This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2662804, IEEE Transactions on Automatic Control

Multivalued robust tracking control of Lagrange systems: Continuous and discrete-time algorithms.

Félix A. Miranda-Villatoro †, Bernard Brogliato ‡, and Fernando Castaños †

***Abstract*—The robust trajectory tracking of fully actuated Lagrange systems is studied. Exogenous perturbations as well as parameter uncertainties are taken into account. A family of set-valued passivity-based controllers is proposed, including first-order sliding-mode schemes. The existence of solutions and the stability of the closed-loop system are established in continuous-time. An implicit discretization approach is proposed and the well posedness and the stability of the closed-loop system are studied. Numerical simulations illustrate the effectiveness of the proposed discrete-time controller.**

***Index Terms*—Robust control, sliding-mode control, differential inclusion, discrete-time, trajectory tracking, Euler-Lagrange systems, finite-time convergence, Lyapunov stability, convex analysis, passivity-based controller.**

I. Introduction

THERE exists a vast literature on the control of Euler-Lagrange systems (see, e.g., [12], [39], [21] and references therein). In the early eighties, the problem of robust tracking for nonlinear robotic systems was approached in [36] by using sliding-mode control techniques. Later in [34], [35] the methodology was improved by the use of adaptive schemes. The main idea was to use a passivity-based approach to render the closed-loop passive and globally asymptotically stable [30]. As it is well known, the implementation of robust controllers based on sliding-mode techniques, suffers from the so-called chattering problem [39], i.e., the output and the control input switch at a high frequency between a finite number of values. Chattering becomes dangerous in mechanical systems, specially when the discrete-time controller is implemented without the proper discretization scheme. Recently, implicit discretization schemes for linear systems with sliding modes were proposed in [1], [2], [20] (see also [23] for a similar approach), and experimentally tested with success in [18], [19], [20], [40], where it is shown that chattering in both, the output and the input, is almost totally suppressed.

We begin with an Euler-Lagrange system for which a desired trajectory $q_{\mathrm{d}}, \dot{q}_{\mathrm{d}}, \ddot{q}_{\mathrm{d}}$ is given and we propose a family of multivalued control laws such that robust tracking is obtained, both in continuous and discrete-time. Robustness is obtained in the presence of bounded external disturbances and parametric uncertainties. The results we present encompass [1], [2], [4], [5], [20] in the sense that none of the previous papers deals with parametric uncertainties and [1], [2], [20] limit their study to linear systems. From a mathematical point of view, the problem with parametric uncertainties is that they appear as a term which cannot be uniformly upper bounded by a constant.

The part on continuous time is a nontrivial extension of the results in [4], [34], [35], [36] and is strongly based on the theories of maximal monotone operators and convex analysis. The time-discretization of set-valued sliding-mode control laws requires particular care, as it may yield numerical chattering if the set-valued part of the controller is discretized using an explicit scheme [1], [2], [15], [18], [19], [20], [40], [41]. Moreover, an explicit discretization may yield unstable closed-loop systems in the nonlinear case [27] while, on the other hand, the implicit method advocated in [1], [2], [5], [20] retains the continuous-time stability properties of the system in question [11], [20].

Set-valued control laws are common in sliding-mode control theory, where the sign multifunction plays a particularly important role. However, little attention has been granted to other possible multifunctions. Only until recently, more general set-valued maximal monotone operators were studied in a control context [4], [10], [28], [38]. In this paper we study the use of other multifunctions for the robust control of dynamical systems and their implementation in discrete-time. The main objective is to diminish the chattering phenomenon.

Contributions: We generalize the implicit method for discrete-time sliding mode controllers proposed in [1], [2], [20] in several ways: First, we take into account the lack of complete knowledge on system parameters and propose a family of set-valued controllers for the robust tracking problem. The family contains the signum multifunction but it is not limited to it. Second, we provide an algorithm for the computation of the control that will achieve the robust tracking with virtually no chattering.

Paper structure: Section II contains mathematical preliminaries while Section III recalls some basic properties of Lagrangian dynamics. We present in Section IV the well-posedness analysis of the closed-loop system with set-valued controllers (existence of solutions), relaxing a stringent assumption made in [4] (see Remark 2). The stability analysis is made in Section V. We do not establish uniqueness of solutions, but we do prove that all of them yield a tracking error with suitable stability properties. Section VI is dedicated to the analysis of the discrete-time controller. Due to the nonlinearities of the Euler-Lagrange dynamics, the design of the implicit discrete-time controller is made from an inexact discretization of the continuous plant. The design of the discrete-time nonlinear passivity-based controller is made in Section VI-A and the stability analysis in Section VI-B. Numerical simulations illustrate the theoretical developments

† Automatic Control Department, Cinvestav-IPN, Av. Instituto Politécnico Nacional 2508, 07360, Mexico City, Mexico.

‡ INRIA Grenoble Rhône-Alpes, University of Grenoble-Alpes, Inovalleé, 655 Av. de l'Europe, 38334, Saint-Ismier, France. bernard.brogliato@inria.fr

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2662804, IEEE Transactions on Automatic Control

in section VII. Conclusions end the article in section VIII.

II. Preliminaries

Some preliminary results are presented together with the notation used through this article. Let X be an n-dimensional real space with the usual Euclidean inner product denoted as $\langle\cdot,\cdot\rangle$ and the corresponding norm as $\|\cdot\|$. A symmetric matrix $M = M^\top$ is called positive definite if $x^\top Mx > 0$ for any $x \in X \setminus \{0\}$. For any matrix $M \in \mathbb{R}^{n\times n}$ the norm $\|M\|_m$ is the induced norm given by $\|M\|_m = \sup_{\|x\|=1}\|Mx\|$. The unitary ball of $\mathbb{R}^n$ with center in 0 is denoted by $\mathbb{B}_n$. We denote by $\Gamma_0(X)$ the set of all proper, convex, and lower semi-continuous (lsc) functions from X to $\mathbb{R}\cup\{+\infty\}$. The indicator of a set $C \subset X$ is the function $\Psi_C : X \to \mathbb{R}\cup\{+\infty\}$ which satisfies $\Psi_C(x) = 0$ for $x \in C$ and $\Psi(x) = +\infty$ otherwise. The boundary of a set A is denoted as $\mathrm{bd}(A)$. The following definitions are rather standard in the convex analysis literature. The interested reader can refer to [8], [16], [17], [31], [32] for further details.

Definition 1. Let $\Phi \in \Gamma_0(X)$. The subdifferential of Φ at x, denoted as $\partial\Phi(x)$, is the set-valued map given by

$$\partial\Phi(x) := \{\zeta \in \mathbb{R}^n \,|\, \langle\zeta, \eta - x\rangle \le \Phi(\eta) - \Phi(x) \quad \text{for all } \eta \in X\}.$$

Definition 2. Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be a proper function and let $\lambda > 0$. The Moreau envelope of f of index λ is

$$f^\lambda(x) := \inf_{w\in X}\left\{f(w) + \frac{1}{2\lambda}\|w - x\|^2\right\}. \quad (1)$$

Remark 1. When $f \in \Gamma_0(X)$, the Moreau envelope is known as the Moreau-Yosida approximation of f of index λ and it is known that $f^\lambda \in \Gamma_0(X)$. Furthermore, the gradient of f^λ exists and f^λ is Lipschitz continuous with constant $1/\lambda$ [32, Exercise 12.23].

Definition 3. Let $f \in \Gamma_0(X)$ and let $x \in X$. Then, the proximal map of f at x, denoted as $\mathrm{Prox}_f(x)$, is the unique minimizer of $f(w) + \frac{1}{2}\|w - x\|^2$, i.e.,

$$f(\mathrm{Prox}_f(x)) + \frac{1}{2}\|\mathrm{Prox}_f(x) - x\|^2 = \min_{w\in X}\left\{f(w) + \frac{1}{2}\|w - x\|^2\right\} = f^1(x). \quad (2)$$

It is important to notice that, when $f = \Psi_C$ (the indicator of the set C), the proximal map agrees with the classical projection operator $\mathrm{Proj}_C(\cdot)$ given by

$$\mathrm{Proj}_C(x) = \arg\min_{w\in C}\frac{1}{2}\|w - x\|^2. \quad (3)$$

The distance between a point $w \in \mathbb{R}^n$ and a closed convex set A is given by the expression

$$\mathrm{dist}(x, C) = \min_{w\in C}\|x - w\| = \|x - \mathrm{Proj}_C(x)\|. \quad (4)$$

Definition 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper and lsc function. The conjugate function of f is

$$f^\star(x^\star) = \sup_{w\in X}\{\langle w, x^\star\rangle - f(w)\}.$$

It follows from the definition of conjugate function that, for any two functions $f, g \in \Gamma_0(X)$ such that $f \ge g$, we have $g^\star \ge f^\star$. The following facts will be useful in the sequel.

Lemma 1 ([16, Lemma 5.2.1])**.** *Let $f \in \Gamma_0(X)$ and let $\mathcal{A} : X \to X$ be a continuous and strongly monotone operator. That is, for any $x_1, x_2 \in X$,*

$$\langle\mathcal{A}(x_1) - \mathcal{A}(x_2), x_1 - x_2\rangle \ge \alpha\|x_1 - x_2\|$$

for some $\alpha > 0$. Then, for each $v \in X$, there exists a unique solution $x \in X$ to the variational inequality

$$\langle\mathcal{A}x - v, \eta - x\rangle + f(\eta) - f(x) \ge 0 \quad \text{for all } \eta \in X.$$

Proposition 1 ([8, Th. 14.3] Moreau's decomposition)**.** *Let $f \in \Gamma_0(X)$ and $\lambda > 0$. For any $x \in X$ we have*

$$x = \mathrm{Prox}_{\lambda f}(x) + \lambda\,\mathrm{Prox}_{f^\star/\lambda}(x/\lambda).$$

We will use Proposition 1 in order to compute explicitly the proximal map of the norm function that will be used in Section IV.

Lemma 2 ([8, Example 14.5])**.** *Consider a function $f : \mathbb{R}^n \to \mathbb{R}_+$, $x \mapsto c\|x\|$ with $c > 0$. The proximal map of index λ at x, $\mathrm{Prox}_{\lambda f}(x)$, is given by*

$$\mathrm{Prox}_{\lambda f}(x) = \begin{cases}\left(1 - \frac{\lambda c}{\|x\|}\right)x & \text{if } \|x\| > \lambda c,\\ 0 & \text{if } \|x\| \le \lambda c.\end{cases} \quad (5)$$

III. Lagrangian mechanics

Let us introduce the class of dynamical systems on which we will focus. We start with a nonlinear system described by Euler-Lagrange equations,

$$M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + G(q(t)) + F(t, q(t), \dot{q}(t)) = \tau(t), \quad (6)$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the vectors of generalized positions, velocities and accelerations, respectively. The matrix $M(q) \in \mathbb{R}^{n\times n}$, $M(q) = M(q)^\top > 0$, denotes the inertia matrix of the system. The term $C(q,\dot{q})\dot{q} \in \mathbb{R}^n$ represents the centripetal and Coriolis forces acting on the system. The term $G(q) \in \mathbb{R}^n$ is the vector of gravitational forces. The vector $F(t, q, \dot{q}) \in \mathbb{R}^n$ accounts for unmodeled dynamics and external disturbances. Finally, the vector $\tau \in \mathbb{R}^n$ represents the control input forces. We assume that $C(q,\dot{q})$ is defined using the so-called *Christoffel's symbols* [21, Chapter 4].

Property 1. *For all differentiable functions q, the matrices $M(q)$ and $C(q,\dot{q})$ satisfy*

$$\frac{d}{dt}M(q(t)) = C(q(t),\dot{q}(t)) + C^\top(q(t),\dot{q}(t)).$$

Notice that the previous property implies that $\dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric.

The following assumptions are standard [21], [12].

Assumption 1. *The matrices $M(q)$, $C(q,\dot{q})$ together with the vectors $G(q)$ and $F(t, q, \dot{q})$ satisfy the following inequalities* *for all* $(t, q, \dot{q}) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$ *and some known positive constants* k_1, k_2, k_C, k_G *and* k_F*:*

$$0 < k_1 \le \|M(q)\|_m \le k_2, \qquad \|C(q,\dot{q})\|_m \le k_C\|\dot{q}\|,$$
$$\|G(q)\| \le k_G\|q\|, \qquad \|F(t,q,\dot{q})\| \le k_F.$$

Assumption 2. *There exists a constant* k_3 *such that, for all* $x, y \in \mathbb{R}^n$*,* $\|M(x) - M(y)\|_m \le k_3\|x - y\|$*.*

Assumption 3. *The function* $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $h(x_1, x_2, x_3) := C(x_1, x_2)x_3$ *is locally Lipschitz.*

Assumption 4. *The function* $F(t, x_1, x_2)$ *is continuous in* t *and uniformly locally Lipschitz in* (x_1, x_2)*, (i.e., the Lipschitz constant is independent of* t*).*

Assumption 5. *The function* $G(\cdot)$ *is Lipschitz continuous and satisfies* $0 = G(0) \le G(x)$ *for all* $x \in \mathbb{R}^n$*.*

IV. Well-posedness of the continuous-time closed-loop dynamics

A. Multivalued control law

In this section we present the multivalued control law inspired by the controller proposed in [34], [35] for the case when the parameters are known. Subsequently we will establish the existence of solutions and stability of the closed-loop.

Let us introduce the position error $\tilde{q} = q - q_{\mathrm{d}}$ and the sliding surface $\sigma = \dot{\tilde{q}} + \Lambda\tilde{q}$, which will be used in order to maintain the error signal around zero. Here, the matrix $-\Lambda \in \mathbb{R}^{n\times n}$ is Hurwitz and satisfies $K_p\Lambda = \Lambda^\top K_p > 0$ for a symmetric and positive definite matrix $K_p \in \mathbb{R}^{n\times n}$. The proposed control law has the following form:

$$\tau(q,\dot{q}) = \hat{M}(q)\ddot{q}_{\mathrm{r}} + \hat{C}(q,\dot{q})\dot{q}_{\mathrm{r}} + \hat{G}(q) - K_p\tilde{q} + u, \qquad (7)$$

where $\dot{q}_{\mathrm{r}} = \dot{q}_{\mathrm{d}} - \Lambda\tilde{q}$, $K_p \in \mathbb{R}^{n\times n}$, $K_p = K_p^\top > 0$. The term u accounts for the multivalued part of the controller and is specified below. The matrices $\hat{M}(q)$, $\hat{C}(q,\dot{q})$ and $\hat{G}(q)$ describe the nominal system and are assumed to fulfill Assumptions 1 to 5 (although with different bounds). In other words, we assume that all the uncertainties are in the system parameters and not in the structure of the matrices.

Assumption 6. *The matrices* $\hat{M}(q)$*,* $\hat{C}(q,\dot{q})$ *together with the vector* $\hat{G}(q)$ *satisfy the following inequalities for all* $(t,q,\dot{q}) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$ *and some known positive constants* $\hat{k}_1, \hat{k}_2, \hat{k}_C$ *and* $\hat{k}_G$

$$0 < \hat{k}_1 \le \|\hat{M}(q)\|_m \le \hat{k}_2, \qquad \|\hat{C}(q,\dot{q})\|_m \le \hat{k}_C\|\dot{q}\|,$$
$$\|\hat{G}(q)\| \le \hat{k}_G\|q\|.$$

After some simple manipulations on (6) and (7), the closed-loop system results in

$$M(q)\dot{\sigma} + C(q,\dot{q})\sigma + K_p\tilde{q} + \xi(t,\sigma,\tilde{q}) = u, \qquad (8a)$$
$$\dot{\tilde{q}} = \sigma - \Lambda\tilde{q}, \qquad (8b)$$

where the new function $\xi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ accounts for all the uncertainties in the system and is given by

$$\xi(t,\sigma,\tilde{q}) = F(t,q,\dot{q}) + \Delta M(q)\ddot{q}_{\mathrm{r}} + \Delta C(q,\dot{q})\dot{q}_{\mathrm{r}} + \Delta G(q), \quad (9)$$

where $\Delta M(q) = M(q) - \hat{M}(q)$, $\Delta C(q,\dot{q}) = C(q,\dot{q}) - \hat{C}(q,\dot{q})$ and $\Delta G(q) = G(q) - \hat{G}(q)$. Note that the closed-loop system (8) slightly differs from the closed-loop system in [34], [35], since we have omitted the term proportional to σ and we have added the term $K_p\tilde{q}$ instead. Additionally, it is worth to mention that the function ξ is not uniformly bounded but it is still upper-bounded by a locally Lipschitz continuous function of positions and velocities, as the following proposition reveals. This fact is a nice feature of the passivity-based control not shared by other nonlinear control techniques like feedback linearization.

Proposition 2. *The function* $\xi(t,\sigma,\tilde{q})$ *satisfies*

$$\|\xi(t,\sigma,\tilde{q})\| \le \beta(\sigma,\tilde{q}),$$

where $\beta(\sigma,\tilde{q}) = c_1 + c_2\|\sigma\| + c_3\|\tilde{q}\| + c_4\|\tilde{q}\|\|\sigma\| + c_5\|\tilde{q}\|^2$*, for known positive constants* c_i*,* $i = 1,\dots,5$*.*

Proof. From (9) we have

$$\|\xi(t,\sigma,\tilde{q})\| \le \|F(t,q,\dot{q})\| + \|\Delta M(q)\ddot{q}_{\mathrm{r}}\| + \|\Delta C(q,\dot{q})\dot{q}_{\mathrm{r}})\| + \|\Delta G(q)\|. \quad (10)$$

It follows from Assumption 1 that the first term on the right-hand side of (10) is bounded by a constant k_F. The following terms satisfy

$$\|\Delta M(q)\ddot{q}_{\mathrm{r}}\| \le (k_2 + \hat{k}_2)(\|\ddot{q}_{\mathrm{d}}\| + \|\Lambda\|_m\|\sigma\| + \|\Lambda\|_m^2\|\tilde{q}\|).$$
$$\|\Delta C(q,\dot{q})\dot{q}_{\mathrm{r}}\| \le (k_C + \hat{k}_C)\left[(\|\dot{q}_{\mathrm{d}}\| + \|\Lambda\|_m\|\tilde{q}\|)^2 + (\|\dot{q}_{\mathrm{d}}\| + \|\Lambda\|_m\|\tilde{q}\|)\|\sigma\|\right],$$
$$\|\Delta G(q)\| \le (k_G + \hat{k}_G)(\|q_d\| + \|\tilde{q}\|).$$

Inasmuch as the variables $q_{\mathrm{d}}, \dot{q}_{\mathrm{d}}$ and $\ddot{q}_{\mathrm{d}}$ are known and bounded, we obtain the desired result. □

Now we define the multivalued part of the control law τ as

$$-u \in \gamma(\sigma,\tilde{q})\partial\Phi(\sigma), \qquad (11)$$

where the function $\gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is locally Lipschitz continuous and is specified below in Theorem 1. Additionally, $\Phi \in \Gamma_0(\mathbb{R}^n)$ and is selected in such a way that the following assumption is fulfilled.

Assumption 7. *The function* $\Phi \in \Gamma_0(\mathbb{R}^n)$ *has effective domain equal to* $\mathbb{R}^n$ *and satisfies* $0 = \Phi(0) \le \Phi(w)$ *for all* $w \in \mathbb{R}^n$*. Also, we have that* $0 \in \operatorname{int}\partial\Phi(0)$*.*

Notice that Assumption 7 rules out linear controllers. Indeed, we require Φ to be convex and to be non differentiable at the origin, e.g., the norm function $\|\cdot\|$. It is noteworthy that the condition $0 \in \operatorname{int}\partial\Phi(0)$ is essential for the finite-time convergence of the sliding variable σ (see the proofs of Theorem 2 and Lemma 6). The key property is established formally in the following proposition.

Proposition 3. *The following assertions are equivalent:*

- $0 \in \operatorname{int}\partial\Phi(0)$,
- *There exists* $\alpha > 0$ *such that,* $\Phi(\cdot) \ge \alpha\|\cdot\|$*.*

Proof. Let $0 \in \operatorname{int}\partial\Phi(0)$, i.e., there exists $\alpha > 0$ such that for all $\rho \in \alpha\mathbb{B}_n$ and all $\eta \in \mathbb{R}^n$, we have $\langle\rho,\eta\rangle \le \Phi(\eta)$. Equivalently, $\sup\{\langle\rho,\eta\rangle \mid \rho \in \alpha\mathbb{B}_n\} \le \Phi(\eta)$ for all $\eta \in \mathbb{R}^n$ and consequently $\alpha\|\eta\| \le \Phi(\eta)$ for all $\eta \in \mathbb{R}^n$. □

B. Existence of solutions

The next step consists in establishing the existence of solutions for the closed-loop system (8) when the multivalued control (11) is applied. To this end we start using the Yosida approximation of the set-valued map $\partial\Phi(\cdot)$ and after proving the boundedness of σ and $\tilde{q}$, we can conclude the existence of solutions of the differential equation taking a limit process. Let $\Phi^\lambda(\sigma)$ be the Moreau-Yosida approximation of Φ of index λ at the point $\sigma \in \mathbb{R}^n$. As is pointed out in [6, Th. 4 Sec. 3.4], the gradient $\nabla\Phi^\lambda(\sigma)$ corresponds to the Yosida approximation of $\partial\Phi(\sigma)$. Thus, the *approximated* closed-loop dynamics satisfy

$$\begin{aligned} M(q_\lambda)\dot{\sigma}_\lambda + C(q_\lambda, \dot{q}_\lambda)\sigma_\lambda \\ +K_p\tilde{q}_\lambda + \xi(t, \sigma_\lambda, \tilde{q}_\lambda) = -\gamma(\sigma_\lambda, \tilde{q}_\lambda)\nabla\Phi^\lambda(\sigma_\lambda), \quad &(12a)\\ \dot{\tilde{q}}_\lambda = \sigma_\lambda - \Lambda\tilde{q}_\lambda. \quad &(12b)\end{aligned}$$

As a first step we prove that the closed-loop system (12) is well-posed. Once the well-posedness of the approximated system is established, the existence of solutions for the differential inclusion (8)-(11) can be established by applying [4, Theorem 4.2]. This procedure is formalized in Theorem 1. The proof is only outlined, but can be found in its complete version in [29].

Theorem 1. *Let Assumptions 1-7 hold. Then, there exists a solution* $\sigma : [0, +\infty) \to \mathbb{R}^n$, $\tilde{q} : [0, +\infty) \to \mathbb{R}^n$ *of* (8)-(11) *for every* $(\sigma_0, \tilde{q}_0) \in \mathbb{R}^n \times \mathbb{R}^n$*, whenever:*

$$\frac{\alpha}{2}\gamma(\sigma, \tilde{q}) \geq \beta(\sigma, \tilde{q}), \tag{13}$$

where β *is specified in Proposition 2 and* α *is given in Proposition 3. The notion of* solution *is taken in the following sense:*

- σ *is continuous with derivative* $\dot{\sigma}$ *continuous and bounded in bounded sets.*
- $\tilde{q}$ *is continuous with derivative* $\dot{\tilde{q}}$ *continuous and bounded in bounded sets.*
- *Equations* (8)-(11) *are satisfied for almost all* $t \in [0, +\infty)$.
- $\sigma(0) = \sigma_0$ *and* $\tilde{q}(0) = \tilde{q}_0$.

Sketch of the proof. Clearly, equation (12) is equivalent to

$$\begin{bmatrix} \dot{\sigma}_\lambda \\ \dot{\tilde{q}}_\lambda \end{bmatrix} = \begin{bmatrix} -M^{-1}(q_\lambda)G(t, q_\lambda, \dot{q}_\lambda, \sigma_\lambda) \\ \sigma_\lambda - \Lambda\tilde{q}_\lambda \end{bmatrix}, \tag{14}$$

where

$$\begin{aligned} G(t, q_\lambda, \dot{q}_\lambda, \sigma_\lambda) = C(q_\lambda, \dot{q}_\lambda)\sigma_\lambda + K_p\tilde{q}_\lambda \\ + \xi_\lambda + \gamma(\sigma_\lambda, \tilde{q}_\lambda)\nabla\Phi^\lambda(\sigma_\lambda)\end{aligned}$$

and $\xi_\lambda = \xi(t, \sigma_\lambda, \tilde{q}_\lambda)$. Since all the terms in ξ_λ are locally Lipschitz together with the function γ and the gradient $\nabla\Phi^\lambda$, it is easy to see that the term

$$\begin{aligned} g(\sigma_\lambda, \tilde{q}_\lambda) := C(q_\lambda, \dot{q}_\lambda)\sigma_\lambda + K_p\tilde{q}_\lambda + \xi_\lambda \\ + \gamma(\sigma_\lambda, \tilde{q}_\lambda)\nabla\Phi^\lambda(\sigma_\lambda)\end{aligned}$$

is locally Lipschitz as well. It is not difficult to show that Assumption 2 ensures the local Lipschitz property for $M^{-1}(q_\lambda)g(\sigma_\lambda, \tilde{q}_\lambda)$. In conclusion, the right-hand side of (14) is locally Lipschitz continuous and, therefore, there exists a unique solution in an interval $[0, T)$ in the sense of Caratheodory.

Now we prove that the solution exists for all $T > 0$. Following [37] and [12, p. 403], we consider the energy function $H(\sigma_\lambda, \tilde{q}_\lambda) = \frac{1}{2}\sigma_\lambda^\top M(q_\lambda)\sigma_\lambda + \frac{1}{2}\tilde{q}_\lambda^\top K_p\tilde{q}_\lambda$. The derivative of H along trajectories of (14) takes the form

$$\begin{aligned} \dot{H}(\sigma_\lambda, \tilde{q}_\lambda) = -\tilde{q}_\lambda^\top K_p\Lambda\tilde{q}_\lambda \\ - \gamma(\sigma_\lambda, \tilde{q}_\lambda)\langle\nabla\Phi^\lambda(\sigma_\lambda), \sigma_\lambda\rangle - \langle\xi_\lambda, \sigma_\lambda\rangle. \end{aligned} \tag{15}$$

The following step consists in describing more precisely the term $\langle\nabla\Phi^\lambda(\sigma_\lambda), \sigma_\lambda\rangle$. As pointed out in Remark 1, the function Φ^λ is convex and differentiable. Therefore, its gradient satisfies

$$\begin{aligned} -\langle\nabla\Phi^\lambda(\sigma_\lambda), \sigma_\lambda\rangle \leq -f^\lambda(\sigma_\lambda) \\ = -\alpha\|\operatorname{Prox}_{\lambda f}(\sigma_\lambda)\| - \frac{1}{2\lambda}\left\|\operatorname{Prox}_{\lambda f}(\sigma_\lambda) - \sigma_\lambda\right\|^2, \end{aligned} \tag{16}$$

where we have used [17, Th. 4.1.1] and Proposition 3 with $f(\cdot) = \alpha\|\cdot\|$. The substitution of (16) into (15) together with Lemma 2 leads to

$$\dot{H}(\sigma_\lambda, \tilde{q}_\lambda) \leq -\tilde{q}_\lambda^\top K_p\Lambda\tilde{q}_\lambda - \left(\frac{\alpha}{2}\gamma(\sigma_\lambda, \tilde{q}_\lambda) - \beta(\sigma_\lambda, \tilde{q}_\lambda)\right)\|\sigma_\lambda\| \tag{17}$$

for all $\|\sigma_\lambda\| > \alpha\lambda$, which is strictly negative for all $\tilde{q}_\lambda$ in view of (13). On the other hand, considering the case when $\|\sigma_\lambda\| \leq \alpha\lambda$ and $\|\tilde{q}\| \geq r$, we have that

$$\dot{H}(\sigma_\lambda, \tilde{q}_\lambda) \leq -\left[\lambda_{\min}(K_p\Lambda) - \alpha\lambda\rho\right]\|\tilde{q}_\lambda\|^2 - \frac{1}{2\lambda}\gamma\|\sigma_\lambda\|^2, \tag{18}$$

where $\rho = (c_1 + c_2\alpha\lambda/r^2 + (c_3 + c_4\alpha\lambda)/r + c_5$ and the positive constants c_i, $i \in \{1, \ldots, 5\}$ are defined in Proposition 2. Thus, it becomes clear that (18) is negative for all $\lambda > 0$ small enough. It follows that any level set $V_c := \{(\sigma_\lambda, \tilde{q}_\lambda) \in \mathbb{R}^{2n} | H(\sigma_\lambda, \tilde{q}_\lambda) \leq c\}$ that contains the ball $\sqrt{\alpha^2\lambda^2 + r^2}\mathbb{B}_{2n}$, is positively invariant and attractive since $\dot{H} < 0$ outside V_c. Hence, the pair $(\sigma_\lambda, \tilde{q}_\lambda)$ is ultimately bounded (see, e.g., [22, Sec. 4.8]). Finally, the existence of a solution of (8)-(11) is established by taking the limit as $\lambda \to 0$. Formally, this is a direct application of [4, Theorem 4.2] (taking into account that $\ddot{q}$ corresponds to $\dot{\sigma}$ in our setting). □

Remark 2. The previous proof differs from the proof of Lemma 4.1 in [4] in that the stringent Assumption $\mathcal{H}_{\Phi,F}$ is relaxed. Such assumption imposed a severe restriction on the relation between the Moreau-Yosida approximation of Φ and the disturbance. Instead, we only require (13) and $0 \in \operatorname{int}\partial\Phi(0)$ for the design function. It is thus possible to consider a much larger and realistic class of disturbances.

Remark 3. Theorem 1 does not guarantee uniqueness of solutions. An analytic proof of uniqueness requires more stringent properties which are rarely satisfied in practical cases [4, Section 5].

C. Case with a constant gain γ

The well-posedness of the closed-loop system (8)-(11) has been proved above making use of a state dependent gain

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2662804, IEEE Transactions on Automatic Control

$\gamma(\sigma, \tilde{q})$ which satisfies (13). An important case of study is when the control gain is set constant, which simplifies the implementation.

Corollary 1. *Let Assumptions 1-7 hold. Consider the system*

$$\begin{aligned} M(q_\lambda)\dot{\sigma}_\lambda + C(q_\lambda, \dot{q}_\lambda)\sigma_\lambda & \\ + K_p\tilde{q}_\lambda + \xi(t, \sigma_\lambda, \tilde{q}_\lambda) &= -\gamma\nabla\Phi^\lambda(\sigma_\lambda), \quad (19a) \\ \dot{\tilde{q}}_\lambda &= \sigma_\lambda - \Lambda\tilde{q}_\lambda. \quad (19b) \end{aligned}$$

Fix $R > 0$ and define the set

$$W_{R,K_p} := \{(\sigma_\lambda, \tilde{q}_\lambda) \in \mathbb{R}^n \times \mathbb{R}^n \mid H(\sigma_\lambda, \tilde{q}_\lambda) \leq R\}, \quad (20)$$

where $H(\sigma_\lambda, \tilde{q}_\lambda) = \frac{1}{2}\sigma_\lambda^\top M(q_\lambda)\sigma_\lambda + \frac{1}{2}\tilde{q}_\lambda^\top K_p\tilde{q}_\lambda$. Let k_{p_2} be the maximum eigenvalue of K_p and let R_ξ be a positive constant satisfying

$$R_\xi = \max_{(\sigma,\tilde{q})\in W_{R,K_p}} \beta(\sigma, \tilde{q}). \quad (21)$$

Then, for any initial condition $(\sigma_\lambda(0), \tilde{q}_\lambda(0)) \in W_{R,K_p}$, the unique solution of (19) is bounded for all $t \geq 0$ whenever

$$\gamma > 2\frac{R_\xi}{\alpha} \quad \text{and} \quad \alpha^2\lambda^2 + r^2 \leq 2\frac{R}{\max\{k_2, k_{p_2}\}},$$

for some $r > 0$. Moreover, the system is semi-globally practically asymptotically stable.

Proof. The proof is a consequence of Theorem 1. Indeed, the condition $\alpha^2\lambda^2 + r^2 \leq 2R/\max\{k_2, k_{p_2}\}$ implies that $\sqrt{\alpha^2\lambda^2 + r^2}\mathbb{B}_{2n} \subset W_{R,K_p}$ and, using the same arguments as those in the proof of Theorem 1, it follows that $\dot{H} < 0$ for all $(\sigma_\lambda, \tilde{q}_\lambda) \in \mathrm{bd}(W_{R,K_p})$. This proves the positive invariance of W_{R,K_p}. Moreover, we have semi-global practical stability since the trajectories converge to the smallest level set which contains the ball $\sqrt{\alpha^2\lambda^2 + r^2}\mathbb{B}_{2n}$, and this set can be made arbitrary small by decreasing the value of λ (consider, for example, $r = \lambda^{1/3}$ in the proof of Theorem 1). □

It is clear that, for any fixed $R > 0$ and the conditions of Corollary 1 satisfied, Theorem 1 ensures the existence of solutions of the differential inclusion (8) with the multivalued controller $u \in -\gamma\partial\Phi(\sigma)$.

V. Robust stability of the closed-loop system

In this section we prove how the trajectory tracking is robustly achieved in the presence of external bounded disturbances and parametric uncertainties. Additionally, we show that the variable σ reaches zero in finite time. In order to obtain an upper bound of the reaching time, we shall take only the dynamic equation related to σ and make $K_p = 0$, i.e.,

$$M(q)\dot{\sigma} + C(q, \dot{q})\sigma + \xi(t, \sigma, \tilde{q}) \in -\gamma(\sigma, \tilde{q})\partial\Phi(\sigma). \quad (22)$$

From the closed-loop equation (8b) we see that the finite-time stabilization of the variable σ implies the asymptotic stability of the error $\tilde{q}$ and its derivatives.

Theorem 2. *Consider system (22). Let the assumptions of Theorem 1 hold. Set $\gamma(\sigma, \tilde{q}) = (2\beta(\sigma, \tilde{q}) + \delta)/\alpha$, where $\delta > 0$ is constant and β is defined as in Proposition 2. Then, the sliding surface $\sigma = 0$ is reached in finite time.*

Proof. Consider the function $V(\sigma, t) = \frac{1}{2}\sigma^\top M(q(t))\sigma$, which is positive definite as a function of σ alone. Taking the time-derivative of V along the trajectories of (22) leads to

$$\dot{V} \leq -\gamma(\sigma, \tilde{q})\langle\zeta, \sigma\rangle + \|\xi(t, \sigma, \tilde{q})\|\|\sigma\|,$$

where $\zeta \in \partial\Phi(\sigma)$ and Property 1 was used. From the definition of the subdifferential and from Proposition 3, it follows that $-\langle\zeta, \sigma\rangle \leq -\Phi(\sigma) \leq -\alpha\|\sigma\|$, which yields

$$\dot{V} \leq -\left[\alpha\gamma(\sigma, \tilde{q}) - \beta(\sigma, \tilde{q})\right]\|\sigma\|.$$

Hence, if $\alpha\gamma(\sigma, \tilde{q}) = \beta(\sigma, \tilde{q}) + \delta$ where δ is a positive constant, we obtain $\dot{V} \leq -\delta\|\sigma\| = -\delta\sqrt{\frac{2}{k_2}}V^{1/2}$. By applying the Comparison Lemma and integrating over the time-interval $[0, t]$ we obtain $V^{1/2}(t) \leq V^{1/2}(0) - \frac{\delta}{\sqrt{2k_2}}t$. Consequently, V reaches zero in a finite time t^* bounded by $t^* \leq \frac{\sqrt{2k_2}}{\delta}V^{1/2}(0)$. □

Remark 4. The case $K_p \neq 0$ makes the computations more laborious and is left to the reader. The main difficulty resides in the fact that it is no longer possible to analyze the sliding and the error dynamics separately. It is then necessary to consider the *full* Lyapunov function $V = \sigma^\top M\sigma + \tilde{q}^\top K_p\tilde{q}$ or to use a small-gain-theorem approach.

Now that the global asymptotic stability of the origin has been established in the presence of parametric uncertainty and external disturbances using a state-dependent gain $\gamma(\sigma, \tilde{q})$, we will derive stability conditions for the case when γ is constant.

Theorem 3. *Let the assumptions of Theorem 1 hold. Consider system (8) with the multivalued control law $u \in -\gamma\partial\Phi(\sigma)$ and consider a compact set W_{R,K_p} as in (20) with $R > 0$ fixed. The origin of the closed-loop system is semi-globally asymptotically stable. Moreover, the basin of attraction contains W_{R,K_p} whenever*

$$\gamma > \frac{R_\xi}{\alpha}, \quad (23)$$

with R_ξ as in (21).

Proof. The result follows from the fact that W_{R,K_p} is positively invariant (the proof is similar to the one of Corollary 1) and the fact that, for $\zeta \in \partial\Phi(\sigma)$, we have that $\dot{H} \leq -\tilde{q}^\top K_p\Lambda\tilde{q} - (\gamma\alpha - R_\xi)\|\sigma\|$. It is clear that $\dot{H}$ is negative definite whenever (23) holds. □

VI. Implicit discrete-time sliding-mode trajectory tracking control

A. Discrete-time controller design

This section is devoted to the analysis of the discrete-time version of the above robust set-valued passivity-based control algorithms. We consider an implicit time discretization similar to the one proposed in [1], [2], [20] for linear time-invariant systems with known parameters. The difficulty in extending the above-mentioned method resides in the facts that the plant is now nonlinear (which prevents us from using an exact integration like the zero-order-hold method), the controller is also nonlinear and, most importantly, we allow for parametric uncertainties.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2662804, IEEE Transactions on Automatic Control

Let us start with the following Euler discretization of the plant (6):

$$M(q_k)\frac{\dot q_{k+1}-\dot q_k}{h}+C(q_k,\dot q_k)\dot q_{k+1} \\ +G(q_k)+F(t_k,q_k,\dot q_k)=\tau_k, \quad (24a)$$

$$q_{k+1}=q_k+h\dot q_k. \quad (24b)$$

(Henceforth, for a given function F, F_k denotes $F(t_k)$.)

Assuming that the matrix $\hat C(q,\dot q)$ is also computed using the Christoffel's symbols from $\hat M(q)$, then the pair $\hat M(q)$, $\hat C(q,\dot q)$ satisfies a property similar to Property 1.

Property 2. *The matrices $\hat M(q)$ and $\hat C(q,\dot q)$ satisfy*

$$\frac{d}{dt}\hat M(q(t))=\hat C(q(t),\dot q(t))+\hat C^\top(q(t),\dot q(t)).$$

Notice that Property 2 is not necessary in the continuous-time case since the explicit selection of the controller was not specified. This stands in contrast to the discrete-time setting, where one of our main concerns is obtaining a numerical value for the control input at each time step (see (32) below).

Lemma 3. *For any $k\geq 0$ we have*

$$\hat M_{k+1}-\hat M_k=h\hat C_k+h\hat C_k^\top+\hat\epsilon_k, \quad (25a)$$

$$M_{k+1}-M_k=hC_k+hC_k^\top+\epsilon_k, \quad (25b)$$

where h is the time step and satisfies $t_{k+1}-t_k=h$, and $\epsilon_k,\hat\epsilon_k\in\mathbb{R}^{n\times n}$ are $o(h)$ ('little-o') matrix functions, i.e.,

$$\lim_{h\downarrow 0}\frac{\|\hat\epsilon_k\|_m}{h}=\lim_{h\downarrow 0}\frac{\|\epsilon_k\|_m}{h}=0.$$

Proof. Obtained from the expansion in Taylor series of Properties 1 and 2 [5]. □

Following the same methodology as in the continuous-time problem, we introduce the position error $\tilde q_k=q_k-q_k^{\mathrm d}$ as well as the sliding surface $\sigma_k=\dot{\tilde q}_k+\Lambda\tilde q_k$, where $\tilde q_{k+1}=\tilde q_k+h\dot{\tilde q}_k$, $\Lambda\in\mathbb{R}^{n\times n}$ is a Hurwitz matrix as in the continuous-time case, and $q_k^{\mathrm d}$ refers to the sample of the reference trajectory at time t_k. We propose the control law τ_k as

$$\tau_k=\hat M_k\frac{\dot q^r_{k+1}-\dot q^r_k}{h}+\hat C_k\dot q^r_{k+1}+\hat G_k+u_k, \quad (26a)$$

$$q^r_{k+1}=q^r_k+h\dot q^r_k, \quad (26b)$$

where $\dot q^r_k=\dot q^{\mathrm d}_k-\Lambda\tilde q_k$ and u_k refers to the multivalued part of the controller plus an additional dissipation term specified below. After some simple algebraic manipulations, the closed-loop system is obtained from (24) and (26) as

$$M_k\sigma_{k+1}-M_k\sigma_k+hC_k\sigma_{k+1}=-h\xi_k+hu_k, \quad (27a)$$

$$\tilde q_{k+1}=(I-h\Lambda)\,\tilde q_k+h\sigma_k, \quad (27b)$$

where $\sigma_{k+1}=\sigma_k+h\dot\sigma_k$, $\tilde q_{k+1}=\tilde q_k+h\dot{\tilde q}_k$ and the equivalent disturbance $\xi_k:=\xi(t_k,\sigma_k,\tilde q_k)$ is given by

$$\xi_k=F_k+\left(M_k-\hat M_k\right)\left(\ddot q^d_k-\Lambda\left(\sigma_k-\Lambda\tilde q_k\right)\right)+G_k-\hat G_k \\ +\left(C_k-\hat C_k\right)\left(\dot q^{\mathrm d}_{k+1}-\Lambda\left[(I-h\Lambda)\,\tilde q_k+h\sigma_k\right]\right). \quad (28)$$

It is easy to prove that the discrete-time version of the disturbance ξ_k satisfies an analogue version of Proposition 2:

Proposition 4. *The function $\xi(t_k,\sigma_k,\tilde q_k)$ satisfies*

$$\|\xi(t_k,\sigma_k,\tilde q_k)\|\leq\beta(\sigma_k,\tilde q_k),$$

where

$$\beta(\sigma_k,\tilde q_k)=c_1+c_2\|\sigma_k\|+c_3\|\tilde q_k\|+c_4\|\tilde q_k\|\|\sigma_k\|+c_5\|\tilde q_k\|^2$$

and c_i, $i=1,\dots,5$ are known positive constants.

Proof. The result is obtained by following the same steps as in the proof of Proposition 2. □

If u_k is well-posed and non anticipative, i.e., if it depends only on the data available at time t_k, then the control law τ_k will be non anticipative as well. Simple computations reveal that (26) is equivalent to

$$\tau_k=\hat M_k\left(\ddot q^{\mathrm d}_k-\Lambda\left(\sigma_k-\Lambda\tilde q_k\right)\right)+\hat G_k+u_k \\ +\hat C_k\left(\dot q^{\mathrm d}_{k+1}-\Lambda\left[(I-h\Lambda)\,\tilde q_k+h\sigma_k\right]\right). \quad (29)$$

Equation (27) leads us to the following.

Assumption 8. *The step length $h>0$ is small enough such that the spectrum of $I-h\Lambda$ is contained in the interior of the complex unitary circle.*

At this point we specify the remaining term u_k in a similar way as its counterpart in continuous-time (11),

$$-u_k\in K_\sigma\hat\sigma_{k+1}+\gamma\partial\Phi(\hat\sigma_{k+1}), \quad (30)$$

where $K_\sigma=K_\sigma^\top>0$. The gain $\gamma>0$ is considered constant and $\hat\sigma_{k+1}$ is defined by the nominal version of (27a),

$$\hat M_k\hat\sigma_{k+1}-\hat M_k\sigma_k+h\hat C_k\hat\sigma_{k+1}+hK_\sigma\hat\sigma_{k+1}\in-h\gamma\partial\Phi(\hat\sigma_{k+1}). \quad (31)$$

Since the equivalent disturbance ξ_k is unknown, we will compute the controller from the nominal unperturbed plant (31) with state $\hat\sigma_k$ and using (27) as follows:

$$M_k\sigma_{k+1}-M_k\sigma_k+hC_k\sigma_{k+1} \\ +hK_\sigma\hat\sigma_{k+1}-h\xi_k=-h\gamma\zeta_{k+1}, \quad (32a)$$

$$\zeta_{k+1}\in\partial\Phi(\hat\sigma_{k+1}), \quad (32b)$$

$$\hat M_k\hat\sigma_{k+1}-\hat M_k\sigma_k+h\hat C_k\hat\sigma_{k+1} \\ +hK_\sigma\hat\sigma_{k+1}=-h\gamma\zeta_{k+1}, \quad (32c)$$

$$\tilde q_{k+1}=(I-h\Lambda)\,\tilde q_k+h\sigma_k, \quad (32d)$$

Notice that the discrete-time closed-loop system (32) is slightly different from the direct discretization of the continuous-time closed-loop system (8), since it contains a new term $K_\sigma\hat\sigma_{k+1}$ and we have made $K_p=0$. The additional term $K_\sigma\hat\sigma_{k+1}$ will assure the stability of the closed-loop system by adding dissipation, as is shown in the proofs of Theorems 4 and 5. From now on we will concentrate our attention on equations (32a)-(32c), for which, if some stability properties are preserved, then the boundedness of the solutions of the difference equation (32d) follows. Moreover, from Assumption 8 we have that $\tilde q_k\to 0$ as $\sigma_k\to 0$ and $k\to+\infty$.

System (32a)-(32d) may be viewed as follows: Equations (32a) and (32d) are the Euler discretization of the plant with a pre-feedback, (32c) is a nominal unperturbed system and (32b) is the discretized set-valued controller to be calculated from (32c). From (32) it becomes clear that, when all uncertainties and disturbances vanish, $\hat{\sigma}_k = \sigma_k$ whenever $\hat{\sigma}_0 = \sigma_0$.

First, we prove the well-posedness of the general scheme (32), i.e., we prove that we can compute a selection of the multivalued controller (32b) in a unique fashion, using only the information available at the time step k. We note first that (32c) and (32b) imply

$$(\hat{M}_k + h\hat{C}_k + hK_\sigma)\hat{\sigma}_{k+1} - \hat{M}_k\sigma_k \in -h\gamma\partial\Phi(\hat{\sigma}_{k+1}). \quad (33)$$

Equivalently,

$$\left\langle \hat{\mathcal{A}}_k\hat{\sigma}_{k+1} - \hat{M}_k\sigma_k, \eta - \hat{\sigma}_{k+1}\right\rangle + h\gamma\Phi(\eta) - h\gamma\Phi(\hat{\sigma}_{k+1}) \geq 0, \quad (34)$$

for all $\eta \in \mathbb{R}^n$, where $\hat{\mathcal{A}}_k := \left(\hat{M}_k + h\hat{C}_k + hK_\sigma\right)$. It is clear from Lemma 1 that $\hat{\sigma}_{k+1}$ is uniquely determined if the operator $\hat{\mathcal{A}}_k$ is strongly monotone. Additionally, note that $\hat{\sigma}_{k+1}$ depends on $\hat{\mathcal{A}}_k, \hat{M}_k, \sigma_k, h, \gamma$ and Φ only (all of them available at time step k). In order to obtain conditions for the strong monotonicity of $\hat{\mathcal{A}}_k$ we note that, for any $w \in \mathbb{R}^n$,

$$\left\langle \hat{\mathcal{A}}_k w, w\right\rangle \geq \left(\hat{k}_1 + h\kappa_1 - \frac{\|\hat{\epsilon}_k\|_m}{2}\right)\|w\|^2, \quad (35)$$

where κ_1 is the minimum eigenvalue of K_σ and we have made use of Assumption 6 and Lemma 3. Hence, $\hat{\mathcal{A}}_k$ is strongly monotone for any h small enough such that

$$\frac{\hat{k}_1}{2} + h\kappa_1 - \frac{\|\hat{\epsilon}_k\|_m}{2} \geq 0. \quad (36)$$

By applying Lemma 1 we obtain the uniqueness of $\hat{\sigma}_{k+1}$. Moreover, the solution $\hat{\sigma}_{k+1}$ is Lipschitz continuous with respect to σ_k. It is noteworthy that the condition on the strong monotonicity of $\hat{\mathcal{A}}_k$ of Lemma 1 can be relaxed using the approach developed in [3, §2.7]. It is possible to derive an implicit formulation for the solution of (34), so that it can be easily found numerically. The following Lemma provides the means to accomplish that.

Lemma 4. *Consider the following variational inequality of the second kind,*

$$\langle Px - r, \eta - x\rangle + \phi(\eta) - \phi(x) \geq 0 \quad \text{for all } \eta \in \mathbb{R}^n \quad (37)$$

with $P \in \mathbb{R}^{n\times n}$ a strongly monotone operator (but not necessarily symmetric). Then, the unique solution of (37) satisfies

$$x = \mathrm{Prox}_{\mu\phi}\left((I - \mu P)\,x + \mu r\right) \quad (38a)$$
$$= \left(Id - \mu\,\mathrm{Prox}_{\phi^\star/\mu} \circ \mu^{-1} Id\right)\left((I - \mu P)x + \mu r\right) \quad (38b)$$

for some $\mu > 0$. Moreover, there exists $\mu > 0$ such that the map $x \mapsto \mathrm{Prox}_{\mu\phi}\left((I - \mu P)\,x + \mu r\right)$ is a contraction.

Proof. Let x be the solution of (37). Then, for any $\mu > 0$, we have $\mu r - \mu Px \in \partial(\mu\phi)(x)$ or, equivalently, $(I - \mu P)x + \mu r - x \in \partial(\mu\phi)(x)$. Hence, $x = \mathrm{Prox}_{\mu\phi}((I - \mu P)x + \mu r)$. The second equality in (38) is a direct consequence of Moreau's decomposition Theorem (Proposition 1). Recalling that $\mathrm{Prox}_{\mu\phi}$ is a non expansive operator, we have that

$$\|\mathrm{Prox}_{\mu\phi}(y_1) - \mathrm{Prox}_{\mu\phi}(y_2)\| \leq \|I - \mu P\|_m \|x_1 - x_2\|,$$

where $y_i = (I - \mu P)x_i + \mu r$, $i = 1, 2$. Now, because we are using the Euclidean norm we have that the induced norm of a matrix A satisfies $\|A\|_m = \sqrt{\lambda_{\max}(A^\top A)}$ [26, p. 365 Exercise 5]. Thus, if $I - (I - \mu P)^\top (I - \mu P)$ is positive definite, then the map defined by $x \mapsto \mathrm{Prox}_{\mu\phi}((I - \mu P)x + \mu r)$ is a contraction. The condition for positive definiteness reads

$$0 < P + P^\top - \mu P^\top P$$

which, by the strong monotonicity of P, is readily satisfied by selecting μ small enough. □

Remark 5. There are several ways to numerically solve problems of the form (37), like the semi-smooth Newton method [14, §7.5] advocated in [5, Section 6]. For control applications this method may be too time-consuming since it involves the computation of inverse matrices and proximal maps of composite functions. In contrast, the simple method of successive approximations [24, §14] can quickly find the fixed point or (38). Details about the implementation are given in Section VII.

According to Lemma 4, the selection of the control value can be obtained from (32b), (32c) as

$$\zeta_{k+1} = -\frac{1}{h\gamma}\left(\hat{\mathcal{A}}_k\hat{\sigma}_{k+1} - \hat{M}_k\sigma_k\right) \quad (39a)$$
$$\hat{\sigma}_{k+1} = \mathrm{Prox}_{\mu h\gamma\Phi}((I - \mu\hat{\mathcal{A}}_k)\hat{\sigma}_{k+1} + \mu\hat{M}_k\sigma_k), \quad (39b)$$

where $\mu > 0$ is such that $0 < \hat{\mathcal{A}}_k + \hat{\mathcal{A}}_k^\top - \mu\hat{\mathcal{A}}_k^\top\hat{\mathcal{A}}_k$. The solution of the implicit equation (39b) with unknown $\hat{\sigma}_{k+1}$ is a function of σ_k and h, and it is clear from (39a) that the controller is non-anticipative. Let us now present conditions that guarantee (36) and, consequently, the possible application of Lemma 4 to (34).

Lemma 5. *There exists $\delta^* > 0$ (depending on $\tilde{q}_0$ and σ_0) such that, for any $h \in (0, \delta^*]$ the following inequalities hold:*

$$\|\hat{\epsilon}_k\|_m \leq \min\left\{\hat{k}_1, 2h\kappa_1\right\}, \quad (40a)$$
$$\|\epsilon_k\|_m \leq \min\left\{k_1, 2h\kappa_1\right\}, \quad (40b)$$

where κ_1 is the minimum eigenvalue of K_σ and $\hat{\epsilon}_k, \epsilon_k$ satisfy (25).

Proof. It follows from Lemma 3 that (40) is always solvable. Indeed, since $\hat{\epsilon}_k$ (ϵ_k) is $o(h)$ we have that, for any $\hat{\varepsilon} > 0$ ($\varepsilon > 0$), there exists $\hat{\delta} > 0$ ($\delta > 0$) such that $\|\hat{\epsilon}_k\|_m < \hat{\varepsilon}h$ ($\|\varepsilon\| < \varepsilon h$) for all $h < \hat{\delta}$ ($h < \delta$). Therefore, by choosing $\hat{\varepsilon}$ and ε small enough, we have that both inequalities in (40) are fulfilled for all $h \in (0, \min\{\hat{\delta}, \delta\}]$. □

The previous reasoning calls our attention to a detail regarding the uniformity of h. That is, whether or not δ^* and $\hat{\delta}$ can be selected independently of the time step k. The fact that it does becomes more clear after proving that all solutions of (32) are bounded. In the mean time, the rigorous reader can set $h = h_k$. Note also that, by Lemma 5, equation (40a) implies (36).

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2662804, IEEE Transactions on Automatic Control

B. Stability of the discrete-time closed-loop system

Once the solvability of the control law has been established for each time step, we turn to the question about the stability of the closed-loop discrete-time system (32). To this end we present two cases. The first one addresses the stability issue without parametric uncertainty, whereas in the second case the full perturbation case (i.e., external disturbance and parametric uncertainty) is considered.

The following bounds will be useful.

Proposition 5. *Let Assumption 6 hold and assume that the time step $h > 0$ is such that* (40a) *is satisfied. Then, for all $k \in \mathbb{N}$ the following bounds hold:*

$$\|\hat{\mathcal{A}}_k^{-1}\|_m \leq \frac{1}{\hat{k}_1}, \tag{41}$$

$$\|\hat{\mathcal{B}}_k^{-1}\|_m \leq \frac{2}{\hat{k}_1}, \tag{42}$$

where $\hat{\mathcal{A}}_k := \hat{M}_k + h\hat{C}_k + hK_\sigma$ and $\hat{\mathcal{B}}_k := \hat{M}_k + h\hat{C}_k$.

Proof. From (35) and (40a) it follows that, for any vector $w \in \mathbb{R}^n \setminus \{0\}$, we have $\|\hat{\mathcal{A}}_k w\|\|w\| \geq \hat{k}_1\|w\|^2$, so

$$\|w\| \leq \frac{1}{\hat{k}_1}\left\|\left(\hat{M}_k + h\hat{C}_k + hK_\sigma\right)w\right\|.$$

In particular, $w = \left(\hat{M}_k + h\hat{C}_k + hK_\sigma\right)^{-1}x$ with $x \in \mathbb{R}^n \setminus \{0\}$ yields the desired result. The proof of the second inequality follows the same steps and takes into account the fact that $\hat{k}_1 - \|\hat{\epsilon}\|_m \geq 0$ (see (40a)). □

Remark 6. Bounds for matrices $\mathcal{A}_k^{-1}$ and $\mathcal{B}_k^{-1}$ (depending on M_k and C_k) can be obtained as $\|\mathcal{A}_k^{-1}\|_m \leq 1/k_1$ and $\|\mathcal{B}_k^{-1}\|_m \leq 2/k_1$ in a similar way by making use of (40b) in Proposition 5.

Before presenting the main results on the stability of the closed-loop system we show that, even in the presence of an external perturbation ξ_k, the variable $\hat{\sigma}_k$, which is a state of the nominal unperturbed system (32b)-(32c), is maintained at zero.

Lemma 6. *Let $h > 0$ be small enough such that* (40a) *holds. If*

$$\left\|\frac{\hat{M}_k\sigma_k}{h}\right\| \leq \gamma\alpha,$$

then $\hat{\sigma}_{k+1} = 0$. Moreover, suppose that $M_k = \hat{M}_k$, $C_k = \hat{C}_k$ (no parametric uncertainty), that ξ_k is uniformly bounded by some constant $0 < \bar{F} < +\infty$ and that the gain satisfies

$$2\frac{\hat{k}_2}{\hat{k}_1}\bar{F} \leq \gamma\alpha. \tag{43}$$

Then, $\hat{\sigma}_{k_0+1} = 0$ for some $k = k_0$ implies that $\hat{\sigma}_{k_0+n} = 0$ for all $n \geq 1$.

Proof. Since the solution of the variational inequality (34) is unique, we have that $\hat{\sigma}_{k+1} = 0$ if, and only if, $\hat{M}_k\sigma_k/h\gamma$ belongs to the set of minimizers of the conjugate function $\Phi^\star$. Indeed, from (33) we have the following chain of equivalences:

$$\hat{\sigma}_{k+1} = 0 \iff \frac{\hat{M}_k\sigma_k}{h\gamma} \in \partial\Phi(0) \iff$$
$$0 \in \partial\Phi^\star\left(\frac{\hat{M}_k\sigma_k}{h\gamma}\right) \iff \frac{\hat{M}_k\sigma_k}{h\gamma} \in \operatorname{Arg\,min}\Phi^\star.$$

Now, according to Assumption 7, $\Phi(\cdot) \geq \alpha\|\cdot\|$, which in fact implies $\Psi_{\alpha\mathbb{B}_n}(\cdot) \geq \Phi^\star(\cdot)$ for, recall that the conjugate function of $\alpha\|\cdot\|$ is the indicator function of the set $\alpha\mathbb{B}_n$ and $f \geq g$ implies $g^\star \geq f^\star$. Hence, we have that $\Phi^\star(w) \leq 0$ for any $w \in \alpha\mathbb{B}_n$. On the other hand, from the definition of the conjugate function, the fact that $\Phi \in \Gamma_0(\mathbb{R}^n)$ and using the Fenchel-Moreau Theorem [9, Theorem I.10], it is easy to deduce that $0 = \Phi(0) = \Phi^{\star\star}(0) = -\inf\Phi^\star$, and we have $0 \leq \Phi^\star(w)$ for all $w \in \mathbb{R}^n$. Therefore we have proved that, for any $w \in \alpha\mathbb{B}_n$, one has $\Phi^\star(w) = 0$, while $\Phi^\star(\cdot) \geq 0$ everywhere. In other words, $\alpha\mathbb{B}_n \subseteq \operatorname{Arg\,min}\Phi^\star$.

For the second part of the proof, let k_0 be such that $\hat{\sigma}_{k_0+1} = 0$. We know from (32c) that $-\hat{M}_k\sigma_{k_0} = -h\gamma\zeta_{k_0+1}$ for some $\zeta_{k_0+1} \in \partial\Phi(0)$. Substitution of σ_{k_0} in (32a) gives

$$\sigma_{k_0+1} = -h\hat{\mathcal{B}}_{k_0}^{-1}\xi_{k_0}. \tag{44}$$

Equations (42) and (43) then yield

$$\left\|\frac{\hat{M}_{k_0+1}\sigma_{k_0+1}}{h}\right\| = \left\|\hat{M}_{k_0+1}\hat{\mathcal{B}}_{k_0}^{-1}\xi_{k_0}\right\| \leq \frac{2\hat{k}_2}{\hat{k}_1}\bar{F} \leq \gamma\alpha.$$

From the inequality above we obtain $\hat{\sigma}_{k_0+2} = 0$. An induction argument allows us to conclude that $\hat{\sigma}_{k_0+n} = 0$ for all $n \geq 1$. □

In continuous-time, the selection of the set-valued controller exactly compensates for the perturbation on the sliding surface $\sigma = 0$, see (22). This is not possible in discrete-time. The following corollary gives the value of the controller once the *nominal sliding surface* $\hat{\sigma}_k = 0$ has been reached.

Corollary 2. *Under the assumptions of Lemma 6, the equivalent control which maintains the constraint $\hat{\sigma}_{k+n} = 0$ for all $n \geq 1$ is given by*

$$\zeta_{k+2}^{eq} = \frac{1}{h\gamma}\hat{M}_{k+1}\mathcal{B}_k^{-1}\left((M_k - \hat{M}_k)\sigma_k - h\xi_k\right) \tag{45}$$

with $\mathcal{B}_k = M_k + hC_k$.

Proof. According to (32c), the condition $\hat{\sigma}_{k+n} = 0$ for all $n \geq 1$ implies $\hat{M}_k\sigma_k = -h\gamma\zeta_{k+1}^{eq}$. Substitution of ζ_{k+1}^{eq} in (32a) then yields

$$(M_k + hC_k)\,\sigma_{k+1} - (M_k - \hat{M}_k)\sigma_k = -h\xi_k.$$

It follows that $\sigma_{k+1} = \mathcal{B}_k^{-1}\left((M_k - \hat{M}_k)\sigma_k - h\xi_k\right)$. Another iteration on (32c) results in $\hat{\mathcal{A}}_k\hat{\sigma}_{k+2} - \hat{M}_{k+1}\sigma_{k+1} = -h\gamma\zeta_{k+2}^{eq}$ and the result follows. □

The previous corollary has the following interpretation: The scheme in (39) for computing the controller for the nominal system (32b)-(32c) allows to compensate for the disturbance of the actual system with a delay of one time step. Obviously, the equivalent control in (45) is not directly implementable since the disturbance is unknown.

It is noteworthy that the magnitude of the equivalent control in (45) does not depend on h (cf. (44)).

Theorem 4 (Known parameters). *Let Assumptions 1-8 hold. Consider the discrete-time dynamical system* (32a)-(32c) *without parametric uncertainty ($M_k = \hat{M}_k$, $C_k = \hat{C}_k$) and ξ_k uniformly bounded by $\bar{F}$. Then, the origin $(\sigma, \hat{\sigma}) = 0$ is globally practically stable whenever*

$$\gamma\alpha \geq \max\left\{\frac{2\hat{k}_2}{\hat{k}_1}\bar{F}\left(1+\frac{\bar{F}}{\hat{k}_1\hat{r}}\right), 2\hat{k}_2\sqrt{\frac{\hat{k}_2}{\hat{k}_1}}\left(\hat{r}+\frac{2\bar{F}}{\hat{k}_1}\right)\right\} \tag{46}$$

for some $0 < \hat{r}$ small enough and fixed. Moreover, $\hat{\sigma}_k$ reaches the origin in a finite number of steps k^*, and $\hat{\sigma}_k = 0$ for all $k \geq k^* + 1$.

Proof. Consider the functions $V_{1,k} := \hat{\sigma}_k^\top \hat{M}_k \hat{\sigma}_k$ and $V_{2,k} := \sigma_k^\top \hat{M}_k \sigma_k$ and their respective differences $\Delta V_i := V_{i,k+1} - V_{i,k}$, for $i = 1, 2$. The following is due to (25) and (32c):

$$\begin{aligned}\Delta V_1 &= \hat{\sigma}_{k+1}^\top \hat{M}_{k+1}\hat{\sigma}_{k+1} - \hat{\sigma}_k^\top \hat{M}_k \hat{\sigma}_k \\ &= \hat{\sigma}_{k+1}^\top\big(\hat{M}_{k+1} - \hat{M}_k\big)\hat{\sigma}_{k+1} + 2\hat{\sigma}_{k+1}^\top \hat{M}_k\left(\hat{\sigma}_{k+1} - \sigma_k\right) \\ &\quad - \hat{\sigma}_{k+1}^\top \hat{M}_k \hat{\sigma}_{k+1} + 2\hat{\sigma}_{k+1}^\top \hat{M}_k \sigma_k - \hat{\sigma}_k^\top \hat{M}_k\hat{\sigma}_k \\ &\leq \hat{\sigma}_{k+1}^\top\big(2h\hat{C}_k + \hat{\epsilon}_k\big)\hat{\sigma}_{k+1} + \sigma_k^\top \hat{M}_k \sigma_k - \hat{\sigma}_k^\top \hat{M}_k \hat{\sigma}_k \\ &\quad + 2\hat{\sigma}_{k+1}^\top\big(-h\hat{C}_k\hat{\sigma}_{k+1} - hK_\sigma\hat{\sigma}_{k+1} - h\gamma\zeta_{k+1}\big).\end{aligned} \tag{47}$$

Now, adding and subtracting the term $\sigma_{k+1}^\top \hat{M}_{k+1}\sigma_{k+1} + \hat{\sigma}_{k+1}^\top \hat{M}_{k+1}\hat{\sigma}_{k+1}$ to the right-hand side of (47) results in

$$\begin{aligned}\Delta V_1 &\leq \hat{\sigma}_{k+1}^\top \hat{\epsilon}_k \hat{\sigma}_{k+1} - 2h\hat{\sigma}_{k+1}^\top\left(K_\sigma\hat{\sigma}_{k+1} + \gamma\zeta_{k+1}\right) \\ &\quad + \Delta V_1 - \Delta V_2 + \sigma_{k+1}^\top \hat{M}_{k+1}\sigma_{k+1} - \hat{\sigma}_{k+1}^\top \hat{M}_{k+1}\hat{\sigma}_{k+1},\end{aligned}$$

and it follows that

$$\begin{aligned}\Delta V_2 &\leq \hat{\sigma}_{k+1}^\top \hat{\epsilon}_k \hat{\sigma}_{k+1} - 2h\hat{\sigma}_{k+1}^\top\left(K_\sigma\hat{\sigma}_{k+1} + \gamma\zeta_{k+1}\right) \\ &\quad + \sigma_{k+1}^\top \hat{M}_{k+1}\sigma_{k+1} - \hat{\sigma}_{k+1}^\top \hat{M}_{k+1}\hat{\sigma}_{k+1}.\end{aligned} \tag{48}$$

Substitution of (32c) into (32a) yields (recall that here $M_k = \hat{M}_k$, $C_k = \hat{C}_k$ and $\hat{\mathcal{B}}_k = \hat{M}_k + h\hat{C}_k$)

$$\sigma_{k+1} = \hat{\sigma}_{k+1} - h\hat{\mathcal{B}}_k^{-1}\xi_k, \tag{49}$$

from which we derive

$$\begin{aligned}\sigma_{k+1}^\top \hat{M}_{k+1}\sigma_{k+1} &= \hat{\sigma}_{k+1}^\top \hat{M}_{k+1}\hat{\sigma}_{k+1} - 2h\hat{\sigma}_{k+1}^\top \hat{M}_{k+1}\mathcal{B}_k^{-1}\xi_k \\ &\quad + h^2\xi_k^\top \mathcal{B}_k^{-\top}\hat{M}_{k+1}\mathcal{B}_k^{-1}\xi_k.\end{aligned} \tag{50}$$

After substitution of (50) into (48) we arrive at (recall that $\kappa_1 = \lambda_{\min}(K_\sigma)$)

$$\begin{aligned}\Delta V_2 &\leq -\left(2h\kappa_1 - \|\hat{\epsilon}\|_m\right)\|\hat{\sigma}_{k+1}\|^2 - 2h\gamma\hat{\sigma}_{k+1}^\top\zeta_{k+1} \\ &\quad + 4h\frac{\hat{k}_2}{\hat{k}_1}\|\xi_k\|\|\hat{\sigma}_{k+1}\| + 4h^2\frac{\hat{k}_2}{\hat{k}_1^2}\|\xi_k\|^2 \\ &\leq -\left(2h\kappa_1 - \|\hat{\epsilon}\|_m\right)\|\hat{\sigma}_{k+1}\|^2 \\ &\quad - 2h\left(\gamma\alpha - \frac{2\hat{k}_2}{\hat{k}_1}\bar{F}\right)\|\hat{\sigma}_{k+1}\| + 4h^2\frac{\hat{k}_2}{\hat{k}_1^2}\bar{F}^2,\end{aligned} \tag{51}$$

where we used the fact that $\zeta_{k+1} \in \partial\Phi(\hat{\sigma}_{k+1})$ together with Proposition 3 in the last inequality. Now, assume that $\|\sigma_{k+1}\| > \left(\hat{r} + \frac{2\bar{F}}{\hat{k}_1}\right)h$ for some $0 < \hat{r} < +\infty$. Equations (49) and (42) ensure that $\|\sigma_{k+1}\| > \left(\hat{r} + \frac{2\bar{F}}{\hat{k}_1}\right)h$ implies $\|\hat{\sigma}_{k+1}\| > \hat{r}h$. Hence,

$$\begin{aligned}\Delta V_2 &\leq -\left(2h\kappa_1 - \|\hat{\epsilon}\|_m\right)\|\hat{\sigma}_{k+1}\|^2 \\ &\quad - 2h\left(\gamma\alpha - \frac{2\hat{k}_2}{\hat{k}_1}\bar{F}\left(1 + \frac{\bar{F}}{\hat{k}_1\hat{r}}\right)\right)\|\hat{\sigma}_{k+1}\|.\end{aligned}$$

Finally, from (46) and (40a) we conclude that $\Delta V_2 < 0$ whenever $\|\sigma_{k+1}\| > \left(\hat{r} + \frac{2\bar{F}}{\hat{k}_1}\right)h$. Therefore, we obtain the ultimate boundedness of the solution of (32a), i.e., for any initial condition $\sigma_0 \in \mathbb{R}^n$, we have that

$$\operatorname{dist}\left(\sigma_k, \sqrt{\frac{\hat{k}_2}{\hat{k}_1}}\left(\hat{r} + \frac{2\bar{F}}{\hat{k}_1}\right)h\mathbb{B}_n\right) \to 0$$

as $k \to \infty$. More precisely, we have proved the global practical stability of the origin, since the set $\sqrt{\frac{\hat{k}_2}{\hat{k}_1}}\left(\hat{r} + \frac{2\bar{F}}{\hat{k}_1}\right)h\mathbb{B}_n$ can be made arbitrary small by letting h approach zero.

Now we proceed with the proof of the finite-time convergence of $\hat{\sigma}_k$. Because of the ultimate boundedness of the solution of (32a) we know that there exists a finite number of steps k^* such that $\|\sigma_k\| \leq 2\sqrt{\hat{k}_2/\hat{k}_1}\left(\hat{r} + 2\bar{F}/\hat{k}_1\right)h$ for all $k \geq k^*$. Then, from (46) we have that

$$\left\|\frac{\hat{M}_k\sigma_k}{h}\right\| \leq \frac{\hat{k}_2}{h}\|\sigma_k\| \leq 2\hat{k}_2\sqrt{\frac{\hat{k}_2}{\hat{k}_1}}\left(\hat{r} + \frac{2\bar{F}}{\hat{k}_1}\right) \leq \alpha\gamma$$

for all $k \geq k^*$. From Lemma 6 we conclude that $\hat{\sigma}_k$ reaches zero in at most $k^* + 1$ steps. Moreover, $\hat{\sigma}_{k^*+n} = 0$ for all $n \geq 1$, since the ball $2\sqrt{\hat{k}_2/\hat{k}_1}\left(\hat{r} + 2\bar{F}/\hat{k}_1\right)h\mathbb{B}_n$ is positively invariant. Finally, since $\hat{\sigma}_{k^*+n} = 0$ for all $n \geq 1$, it follows from (49) that

$$\|\sigma_{k^*+n}\| = h\|\mathcal{B}_{k^*+n}^{-1}\xi_{k^*+n}\| \leq \frac{2\bar{F}}{\hat{k}_1}h \quad \text{for all } n \geq 1.$$

□

Remark 7. Under the assumptions given in Theorem 4, it is clear that the sliding variable σ_k converges to a ball of radius $r_\sigma = \sqrt{\hat{k}_2/\hat{k}_1}\left(\hat{r} + 2\bar{F}/\hat{k}_1\right)h$, which implies the boundedness of the state variable $\tilde{q}_k$. Recalling that Λ and h are such that Assumption 8 holds, the solution at the step k is given by

$$\tilde{q}_k = (I - h\Lambda)^k\tilde{q}_0 + h\sum_{n=0}^{k-1}(I - h\Lambda)^{(n+1)}\sigma_{k-n}.$$

Hence, if σ_k is bounded by R_σ for all $k \in \mathbb{N}$, we have that

$$\limsup_{k\to\infty}\|\tilde{q}_k\| \leq hR_\sigma\sum_{n=0}^{\infty}\|(I - h\Lambda)^n\| \leq hR_\sigma\rho$$

for some finite $\rho > 0$ [33, Theorem 22.11]. Therefore, $\tilde{q}_k$ is also bounded for all $k \in \mathbb{N}$. In fact, it converges to a ball of radius $hR_\sigma\rho$.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2662804, IEEE Transactions on Automatic Control

Corollary 3. *Let the assumptions of Theorem 4 hold. Then in the case when there is no disturbance ($\xi \equiv 0$), the origin of* (32) *is globally finite-time Lyapunov stable, while $\tilde{q}_k \to 0$ asymptotically.*

Proof. From (49) we have that $\sigma_{k+1} = \hat{\sigma}_{k+1}$ and the result follows since $\hat{\sigma}_k$ reaches the origin after a finite number of steps. The last statement becomes clear by taking $R_\sigma = 0$ in Remark 7. ☐

Let us now consider the parametric uncertainty. In this case we see that the difference equation (32a) is equivalent to

$$\hat{M}_k\sigma_{k+1} - \hat{M}_k\sigma_k + h\hat{C}_k\sigma_{k+1} + hK_\sigma\hat{\sigma}_{k+1} + h(\xi_k + \theta_k + \vartheta_k) = -h\gamma\zeta_{k+1}, \quad (52)$$

where we have defined two new perturbation terms associated with the parametric uncertainty as $\theta_k := (M_k - \hat{M}_k)\dot{\sigma}_k$ and $\vartheta_k := (C_k - \hat{C}_k)\sigma_{k+1}$, where $\sigma_{k+1} = \sigma_k + h\dot{\sigma}_k$. The two additional disturbance terms will add new constraints to both, the gain of the controller and the time step of the discrete-time scheme as stated in the following theorem.

Theorem 5 (Parametric uncertainty)**.** *Let Assumptions 1-8 hold. Consider the discrete-time dynamical system* (32)*. Then, there exist constants $\hat{r}_\sigma > 0$ and $h^* > 0$ such that, for all $h \in (0, \min\{\delta^*, h^*\}]$ with δ^* given by Lemma 5, the origin of* (32a) *is semi-globally practically stable whenever γ and α satisfy*

$$\gamma\alpha > \max\left\{\frac{2\hat{k}_2}{\hat{k}_1}\bar{\beta}\left(1 + \frac{\bar{\beta}}{\hat{k}_1\hat{r}_\sigma}\right), 2\hat{k}_2\sqrt{\frac{\hat{k}_2}{\hat{k}_1}}\left(\hat{r}_\sigma + \frac{2\mathcal{F}}{\hat{k}_1}\right)\right\}. \quad (53)$$

The constants $\bar{\beta}$ and $\mathcal{F}$ are specified in the proof. Moreover, $\hat{\sigma}_k$ reaches the origin in a finite number of steps k^*, and $\hat{\sigma}_k = 0$ for all $k \geq k^* + 1$.

Proof. The analysis made in the proof of Theorem 4 can be repeated for (52) if we aggregate the uncertainty as $\hat{\xi}_k = \xi_k + \vartheta_k + \theta_k$. However, special care must be taken since $\hat{\xi}_k$ is not uniformly bounded anymore. Consider the compact set $W := \{w \in \mathbb{R}^n | w^\top \hat{M}_0 w \leq R\}$ for some $R \in \mathbb{R}_+$. For any $\sigma_0 \in \mathbb{R}^n$ we can always find $R > 0$ such that $\sigma_0 \in W$. Moreover, by Assumption 6, there is a known $R_\sigma > 0$ such that $W \subset R_\sigma\mathbb{B}_n$. Following the same steps as in the proof of Theorem 4 we arrive at an inequality similar to (51),

$$\Delta V_2 \leq -(2h\kappa_1 - \|\hat{\epsilon}\|_m)\|\hat{\sigma}_{k+1}\|^2 - 2h\left(\gamma\alpha - \frac{2\hat{k}_2}{\hat{k}_1}\|\hat{\xi}_k\|\right)\|\hat{\sigma}_{k+1}\| + 4h^2\frac{\hat{k}_2}{\hat{k}_1^2}\|\hat{\xi}_k\|^2. \quad (54)$$

Now, choose R large enough such that $h\hat{r}_\sigma\mathbb{B}_n \subset W$, where $\hat{r}_\sigma > 0$ is a design parameter. We have two cases.

Case 1: First consider the case where $\|\hat{\sigma}_{k+1}\| \geq h\hat{r}_\sigma$. Hence,

$$\Delta V_2 \leq -(2h\kappa_1 - \|\hat{\epsilon}\|_m)\|\hat{\sigma}_{k+1}\|^2 - 2h\left(\gamma\alpha - \frac{2\hat{k}_2}{\hat{k}_1}\|\hat{\xi}_k\| - \frac{2\hat{k}_2}{\hat{k}_1^2\hat{r}_\sigma}\|\hat{\xi}_k\|^2\right)\|\hat{\sigma}_{k+1}\|. \quad (55)$$

The next step consists in finding appropriate bounds for the term $\|\hat{\xi}_k\|$ on W. We have

$$\|\hat{\xi}_k\| \leq \bar{\beta} + \|M_k - \hat{M}_k\|_m\|\dot{\sigma}_k\| + \|C_k - \hat{C}_k\|_m\|\sigma_{k+1}\|,$$

where $\bar{\beta} := \max_{(\sigma_k,\tilde{q}_k)\in W\times\tilde{R}\mathbb{B}_n}\beta(\sigma_k,\tilde{q}_k)$ is an upper bound of $\beta(\sigma_k,\tilde{q}_k)$ (cf. Proposition 4) and $\tilde{R} = \tilde{R}(\sigma_0,\tilde{q}_0)$ is the radius of a closed ball such that $\tilde{q}_k \in \tilde{R}\mathbb{B}_n$ (the radius can always be found in view of Remark 7). Recalling that $h\dot{\sigma}_k = \sigma_{k+1} - \sigma_k$ and using (52) as well as Assumptions 1 and 6, one can see that

$$\|\dot{\sigma}_k\| \leq \frac{1}{\hat{k}_1}\left[k_C\|\dot{q}_k\|\|\sigma_{k+1}\| + \frac{\kappa_2\hat{k}_2}{\hat{k}_1}R_\sigma + \left(1 + h\frac{\kappa_2}{\hat{k}_1}\right)\gamma R_\zeta + \bar{\beta}\right], \quad (56)$$

where κ_2 is the maximum eigenvalue of K_σ and R_ζ is an upper-bound of ζ_{k+1}. Thus, from (56) we obtain

$$\|\hat{\xi}_k\| \leq (1 + a_1)\bar{\beta} + a_1\frac{\kappa_2 k_2}{\hat{k}_1}R_\sigma + \left(1 + h\frac{\kappa_2}{\hat{k}_1}\right)a_1\gamma R_\zeta + (a_1 k_C\|\dot{q}_k\| + a_0)\|\sigma_{k+1}\|, \quad (57)$$

where $a_0 := \left\|C_k - \hat{C}_k\right\|_m$ and $a_1 := \left\|M_k - \hat{M}_k\right\|_m / k_1$. Note that a_0 is in general a function of σ_k and $\tilde{q}_k$, whereas a_1 is a function of $\tilde{q}_k$ only. It follows also from (32a) that

$$\|\sigma_{k+1}\| \leq \frac{2}{k_1}\left(\left(k_2 + h\frac{\kappa_2\hat{k}_2}{\hat{k}_1}\right)R_\sigma + \left(1 + h\frac{\kappa_2}{\hat{k}_1}\right)h\gamma R_\zeta + h\bar{\beta}\right), \quad (58)$$

where we made use of an analog of Proposition 5 for $\mathcal{B}_k^{-1}$ (see Remark 6). After some algebraic operations, the substitution of (58) into (57) results in

$$\left\|\hat{\xi}_k\right\| \leq b_0 + b_1 h + b_2 h^2 =: \mathcal{F}, \quad (59)$$

where each $b_i > 0$ is given by

$$b_0 := \left(\bar{\beta} + \gamma R_\zeta + \left(\frac{\hat{k}_2}{\hat{k}_1}\kappa_2 + 2\frac{k_2}{k_1}k_C R_q\right)R_\sigma\right)a_1 + \bar{\beta} + 2\frac{\kappa_2}{k_1}a_0 R_\sigma \quad (60a)$$

$$b_1 := \frac{\kappa_2}{\hat{k}_1}\gamma R_\zeta a_1 + \left(\bar{\beta} + \gamma R_\zeta + 2\frac{\kappa_2 k_2}{k_1\hat{k}_1}R_\sigma\right)(a_0 + k_C R_q a_1), \quad (60b)$$

$$b_2 := \frac{\kappa_2}{\hat{k}_1}\gamma R_\zeta(a_0 + k_C R_q a_1) \quad (60c)$$

and $R_q < +\infty$ is an upper-bound of $\dot{q}_k$ (which exists because both σ_k and $\tilde{q}_k$ are bounded on W).

It is thus clear from (55) and (59) that ΔV_2 is strictly negative whenever

$$g(h) := -\gamma\alpha + \frac{2\hat{k}_2}{\hat{k}_1}(b_0 + b_1 h + b_2 h^2) + \frac{2\hat{k}_2}{\hat{k}_1^2\hat{r}_\sigma}(b_0 + b_1 h + b_2 h^2)^2 < 0. \quad (61)$$

Condition (61) can be written in the form $g(h) := d_4 h^4 + d_3 h^3 + d_2 h^2 + d_1 h + d_0$, where $d_i > 0$ for $i = 1, \ldots, 4$. It is clear that, if $d_0 < 0$, then there exists $h^* > 0$ such that $g(h) < 0$ for all $h \in (0, h^*]$.

From (61) and (60a) it follows that d_0 is given by

$$d_0(b_0) := -\gamma\alpha + \frac{2\hat{k}_2}{\hat{k}_1} b_0 + \frac{2\hat{k}_2}{\hat{k}_1^2 \hat{r}_\sigma} b_0^2.$$

Notice that in the case without parametric uncertainty we have $a_1 = a_0 = 0$ and, from (59)-(60), we have that the polynomial $g(h)$ reduces to $d_0(\bar{\beta})$, which is strictly negative in the light of (53). Thus, by continuity there exists $b_0^* > \bar{\beta}$ such that $d_0 < 0$ for all $b_0 \in [\bar{\beta}, b_0^*)$. Indeed, let us write the polynomial d_0 as $d_0(b_0) = \bar{c}_0 b_0^2 + \bar{c}_1 b_0 + \bar{c}_2$, where the values of the $\bar{c}_i$'s are easily obtained from the definition of d_0. Thus, from the continuity of d_0 together with the fact that $\bar{c}_2 < 0$ it becomes clear that for all b_0 such that $b_0 \in \left[\bar{\beta}, \bar{\beta} + (-\bar{c}_1 + \sqrt{\bar{c}_1^2 - 4\bar{c}_0\bar{c}_2})/\bar{c}_0\right)$, then $d_0(b_0) < 0$. It is noteworthy that the previous condition imposed in b_0 can be always made feasible by increasing the value of $\bar{c}_2$ through an increment in the gain α. This last fact implies that there exists $h^* > 0$ such that for any $h \in (0, h^*]$ we have that $g(h) < 0$ and therefore $\Delta V_2 < 0$.

Case 2: For the second case, (i.e., $\|\hat{\sigma}_{k+1}\| \leq h\hat{r}_\sigma$), the Lyapunov difference ΔV_2 could fail to be negative, but instead we prove that if it increases, it will be in *small* quantitites in such a way that σ_k remains in W. Formally, from (54) it follows that (using (53) and Lemma 5)

$$V_2(\sigma_{k+1}) \leq V_2(\sigma_k) + 4h^2 \frac{\hat{k}_2}{\hat{k}_1} \|\hat{\xi}_k\|^2.$$

Hence, letting $h > 0$ be such that

$$R > \max_{\|w\| \leq r_\sigma} V(w) + 4h^2 \frac{\hat{k}_2}{\hat{k}_1} \mathcal{F},$$

we get $V_2(\sigma_{k+1}) < R$, i.e., $\sigma_{k+1} \in W$, (where r_σ is the appropriate bound of σ_k, consequence of $\|\hat{\sigma}_{k+1}\| \leq h\hat{r}_\sigma$). Therefore, in both cases the next iteration σ_{k+1} remains in W and the positive invariance of W follows.

Let us now pass to the last part of the theorem. Assume that we start at $k = 0$ with an initial condition $\sigma_0 \in \mathbb{R}^n$. We have shown that there exists $R > 0$ such that $\sigma_0 \in W$. Moreover, there exists $R_\sigma > 0$ such that $W \subset R_\sigma \mathbb{B}_n$. Since W is invariant, it follows that $\|\xi_k\|$ is bounded by $\bar{\beta}$ for all $k \in \mathbb{N}$. These statements imply that that the bound (59) is valid for all $k \in \mathbb{N}$. Upon examination of (54) and by considering that $\|\sigma_{k+1}\| > (\hat{r}_\sigma + 2\mathcal{F}/\hat{k}_1)h$ with $\hat{r}_\sigma > 0$ fixed and $\mathcal{F}$ defined in (59), we have that

$$\begin{aligned}\Delta V_2 \leq & -(2h\kappa_1 - \|\hat{\epsilon}\|_m)\|\hat{\sigma}_{k+1}\|^2 \\ & - 2h\left(\gamma\alpha - \frac{\hat{k}_2}{\hat{k}_1}\|\hat{\xi}_k\| - \frac{\hat{k}_2}{2\hat{k}_1^2\hat{r}_\sigma}\|\hat{\xi}_k\|^2\right)\|\hat{\sigma}_{k+1}\|\end{aligned}$$

(as in the proof of Theorem 4, the constraint $\|\sigma_{k+1}\| > (\hat{r}_\sigma + 2\mathcal{F}/\hat{k}_1)h$ implies that $\|\hat{\sigma}_{k+1}\| \geq \hat{r}_\sigma h$). For semi-global practical stability we need to prove that the term within parenthesis that pre-multiplies $\|\hat{\sigma}_{k+1}\|$ is negative. In the first part of the proof we have already shown that

$$-\gamma\alpha + 2\frac{\hat{k}_2}{\hat{k}_1}\|\hat{\xi}_k\| + 2\frac{\hat{k}_2}{\hat{k}_1^2\hat{r}_\sigma}\|\hat{\xi}_k\|^2 < g(h) < 0$$

whenever b_0 and h are small enough. Therefore, $\Delta V_2 < 0$ for all $\|\sigma_{k+1}\| > (\hat{r}_\sigma + 2\mathcal{F}/\hat{k}_1)h$. The proof for the finite-time convergence mimics the corresponding part of the proof of Theorem 4. □

The following theorem relates the solutions of the discrete-time system (32) to the ones of an associated continuous-time system. A detailed proof may be found in [29, §6.3]

Theorem 6 (Convergence of the discrete-time solutions)**.** *Let $(\sigma_k, \tilde{q}_k)$ be a solution of the closed-loop discrete-time system* (32) *and let the functions*

$$\begin{aligned}\sigma_h(t) &:= \sigma_{k+1} + \frac{t_{k+1} - t}{h}(\sigma_k - \sigma_{k+1}), \\ \tilde{q}_h(t) &:= \tilde{q}_{k+1} + \frac{t_{k+1} - t}{h}(\tilde{q}_k - \tilde{q}_{k+1}),\end{aligned}$$

for all $t \in [t_k, t_{k+1})$, be the piecewise-linear approximations of σ_k and $\tilde{q}_k$ respectively. Then, $(\sigma_h, \tilde{q}_h)$ converges to $(\sigma, \tilde{q})$ as the sampling time h decreases to zero, where $(\sigma, \tilde{q})$ is a solution of

$$\begin{aligned}M(q(t))\dot{\sigma}(t) + C(q(t), \dot{q}(t))\sigma(t) & \\ +K_\sigma\sigma(t) + \xi(t, \sigma(t), \tilde{q}(t)) &= -\gamma\zeta(t), \quad (62a)\\ \zeta(t) &\in \partial\Phi(\sigma(t)), \quad (62b)\\ \dot{\tilde{q}}(t) &= \sigma(t) - \Lambda\tilde{q}(t) \quad (62c)\end{aligned}$$

with $\sigma(0) = \sigma_0$ and $\tilde{q}(0) = \tilde{q}_0$.

Sketch of the proof. By the boundedness of the discrete iterations, it is possible to approximate σ_k and q_k by piecewise linear and step functions. By the Arzela-Ascoli and Banach-Alaoglu Theorems [25, Theorems 1.3.8, 2.4.3], these converge (strongly in $L^2([0,T];\mathbb{R}^n)$) to limit functions σ and q. The relevant assumptions are that the derivative of $F(\cdot, q, \dot{q})$ maps bounded sets of $L^2([0,T];\mathbb{R}^n)$ into bounded sets of $L^2([0,T];\mathbb{R}^n)$, and that the operator $\partial\Phi$ is maximal monotone. Under these assumptions, a direct application of [6, Chapter 3.1, Proposition 2] guarantees that the limits are solutions of (62). □

In other words, the time-discretization chosen in Section VI-A is a suitable approximation of the continuous-time dynamics. This is an important conclusion since the discrete-time controller is designed from an approximation of the continuous-time plant (24). More on the closed-loop behaviour depending on h is illustrated by examples in the next section.

VII. Numerical Example

Consider the two-link planar elbow manipulator depicted in Fig. 1. Its dynamics are given by (6) with

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2662804, IEEE Transactions on Automatic Control

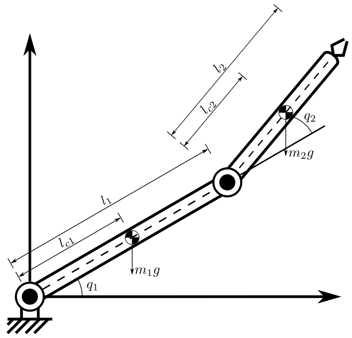

Fig. 1: A two-link planar elbow manipulator.

'Real' parameters			
m_1	1.5 kg	m_2	1.0 kg
l_1	0.4 m	l_2	0.3 m
l_{c1}	0.2 m	l_{c2}	0.2 m
I_1	0.08 kg m^2	I_2	0.03 kg m^2
Nominal parameters			
$\hat{m}_1$	1.6 kg	$\hat{m}_2$	0.8 kg
$\hat{l}_1$	0.4 m	$\hat{l}_2$	0.3 m
$\hat{l}_{c1}$	0.25 m	$\hat{l}_{c2}$	0.15 m
$\hat{I}_1$	0.0853 kg m^2	$\hat{I}_2$	0.0240 kg m^2

TABLE I: Parameters of the two-link planar elbow

$$M(q) = \begin{bmatrix} m_1 l_{c1}^2 + m_2 a + I_1 + I_2 & m_{12} \\ m_{12} & m_2 l_{c2}^2 + I_2 \end{bmatrix},$$
$$C(q,\dot{q}) = -m_2 l_1 l_{c2} \sin(q_2) \begin{bmatrix} \dot{q}_2 & \dot{q}_1 + \dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix},$$
$$G(q) = \begin{bmatrix} (m_1 l_{c1} + m_2 l_1) g \cos(q_1) + m_2 l_{c2} g \cos(q_1 + q_2) \\ m_2 l_{c2} \cos(q_1 + q_2) \end{bmatrix},$$

where m_i represents the mass of the i-th link; $m_{12} = m_2(l_{c2}^2 + l_1 l_{c2} \cos(q_2)) + I_2$ and $a := l_1^2 + l_{c2}^2 + 2 l_1 l_{c2}(l_{c2} + \cos(q_2))$; l_i and l_{ci}, are the length of the i-th link and the distance from the base of the i-th link to its center of mass, respectively; I_i is the inertia moment of the i-th link, $i = 1, 2$. The constant $g = 9.81 m/s^2$ is the acceleration due to gravity. The parameters of the 'real' plant and of the nominal model are as shown in Tab. I.

Our control objective is to track the trajectory $q_{\mathrm{d}}(t) = \frac{\pi}{2}\begin{bmatrix}\sin(t) + 1, & -\cos(t)\end{bmatrix}^\top$, We suppose that the system is subject to the disturbance

$$F(t,q,\dot{q}) = 0.25 \begin{bmatrix} \cos(\pi t)\sin(t) \\ 0.5 \sin(\sqrt{2}t)\sin(t/3)\cos(t) \end{bmatrix} + 0.5 \begin{bmatrix} \tanh(\dot{q}_1) \\ \tanh(q_2)\cos(q_1 + q_2) \end{bmatrix},$$

The gains of the controller are set as:

$$K_\sigma = 2 \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 5 & -0.5 \\ -0.5 & 8 \end{bmatrix}, \quad \gamma = 0.5, \quad \alpha = 1.$$

In all this section we set $\Phi(x) = \alpha \|x\|_1$ (but other choices are possible). Hence,

$$\partial \Phi(x) = \begin{bmatrix} \mathrm{sgn}(x_1) & \mathrm{sgn}(x_2) & \dots & \mathrm{sgn}(x_n)) \end{bmatrix}^T$$

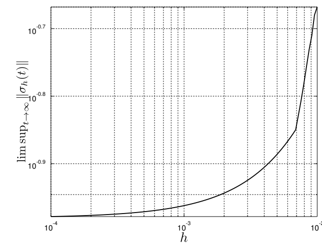

Fig. 2: Evolution of the $\limsup_{t\to\infty} \|\sigma_h(t)\|$ of the sliding variable σ_h as a function of h in a logarithmic scale.

with $\mathrm{sgn}(0) = [-1, 1]$. With this choice of Φ, the algorithm (39) for the computation of the control law is implemented for each $k \in \mathbb{N}$ using the successive approximations method as follows:

1) Set $\mu > 0$ small enough such that $0 < \hat{\mathcal{A}}_k + \hat{\mathcal{A}}_k^\top - \mu \hat{\mathcal{A}}_k^\top \hat{\mathcal{A}}_k$ holds.
2) Set $j = 0$ and set $x^0 \in \mathbb{R}^n$.
3) Compute x^{j+1} as
$$v^j = (I - \mu \hat{\mathcal{A}}_k) x^j + \mu \hat{M}_k \sigma_k,$$
$$x^{j+1} = v^j - \mu\, \mathrm{Proj}_{[-c,c]^n}\left(\frac{v^j}{\mu}\right),$$
where $c = h\gamma\alpha$ and the set $[-c, c]^n$ represents the n-cube in $\mathbb{R}^n$ centered at the origin with edge length equal to $2c$.
4) If $\|x^{j+1} - x^j\| > \varepsilon$, then increase j and go to step 3. Else, set $\hat{\sigma}_{k+1} = x^{j+1}$ and stop.

In the previous algorithm, the variable ε represents the precision of the algorithm; it was set to 10^{-9} in all the simulations. The simulations were performed using the continuous-time plant (6), but the feedback control was implemented using the stepwise discrete-time scheme (29)-(30). The initial conditions were set as $[q_0^\top, \dot{q}_0^\top] = [\pi/3, -\pi/4, \pi/3, \pi/8]^\top$. The trajectories obtained in simulation are shown in Figs. 3-5 for several sampling times. Note that the discrete-time sliding phase, $\hat{\sigma}_k = 0$ for all k large enough, cannot be reached (even with precise knowledge of system parameters) because of the plant's discretization error. This fact induces an error in the set-valued input, which explains the appearance of chattering that is absent in the discrete/discrete setting. It is also worth mentioning that this numerical chattering appears when $h = 10^{-2}$ s (Fig. 3) but vanishes for smaller sampling periods. Fig. 2 depicts how the norm of the sliding variable σ_h, associated with the continuous plant/discrete controller setting, evolves as a function of the sampling time $h > 0$. We can see that the order of convergence is not constant and, moreover, it tends to zero as h decreases to zero.

Finally, in order to set-up a benchmark for evaluating the implicit discretization scheme, we present the case when the controller is discretized in an explicit way, i.e., when (30) is replaced by

$$-u_k \in K_\sigma \sigma_k + \gamma \partial \Phi(\sigma_k). \tag{63}$$

Notice that in the explicit case there is no need for the scheme (32), since the variable σ_k is assumed to be known at time t_k.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2662804, IEEE Transactions on Automatic Control

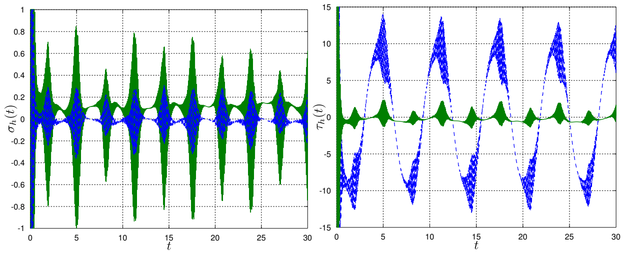

Fig. 3: Evolution of the sliding variable σ_h (left) and the control input τ_h (right) for the closed-loop system (6), (29) and (30) with sampling time $h = 10^{-2}$ s.

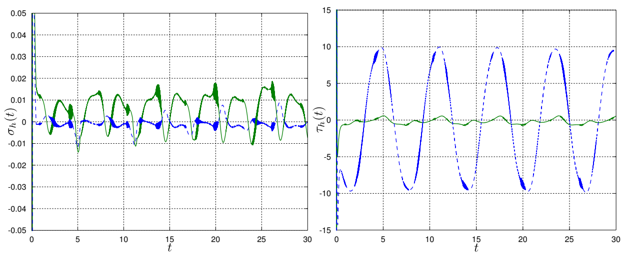

Fig. 4: Evolution of the sliding variable σ_h (left) and the control input τ_h (right) for the closed-loop system (6), (29) and (30) with sampling time $h = 10^{-3}$ s.

Figure 6 illustrates how the numerical chattering effect appears in the input and the output of the closed-loop system when the explicit method is used. On the other hand, chattering is almost suppressed with the implicit controller, even for the same values of h (see Figs. 4 and 5). Moreover, we see that the error in the sliding variable σ_h and the chattering effect in both the input τ_h and the output σ_h, is much larger (under the same sampling rate) with the explicit algorithm (63). Finally, it is worth to mention that, when $h = 10^{-2}$ s, the resulting closed-loop system with the explicit controller (63) shows an unstable behavior (a phenomenon already observed for different plants and controllers in [20], [27]), while the implicit algorithm keeps the input and output bounded (see Fig. 3). The implicit discrete-time controller (30) supersedes the explicit one (63), since the former allows much smaller sampling rates and exhibits a significantly better chattering

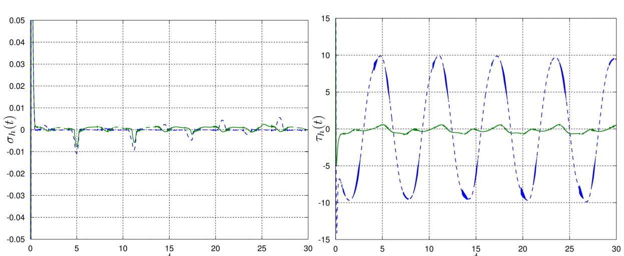

Fig. 5: Evolution of the sliding variable σ_h (left) and the control input τ_h (right) for the closed-loop system (6), (29) and (30) with sampling time $h = 10^{-4}$ s.

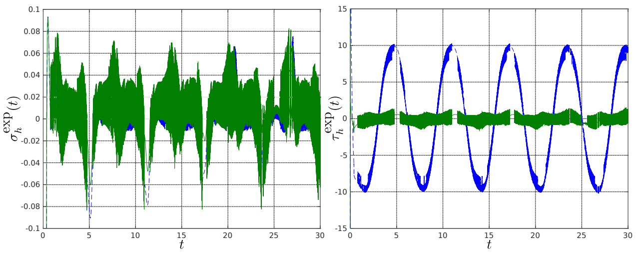

Fig. 6: Evolution of sliding variable (left) and the control input (right) for the closed-loop system (6) and (29) (63) with sampling time $h = 10^{-3}$ s.

alleviation together with smaller error amplitudes. Thus, the conclusions drawn in [18], [19], [20] from experimental data extend to the nonlinear uncertain case analyzed in this article.

VIII. Conclusions and Further Research

The main objective of this article was the analysis of a family of implicit discrete-time set-valued sliding-mode controllers for trajectory tracking in fully actuated Euler-Lagrange systems. First, continuous-time controllers were studied for systems with exogenous disturbances and parametric uncertainties. Well-posedness together with stability results were established. Subsequently, the analysis of the implicit discrete-time scheme was carried out. Interesting features were obtained: finite-time convergence for the nominal (unperturbed) sliding variable, robustness against external and parametric uncertainties, convergence of solutions of the discrete-time system to solutions of the continuous-time closed-loop system and input and output chattering alleviation.

Simulations validate the theoretical results and allow to better understand the limitations of the proposed scheme.

References

[1] V. Acary and B. Brogliato. Implicit Euler numerical scheme and chattering-free implementation of sliding mode systems. *Systems & Control Letters*, 59(5):284-293, 2010.

[2] V. Acary, B. Brogliato, and Y. V. Orlov. Chattering-free digital sliding-mode control with state observer and disturbance rejection. *Automatic Control, IEEE Transactions on*, 57(5):1087-1101, 2012.

[3] K. Addi, B. Brogliato, and D. Goeleven. A qualitative mathematical analysis of a class of linear variational inequalities via semi-complementarity problems: applications in electronics. *Mathematical Programming*, 126(1):31-67, 2011.

[4] S. Adly, B. Brogliato, and B. K. Le. Well-posedness, robustness, and stability analysis of a set-valued controller for Lagrangian systems. *SIAM Journal on Control and Optimization*, 51(2):1592-1614, 2013.

[5] S. Adly, B. Brogliato, and B. K. Le. Implicit Euler time-discretization of a class of Lagrangian systems with set-valued robust controller. *Journal of Convex Analysis*, 23(1-2-3), 2016.

[6] J. Aubin and A. Cellina. *Differential Inclusions: Set-Valued Maps and Viability Theory*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 1984.

[7] J. Bastien and M. Schatzman. Numerical precision for differential inclusions with uniqueness. *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, 36(3):427-460, 2002.

[8] H. Bauschke and P. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. CMS Books in Mathematics. Springer New York, 2011.

[9] H. Brézis. *Analyse Fonctionnelle: Théorie et Applications*. Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1993.

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2662804, IEEE Transactions on Automatic Control

[10] B. Brogliato and D. Goeleven. Well-posedness, stability and invariance results for a class of multivalued Lur'e dynamical systems. *Nonlinear Analysis: Theory, Methods and Applications*, 74(1):195 - 212, 2011.

[11] B. Brogliato and A. Polyakov. Globally stable implicit Euler time-discretization of a nonlinear single-input sliding-mode control system. *54th IEEE Conference on Decision and Control*, Osaka, Japan, December 2015, 5426-5431.

[12] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland. *Dissipative Systems Analysis and Control: Theory and Applications*. Communications and Control Engineering. Springer Verlag London, 2nd edition, 2007.

[13] J. Cortés. Discontinuous dynamical systems: A tutorial on solutions, nonsmooth analysis and stability. *Control Systems Magazine, IEEE*, 28(3):36-73, 2008.

[14] F. Facchinei and J. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*, volume II. Springer-Verlag, New York, 2003.

[15] Z. Galias and X. Yu. Euler's discretization of single input sliding-mode control systems. *Automatic Control, IEEE Transactions on*, 52(9):1726-1730, Sept 2007.

[16] D. Goeleven, D. Motreanu, Y. Dumont, and M. Rochdi. *Variational and Hemivariational Inequalities: Theory, Methods and Applications*, volume 1. Springer, 2003.

[17] J. B. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms I*. Springer-Verlag, New York, 1993.

[18] O. Huber, V. Acary, B. Brogliato, and F. Plestan. Discrete-time twisting controller without numerical chattering: analysis and experimental results with an implicit method. *Control Engineering Practice*, 46(1): 129-141, January 2016.

[19] O. Huber, B. Brogliato, V. Acary, A. Boubakir, F. Plestan, and B. Wang. Experimental results on implicit and explicit time-discretization of equivalent-control-based sliding mode control. In L. Fridman, J.-P. Barbot, and F. Plestan, editors, *Recent Trends in Sliding Mode Control*, The IET, ISBN: 978-1-78561-076-9, 2016.

[20] O. Huber, V. Acary, B. Brogliato. Lyapunov stability and performance analysis of the implicit discrete sliding mode control. *Automatic Control, IEEE Transactions on*, 2016, 61(10): 3016-3030, October 2016.

[21] R. Kelly, V. Santibáñez, and A. Loría. *Control of Robot Manipulators in Joint Space*. Springer-Verlag, 2005.

[22] H. Khalil. *Nonlinear Systems*. Prentice Hall, third edition, 2002.

[23] R. Kikuuwe, S. Yasukouchi, H. Fujimoto, M. Yamamoto. Proxy-based sliding mode control: A safer extension of PID position control. *Robotics, IEEE Transactions on*, 26(4): 670-683, 2010.

[24] A. N. Kolmogorov and S. V. Fomin. *Elements of the Theory of Funtions and Functional Analysis*, volume 1. Graylock Press, New York, 1957.

[25] A. J. Kurdila and M. Zabarankin. *Convex Functional Analysis*. Systems & Control: Foundations & Applications. Birkhäuser Verlag, Germany, 2005.

[26] P. Lancaster and M. Tismenetsky. *The Theory of Matrices: With Applications*. Academic Press, San Diego, CA, 1985.

[27] A. Levant. On fixed and finite time stability in sliding mode control. *Proc. of 52th IEEE Conference on Decision and Control*, 4260-4265, Firenze, Italy, December 2013.

[28] F. A. Miranda-Villatoro and F. Castaños. Robust output regulation of strongly linear passive systems with multivalued maximally monotone controls. *Automatic Control, IEEE Transactions on*, 2016, DoI: 10.1109/TAC.2016.2544926

[29] F. A. Miranda-Villatoro, B. Brogliato and F. Castaños. Multivalued robust tracking control of fully actuated Lagrange systems: Continuous and discrete-time algorithms. Preprint, https://hal.inria.fr/hal-01254303, January 2016.

[30] R. Ortega and M. W. Spong. Adaptive motion control of rigid robots: A tutorial *Automatica*, 25(6):877-888, 1989.

[31] R. T. Rockafellar. *Convex Analysis*. Convex Analysis. Princeton University Press, 1970.

[32] R. T. Rockafellar and R. Wets. *Variational Analysis*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen. Springer, 2009.

[33] W. Rugh. *Linear System Theory*. Prentice-Hall, second edition, 1996.

[34] J.-J. E. Slotine and W. Li. On the adaptive control of robot manipulators. *The International Journal of Robotics Research*, 6(3):49-59, 1987.

[35] J.-J. E. Slotine and W. Li. Adaptive manipulator control: A case study. *Automatic Control, IEEE Transactions on*, 33(11):995-1003, Nov 1988.

[36] J.-J. E. Slotine and S. S. Sastry. Tracking control of non-linear systems using sliding surfaces, with application to robot manipulators. *International Journal of Control*, 38(2):465-492, 1983.

[37] M. Spong, R. Ortega, and R. Kelly. Comments on "Adaptive manipulator control: a case study" by J. Slotine and W. Li. *Automatic Control, IEEE Transactions on*, 35(6):761-762, June 1990.

[38] A. Tanwani, B. Brogliato, and C. Prieur. On output regulation in state-constrained systems: An application to polyhedral case. In *Proc. 19th IFAC World Congress, South Africa*, pages 1513-1518, 2014.

[39] V. Utkin, J. Guldner, and J. Shi. *Sliding Mode Control in Electro-Mechanical Systems*, volume 34. CRC press, 2009.

[40] B. Wang, B. Brogliato, V. Acary, A. Boubakir, and F. Plestan. Experimental comparisons between implicit and explicit implementations of discrete-time sliding mode controllers: Toward input and output chattering suppression. *Control Systems Technology, IEEE Transactions on*, 23(5):2071-2075, Sept 2015.

[41] X. Yu, B. Wang, Z. Galias, G. Chen. Discretization effect on equivalent control-based multi-input sliding-mode control systems. *Automatic Control, IEEE Transactions on*, 53(6): 1563-1569, July 2008.

Félix A. Miranda-Villatoro (S'15) received the B.Eng. degree in Electronic Engineering from the Universidad Autónoma del Estado de México (UAE-Mex), in 2010. After that, he obtained the M.Sci. degree and the PhD. degree from CINVESTAV-IPN in 2012 and 2016 respectively. His research interests include robust control, differential inclusions, and the application of nonsmooth tools in control theory.

Bernard Brogliato was born in 1963 in Saint-Symphorien de Lay, France. He graduated from the Ecole Normale Supérieure de Cachan, Mechanical Engineering Dept., and he obtained his PhD in Automatic Control in 1991 at the National Polytechnic Institute of Grenoble (INPG). He is working at INRIA Grenoble Rhône-Alpes since 2001. His research works concern nonsmooth dynamical systems and dissipative systems (modelling, control, analysis), with applications in mechanics, circuits, optimal control and variable-structure systems. He authored and co-authored five monographs: *Nonsmooth Mechanics* (Springer CCE series, 3rd ed. 2016), *Dissipative Systems Analysis and Control* (with R. Lozano, B. Maschke, O. Egeland, Springer CCE series, 2nd ed. 2007), *Numerical Methods for Nonsmooth Dynamical Systems* (with V. Acary, Springer LNACM series vol.35, 2008), *Nonsmooth Modelling and Simulation for Switched Circuits* (with V. Acary, O. Bonnefon, Springer Verlag LNEE series vol.69, 2011), *Multiple Impacts in Dissipative Granular Chains* (with N.S. Nguyen, Springer Verlag LNACM series vol.72, 2014).

Fernando Castaños (S'06-M'10) received the B.Eng. in Electric and Electronic Engineering from Universidad Nacional Autónoma de México (UNAM). He received the M.Eng. in Control Engineering from UNAM and the Ph.D degree from Université Paris-Sud XI. He was a postdoctoral fellow at McGill's Center for intelligent machines for two years. He joined the Automatic Control Department, Cinvestav, Mexico, in 2011. His research interests are passivity-based control, nonlinear control, port-Hamiltonian systems, robust control and variable structure systems.