_0 = \{0\}$. And thus the geometric condition (11) is not satisfied. In order to get closer to such a condition, let us add an external integrator to system (2), namely: [12]

$$\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&1&0\\ \hline 0&0&0&0\end{array}\right]\dot{x} = \left[\begin{array}{cccc}0&0&1&-1\\0&0&0&0\\0&0&0&-1\\ \hline \alpha&0&\beta&1\end{array}\right]x + \left[\begin{array}{cc}0&0\\1&0\\0&1\\ \hline 0&0\end{array}\right]u \ ; \ \ y = \begin{bmatrix}0&0&0&1\end{bmatrix}x \tag{35}$$

For this system, we have: $\mathcal{V}^* = \{e_1,\ e_2,\ e_3\}$, $\mathcal{K}_E = \{e_4\}$ and $E^{-1}\mathcal{B} = \{e_2,\ e_3, e_4\}$; and thus: $\mathcal{X}_{\mathcal{V}^*} = \{e_1\}$, $\mathcal{X}_1 = \{e_2\}$, $\mathcal{X}_2 = \{e_3\}$, $\mathcal{X}_{\mathcal{K}_E} = \{e_4\}$, $\mathcal{X}_1 = \mathcal{V}^* \cap \mathcal{K}_E = \mathcal{X}_3 = \mathcal{X}_0 = \{0\}$. And now the geometric condition (11) is satisfied. Also from (35) we get: $L_1 = L_2 = T_V = T_1 = K_1 = T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} = T^{\mathcal{X}_1}_{\mathcal{X}_2} = T^{\mathcal{X}_1}_{X_{\mathcal{K}_E}} = T^{\mathcal{X}_2}_{\mathcal{X}_1} = \bar{Q}_{\mathcal{X}_{\mathcal{K}_E}} = 1$, $N_1 = 0$, $A_1 = -1$ $R_1 = [1\ 0]$, $R_2 = [0\ 1]$, $Q_{\mathcal{X}_{\mathcal{V}^*}} = [1\ 0\ 0\ 0]$, $Q_{\mathcal{X}_1} = [0\ 1\ 0\ 0]$, $Q_{\mathcal{X}_2} = [0\ 0\ 1\ 0]$, $Q_{\mathcal{X}_{\mathcal{K}_E}} = Q_{E30} = [0\ 0\ 0\ 1]$, $P_{\mathcal{X}_{\mathcal{V}^*}} = [1\ 0\ 0]$.

P.D. Feedback: From (25) we get the derivative feedback: $L_2R_2u^* = -[\,0\ 0\ -1\ 1\,]\dot{x} + [\,0\ 1\,]\bar{u}$; the proportional feedback is: $L_1R_1u^* = -(1/\varepsilon)[\,0\ 1\ 0\ 0\,]x$. The closed loop system is:

$$\left[\begin{array}{cccc}1&0&0&0\\0&1&0&0\\0&0&0&\boxed{1}\\ \hline 0&0&0&0\end{array}\right]\dot{x} = \left[\begin{array}{cccc}0&0&1&-1\\0&-(1/\varepsilon)&0&0\\0&0&0&\boxed{-1}\\ \hline \alpha&0&\beta&1\end{array}\right]x + \left[\begin{array}{c}0\\0\\ \boxed{1}\\ \hline 0\end{array}\right]\bar{u}_2 \ ; \ \ y^* = \begin{bmatrix}0&0&0&\boxed{1}\end{bmatrix}x \tag{36}$$

The characteristic polynomial of the closed loop system (36) is: $\det[\lambda((\mathbb{E}-\mathbb{B}F_d)-(\mathbb{A}_i+\mathbb{B}F_p)] = (\lambda+1)(\lambda+1/\varepsilon)(\beta\lambda+\alpha)$. And then ($\overline{D} = \begin{bmatrix}\alpha & 0 & \beta & 1\end{bmatrix}$):

$$\Gamma_{(F_p,F_d)}(\overline{D}) = \{(\alpha,\ \beta)\ \mid \alpha \cdot \beta > 0 \text{ or } (\beta = 0\ \&\ \alpha \neq 0)\} \tag{37}$$

P. Feedback: From (27), (28) and (29) we get: $L_2R_2u = g + [0\ 1]\bar{u}$, $g = -\frac{1}{\varepsilon}[0\ 1\ -1\ 1]x$, and $L_1R_1u = -\frac{1}{\varepsilon}[0\ 1\ -1\ 1]x$. The closed loop system is (c.f. (33)):

$$\left[\begin{array}{cccccc}1&0&0&0&0&0\\0&\boxed{1}&0&0&0&\boxed{-1}\\0&0&0&\boxed{1}&0&0\\0&0&0&0&0&0\\0&0&1&-1&0&0\\ \hline 0&0&0&0&0&0\end{array}\right]\dot{z} = \left[\begin{array}{cccccc}0&0&1&-1&0&0\\0&\boxed{-(1/\varepsilon)}&0&0&0&0\\0&\boxed{-1}&0&\boxed{-1}&0&0\\0&1&0&0&1&-1\\0&0&0&0&0&1\\ \hline \alpha&0&\beta&1&0&0\end{array}\right]z + \left[\begin{array}{c}0\\0\\ \boxed{1}\\0\\0\\ \hline 0\end{array}\right]\bar{u}_2 \ ; \ \ y = \begin{bmatrix}0&0&0&\boxed{1}&0&0\end{bmatrix}z \tag{38}$$

The characteristic polynomial of the closed loop system (38) is: $\det[\lambda(\mathbb{E} - (\mathbb{A}_i + \mathbb{B}F_p)] = (\lambda + 1)(\lambda + 1/\varepsilon)(\beta\lambda + \alpha) - (\beta + 1)\lambda^3$. And then ($\widetilde{D} = \begin{bmatrix}\alpha & 0 & \beta & 1 & 0 & 0\end{bmatrix}$):

$$\Gamma_F(\widetilde{D}) = \left\{(\alpha,\beta)\,\middle|\,\alpha < \min\left\{-\left(\tfrac{1}{\varepsilon}+1\right)\beta, -\left(\tfrac{\beta}{1+\varepsilon}\right), 0\right\}\right\} \tag{39}$$

[12] (i) The second row and second column correspond to the added external integrator, (ii) we have applied a previous proportional feedback to get (12) and (iii) in the bottom we have added the algebraic equation $0 = D_i x$ (c.f. (3)).

From this illustrative example, we can remark that:

(1) With the proportional and derivative feedback we obtain a controllable state space description decoupled from the internal variable structure (see in (36) the subsystem enclosed by the solid line boxes).
(2) If, in addition, we want internal stability, the variation of structure (in the P.D. feedback case) must take place in the region (37). Let us note that the four cases considered at the beginning of the introductory Section, $(\alpha, \beta) \in \{(-1,-1),\ (-1,0),\ (-1,-5),\ (1,1)\}$, are inside of $\Gamma_{(F_p,F_d)}(D_i)$.
(3) For the proportional feedback the *ARISV-Problem* has a solution if the variation of the internal structure takes place in the region (39). For this case, only the structures $(\alpha, \beta) \in \{(-1,-1),\ (-1,0),\ (-1,-5)\}$ can be considered; the case $(\alpha, \beta) = (1,1)$ cannot be kept.
(4) Since the subsystem enclosed by solid lines in systems (36) and (38) are the same, the structural properties are also the same.
(5) The region of the possible variations preserving the internal stability is reduced when the P.D. feedback is approximated. For the P.D. case this region is the first and third orthant; for the P. case the region is the third orthant; the case $(\alpha, \beta) = (1,1)$ cannot be kept.
(6) If the geometric condition (4.b) is satisfied but not (11), we only need to add some integrators until satisfying (11).

6 Concluding Remarks

In this paper we have given a suitable approximation of the Proportional and Derivative Feedback proposed in (Bonilla and Malabre 2003) for rejecting the internal structure variations in implicit linear systems. The proposed Proportional Feedback solves Problem 5, which is an almost rejection version. The synthesized control law is very close to the exact PD one.

We show in Theorem 16 that there exists a finite time t^* and an upper positive bound ε^* for reaching the desired size gap δ. To solve the almost version we have changed the geometric condition (4.b) into the less restrictive one (11). We have also shown by means of one illustrative example that, in case that the geometric condition (11) is not satisfied (but (4.b) is), it is enough to add to the implicit system (1) some chains of integrators in order to get this condition fulfilled. The price to be paid for this approximation lies in the reduction of the set (34) for the accepted internal variations (c.f. (37) with (39)).

As an interesting by-product, a system theoretical interpretation of the classical process of "integration by parts" is given and shown to be equivalent to some particular generalized changes of bases.

A Appendix

A.1 Proof of Lemma 14

This proof is done in 8 steps:

(1) From (15) and (17), any $x \in \mathcal{X}$ and any $u \in \mathcal{U}$ can be expressed as: $x = (V_{\mathcal{X}_{\mathcal{V}^*}} Q_{\mathcal{X}_{\mathcal{V}^*}} + V_{\mathcal{X}_1} Q_{\mathcal{X}_1} + V_{\mathcal{X}_2} Q_{\mathcal{X}_2} + V_{VE} Q_{VE} + V_{E30} Q_{E30})x$ and $u = (W_1 R_1 + W_2 R_2 + W_3 R_3)u$.
(2) From (16.c) and (13.a) we get: $P_{\mathcal{X}_{\mathcal{V}^*}} E V_i = 0$, for $i \in \{\mathcal{X}_1, \mathcal{X}_2, VE, E30\}$; $P_1 E V_i = 0$, for $i \in \{\mathcal{X}_{\mathcal{V}^*}, \mathcal{X}_2, VE, E30\}$; $P_{230} E V_i = 0$, for $i \in \{\mathcal{X}_{\mathcal{V}^*}, \mathcal{X}_1, VE\}$.
(3) From (16.c), (12) and (16.b) we get: $P_1 A = 0$ and $P_{230} A V_i = 0$ for $i \in \{\mathcal{X}_{\mathcal{V}^*}, \mathcal{X}_1, \mathcal{X}_2, VE\}$.
(4) From (16.c), (16.a) and (17.c) we get: $P_{\mathcal{X}_{\mathcal{V}^*}} B = 0$; $P_1 B W_i = 0$, for $i \in \{2,3\}$; $P_{230} B W_1 = 0$.
(5) From (15.a) and (15.b) we get: $C V_i = 0$, for $i \in \{\mathcal{X}_{\mathcal{V}^*}, \mathcal{X}_1, \mathcal{X}_2, VE\}$.
(6) From (18), (16.c), (16.a), (17), (13.a) and (15.a), we get: $\operatorname{Im} K_1 = P_{230} E \mathcal{X}_2 = E \mathcal{X}_2$ and $\mathcal{K}_{K_1} = \mathcal{X}_2 \cap E^{-1} \mathcal{K}_{P_{230}} = \mathcal{X}_2 \cap E^{-1} E(\mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1) = \mathcal{X}_2 \cap (\mathcal{K}_E \oplus \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1) = \{0\}$.
(7) From (19), (18), (16.c), (13.a) and (15.a), we get: $\operatorname{Im} N_1 = P_{230} E(\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0) = E(\mathcal{X}_3 \oplus \mathcal{X}_0)$ and $\mathcal{K}_{N_1} = (\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0) \cap E^{-1} \mathcal{K}_{P_{230}} = (\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0) \cap E^{-1} E(\mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1) = (\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0) \cap (\mathcal{K}_E \oplus \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1) = \mathcal{X}_{\mathcal{K}_E}$.
(8) From (19), (17) and (18), we get: $\operatorname{Im} L_1 = P_1 B B^{-1} E \mathcal{X}_1 = P_1 E \mathcal{X}_1 = E \mathcal{X}_1$ and $\mathcal{K}_{L_1} = B^{-1} E \mathcal{X}_1 \cap B^{-1} \mathcal{K}_{P_1} = B^{-1}(E \mathcal{X}_1 \cap \mathcal{K}_{P_1}) = \mathcal{K}_B = \{0\}$. And for $i \in \{2,3\}$: $\operatorname{Im} L_i = P_{230} B B^{-1} E \mathcal{X}_i = P_{230} E \mathcal{X}_i = E \mathcal{X}_i$ and $\mathcal{K}_{L_i} = B^{-1} E \mathcal{X}_i \cap B^{-1} \mathcal{K}_{P_{230}} = B^{-1}(E \mathcal{X}_i \cap \mathcal{K}_{P_{230}}) = \mathcal{K}_B = \{0\}$.

A.2 Proof of Lemma 15

This proof is done in 4 steps:

(1) From (23), (21), (22) and (18), we get: $\left(N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right)$: $\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \to E(\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_0)$.

(2) The domain, $\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0$, and the co-domain, $E(\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_0)$, are isomorphic. Indeed, from (24), (13.a) and (15.a), we get: $\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \approx \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \approx E(\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_0)$.

(3) The map $\left(N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right)$ is monic. This item is proved in 3 steps:

(*i*) Let us first note that (21), (22) and (23) imply that: Im $N_1 = E(\mathcal{X}_3 \oplus \mathcal{X}_0)$ and Im $\left(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) = K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \mathcal{X}_{\mathcal{K}_E} = K_1 \mathcal{X}_2 = \text{Im } K_1 = E\mathcal{X}_2$, and then:

$$E(\mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_0) = \text{Im } N_1 \oplus \text{Im } \left(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) \tag{A.1}$$

(*ii*) Let us next note that (22), (21) and (23) imply that: $\mathcal{K}_{N_1} = \mathcal{X}_{\mathcal{K}_E}$ and Ker $\left(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) = \overline{Q}^{-1}_{\mathcal{X}_{\mathcal{K}_E}} T^{\mathcal{X}_{\mathcal{K}_E}}_{\mathcal{X}_2} \mathcal{K}_{K_1} = \text{Ker } \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} = \mathcal{X}_3 \oplus \mathcal{X}_0$, and then:

$$\mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 = \mathcal{K}_{N_1} \oplus \text{Ker } (K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}) \tag{A.2}$$

(*iii*) Let us now take a $x \in \text{Ker } (N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}})$, i.e. $(N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}})x = 0$. In view of (A.2) there exist unique $x_1 \in \mathcal{K}_{N_1}$ and $x_2 \in \text{Ker } \left(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right)$ such that $x = x_1 + x_2$. Then: $N_1 x_2 = -K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}} x_1$. From this last equality and from (A.1) we have: $N_1 x_2 \in \text{Im } N_1 \cap \text{Im } (K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}) = \{0\}$, which together with (A.2) imply $x_2 \in \mathcal{K}_{N_1} \cap \text{Ker } \left(K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) = \{0\}$. As $x_2 = \{0\}$, we get from (A.2): $x_1 \in \text{Ker } (K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}) \cap \mathcal{K}_{N_1} = \{0\}$. Therefore: Ker $\left(N_1 + K_1 T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} \overline{Q}_{\mathcal{X}_{\mathcal{K}_E}}\right) = \{0\}$.

(4) T_{230} is an isomorphism because it is a square monic map.

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February 1, 2004

Professor Tempo
Automatica Editor
System and Control Theory

Dear Professor Tempo,

Recently we have found a typo in the paper (with reference number: 3641) that we have submitted in November 21, 2003 to Automatica.

The last equalities in equations (30) and (31) have to be modified as follows:

IT IS WRITTEN

$$\begin{array}{l} T_V Q_{\mathcal{X}_{\mathcal{V}*}} \dot{x} = P_{\mathcal{X}_{\mathcal{V}*}} A x \\ T_1 Q_{\mathcal{X}_1} \dot{x} = -\frac{1}{\varepsilon} T_1 Q_{\mathcal{X}_1} x + \frac{1}{\varepsilon} T_1 \left(T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} \right) x \\ \left(K_1 Q_{\mathcal{X}_2} + N_1 Q_{E30} \right) \dot{x} = A_1 Q_{E30} x + K_1 g + L_2 R_2 \bar{u} \\ 0 = \frac{1}{\varepsilon} T^{\mathcal{X}_2}_{\mathcal{X}_1} Q_{\mathcal{X}_1} x - \left(Q_{\mathcal{X}_2} - T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} \right) x + g \end{array} \quad (30)$$

$$\begin{array}{l} Q_i z = Q_i x \quad , \quad \forall\, i \neq \mathcal{X}_1 \quad \text{and} \\ Q_{\mathcal{X}_1} z = Q_{\mathcal{X}_1} x - \frac{1}{\varepsilon} (T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) x + (T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) \dot{x} \end{array} \quad (31)$$

IT SHOULD BE CHANGED TO

$$\begin{array}{l} T_V Q_{\mathcal{X}_{\mathcal{V}*}} \dot{x} = P_{\mathcal{X}_{\mathcal{V}*}} A x \\ T_1 Q_{\mathcal{X}_1} \dot{x} = -\frac{1}{\varepsilon} T_1 Q_{\mathcal{X}_1} x + \frac{1}{\varepsilon} T_1 \left(T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} \right) x \\ \left(K_1 Q_{\mathcal{X}_2} + N_1 Q_{E30} \right) \dot{x} = A_1 Q_{E30} x + K_1 g + L_2 R_2 \bar{u} \\ 0 = \frac{1}{\varepsilon} T^{\mathcal{X}_2}_{\mathcal{X}_1} Q_{\mathcal{X}_1} x - \frac{1}{\varepsilon} \left(Q_{\mathcal{X}_2} - T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}} \right) x + g \end{array} \quad (30)$$

$$\begin{array}{l} Q_i z = Q_i x \quad , \quad \forall\, i \neq \mathcal{X}_1 \quad \text{and} \\ Q_{\mathcal{X}_1} z = \frac{1}{\varepsilon} Q_{\mathcal{X}_1} x - \frac{1}{\varepsilon} (T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) x + (T^{\mathcal{X}_1}_{\mathcal{X}_2} Q_{\mathcal{X}_2} - T^{\mathcal{X}_1}_{\mathcal{X}_{\mathcal{K}_E}} Q_{\mathcal{X}_{\mathcal{K}_E}}) \dot{x} \end{array} \quad (31)$$

These are just typos, with no consequence on the rest of the development and results.

We apologize for our lack of careful reading.

Sincerely yours,

Moisés Bonilla Estrada

Apéndice E

Optimisation de Systèmes Implicites Rectangulaires (JDA-Valenciennes, France)

Optimisation de Systèmes Implicites Rectangulaires

J. Pacheco M.*, M. Bonilla E.†, M. Malabre‡

Résumé

We present here some recent results concerning the control of linear descriptor systems. We consider non square internal descriptions (the number of dynamic equations is less than the number of descriptor variables) with additional algebraic constraints which can switch within a finite dimensional set. We show how to design suitable approximations of PD feedbacks which force this set of models to behave as a fixed strictly proper system. We also give a system theoretic interpretation of the process of *integration by parts*, which is presently used to design an alternative proportional feedback approximation of the previous PD laws. Several examples are included for illustration.

Thème clé principal : Systèmes continus, commande.

1 Introduction

Le cadre du présent travail est une extension des systèmes *classiques* du type strictement propres (décrits par des modèles du type : $\dot{x} = Ax + Bu$, $y = Cx$, à savoir les systèmes généralisés, aussi connus sous le nom de systèmes différentiels et algébriques, systèmes singuliers ou *systèmes implicites.* Ces systèmes sont décrits par des modèles du type : $E\dot{x} = Ax + Bu$, $y = Cx$. Cette classe de modèles permet de décrire des comportements beaucoup plus nombreux et plus riches que dans le cas classique strictement propre. On peut par exemple mentionner (et la liste n'est pas exhaustive) : des systèmes ayant des comportements impulsionnels (liés par exemple à des dérivations) ; des systèmes controlés par des lois de commande proportionnelles et dérivées ; des systèmes avec des contraintes algébriques sur l'état ; des systèmes avec des restrictions sur les commandes ; etc ...

Dans ce contexte généralisé, on s'intéresse tout particulièrement à des modélés implicites pour lesquels le nombre des variables internes est supérieur au nombre d'équations d'état. Ces *systèmes implicites rectangulaires* ont ceci de particulier que pour une condition initiale donnée, et pour une loi de commande donnée, la solution en termes de trajectoire d'état n'est pas unique. Ces systèmes possèdent un degré de liberté interne (en quelque sorte à l'origine du comportement non *unique*). On peut ainsi décrire des *systèmes à structure variable* (par exemple dont le comportement est également fixé par la position de certains commutateurs). Plus précisément, on décrit ici une famille de systèmes dont les changements potentiels proviennent d'une équation de contrainte algébrique, du type $D_i x = 0$ en superposition avec un ensemble *constant* d'équations dynamiques du type $E\dot{x} = Ax+Bu$. Cette famille de systèmes peut être décrite par un modèle implicite unique designé sous le nom de *système global* :

$$\underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix}}_{\mathbb{E}}\dot{x} = \underbrace{\begin{bmatrix} A \\ D_i \end{bmatrix}}_{\mathbb{A}_i}x + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\mathbb{B}}u; \quad y = Cx \tag{1}$$

où $\mathbb{E} : \mathcal{X} \rightarrow \underline{\mathcal{X}}_{g_i}, \mathbb{A}_i = \mathcal{X} \rightarrow \underline{\mathcal{X}}_{g_i}$, $\mathbb{B} : \mathcal{U} \rightarrow \underline{\mathcal{X}}_{g_i}$ et $C : \mathcal{X} \rightarrow \mathcal{Y}$ sont opérateurs linéaires avec $\underline{\mathcal{X}}_{g_i} = \underline{\mathcal{X}} \oplus \underline{\mathcal{X}}_i$. Cette représentation globale est très importante, comme argumenté dans [5] ; notamment il y a été montré que sous certaines conditions (assez peu restrictives, principalement de type commandabilité) il est possible de commander cette famille de systèmes, à l'aide de lois de commande du type Proportionnel et Dérivé, de manière à obtenir une réponse bouclée unique, à la dynamique imposée, quelle que soit la valeur des paramètres liés à la partie algébrique des équations (liés à $D_i x = 0$) dans l'ensemble des modèles possibles ainsi décrits.

A titre d'exemple illustratif (mais nous ne pouvons le détailler ici) une classe particulièrement intéressante dans la famille de tels modèles est celle des *systèmes en escaliers* (voir [4]) qui décrit les changements liés à la position de plusieurs commutateurs. Une application récente de ces systèmes en escalier a permis de décrire par un modèle linéaire à commutations (avec trois composantes linéaires) le processus de croissance végétale décrit par la fonction (non linéaire) dite fonction logistique (et classique dans ce domaine).

Nous allons dans ce qui suit mettre l'accent sur la richesse de description de ces modèles implicites à structure variable (Section 2) et présenter quelques contributions visant à controller ces systèmes. La Section 3 concerne un aspect pratique lié à la mise en œuvre de compensateurs impropres déduits d'une analyse simplifiée par le cadre implicite, à savoir leurs approximations par des systèmes propres à grand gain. Ces résultats sont tout particulièrement bien adaptés dans le contexte des systèmes classiques (strictement propres) lorsque l'on utilise des lois de commande généralisées (Proportionnelles et Dérivées). La section suivante présente une partie de résultats récents sur une alternative consistant à rechercher d'emblée une loi de commande Proportionnelle pour une version approximative du problème de départ. Cette démarche nous semble particulièrement adaptée à la commande des systèmes à structure variable. Dans la Section 4, nous mettons l'accent sur un ingrédient de la solution qui nous semble intéressant : une interprétation système du procédé mathématique d'intégration par parties formulée en termes de changements de bases généralisés (incluant des dérivées des variables). La présentation se termine par quelques perspectives à la suite des résultats ainsi obtenus.

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2 Exemple illustratif

De manière à illustrer plus précisément notre contexte de travail, considérons un système décrit par les équations dynamiques :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad (2)$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

et contraint par la relation suivante :

$$\begin{bmatrix} \alpha & \beta & 1 \end{bmatrix} x = 0 \qquad (3)$$

Suivant les valeurs des paramètres α et β, ce système peut faire avoir des comportements très différents.

Par exemple, si $(\alpha, \beta) = (-1, -1)$, i.e., $x_3 = x_1 + x_2$, alors, la description entrée-sortie est $\dot{y} + y = u$. Si $(\alpha, \beta) = (-1, 0)$, i.e., $x_3 = x_1$, alors, la description entrée-sortie est $\ddot{y} + \dot{y} = u$. Si $(\alpha, \beta) = (-1, -5)$, i.e., $x_3 = x_1 + 5x_2$, alors, la description entrée-sortie est $\ddot{y} + 6\dot{y} + 5y = 5\dot{u} + u$. Finalement, si $(\alpha, \beta) = (1, 1)$, i.e., $x_3 = -(x_1 + x_2)$, alors, la description entrée-sortie est $\dot{y} - 3y = -u$.

Une réalisation implicite globale de (2) et (3) est :

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \dot{x} = \left[\begin{array}{ccc} 0 & 1 & -1 \\ 1 & 0 & -1 \\ \hline \alpha & \beta & 1 \end{array}\right] x + \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right] u$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

En appliquant la procédure proposée dans [6], on obtient une loi de commande Proportionnelle-Dérivée qui rend inobservable sur la sortie le changement de structure provoqué par les variations des paramètres (α, β) :$F^*_{d_1} = \begin{bmatrix} 0 & -1 & -1 \end{bmatrix}$ et $F^*_{p_1} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$. On peut ensuite placer le pôle de la dynamique externe, en superposant un second retour proportionnel : $F^*_{p_2} = \begin{bmatrix} 0 & 0 & (1 - 1/\tau_o) \end{bmatrix}$. Ainsi, la loi de commande : $u = F^*_{d_1}\dot{x} + \left(F^*_{p_1} + F^*_{p_2}\right)x + R/\tau_o$ conduit au système bouclé :

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -1/\tau_o \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/\tau_o \end{bmatrix} R$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$$

De cette manière, le degré de liberté interne (dû aux changements possibles des valeurs des paramètres) est rendu inobservable. En d'autres termes, la *variation interne de structure* n'est plus présente sur la sortie. Le système en boucle fermée se comporte comme $\tau_o\dot{y} + y = R$, quelle que soit la restriction active : $0 = \begin{bmatrix} \alpha & \beta & 1 \end{bmatrix} x$, *.i.e.* quelles que soient les valeurs des paramètres α et β.

La mise en œuvre de telles lois de commande présuppose que les variables internes x, et sa dérivée $\dot{x}$, sont disponibles, ce qui n'est généralement pas le cas. Pour contourner cette difficulté, on peut utiliser les résultats proposés par [1] et où est développé un *detecteur de structure* s'appuyant sur un algorithme adaptatif à base de gradient normalisé, et dont la finalité est de déterminer quelle est la structure *active* parmi celles qui ont été décrites dans le modèle : c'est à dire, sur l'exemple précédent, de préciser si le comportement est-il celui du premier ordre, du second ordre ou du second ordre avec un zéro dominant.

Une seconde difficulté concerne le caractère impropre de la loi de commande retenue (action dérivée). Il faut alors passer par une phase d'approximation. C'est l'objet de la section suivante.

3 Approximation exponentiellement propre de lois de commande impropres

Nous allons présenter ici les résultats principaux détaillés dans l'article [12].

De manière à simplifier quelques expressions ultérieures, nous adoptons les notations suivantes.

Notation 1 *$\underline{\chi}^i_k$ désigne un vecteur kx1 dont la i-ème composante est 1 et les autres sont zéro. $L\{v\}$ désigne une matrice de Toeplitz triangulaire basse dont la première colonne est constituée du vecteur v. $D\{X_1, ..., X_k\}$ désigne une matrice diagonale par blocs dont les blocs diagonaux sont les matrices $X_1, ..., X_k$.*

Exemple 1

$$\underline{\chi}^1_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, L\left\{\begin{bmatrix} a \\ b \end{bmatrix}\right\} = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}, D\{X_1, X_2\} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

3.1 Introduction

Nous sommes intéressés par la résolution du problème suivant :

Problème 1 *Etant donné le compensateur impropre, $\Sigma^c : \mathcal{U} \to \mathcal{Y}$, dont la réalisation est donnée par :*

$$N\dot{\omega} = \omega + \Gamma u; \quad y^* = \Delta\omega \qquad (4)$$

avec $N : \mathcal{W} \to \mathcal{W}$, $\Gamma : \mathcal{U} \to \mathcal{W}$, $\Delta : \mathcal{W} \to \mathcal{Y}$ des opérateurs linéaires et où N est nilpotente, on souhaite concevoir un filtre strictement propre, $\Sigma^f : \mathcal{Y} \to \mathcal{Y}$, de réalisation $\dot{z} = A(\varepsilon)z + B(\varepsilon)y^*$ et $y = Cz$ tel que :

1. *$\lim_{\varepsilon \to 0} \|y(t) - y^*(t)\| \leq e^{-\beta t}\|z(0)\|$, avec $\beta > 0$ et Σ^f internement stable pour tout $\varepsilon > 0$*
2. *la matrice de transfert du système global, $\Sigma^f \circ \Sigma^c$, soit propre.*

On peut, sans perte de généralité, supposer que (4) est complètement observable, et par conséquent, sa forme canonique de Kronecker ne possède que des blocs d'indices minimaux par les lignes (voir [9], [10] et [11]). Nous allons donc supposer que (4) est sous la forme canonique de Kronecker suivante :

$$\begin{cases} N = D\{N_1, N_2, ..., N_n\}; \Delta = D\left\{\Delta_1^T, \Delta_2^T, ..., \Delta_n^T\right\} \\ N_i = L\left\{\underline{\chi}^2_{(k_i+1)}\right\}; \Delta_i = \underline{\chi}^{(k_i+1)}_{(k_i+1)}, \text{avec } i = 1, ..., n \end{cases}$$

A noter que les entiers $k_i \geq 0$, $i = 1, ..., n$ représentent les ordres des pôles à l'infini du compensateur (4).

Nous pouvons dès à présent résoudre le premier point du problème précédent, *i.e.*, obtenir une réalisation strictement propre et internement stable, Σ^f, et dont le comportement externe approxime de manière exponentielle le compensateur impropre Σ^c.

3.2 Approximation exponentielle

Pour réaliser l'approximation exponentielle (point 1 du Problème 1), nous utilisons le Lemme suivant :

Lemme 1 *Soit le système, $\Sigma^f : \mathcal{Y} \to \mathcal{Y}$,*

$$\begin{cases} \dot{\bar{x}} = A_\beta \bar{x} - \varepsilon^{k+1} y \\ \varepsilon\dot{\hat{x}} = A_o\hat{x} + B_o(\bar{x} + y^*) \\ y = C_o\hat{x} \end{cases} \qquad (5)$$

avec $\hat{x} \in \widehat{\mathcal{X}}$; $\bar{x}, y, y^* \in \mathcal{Y}$, $\varepsilon > 0$ tel que :

1. *A_β et A_o sont Hurwitz-stables,*

2. $\mathcal{L}^{-1}\left\{(\varepsilon sI-A_o)^{-1}\right\} = (1/\varepsilon)\overline{A}_o(t,\varepsilon)e^{-t/\varepsilon}$, *où* $\mathcal{L}^{-1}\{\cdot\}$ *désigne la Transformée Inverse de Laplace,*
3. *Les éléments de* $\overline{A}_o(t,\varepsilon)$ *sont des polynômes en la variable* t/ε*,*
4. $\int_0^\infty C_o\overline{A}_o(\lambda)e^{-\lambda}B_o d\lambda = I$, *où* $\overline{A}_o(\lambda) = \overline{A}_o(\varepsilon\lambda,\varepsilon)$.

Alors $\lim_{\varepsilon\to 0}[y-y^*] = e^{A_\beta t}\bar{x}(0)\,;t > 0$ *et* $\det\begin{bmatrix}(\varepsilon sI-A_o) & -B_o\\ \varepsilon^{k+1}C_o & (sI-A_\beta)\end{bmatrix}$ *est Hurwitz-stable.*

Il faut insister à ce niveau sur le fait qu'il est *toujours* possible de choisir A_o, B_o et C_o de manière à satisfaire les hypothèses du lemme précédent.

Nous allons maintenant rapidement examiner, en termes géométriques, la propreté externe des systèmes implicites.

3.3 Propreté

3.3.1 Propreté Interne

Définition 1 (Voir [5]) *Le système implicite* $\mathbb{E}\dot{x} = \mathbb{A}x + \mathbb{B}u$, *où* $\mathbb{E},\mathbb{A},\mathbb{B}$ *sont définis comme (1), est internement propre si et seulement si le faisceau de ses pôles* $[\lambda\mathbb{E}-\mathbb{A}]$ *est carré, régulier (i.e. satisfait* $\det[\lambda\mathbb{E}-\mathbb{A}]\neq 0$*) et n'a pas de zéro infini d'ordre supérieur à 1.*

Proposition 1 (Voir [5]) *Le système décrit par (1) est internement propre si et seulement si* $\mathcal{X} = \ker D_i \oplus \ker E$.

Considérons maintenant la propreté externe.

3.3.2 Propreté Externe

Définition 2 *Le système implicite*

$$E\dot{x} = Ax + Bu; \quad y = Cx \tag{6}$$

où $E:\mathcal{X}\to\underline{\mathcal{X}}$, $A:\mathcal{X}\to\underline{\mathcal{X}}$, $B:\mathcal{U}\to\mathcal{X}$, *et* $C:\mathcal{X}\to\mathcal{Y}$ *sont des opérateurs linéaires, est externement propre, si et seulement si, sa partie externe minimale est internement propre.*

Le sous-espace $\mathcal{V}^*_{\mathcal{X}}$ caractérise (conjointement avec $E\mathcal{V}^*_{\mathcal{X}} + \mathrm{Im}B$ dans le co-domaine) l'ensemble de toutes les trajectoires qui sont identiquement nulles, quelle que soit l'entrée u. Ce sous-espace s'obtient comme la limite de l'algorithme non croissant $\mathcal{V}^o_{\mathcal{X}} = \mathcal{X}$, $\mathcal{V}^{\mu+1}_{\mathcal{X}} = A^{-1}(E\mathcal{V}^\mu_{\mathcal{X}} + \mathrm{Im}B)$.

Le sous-espace $\mathcal{V}^*_o$ caractérise (conjointement avec $E\mathcal{V}^*_o$) l'ensemble de toutes les trajectoires qui sont inobservables sur la sortie y. Ce sous-espace s'obtient comme la limite de l'algorithme non croissant $\mathcal{V}^o_o = \mathcal{X}$, $\mathcal{V}^{\mu+1}_o = \ker C \cap A^{-1}E\mathcal{V}^\mu_o$.

Le sous-espace $\mathcal{R}^*_{ao}$ caractérise (conjointement avec $A\mathcal{R}^*_{ao}$) l'ensemble de toutes les trajectoires *différentiellement redondantes, i.e.* avec phénomènes impulsionels sans influence sur le comportement entrée-sortie. Ce sous-espace s'obtient comme la limite de l'algorithme non décroissant suivant :

$$\mathcal{R}^o_{ao} = \ker C\cap\ker E\ ; \quad \mathcal{R}^{\mu+1}_{ao} = \ker C\cap E^{-1}A\mathcal{R}^\mu_{ao} \tag{7}$$

Nous pouvons maintenant caractériser la propreté externe d'un système implicite :

Théorème 1 (Voir [5]) *Si le système implicite (6) est observable et ne possède pas de trajectoire identiquement nulle pour toute entrée, à savoir si* $\mathcal{V}^*_o = \{0\}$ *et* $\mathcal{V}^*_{\mathcal{X}} = \mathcal{X}$, *alors (6) est externement propre si et seulement si* $\mathcal{V}^*_{\mathcal{X}_o} + \mathcal{S}^*_{\mathcal{X}_o} = \mathcal{X}$, $\mathcal{V}^*_{\mathcal{X}_o}\cap\mathcal{S}^*_{\mathcal{X}_o}\subset\mathcal{R}^*_{ao}$ *et* $\dim\left(\left(\mathcal{V}^*_{\mathcal{X}_o}+\mathcal{R}^*_{ao}+\overline{\mathcal{T}}^2_1\right)\Big/\left(\mathcal{V}^*_{\mathcal{X}_o}+\mathcal{R}^*_{ao}+\overline{\mathcal{T}}^1_1\right)\right) = 0$ *où* $\mathcal{V}^*_{\mathcal{X}_o}$ *et* $\mathcal{S}^*_{\mathcal{X}_o}$ *sont respectivement les limites des deux algorithmes suivants :*

$$\begin{array}{l}\mathcal{V}^o_{\mathcal{X}_o} = \mathcal{X}\ ,\mathcal{V}^{\mu+1}_{\mathcal{X}_o} = A^{-1}E\mathcal{V}^\mu_{\mathcal{X}_o}\\ \mathcal{S}^o_{\mathcal{X}_o} = \ker E\ ,\mathcal{S}^{\mu+1}_{\mathcal{X}_o} = E^{-1}A\mathcal{S}^\mu_{\mathcal{X}_o}\end{array} \tag{8}$$

et $\overline{\mathcal{T}}^\mu_1$ *et* $\overline{\mathcal{T}}^\mu_2$ *sont obtenus à partir des deux algorithmes* $\overline{\mathcal{T}}^o_1 = \mathcal{R}^*_{ao}, \overline{\mathcal{T}}^{\mu+1}_1 = E^{-1}A\left(\overline{\mathcal{T}}^\mu_1+\mathcal{R}^*_{ao}\right)$ *et* $\overline{\mathcal{T}}^o_2 = \mathcal{X}, \overline{\mathcal{T}}^{\mu+1}_2 = A^{-1}E\overline{\mathcal{T}}^\mu_2 + \mathcal{R}^*_{ao}$

Corollaire 1 (Voir [5]) *Si le système implicite (6) est observable, ne possède pas de trajectoire identiquement nulle pour toute entrée, et si l'équation dynamique interne ne fait apparaître que des modes exponentiels et des modes impulsionnels, à savoir si :* $\mathcal{V}^*_o = \{0\}$, $\mathcal{V}^*_{\mathcal{X}} = \mathcal{X}$ *et* $\mathcal{X} = \mathcal{V}^*_{\mathcal{X}_o}\oplus\mathcal{S}^*_{\mathcal{X}_o}$, *alors (6) est externement propre si et seulement si* $E^{-1}A\mathcal{R}^*_{ao} = \mathcal{S}^*_{\mathcal{X}_o}$.

Finalement, la combinaison des deux resultats précédents résoud notre problème d'approximation, ce qui est décrit dans la sous-section suivante.

3.4 Approximation exponentiellement propre

On peut rassembler le compensateur impropre (4) et le filtre strictement propre (5) et les décrire sous forme implicite à l'aide du système global $(\Sigma^f\circ\Sigma^c)$:

$$E\dot{x} = Ax + Bu; \quad y = Cx \tag{9}$$

où :

$$\begin{array}{l}E = \begin{bmatrix} I & 0\\ 0 & N\end{bmatrix}, A = \begin{bmatrix} A_p & B_p\\ 0 & I\end{bmatrix}\\ B = \begin{bmatrix} 0\\ \Gamma\end{bmatrix}, C = \begin{bmatrix} C_p & 0\end{bmatrix}\end{array} \tag{10}$$

avec $A_p = \begin{bmatrix} A_\beta & -\varepsilon^{k+1}C_o\\ \frac{1}{\varepsilon}B_o & \frac{1}{\varepsilon}A_o\end{bmatrix}, B_p = \begin{bmatrix} 0\\ \frac{1}{\varepsilon}B_o\Delta\end{bmatrix}$, $C_p = \begin{bmatrix} 0 & C_o\end{bmatrix}$,et $x^T = \begin{bmatrix}\overline{x}^T & \widehat{x}^T & \omega^T\end{bmatrix}$. Du fait des formes particulières de Σ^c et Σ^f, on obtient :

$$\left\{\begin{array}{l} B_p = [B_{p_1}|B_{p_2}|\cdots|B_{p_n}]\,;\ \text{avec } B_{p_i} = [0|\cdots|0|b_{p_i}]\\ b^T_{p_i} = \left[0|\,0|\cdots|0|\,\frac{1}{\varepsilon}b^T_i\right|0|\cdots|0\right], i = 1,...,n\end{array}\right. \tag{11}$$

Lemme 2 *Considérons les changements de bases :* $R = \begin{bmatrix} I & R_p\\ 0 & I\end{bmatrix}$, *et* $L = \begin{bmatrix} I & L_p\\ 0 & I\end{bmatrix}$, *avec* $R_p = \begin{bmatrix} R_{p_1}| & R_{p_2}| & \cdots & |R_{p_n}\end{bmatrix}$, $R_{p_i} = \begin{bmatrix} A^{k_i-1}_p b_{p_i}\big| & \cdots & |A_p b_{p_i} & |b_{p i} & |0\end{bmatrix}$, $L_p = -(A_pR_p + B_p)$. *Alors* $R_p + L_pN = 0$, $C_pA^j_pb_{p_i} = 0$, *pour* $j = 0,1,\ldots,k_i-2$, *&* $i = 1,\ldots,n$ *et* $C_pA^{k_i-1}_pb_{p_i} = \frac{1}{\varepsilon^{k_i}}\underline{\chi}^i_n$, *pour* $i = 1,\ldots,n$.

Comme conséquence de ce Lemme 2, on peut écrire :

$$LER = \begin{bmatrix} I & 0\\ 0 & N\end{bmatrix}, LAR = \begin{bmatrix} A_p & 0\\ 0 & I\end{bmatrix}, CR = \begin{bmatrix} C_p & C_n\end{bmatrix} \tag{12}$$

avec :

$$C_n = D\left\{\nu^T_1,...,\nu^T_n\right\};\nu_i = \frac{1}{\varepsilon^{k_i}}\underline{\chi}^1_{(k_i+1)},\ i = 1,...,n \tag{13}$$

Il est alors facile de vérifier, à partir de (12) et (13), que $E^{-1}A\mathcal{R}^*_{ao} = \mathcal{S}^*_{\mathcal{X}_o}$, c'est à dire, que le système (9)-(10) est externement propre et satisfait le Lemme 1.

Théorème 2 *Soit le filtre strictement propre* Σ^f, *conçu comme dans le Lemme 1 pour approximer de manière exponentielle le compensateur impropre* Σ^c. *Alors, le système global* $(\Sigma^f\circ\Sigma^c)$ *est externement propre si et seulement si les ordres des zéros à l'infini de* Σ^f *sont respectivement supérieurs ou égaux à* k_i.

Ce résultat généralise au cas multivariable le filtre introduit précédemment dans [2] pour les systèmes monovariables.

Pour illustrer ces résultats, considérons un système impropre qui possède un pôle à l'infini d'ordre 2 :

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \dot{\xi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xi + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y^* = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \xi$$

Dans les deux exemples qui suivent, on applique l'approximation proposée dans le Lemme 1. Pour le premier exemple, le filtre a un zéro à l'infini d'ordre 1 ; pour le second le filtre a un zéro à l'infini d'ordre 2.

Exemple 2 *Considérons le système décrit par :*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \dot{\bar{x}} = \begin{bmatrix} -\beta & -\varepsilon^2 & 0 & 0 & 0 \\ 1/\varepsilon & -1/\varepsilon & 0 & 0 & 1/\varepsilon \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bar{x} + $$
$$+ \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \end{bmatrix}^T u$$
$$y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \bar{x} \tag{14}$$

avec $\bar{x} = \begin{bmatrix} x & z & \xi_1 & \xi_2 & \xi_3 \end{bmatrix}^T$*. En comparant le système (14) avec la forme (9)-(11), on obtient* $A_p = \begin{bmatrix} -\beta & -\varepsilon^2 \\ 1/\varepsilon & -1/\varepsilon \end{bmatrix}$, $b_{p1} = \begin{bmatrix} 0 \\ 1/\varepsilon \end{bmatrix}$, $B_p = B_{p1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\varepsilon \end{bmatrix}$ *et donc* $R_p = \begin{bmatrix} -\varepsilon & 0 & 0 \\ -\frac{1}{\varepsilon^2} & 1/\varepsilon & 0 \end{bmatrix}$, $L_p = \begin{bmatrix} -\varepsilon\beta - 1 & \varepsilon & 0 \\ 1 - \frac{1}{\varepsilon^3} & \frac{1}{\varepsilon^2} & -\frac{1}{\varepsilon} \end{bmatrix}$, *i.e.,*

$$R = \begin{bmatrix} 1 & 0 & -\varepsilon & 0 & 0 \\ 0 & 1 & -1/\varepsilon^2 & 1/\varepsilon & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$L = \begin{bmatrix} 1 & 0 & -(\varepsilon\beta + 1) & \varepsilon & 0 \\ 0 & 1 & 1 - \frac{1}{\varepsilon^3} & \frac{1}{\varepsilon^2} & -\frac{1}{\varepsilon} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Dans ces nouvelles bases, le système est décrit par :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \dot{\tilde{x}} = \begin{bmatrix} -\beta & -\varepsilon^2 & 0 & 0 & 0 \\ 1/\varepsilon & -1/\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tilde{x} + $$
$$+ \begin{bmatrix} (\varepsilon\beta + 1) & (1/\varepsilon^3 - 1) & -1 & 0 & 0 \end{bmatrix}^T u$$
$$y = \begin{bmatrix} 0 & 1 & -\frac{1}{\varepsilon^2} & \frac{1}{\varepsilon} & 0 \end{bmatrix} \tilde{x}$$

qui correspondent à l'expression (12), à partir de laquelle on peut aisément identifier N, A_p, C_p *et* C_n*. L'application des algorithmes (7) et (8) conduit à* $\mathcal{R}^*_{ao} = \{e_5\}$, $E^{-1}A\mathcal{R}^*_{ao} = \{e_4, e_5\}$ *et* $\mathcal{S}^*_{\mathcal{X}_o} = \{e_3, e_4, e_5\}$ *d'où l'on constate que* $\mathcal{S}^*_{\mathcal{X}_o} \neq E^{-1}A\mathcal{R}^*_{ao}$*. Ainsi, d'après le Corollaire 1, (14) n'est pas externement propre.*

Exemple 3 *Considérons maintenant le système :*

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \dot{\underline{x}} =$$
$$= \begin{bmatrix} -\beta & -\varepsilon^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon & 0 & 0 & 1/\varepsilon^2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \underline{x} \tag{15}$$

avec $\underline{x} = \begin{bmatrix} x & z_1 & z_2 & \xi_1 & \xi_2 & \xi_3 \end{bmatrix}^T$*. En comparant le système (15) avec la forme (9)-(11), on obtient* $A_p = \begin{bmatrix} -\beta & -\varepsilon^2 & 0 \\ 0 & 0 & 1 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon \end{bmatrix}$, $b_{p1} = \begin{bmatrix} 0 \\ 0 \\ 1/\varepsilon^2 \end{bmatrix}$, $B_p = B_{p1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\varepsilon^2 \end{bmatrix}$ *et donc* $R_p = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\varepsilon^2} & 0 & 0 \\ -\frac{2}{\varepsilon^3} & 1/\varepsilon^2 & 0 \end{bmatrix}$,

$L_p = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{\varepsilon^3} & -\frac{1}{\varepsilon^2} & 0 \\ -\frac{3}{\varepsilon^4} & \frac{2}{\varepsilon^3} & -\frac{1}{\varepsilon^2} \end{bmatrix}$, *i.e.,*

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/\varepsilon^2 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{\varepsilon^3} & 1/\varepsilon^2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$L = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{\varepsilon^3} & -\frac{1}{\varepsilon^2} & 0 \\ 0 & 0 & 1 & -\frac{3}{\varepsilon^4} & \frac{2}{\varepsilon^3} & -\frac{1}{\varepsilon^2} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Dans ces nouvelles bases, le système est décrit par :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \dot{\tilde{x}} =$$
$$= \begin{bmatrix} -\beta & -\varepsilon^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tilde{x} + \begin{bmatrix} -1 \\ -2/\varepsilon^3 \\ 3/\varepsilon^4 \\ -1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 0 & 1/\varepsilon^2 & 0 & 0 \end{bmatrix} \tilde{x}$$

qui correspondent à l'expression (12), à partir de laquelle on peut aisément identifier N, A_p, C_p *et* C_n*. L'application des algorithmes (7) et (8) conduit à* $\mathcal{R}^*_{ao} = \{e_5, e_6\}$, $E^{-1}A\mathcal{R}^*_{ao} = \{e_4, e_5, e_6\}$ *et* $\mathcal{S}^*_{\mathcal{X}_o} = \{e_4, e_5, e_6\}$ *d'où l'on constate que* $\mathcal{S}^*_{\mathcal{X}_o} = E^{-1}A\mathcal{R}^*_{ao}$*. Ainsi, d'après le Corollaire 1, le système (15) est externement propre.*

A ce niveau d'avancement, on peut se poser une question complémentaire : au lieu de concevoir tout d'abord une solution à base de loi de commande Proportionnelle-Dérivée, pour ensuite l'approximer par un Proportionnel (à grand gain), comme cela vient d'être décrit, existe-t-il une alternative consistant à formuler d'emblée un problème *approché* et à chercher directement une loi de commande de type Proportionnelle ? La réponse est affirmative, et ceci a été décrit dans la contribution [7]. Basiquement, la démarche que nous y avons retenue est de nature géométrique. Nous ne la détaillerons pas ici, pour des raisons de place. Elle s'inspire fortement de celle qui avait été proposée pour les solutions du type Proportionnelles-Dérivées, mais avec les adaptations nécessaires (notamment en termes des décompositions adéquates des sous-espaces concernés). A l'occasion de ces développements est apparue une contribution qui nous semble mériter d'être un peu plus détaillée ici. Pour disposer d'arguments rigoureux permettant de conclure sur des passages à la limite, nous avons en effet été amenés à considérer la technique *classique* de l'intégration par parties. Il se trouve que nous avons pu en donner une interprétation *système* assez intéressante (en termes de changements de bases *généralisés*) qui nous a permis de conclure beaucoup plus facilement dans [7]. Ensuite, nous décrivons succinctement cette interprétation.

4 Intégration par parties

Pour simplifier la présentation, nous allons considérer le cas particulier (mais suffisamment illustratif) décrit pour le système :

$$\begin{cases} \dot{x} = [-1/\varepsilon]\, x + [1/\varepsilon]\, f \\ y = [-1/\varepsilon]\, x + [1/\varepsilon]\, f \end{cases} \tag{16}$$

où y est la sortie et f est une entrée que nous allons supposer au moins deux fois dérivable et à dérivées bornées, *i.e.* $f,\ \dot{f},\ \ddot{f} \in L_\infty$. Nous voulons analyser le comportement externe de ce système lorsque le paramètre positif ε tend vers zéro, et notamment rechercher une argumentation rigoureuse pour le comportement à la limite de y lorsque $\varepsilon \to 0$. Pour cela, écrivons la solution temporelle de (16) (avec $x(0) = x_o$) :

$$\begin{cases} x = e^{-t/\varepsilon}x_o + \frac{1}{\varepsilon}\int_o^t e^{-\frac{t-\tau}{\varepsilon}} f(\tau)d\tau \\ y = -\frac{1}{\varepsilon}e^{-t/\varepsilon}x_o + \frac{1}{\varepsilon}f - \frac{1}{\varepsilon^2}\int_o^t e^{-\frac{t-\tau}{\varepsilon}} f(\tau)d\tau \end{cases}$$

A partir de cette écriture, on peut affirmer que $|y| \leq \frac{1}{\varepsilon}e^{-t/\varepsilon}\,|x_o| + \frac{1}{\varepsilon}\,|f| + \frac{1}{\varepsilon}||f||_\infty$ mais on ne peut rien conclure pour le comportement limite lorsque $\varepsilon \to 0$. En effectuant une intégration par parties ($f(0) = f_o$), on obtient :

$$\begin{cases} x - f = e^{-t/\varepsilon}\,(x_o - f_o) - \int_o^t e^{-(t-\tau)/\varepsilon}\dot{f}(\tau)d\tau \\ y = -\frac{1}{\varepsilon}e^{-t/\varepsilon}\,(x_o - f_o) + \frac{1}{\varepsilon}\int_o^t e^{-\frac{t-\tau}{\varepsilon}}\dot{f}(\tau)d\tau \end{cases}$$

De cette récriture on tire $|y| \leq \frac{1}{\varepsilon}e^{-t/\varepsilon}\,|x_o - f_o| + ||\dot{f}||_\infty$ mais dont on peut seulement conclure que y_2 est bornée lorsque $\varepsilon \to 0$ pour tout $t > 0$.

Si l'on applique une deuxième intégration par parties ($\dot{f}(0) = \dot{f}_o$), on obtient :

$$\begin{cases} x - f + \varepsilon\dot{f} = e^{-t/\varepsilon}\left(x_o - f_o + \varepsilon\dot{f}_o\right) \\ y - \dot{f} = -\frac{1}{\varepsilon}e^{-t/\varepsilon}\left(x_o - f_o + \varepsilon\dot{f}_o\right) - \int_o^t e^{-\frac{t-\tau}{\varepsilon}}\ddot{f}(\tau)d\tau \end{cases}$$

avec comme conséquence $\left|y - \dot{f}\right| \leq \frac{1}{\varepsilon}e^{-t/\varepsilon} \left|x_o - f_o + \varepsilon\dot{f}_o\right| + \varepsilon||\ddot{f}||_\infty$. Ceci permet d'affirmer (le résultat escompté, à savoir) que $y \to \dot{f}$ lorsque $\varepsilon \to 0$ et pour tout $t > 0$.

De cette analyse, on peut constater qu'il a été nécessaire d'appliquer consécutivement deux intégrations par parties pour être à même de conclure sur le passage à la limite.

Nous allons maintenant interpréter ce procédé d'intégrations par parties et montrer qu'il est équivalent à appliquer des changements de bases génaralisés.

4.1 Interprétation *système* de l'intégration par parties

On peut noter que l'écriture du système après la première intégration par parties revient à considérer le modèle de Fliess [8] : $\dot{w} = [-1/\varepsilon]\, w + [1/\varepsilon]\,(-\varepsilon\dot{f})$ et $y = [-1/\varepsilon]\, w$. Cette description *généralisée* s'obtient à partir de (16) à l'aide du changement de variable $w = x - f$.

La seconde intégration par parties correspond à la solution de la description du modèle de Fliess : $\dot{z} = [-1/\varepsilon]\, z + [1/\varepsilon]\,(\varepsilon^2\ddot{f})$ et $y = [-1/\varepsilon]\, z + [1]\,\dot{f}$. Cette description peut être obtenue à partir de (16) après le changement de variable :

$$z = w - (-\varepsilon\dot{f}) = x - f + \varepsilon\dot{f} \tag{17}$$

Les descriptions implicites *classiques* des systèmes (16) et (17) sont données ci-après :

1. Posant $\xi_1 = x$ et $\xi_2 = f$ dans (16), on obtient :

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\dot{\xi} = \begin{bmatrix} -1/\varepsilon & 0 \\ 0 & 1 \end{bmatrix}\xi + \begin{bmatrix} 1/\varepsilon \\ -1 \end{bmatrix} f \qquad y = \begin{bmatrix} -1/\varepsilon & 1/\varepsilon \end{bmatrix}\xi \tag{18}$$

2. Posant $\zeta_1 = z$, $\zeta_2 = f$, $\zeta_3 = -\varepsilon\dot{\zeta}_2$ et $\zeta_4 = \dot{\zeta}_3$ dans (17), on obtient :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\dot{\zeta} = \begin{bmatrix} -\frac{1}{\varepsilon} & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}\zeta + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} f \qquad y = \begin{bmatrix} -\frac{1}{\varepsilon} & 0 & -\frac{1}{\varepsilon} & 0 \end{bmatrix}\zeta \tag{19}$$

Il est possible d'examiner certaines propriétés structurelles de ces représentations, notamment la propreté. Certaines sont résumées dans la sous-section suivante.

4.2 Propriétés structurelles

4.2.1 Propreté interne

D'après la Proposition 1, le système (18) est internement propre, *i.e.*, $\mathcal{X} = \ker\begin{bmatrix} 0 & 1 \end{bmatrix} \oplus \ker\begin{bmatrix} 1 & 0 \end{bmatrix}$. Le système (19) n'est pas internement propre, car

$$\ker\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \cap \ker\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \neq \{0\}$$

4.2.2 Propreté externe

Puisque le système (18) est internement propre, il est *a fortiori* externement propre. Concernant le système (19) il faut effectuer le quotient par $\mathcal{R}^*_{ao}$ dans le domaine, et par $A\mathcal{R}^*_{ao}$ dans le codomaine. L'application de la procédure matricielle proposée dans [3] pour le système (19) conduit à

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\dot{\bar{\zeta}} = \begin{bmatrix} -1/\varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\bar{\zeta} + \begin{bmatrix} 1/\varepsilon \\ -1 \\ 0 \\ 0 \end{bmatrix} f \qquad y = \begin{bmatrix} -1/\varepsilon & 1/\varepsilon & 0 & 0 \end{bmatrix}\bar{\zeta} \tag{20}$$

En appliquant l'algorithme (7) au système (20), on obtient $\mathcal{R}^*_{ao} = \{e_3, e_4\}$. Le système quotient n'est donc rien d'autre que (18). Ainsi, (19) est externement propre et externement équivalent au système (18).

On peut conclure de l'analyse précédente que l'équivalent *système* des deux intégrations par parties est l'application, sur le modèle généralisé de Fliess, du changement de variable[1] $z = x - f + \varepsilon\dot{f}$. Un examen plus poussé montre que le système obtenu après ce changement de base fait apparaître des variables internes (*variable descripteur*) qui sont différentiellement redondantes (modes impulsionnels qui peuvent être éliminés sans modifier le comportement externe du système), et c'est grace à ces composantes rajoutées et à la forme structurellement agréable associée du nouveau modèle ainsi obtenu, que la conclusion s'obtient dans [7], pour la construction alternative d'un retour Proportionnel visant à (approximativement) supprimer l'influence du degré de liberté sur les systèmes à structure variable considérés.

[1] Dans le cas de n integrations par parties, le changement de variable est : $z_n = x + \sum_{i=1}^{n}(-1)^i\varepsilon^{i-1}f^{(i-1)}$

Les sections suivantes résument les résultats déjà obtenus, ainsi que les perspectives.

5 Résultats obtenus

Parmi les principaux résultats obtenus jusqu'à présent dans le cadre de ce travail de doctorat, on peut mentionner :

1. Approximation exponentiellement propre de lois de commande (ou, plus généralement de filtres) impropres : voir [12].
2. Méthode alternative pour obtenir une loi de commande proportionnelle : voir [7]
3. Amélioration des résultats précédemment obtenus par [1] : évaluation d'une borne pour le temps de convergence de l'algorithme (rédaction en cours).

6 Perspectives

Considérons à nouveau le système implicite global $\Sigma^g : (\mathbb{E}, \mathbb{H}, \mathbb{B}, C)$:

$$\Sigma^g : \mathbb{E}\dot{x} = \mathbb{H}x + \mathbb{B}u; \quad y = Cx \tag{21}$$

avec $\mathbb{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}$, $\mathbb{H} = \begin{bmatrix} A \\ D_i \end{bmatrix}$, et $\mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}$ pour $i = 1, ..., n$. Cette classe de modèles est justifiée par l'étude de systèmes à structures variables et nous avons montré comment de tels systèmes pouvaient être controlés, en tirant profit du contexte enrichi des systèmes implicites. En liaison avec l'optimisation des comportements obtenus, nous souhaitons considérer ensuite des problèmes de type *commande optimale*, et notamment le problème suivant :

Problème 2 *A partir d'une condition initiale donnée, on souhaite ramener à l'origine le système (en fait l'ensemble des systèmes) décrit par (21) en minimisant le critère quadratique* $J = \int_0^T \left[x^T Q x + u^T R u\right] dt$ *avec* $Q \geq 0$ *et* $R > 0$.

Pour résoudre ce problème, nous allons considérer plusieurs approches alternatives qui sont esquissées ci-après.

6.1 Approches envisagées

6.1.1 Première approche

Dans une première approche, on peut considérer chaque système possible dans la famille de départ, disons pour un i fixé ; utiliser la contrainte algébrique $D_i x_i = 0$ pour éliminer certaines variables ; puis appliquer la méthodologie *classique* du régulateur quadratique (LQR) à chaque système ; pour $i = 1, ..., n$, on obtient ainsi une loi de commande $u_i = -F_i^* x$ qui minimise le critère $J_i = \int_0^T \left[x_i^T Q_i x_i + u_i^T R_i u_i\right] dt$ en assurant la stabilité exponentielle.

Disposant de toutes ces lois de commande optimales (optimale pour chaque système vu séparément), u_i, on peut appliquer le détecteur de structure que nous avons proposé pour décider quel système est *actif* et lui appliquer sa loi de commande *optimale*.

6.1.2 Seconde approche

A partir du système global (21) on souhaite trouver une loi de commande *optimale* unique, $u = -F^* x$, laquelle sera forcément *moins bonne*, lorsque le système i est actif que la commande optimale qui lui est associée, mais conduisant à un *compromis* global satisfaisant. Ce schéma serait privilégié lorsqu'on ne sait pas avec suffisamment de précision quel système est actif.

6.1.3 Troisième approche

Une troisième approche consiste à supprimer dans un premier temps le degré de liberté (en s'appuyant sur les résultats de [6]) pour obtenir une système propre *unique*. On peut ensuite appliquer au système ainsi obtenu la méthodologie *classique* du régulateur quadratique et analyser les comportements obtenus.

6.1.4 Quatrième approche

Une quatrième approche consiste à utiliser les résultats (malheureusement peu nombreux) développés dans le cadre de la commande optimale des systèmes implicites.

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Apéndice F

Optimisation de Systèmes Implicites (Poster JDOC-Nantes, France)

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Optimisation de Systèmes Implicites

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1. Types de Systèmes

Système Classique

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Système Implicite

$$E\dot{x} = Ax + Bu$$
$$y = Cx$$

Les *systèmes implicites* sont une extension des systèmes classiques. Ils permettent de décrire des comportements beaucoup plus nombreux et plus riches (loi de commande proportionnelle et dérivée, contrainte algébrique sur l'état, restriction sur la commande, comportements impulsionnels, etc.)

2. Système Implicite Rectangulaire

Le nombre des variables internes est supérieur au nombre d'équations d'état. Avec ce type de systèmes , on peut décrire des systèmes à structure variable avec un modèle unique (*système global*)

$$\underbrace{\begin{bmatrix} E \\ 0 \end{bmatrix}}_{\mathbf{E}} \dot{x} = \underbrace{\begin{bmatrix} A \\ D_i \end{bmatrix}}_{\mathbf{A_i}} x + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\mathbf{B}} u$$

$$y = Cx$$

où $D_i x = 0$ est une équation de contrainte algébrique.

3. Exemple Illustratif

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\dot{x} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u;$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}x; \qquad \begin{bmatrix} \alpha & \beta & 1 \end{bmatrix}x = 0$$

Suivant les valeurs des paramètres α et β, ce système peut avoir des comportements très différents . Par exemple, s i $(\alpha,\beta) = (-1,-1)$, i.e., $x_3 = x_1 + x_2$ alors la description entrée-sortie est $\dot{y} + y = u$.

La loi de commande $u = F^*_{d_1}\dot{x} + (F^*_{p_1} + F^*_{p_2})x + R/\tau_0$ où $F^*_{d_1} = \begin{bmatrix} 0 & -1 & -1 \end{bmatrix}$, $F^*_{p_1} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}$ et $F^*_{p_2} = \begin{bmatrix} 0 & 0 & 1-1/\tau_0 \end{bmatrix}$ conduit au système bouclé:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\dot{x} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -1/\tau_0 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1/\tau_0 \end{bmatrix}u; \quad y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

La variation interne de structure n'est plus présente sur la sortie et le système en boucle fermée se comporte comme $\tau_0\dot{y} + y = R$.

Résultats Principaux

1. Approximation exponentiellement propre de lois de commande impropres

Théorème: *Soit le filtre strictement propre:*

$$\Sigma^f : \dot{\bar{x}} = A_\beta \bar{x} - \varepsilon^{k+1} y, \ \ \varepsilon\dot{\hat{x}} = A_0\hat{x} + B_0(\bar{x} + y^*), \ \ y = C_0\hat{x}$$

pour approximer de manière exponentielle le compensateur impropre:

$$\Sigma^c : N\dot{\omega} = \omega + \Gamma u, \ \ y^* = \Delta\omega$$

Alors, le système global $(\Sigma^f o \Sigma^c)$ *est externement propre si et seulement si les ordres des zéros à l'infini de* Σ^f *sont respectivement supérieurs ou égaux à* k_i *(ordres des pôles à l'infini du compensateur)*

Exemple: Pour illustrer ce résultat, considérons un système impropre qui possède un pôle à l'infini d'ordre 2:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\dot{\xi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\xi + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}x$$

Si on prend un filtre avec un zéro à l'infini d'ordre 1 et un filtre avec un zéro à l'infini d'ordre 2, on peut vérifier le théorème précédent. Le premier filtre ne sera pas externement propre mais le deuxième le sera.

2. Interprétation « *système* » de l'intégration par parties

Nous allons considérer le cas particulier:

$$\dot{x} = [-1/\varepsilon]x + [1/\varepsilon]f; \quad y = [-1/\varepsilon]x + [1/\varepsilon]f \qquad (1)$$

où "y" est la sortie et $\exists f, \dot{f}, \ddot{f} \in L_\infty$.

Quel est le comportement externe de ce système quand $\varepsilon \to 0$?

Solution temporelle :

$$x = e^{-t/\varepsilon}x_0 + 1/\varepsilon\int_0^t e^{-(t-\tau)/\varepsilon} f(\tau)d\tau; y = -(1/\varepsilon)e^{-t/\varepsilon}x_0 + 1/\varepsilon f - 1/\varepsilon^2\int_0^t e^{-(t-\tau)/\varepsilon} f(\tau)d\tau$$

•Première intégration par parties:

$$x - f = e^{-t/\varepsilon}(x_0 - f_0) - \int_0^t e^{-(t-\tau)/\varepsilon}\dot{f}(\tau)d\tau; y = -(1/\varepsilon)e^{-t/\varepsilon}(x_0 - f_0) + 1/\varepsilon\int_0^t e^{-(t-\tau)/\varepsilon}\dot{f}(\tau)d\tau$$

•Deuxième intégration par parties:

$$x - f + \varepsilon\dot{f} = e^{-t/\varepsilon}(x_0 - f_0 + \varepsilon\dot{f}_0); y - \dot{f} = -(1/\varepsilon)e^{-t/\varepsilon}(x_0 - f_0 + \varepsilon\dot{f}_0) - \int_0^t e^{-(t-\tau)/\varepsilon}\ddot{f}(\tau)d\tau$$

Et comme conséquence: $\left|y - \dot{f}\right| \le (1/\varepsilon)e^{-t/\varepsilon}\left|x_0 - f_0 + \varepsilon\dot{f}_0\right| + \left\|\dot{f}\right\|_\infty$

Ceci permet d'affirmer que $y \to \dot{f}$ lorsque $\varepsilon \to 0$ pour tout $t > 0$.

L'écriture du système après la 1ère intégration par parties revient à considérer le modèle de Fliess: $\dot{w} = [-1/\varepsilon]w + [1/\varepsilon](-\varepsilon\dot{f})$, $y = [-1/\varepsilon]w$. Cette description généralisée s'obtient à partir de (1) à l'aide du changement $w = x - f$. La 2ème intégration correspond à la solution de $\dot{z} = [-1/\varepsilon]z + [1/\varepsilon](\varepsilon^2\ddot{f})$ et $y = [-1/\varepsilon]z + [1]\dot{f}$. Cette description peut être obtenue à partir de (1) après le changement $z = w - (-\varepsilon\dot{f}) = x - f + \varepsilon\dot{f}$.