SET-VALUED SLIDING-MODE CONTROL OF UNCERTAIN LINEAR SYSTEMS: CONTINUOUS AND DISCRETE-TIME ANALYSIS

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Abstract. In this paper we study the closed-loop dynamics of linear time-invariant systems with feedback control laws that are described by set-valued maximal monotone maps. The class of systems considered in this work is subject to both, unknown exogenous disturbances and parameter uncertainty. It is shown how the design of conventional sliding-mode controllers can be achieved using maximal monotone operators (which include but are not limited to the set-valued signum function). Two cases are analyzed: continuous-time and discrete-time controllers. In both cases well-posedness together with stability results are presented. In discrete time, we show how the implicit scheme proposed for the selection of control actions results in the chattering effect being almost suppressed, even with uncertainty in the system.

Key words. Differential inclusions, robust control, maximal monotone maps, sliding-mode control, discrete-time systems, linear uncertain systems, Lyapunov stability.

AMS subject classifications. 34A60, 93C73, 93C55, 93D09, 34A36, 49J52, 47H05.

1. Introduction. Since its appearance in the late fifties, the so-called sliding modes have been associated with switching control laws. The main idea arises from the behavior of the electrical relay, i.e., the input switches between a finite number of possible values depending on the region of the phase-space in which the system is evolving. This approach works well in principle, but for real-life applications some problems arise due to the intrinsic imperfections in the elements that constitute the controller, as for example: time-delays in the reaction of the components, boundaries in the operation region (finite switching frequency), etc. Among the most dangerous effects resulting from these imperfections we can find the so-called chattering effect. The catastrophic consequences of chattering include component degradation, poor response and, in the worst case, destruction of the system.

On the other hand, the closed-loop features that sliding-mode control offers are very attractive: finite-time convergence, order reduction, robustness against parametric and external disturbances, simple gain tuning. For that reason many research efforts have been directed towards the study of attenuation of the chattering effect. Among these studies we can find adaptive schemes with variable gains [46], high-order sliding modes [33], regularization techniques [49] and suitable discrete-time implementation [1, 2, 25, 26, 27, 48].

Since the work of Filippov [21] sliding-mode control systems have been associated with differential inclusions. More precisely, the solutions of a dynamical system with a *discontinuous* right-hand side are interpreted as solutions of an associated differential inclusion. The work of Filippov provides conditions ensuring the existence of solutions (in the sense of Filippov) for sliding-mode control systems. Surprisingly, there are only a few studies that use the set-valued setting provided by Filippov for the design of the control law that will produce the sliding phenomenon [1, 2, 25, 26, 27, 48].

The objective of this paper is twofold. First, a family of set-valued controllers —which is suitable for the design of sliding-mode controllers— is introduced using

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the so-called maximal monotone operators. The design procedure is revisited for the continuous-time context considering parametric uncertainty and external disturbances. It is shown that the set-valued approach is consistent with the classical design methodology and powerful, allowing us to approach the multivariable problem in a natural way as well as the regularization of the set-valued map. The second aim is to show, step-by-step, the methodology design for the discrete-time case when the set-valued maximal monotone operators are used together with the implicit scheme proposed in [1, 2, 25] (see also [29] for a similar approach of discrete-time sliding mode control). We show how this mathematical formulation is well-posed, providing a better understanding of discrete-time sliding-mode systems.

The main contribution of this paper relies on the inclusion of parametric uncertainty, i.e., we extend the results in [1, 2, 25] by considering the fact that, in most real life applications, the dynamic model of the plant is not accurate. It is noteworthy that the addition of this uncertainty in the plant is not trivial, and that in the aforementioned works the controller depends on the exact knowledge of the parameters. This paper also shows that any maximal monotone set-valued map —different from the commonly used signum set-valued function— can be used in order to achieve the sliding regime. Moreover, the maximal monotone operators allow us to cover, in one setting, several well-known formulations such as the componentwise control or the unit vector control [45]. Thus, to some extent, the tools presented in this paper unify the design of sliding-mode controllers in the framework of set-valued maximal monotone operators. The mathematical framework used in this work for explaining the sliding-mode phenomenon relies on differential inclusions, where (contrary to the conservative thinking of switching) we are giving emphasis to the proper selection of the control values as the main tool towards chattering suppression. Namely, regarding the discrete-time context, the intrinsic properties of maximal monotone operators, together with the differential inclusion formulation of the sliding-mode phenomenon and the implicit discretization approach, allow us to make a *unique* selection for the control values that will compensate for the disturbances and parametric uncertainties with a considerable reduction of chattering in both, the input and the sliding variable, whenever the frequency of sampling is sufficiently high when compared to the external disturbance variations.

The main results, stated in terms of global asymptotic stability and semi-global practical stability of the origin are presented in Theorems 24, 37 and their corollaries for the continuous and discrete-time cases respectively. In addition, a proof of the consistency of the implicit discretization is presented in Section 4.5.

This paper is organized as follows. In Section 2 we recall some preliminaries from convex analysis together with some notation. Section 3 is devoted to the design and well-posedness, in continuous-time, of set-valued controllers using maximal monotone operators. Some results concerning the robustness in the face of parametric and external disturbances of the resulting closed-loop system are presented. The discrete-time counterpart is exposed in Section 4, where the use of the implicit discretization for achieving the discrete-time sliding phase is exposed, together with some stability results and the convergence of the solutions of the discrete-time closed-loop system to a solution of the continuous-time system. Finally, Section 5 depicts the effectiveness of the family of set-valued controllers proposed in Sections 3 and 4 through the use of a numerical example, whereas the Appendix contains most of the proofs.

2. Preliminaries and notation. Let X be a Hilbert space with inner product denoted as $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. A multivalued map $\mathbf{M} : X \rightrightarrows Y$

is a map that is valued over the sets of Y, that is, for any $x \in X$, $\mathbf{M}(x) \subset Y$. The graph of a set-valued map is given as $\operatorname{Graph} \mathbf{M} := \{(x,y) \in X \times Y \mid y \in \mathbf{M}(x)\}$. A set-valued map $\mathbf{M} : X \rightrightarrows X$ is called *monotone* if it satisfies $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ for all $(x_1, y_1), (x_2, y_2) \in \operatorname{Graph} \mathbf{M}$ and it is called *maximal monotone* if its graph is not contained in the graph of any other monotone map. The *resolvent* with index μ, $\mu > 0$, associated with a maximal monotone map $\mathbf{M}$ is a single-valued Lipschitz continuous map $J^\mu_{\mathbf{M}} : X \to X$ given as

$$J^\mu_{\mathbf{M}}(x) := (I + \mu \mathbf{M})^{-1}(x).$$

Moreover, the resolvent $J^\mu_{\mathbf{M}}$ is non-expansive, i.e., $\|J^\mu_{\mathbf{M}}(x_1) - J^\mu_{\mathbf{M}}(x_2)\| \leq \|x_1 - x_2\|$ for all $x_1, x_2 \in X$. A detailed study of the properties of the resolvent can be found in [4, 9, 41]. Related to the resolvent of $\mathbf{M}$ is the so-called Yosida approximation of index μ of the set-valued map $\mathbf{M}$.

Definition 1. *The Yosida approximation of a maximal monotone map is given by*

$$\mathcal{M}^\mu(x) = \frac{1}{\mu}\left(I - J^\mu_{\mathbf{M}}\right)(x). \tag{1}$$

Roughly speaking, the Yosida approximation of $\mathbf{M}$ is a maximal monotone and Lipschitz continuous single-valued function which *approximates the graph of* $\mathbf{M}$ *from below*. Formally we have that for all $x \in \operatorname{Dom} \mathbf{M}$,

$$\|\mathcal{M}^\mu(x)\| \leq \|\operatorname{Proj}_{\mathbf{M}(x)}(0)\| \tag{2}$$

and

$$\mathcal{M}^\mu(x) \to \operatorname{Proj}_{\mathbf{M}(x)}(0) \text{ as } \mu \downarrow 0, \tag{3}$$

where $\operatorname{Proj}_{\mathbf{M}(x)} : X \to \mathbf{M}(x)$ refers to the conventional projection operator, that is,

$$\operatorname{Proj}_{\mathbf{M}(x)}(y) := \underset{\xi \in \mathbf{M}(x)}{\arg\min} \|y - \xi\|.$$

In words, the Yosida approximation of $\mathbf{M}$ converges to the element of minimum norm in the closed convex set $\mathbf{M}(x)$. See, e.g., [4, 9] for a proof of the previous statement and more properties about the Yosida approximation. The next result (taken from [4, Proposition 2, p.141]) states an important topological property concerning the graph of maximal monotone operators.

Proposition 2. *The graph of a set-valued maximal monotone operator* $\mathbf{M} : X \rightrightarrows X$ *is strongly-weakly closed in the sense that if* $x_n \to x$ *strongly in* X *and if* $y_n \in \mathbf{M}(x_n)$ *converges weakly to* y*, then* $y \in \mathbf{M}(x)$*.*

Definition 3. *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a proper, convex, lower semicontinuous function. The subdifferential of* f *at* $x \in \operatorname{Dom} f$ *is given by the set:*

$$\partial f(x) := \{\zeta \in X^* | \langle \zeta, \eta - x \rangle \leq f(\eta) - f(x), \text{ for all } \eta \in X\},$$

where X^* *refers to the dual space of* X*.*

The proof of the following result can be found in [40].

Proposition 4. *The subdifferential of a proper, convex, lower semicontinuous function is a maximal monotone operator.*

DEFINITION 5. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function. The proximal map* $\mathrm{Prox}_f : X \to X$ *is the unique minimizer of* $f(w) + \frac{1}{2}\|x - w\|^2$*, that is,*

$$f(\mathrm{Prox}_f(x)) + \frac{1}{2}\|x - \mathrm{Prox}_f(x)\|^2 = \min_{w \in X} \left\{ f(w) + \frac{1}{2}\|x - w\|^2 \right\}.$$

Along all this work we denote the identity matrix in $\mathbb{R}^{n\times n}$ as I_n. The set $\mathbb{B}_n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ represents the unit closed ball with center at the origin in $\mathbb{R}^n$ with the Euclidean norm. The boundary of a set S is denoted $\mathrm{bd}(S)$. Let $A \in \mathbb{R}^{n\times m}$, the induced norm of A is given by $\|A\| := \sup_{\|x\|=1} \|Ax\| = \sqrt{\lambda_{\max}(A^\top A)}$, where $\lambda_{\max}(B) := \max_{i\in\{1,\dots n\}}\{\lambda_i \in \sigma(B)\}$ and $\sigma(B)$ is the spectrum of the matrix $B \in \mathbb{R}^{n\times n}$. Let $B \in \mathbb{R}^{n\times n}$ be a symmetric matrix, B is called positive definite, $B > 0$, if for any $x \in \mathbb{R}^n \setminus \{0\}$, $x^\top Bx > 0$. It is positive semidefinite, $B \geq 0$, if $x^\top Bx \geq 0$. Let $A = A^\top$ and $B = B^\top$ be square matrices, the inequality $A > B$ stands for $A - B > 0$, i.e., $A - B$ is positive definite. Let $A = A^\top > 0$, the A-norm of a vector $x \in \mathbb{R}^n$ is given by $\|x\|_A^2 = x^\top Ax$. In the case where $1 \leq p \leq \infty$ the norm $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$ for $p \in [1, \infty)$ and $\|x\|_\infty := \max_i |x_i|$.

PROPOSITION 6 (Schur's complement formula). *Let $D_1 = D_1^\top \in \mathbb{R}^{n_1\times n_1}$, $D_2 = D_2^\top \in \mathbb{R}^{n_2\times n_2}$ and $D_3 \in \mathbb{R}^{n_1\times n_2}$ be given matrices. Then, the following three statements are equivalent,*

1. $\begin{bmatrix} D_1 & D_3 \\ D_3^\top & D_2 \end{bmatrix} > 0.$
2. $D_1 > 0$ *and* $D_2 - D_3^\top D_1^{-1} D_3 > 0$.
3. $D_2 > 0$ *and* $D_1 - D_3 D_2^{-1} D_3^\top > 0$.

3. Design of sliding-mode controllers in continuous-time using maximal monotone maps.

3.1. The robust control problem. In this section we make a review of the conventional methodology design for sliding-mode controllers. This review will be useful for two reasons. First, we show that the family of set-valued maximal monotone operators can be used in the design of controllers that guarantee the sliding motion. Second, the concepts recalled here are used for introducing their discrete-time counterpart. We start analyzing a linear time-invariant system with both parametric uncertainty and external disturbances. Specifically, in this work we focus on the case in which the input matrix $B \in \mathbb{R}^{n\times m}$ is known and the dynamics of the plant is affected by a time and state-dependent additive uncertainty $\Delta_A(t, x) \in \mathbb{R}^{n\times n}$, which is a nonlinear time-varying term. The system is characterized in state-space form as

$$\dot{x}(t) = (A + \Delta_A(t, x(t)))x(t) + B\big(u(t) + w(t, x(t))\big),\ x(0) = x_0, \tag{4}$$

where $x(t) \in \mathbb{R}^n$ represents the state variable, $u(t) \in \mathbb{R}^m$ is the control input, whereas $w(t, x(t)) \in \mathbb{R}^m$ accounts for an external disturbance considered unknown but bounded in the L^∞ sense. The matrix A represents the nominal values of the parameters of the plant, which are assumed to be known. Notice that, in general, the addition of the term $\Delta_A(t, x)$ generates a nonlinear, time-varying, and state-dependent mismatched disturbance. Along all this paper, we assume the following.

Assumption 7. The pair (A, B) is stabilizable.

Assumption 8. The matrix $B \in \mathbb{R}^{n\times m}$, where $m < n$, has full column rank.

Assumption 9. For all $t \in [0, +\infty)$ the uncertainty matrix-function $\Delta_A(t, \cdot)$ is locally Lipschitz continuous and satisfies $\Delta_A(t,x)\Lambda\Delta_A^\top(t,x) < I_n$ for all $x \in \mathbb{R}^n$ and for some known symmetric positive definite matrix $\Lambda \in \mathbb{R}^{n\times n}$.

Assumption 10. For all $t \in [0, +\infty)$ the external disturbance $w(t, \cdot)$ is locally Lipschitz continuous. Moreover, there exists $W > 0$ such that $\sup_{t\geq 0} \|w(t,x)\| \leq W < +\infty$.

Notice that Assumption 9 implies that $\Delta_A(t,x)$ is uniformly bounded. Namely, according to Proposition 6 the matrix inequality in Assumption 9 is equivalent to $\Delta_A^\top(t,x)\Delta_A(t,x) < \Lambda^{-1}$. Consequently, $\|\Delta_A(t,x)\|^2 \leq 1/\lambda_{\min}(\Lambda) = \lambda_{\max}(\Lambda^{-1})$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. It is also noteworthy that the kind of parametric disturbances considered in this work embraces time-varying systems and a family of nonlinear systems. The proof of the following proposition can be consulted in [8, Section 7.2.1].

Proposition 11. *Assumption 7 holds if and only if for some $a > 0$ there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ satisfying the following linear matrix inequality (LMI):*

$$B_\perp^\top \left(AP + PA^\top + 2aP\right) B_\perp < 0, \tag{5}$$

where $B_\perp \in \mathbb{R}^{n\times(n-m)}$ denotes an orthogonal complement of the matrix B, i.e., $B_\perp$ is a full column rank matrix whose columns are formed by basis vectors of the null space of $B^\top$.

The design of sliding-mode controllers is accomplished by selecting two central objects: the *sliding surface* and the control law. The former refers to a submanifold on the state-space in which all the trajectories will converge in finite-time by the action of the control law, and the closed-loop system constrained to the sliding surface satisfies the performance requirements. Moreover, once the sliding surface has been reached, the task of the controller is to maintain the trajectories inside it despite the presence of disturbances (sliding phase). In this work the design of the control law is performed using a two-step design methodology. Namely, in the former stage we compute a *nominal* control, denoted as u^{nom}, that guarantees the invariance of the sliding surface $\sigma = 0$ in the absence of the uncertainties, i.e., $w \equiv 0$ and $\Delta_A \equiv 0_{n\times n}$. After that, we propose the set-valued component of the controller, denoted by u^{sv}, which will be responsible for attaining the sliding surface as well as providing robustness against matched disturbances. That is, we have split the control input as $u = u^{\text{nom}} + u^{\text{sv}}$. A crucial point to consider is related to the proper design of the sliding surface which will guarantee the performance of the system in the sliding phase. It was proved in [14, 17, 39] that the correct design of the sliding surface helps to diminish the effects caused by mismatched disturbances and in some special cases (when some structure of the disturbance is imposed) even suppression of the disturbance can be accomplished [18]. More important is the fact that the wrong selection of this surface could increase the effects of the disturbance [14], which in our context implies higher gains. Throughout this work we consider the sliding surface as a hyperplane of the form $H := \{x \in \mathbb{R}^n \mid Cx = 0\}$.

Assumption 12. The matrix $C \in \mathbb{R}^{m\times n}$ is such that the product CB is nonsingular.

Assumption 12 guarantees the uniqueness of the equivalent control as well as the uniqueness of the nominal control. It is noteworthy that the two-step design methodology described above is sometimes called *equivalent-control-based* method and the part of the controller denoted by u^{nom} is called the equivalent control. In this work the concept of equivalent control is used as in [45], i.e., it is the control that maintains the state in sliding motion in the presence of disturbances. It follows that the term u^{nom} is a *nominal equivalent* control, but we prefer to call it merely *nominal* in order to avoid confusion.

3.2. Design of the sliding surface. In this subsection we follow the lines of [14], analyzing the effect of the design of the sliding surface H over the mismatched disturbance. We start studying how the dynamics in sliding phase is affected by the disturbance $\Delta_A(t,x)x$. To this end we use the equivalent control method [44]. Namely, we compute the control that maintains the sliding regime and we will see how the mismatched disturbance affects the closed-loop system. We introduce the so-called *sliding variable* as $\sigma(x) := Cx$. Thus, the equivalent control is computed from the invariance condition $\dot{\sigma} = 0$ as

$$(6)\quad \begin{aligned} &C(Ax^{\mathrm{eq}} + B(u^{\mathrm{eq}} + w) + \Delta_A(t,x^{\mathrm{eq}})x^{\mathrm{eq}}) = 0, \\ &\qquad\Rightarrow u^{\mathrm{eq}} = -(CB)^{-1}C\left(Ax^{\mathrm{eq}} + \Delta_A(t,x^{\mathrm{eq}})x^{\mathrm{eq}}\right) - w. \end{aligned}$$

Substitution of the equivalent control into (4) leads to the expression of the dynamics in sliding phase,

$$(7)\qquad \dot{x}^{\mathrm{eq}} = \left(I_n - B(CB)^{-1}C\right)Ax^{\mathrm{eq}} + \left(I_n - B(CB)^{-1}C\right)\Delta_A(t,x^{\mathrm{eq}})x^{\mathrm{eq}},$$

from which it becomes clear that the matrix characterizing the sliding hyperplane plays a role into the *equivalent disturbance* $\left(I_n - B(CB)^{-1}C\right)\Delta_A(t,x)x$. In [14] the authors proved that the correct design of such hyperplane guarantees that no amplification of the disturbance occurs by using surfaces with $C = B^\top$ or $C = B^+$, where B^+ stands for the left-inverse of the matrix B, i.e., $B^+ = (B^\top B)^{-1}B^\top$. In this work we modify such selection of the surface considering instead $C = B^\top P^{-1}$ and also $C = (B^\top P^{-1}B)^{-1}B^\top P^{-1}$, where P is a solution of (5). First we show that this selection of C gives an equivalent disturbance with minimum P^{-1}-norm. Afterwards we show how the proper choice of P dominates the mismatched disturbance during the sliding phase.

LEMMA 13. *Let $C_1 = B^\top P^{-1}$ and $C_2 = (B^\top P^{-1}B)^{-1}B^\top P^{-1}$, where $P = P^\top > 0$. Then, both C_i, $i = 1,2$, minimize the P^{-1}-norm of the equivalent disturbance $(I_n - B(CB)^{-1}C)\Delta_A(t,x^{\mathrm{eq}})x^{\mathrm{eq}}$.*

Proof. Let $\phi^{\mathrm{eq}} = \Delta_A(t,x^{\mathrm{eq}})x^{\mathrm{eq}}$. Then, the optimization problem

$$(8)\qquad \min_{C\in\mathbb{R}^{m\times n}} \left\|\left(I_n - B(CB)^{-1}C\right)\phi^{\mathrm{eq}}\right\|_{P^{-1}}^2 = \min_{z\in\mathbb{R}^m}\|\phi^{\mathrm{eq}} - Bz\|_{P^{-1}}^2,$$

where $z = (CB)^{-1}C\phi^{\mathrm{eq}}$, has the unique solution $z^* = (B^\top P^{-1}B)^{-1}B^\top P^{-1}\phi^{\mathrm{eq}}$. From the definition of z it follows that $C = B^\top P^{-1}$ achieves the minimum in (8) as well as $C = (B^\top P^{-1}B)^{-1}B^\top P^{-1}$. ∎

Notice that both selections of C stated in Lemma 13 satisfy Assumption 12. Throughout this section we will set $C = (B^\top P^{-1}B)^{-1}B^\top P^{-1}$. In the next subsection we design the control law that assures the sliding motion.

3.3. Design of the control law. Recalling from the above lines that the two-step control design methodology adopted in this paper splits the control input into two components, that is, $u = u^{\mathrm{nom}} + u^{\mathrm{sv}}$, we start with the computation of the nominal control u^{nom}, whereas the set-valued part of the controller is deferred to the next subsection.

The computation of the nominal control u^{nom} is accomplished from the invariance condition $\dot{\sigma} = 0$ in the ideal case, i.e., $w = 0$, $u^{\text{sv}} = 0$ and $\Delta_A = 0$, as

$$\dot{\sigma} = C\dot{x}^{\text{nom}} = C\left(Ax^{\text{nom}} + Bu^{\text{nom}}\right) = 0 \;\Rightarrow\; u^{\text{nom}} = -(CB)^{-1}CAx^{\text{nom}}. \tag{9}$$

Notice that the nominal control is nothing more than a linear feedback law of the form $u^{\text{nom}} = -\Gamma x^{\text{nom}}$ with $\Gamma = (CB)^{-1}CA$. Substitution of the nominal control (9) into the system (4), changing x^{nom} by the real state x, yields,

$$\dot{x} = \left(I_n - B(CB)^{-1}C\right)Ax + B(u^{\text{sv}} + w) + \Delta_A(t,x)x, \tag{10}$$

where u^{sv} is the set-valued part of the controller. In order to obtain the dynamics of the system in the sliding phase, we consider the nonsingular transformation,

$$T = \begin{bmatrix} B_\perp^\top \\ (B^\top P^{-1}B)^{-1}B^\top P^{-1} \end{bmatrix},\; T^{-1} = \begin{bmatrix} PB_\perp(B_\perp^\top PB_\perp)^{-1} & B \end{bmatrix}. \tag{11}$$

Remark 14. It is worth to mention that from the product $T^{-1}T$ we obtain the identity,

$$PB_\perp(B_\perp^\top PB_\perp)^{-1}B_\perp^\top + B(B^\top P^{-1}B)^{-1}B^\top P^{-1} = I_n. \tag{12}$$

From the application of (12) to the term $\phi := \Delta_A(t,x)x$ it follows that

$$\phi = PB_\perp(B_\perp^\top PB_\perp)^{-1}B_\perp^\top\phi + B(B^\top P^{-1}B)^{-1}B^\top P^{-1}\phi = PB_\perp\phi_u + B\phi_m,$$

where $\phi_u := (B_\perp^\top PB_\perp)^{-1}B_\perp^\top\phi$ and $\phi_m := (B^\top P^{-1}B)^{-1}B^\top P^{-1}\phi$ are called the unmatched and the matched components of ϕ respectively.

The next step in our design consists in a change of coordinates of the form $z = Tx$ applied to (10). Notice that, because of the structure of T, we can split the new state variable z as $z = \begin{bmatrix} z_1^\top & z_2^\top \end{bmatrix}^\top$, where $\mathbb{R}^{n-m} \ni z_1 = B_\perp^\top x$ and $\mathbb{R}^m \ni z_2 = (B^\top P^{-1}B)^{-1}B^\top P^{-1}x = Cx = \sigma$. Therefore, recalling that $u = u^{\text{nom}} + u^{\text{sv}}$ with $u^{\text{nom}} = -CAx$, the change of variables $z = Tx$ leads to the regular form [45],

$$\dot{z}_1 = B_\perp^\top\left(A + \hat{\Delta}_A(t,z)\right)PB_\perp\left(B_\perp^\top PB_\perp\right)^{-1}z_1 + B_\perp^\top\left(A + \hat{\Delta}_A(t,z)\right)B\sigma \tag{13a}$$

$$\dot{\sigma} = u^{\text{sv}} + \hat{w}(t,z) + \hat{\phi}_m(t,z), \tag{13b}$$

where, $\hat{\Delta}_A(t,z) := \Delta_A(t,T^{-1}z)$, $\hat{w}(t,z) := w(t,T^{-1}z)$ and $\hat{\phi}_m(t,z) := \phi_m(t,T^{-1}z)$. One comment is in place here. From (13b) it follows that the dynamics of the sliding variable is only affected by the matched part of the original disturbance $\Delta_A(t,x)x$. Hence, in order to achieve the sliding regime it is necessary to take into account only the matched part of the disturbance in the design of u^{sv} [14].

In the next lines provide conditions for the matrix P so that the reduced order dynamics z_1 is asymptotically stable with decay rate a, in the ideal sliding phase, under the influence of the parametric uncertainty Δ_A. To this end, let us consider the reduced order system

$$\dot{z}_1 = B_\perp^\top\left(A + \hat{\Delta}_A(t,z)\right)PB_\perp\left(B_\perp^\top PB_\perp\right)^{-1}z_1 \tag{14}$$

with the Lyapunov-function candidate $V(z_1) = \frac{1}{2} z_1^\top (B_\perp^\top P B_\perp)^{-1} z_1$. Taking the derivative of V along the trajectories of (14) yields

$$\begin{aligned}\dot{V} &= z_1^\top (B_\perp^\top P B_\perp)^{-1} \dot{z}_1 \\ &= \frac{1}{2} \bar{z}_1^\top B_\perp^\top \left(AP + PA^\top\right) B_\perp \bar{z}_1 + \bar{z}_1^\top B_\perp^\top \hat{\Delta}_A P B_\perp \bar{z}_1, \end{aligned} \tag{15}$$

where $\bar{z}_1 = \left(B_\perp^\top P B_\perp\right)^{-1} z_1$. Applying (5), together with the inequality $2p^\top X^\top Y q \leq p^\top X^\top \Psi X p + q^\top Y^\top \Psi^{-1} Y q$, for some $\Psi = \Psi^\top > 0$, it follows that

$$\dot{V} \leq -a \bar{z}_1^\top B_\perp^\top P B_\perp \bar{z}_1 + \frac{1}{2} \bar{z}_1^\top B_\perp^\top \hat{\Delta}_A \Psi \hat{\Delta}_A^\top B_\perp \bar{z}_1 + \frac{1}{2} \bar{z}_1^\top B_\perp^\top P \Psi^{-1} P B_\perp \bar{z}_1. \tag{16}$$

Taking $\Psi = \Lambda$ where $\Lambda = \Lambda^\top > 0$ is defined in Assumption 9 gives,

$$\begin{aligned}\dot{V} &\leq -a \bar{z}_1^\top B_\perp^\top P B_\perp \bar{z}_1 + \frac{1}{2} \bar{z}_1^\top B_\perp^\top B_\perp \bar{z}_1 + \frac{1}{2} \bar{z}_1 B_\perp^\top P \Lambda^{-1} P B_\perp \bar{z}_1 \\ &= -\bar{z}_1^\top B_\perp^\top \left(aP - \frac{1}{2} I_n - \frac{1}{2} P \Lambda^{-1} P\right) B_\perp \bar{z}_1. \end{aligned} \tag{17}$$

From (17) the asymptotic stability of the reduced system (14) in sliding phase follows if

$$B_\perp^\top \left(aP - \frac{1}{2} I_n - \frac{1}{2} P \Lambda^{-1} P\right) B_\perp > 0, \tag{18}$$

Along all this section we will assume that the matrix P satisfies (5) and a stronger version of (18). Namely,

$$Q := \begin{bmatrix} B_\perp^\top \left(aP - I_n - \frac{1}{2} P \Lambda^{-1} P\right) B_\perp & -\frac{1}{2} B_\perp^\top A B \\ -\frac{1}{2} B^\top A^\top B_\perp & K - \frac{1}{2} B^\top \Lambda^{-1} B \end{bmatrix} > 0, \tag{19}$$

where $K = K^\top \in \mathbb{R}^{m \times m}$ is a positive definite matrix. Notice that, as stated, the matrix inequality (19) has to be solved in the variables P and K. Furthermore, from a direct application of the Schur's complement formula (19) it can be expressed as an LMI in the variables P, K and Λ as

$$\begin{bmatrix} B_\perp^\top \left(aP - I_n\right) B_\perp & -\frac{1}{2} B_\perp^\top A B & B_\perp^\top P & 0_{n-m \times n} \\ -\frac{1}{2} B^\top A^\top B_\perp & K & 0_{m \times n} & B^\top \\ P B_\perp & 0_{n \times m} & 2\Lambda & 0_{n \times n} \\ 0_{n \times n-m} & B & 0_{n \times n} & 2\Lambda \end{bmatrix} > 0. \tag{20}$$

The justification for considering (19) instead of (18) comes from the proof of Theorem 22 below, where the complete system (13) is analyzed. Remark that in the case when the pair (A, B) is controllable, the parameter a is free and the LMI (20) is feasible for $a > 0$ large enough and K, Λ sufficiently large too (in the order imposed by the positive definiteness, that is, $K_1 > K_2$ if and only if $K_1 - K_2 > 0$). On the other hand, when the system is only stabilizable, the decay rate a is constrained by the uncontrollable part of the system, setting a lower bound on the norm of the matrices K and Λ. This last condition translates into the consideration of small parametric uncertainties Δ_A, see Assumption 9.

PROPOSITION 15. *The disturbance term $\hat{\phi}_m(t,z)$ satisfies the linear growth condition $\|\hat{\phi}_m(t,z)\| \leq \sqrt{\kappa}\|z\|$, where*

$$\kappa = \frac{\lambda_{\max}(P)\lambda_{\max}(\Lambda^{-1})}{\lambda_{\min}(B^\top P^{-1}B)\lambda_{\min}(P)} \max\left\{\frac{1}{\lambda_{\min}(B_\perp^\top P B_\perp)}, \lambda_{\max}(B^\top P^{-1}B)\right\} \tag{21}$$

Proof. From the definition of $\hat{\phi}_m$ we have that

$$\begin{aligned}\|\hat{\phi}_m(t,z)\| &= \|(B^\top P^{-1}B)^{-1}B^\top P^{-1}\hat{\Delta}_A(t,z)T^{-1}z\| \\ &\leq \|(B^\top P^{-1}B)^{-1}B^\top P^{-1/2}\|\|P^{-1/2}\|\|\hat{\Delta}_A(t,z)\|\|T^{-1}\|\|z\|.\end{aligned}$$

Recalling that the induced Euclidean norm coincides with the spectral norm and making use of the Assumption 9, after simple computations we obtain

$$\|\hat{\phi}_m(t,z)\| \leq \sqrt{\frac{\lambda_{\max}(\Lambda^{-1})}{\lambda_{\min}(B^\top P^{-1}B)\lambda_{\min}(P)}}\|T^{-1}\|\|z\|.$$

On the other hand, recalling that for the matrix norm induced by the Euclidean norm we have that $\|T\| = \|T^\top\|$, see e.g., [32, Theorem 5.4.2], from (11) it follows that

$$\begin{aligned}\|T^{-\top}\|^2 &\leq \left\|\begin{bmatrix}(B_\perp^\top P B_\perp)^{-1}B_\perp^\top P^{1/2} \\ B^\top P^{-1/2}\end{bmatrix}\right\|^2 \|P^{1/2}\|^2 \\ &= \lambda_{\max}(P)\lambda_{\max}\left(\begin{bmatrix}(B_\perp^\top P B_\perp)^{-1} & 0 \\ 0 & B^\top P^{-1}B\end{bmatrix}\right)\end{aligned}$$

and the result follows. ∎

3.3.1. Set-valued controller. In this subsection we study the family of set-valued maximal monotone operators used as feedback control laws for system (13). First, some results about the existence and (in some cases) uniqueness of solutions are presented. Subsequently, we prove how a subfamily of the family of maximal monotone controllers yields finite-time stable sliding modes. We start setting the remaining term u^{sv} in (13b) as

$$-u^{\text{sv}}(t) \in K\sigma(t) + \gamma(z(t))\mathbf{M}(\sigma(t)), \tag{22}$$

where $K \in \mathbb{R}^{m\times m}$ is a positive definite matrix satisfying (20), $\gamma : \mathbb{R}^n \to \mathbb{R}_+$ is a positive function depending on the system state z, and $\mathbf{M} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is a set-valued maximal monotone operator. Thus, from (22) it follows that there exists $\zeta \in \mathbf{M}(\sigma)$ such that $-u^{\text{sv}} = K\sigma + \gamma(z)\zeta$. Hence, the evolution of the sliding variable is dictated by the differential inclusion

$$\begin{cases}\dot{\sigma}(t) = -K\sigma(t) - \gamma(z(t))\zeta(t) + \hat{w}(t,z) + \hat{\phi}_m(t,z), \quad \sigma(0) = \sigma_0 \\ \zeta(t) \in \mathbf{M}(\sigma(t)).\end{cases} \tag{23}$$

In the case when the function γ is constant, the differential inclusion (23) belongs to the class of differential inclusions with maximal monotone right-hand side for which numerous results have been proposed, see e.g., [4, 6, 9, 11, 12, 36, 38] and it embraces several mathematical formulations [10]. The existence and uniqueness of solutions of (23) for the case where γ is constant has been studied assuming the Lipschitz (local) continuity of $\hat{w}(t,\cdot)$ and $\hat{\phi}(t,\cdot)$, see e.g., [9, 12, 15]. For a solution of (23) we mean an absolutely continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}^m$ that satisfies $\sigma(0) = \sigma_0 \in \operatorname{Dom} \mathbf{M}$ together with (23) almost everywhere on $[0, +\infty)$, that is, we consider solutions of differential inclusion (23) in the sense of Caratheodory [19]. It is worth to mention that in the case where γ is a function of the state, the uniqueness of solutions of (23) is not guaranteed, this comes from the fact that, in general, the map $\gamma(z)M(\sigma)$ is not maximal monotone. Here, we present some examples about the different choices of the set-valued map $\mathbf{M}$.

Example 16. Let $\mathbf{M}$ be the subdifferential of $f(\sigma) := \|\sigma\|_1 = \sum_{i=1}^{n} |\sigma_i|$. Then, $\mathbf{M}(\sigma)$, is the vector set-valued signum function,

$$[\mathbf{M}(\sigma)]_i = \begin{cases} 1, & \text{if } \sigma_i > 0, \\ [-1, 1], & \text{if } \sigma_i = 0, \\ -1 & \text{if } \sigma_i < 0. \end{cases}$$

In this case the control scheme agrees with the so-called *componentwise* sliding mode design, see e.g., [45].

Example 17. Let $\mathbf{M}$ be the subdifferential of $f(\sigma) := \|\sigma\|_2$. Then $\mathbf{M}(\sigma)$ is the set-valued vector function,

$$\mathbf{M}(\sigma) = \begin{cases} \mathbb{B}_n, & \text{if } \|\sigma\| = 0, \\ \frac{\sigma}{\|\sigma\|}, & \text{otherwise.} \end{cases}$$

In this case the control scheme coincides with the so-called *unit vector* approach [37, 42].

Example 18. Let Ψ_S be the indicator function of the closed convex set S, i.e., $\Psi_S(\sigma) = 0$, if $\sigma \in S$ and $\Psi_S(\sigma) = +\infty$ otherwise. Let $\sigma(0)$ be inside the set S and let $\mathbf{M}$ be the subdifferential of the indicator function, that is,

$$\mathbf{M}(\sigma) = \{\zeta \in \mathbb{R}^m \mid \langle \zeta, \eta - \sigma \rangle \leq 0, \text{ for all } \eta \in S\} = N_S(\sigma).$$

Here $N_S(\sigma)$ denotes the normal cone to the set S at the point σ. Then the closed-loop system (13b), (22) is well-posed and by Theorem 24 below the sliding mode is reached in finite time. The study of this kind of controllers has been reported in [34, 35]. Moreover, if $S = S(t)$ is a Lipschitz continuous set-valued mapping, then the closed-loop system (13b), (22) represents a perturbed Moreau's sweeping process [13, 20].

In what follows we consider the next condition on the set-valued operator $\mathbf{M}$.

Assumption 19. The set-valued maximal monotone map $\mathbf{M}$ satisfies $0 \in \operatorname{int} \mathbf{M}(0)$.

Remark 20. Assumption 19 is known as a condition for *dry friction* in the mechanics literature. It is strongly linked to the finite-time convergence property, see Theorem 24 and Corollary 40 below. In [3, 5] the same condition was used for proving the finite-time stability of nonlinear oscillators in both, continuous and discrete-time settings.

It is worth to mention that Assumption 19 rules out linear controllers, since we ask for maps $\mathbf{M}$ that must be set-valued at the origin. For example, in the case when $\mathbf{M} = \partial\Phi$ where the function Φ is proper, convex and lower semicontinuous, Assumption 19 asks for functions Φ which are nonsmooth at the origin, so that $\operatorname{int} \mathbf{M}(0) \neq \emptyset$, as for example, the norm function $\|\cdot\|_p$, $1 \leq p \leq \infty$. This last comment reveals that the maximal monotone operators suit perfectly as a tool that unifies the different generalizations of the signum multifunction in the design of sliding mode controllers in the multivariable case.

PROPOSITION 21. *Let Assumption 19 hold. Then for any* $(x, y) \in \operatorname{Graph} \mathbf{M}$ *there exists an* $\varepsilon > 0$ *such that,*

$$\langle x, y\rangle \geq \varepsilon \|x\|. \tag{24}$$

Proof. From Assumption 19, it follows that there exists $\varepsilon > 0$ such that for all $\rho \in \varepsilon\mathbb{B}_m$, $(0, \rho) \in \operatorname{Graph} \mathbf{M}$. Then, from the definition of a maximal monotone map it follows that for any $(x, y) \in \operatorname{Graph} \mathbf{M}$ and any $\rho \in \varepsilon\mathbb{B}_m$, $0 \leq \langle y - \rho, x\rangle$. Consequently, $\sup_{\rho\in\varepsilon\mathbb{B}_m} \langle \rho, x\rangle \leq \langle y, x\rangle$. The conclusion follows. ∎

3.4. Well-posedness and stability of the closed-loop system.

In this subsection we show the well-posedness of the closed-loop system (13), (22) in the case when γ is a state-dependent gain by imposing some conditions on P, in the form of LMI's, such that the unmatched part of the disturbance is dominated, and hence assuring the asymptotic stability of the fixed-point $z_1^* = 0$. After that, we show how the sliding phase is reached in finite time with an appropriate selection of the gain γ. Finally some results about stability and uniqueness of solutions in the case where γ is constant are established.

THEOREM 22. *Let Assumptions 7-10 and 19 hold. Then the closed-loop system* (13), (22)*, where* $\mathbf{M} : \operatorname{Dom} \mathbf{M} \rightrightarrows \mathbb{R}^m$ *is a set-valued maximal monotone map that satisfies* $\operatorname{Dom} \mathbf{M} = \mathbb{R}^m$*, has at least one solution (in Caratheodory's sense [19]), whenever* $P = P^\top > 0$ *satisfies the LMI's* (5), (20) *and, in addition, for some* $\rho > 0$ *we have*

$$\varepsilon\gamma(z) = \rho + W + \sqrt{\kappa}\|z(t)\|, \tag{25}$$

where κ *is as in* (21)*,* W *is the upper bound given in Assumption 10, and* $\varepsilon > 0$ *is as in Proposition 21.*

Proof. See the Appendix. ∎

Remark 23. Notice that the assumption $\operatorname{Dom} \mathbf{M} = \mathbb{R}^m$ rules out multivalued controllers with compact domain as those introduced in Example 18. However, the use of set-valued maps whose domain is not all $\mathbb{R}^m$ is possible using $\gamma > 0$ constant, since we fall in the case of differential inclusion with maximal monotone right-hand side, see e.g., [9, 15].

THEOREM 24. *Let the assumptions of Theorem 22 hold. Then, the origin of the subsystem* (13b) *with the set-valued controller* (22) *is globally finite-time Lyapunov stable whenever*

$$\varepsilon\gamma(z) = \rho + W + \sqrt{\kappa}\|z\|, \tag{26}$$

where ε *is given in* (24) *and* $\rho > 0$ *is an arbitrary constant.*

Proof. We consider the positive definite function of σ, $V(\sigma) = \frac{1}{2}\sigma^\top\sigma$. From the proof of Theorem 22 we have that z_1 is bounded. So, differentiating V along the trajectories of (13b) results in $\dot V = \sigma^\top\dot\sigma = \sigma^\top (u^{\rm sv} + w + \phi_m)$. From (22) there exists a $\zeta \in \mathbf{M}(\sigma)$ such that $u^{\rm sv} = -K\sigma - \gamma(x)\zeta$ and then,

$$\begin{aligned}\dot V &\leq -\sigma^\top K\sigma - \gamma(z)\sigma^\top\zeta + \|w + \phi_m\|\|\sigma\| \\ &\leq -\left(\varepsilon\gamma(z) - W - \sqrt{\kappa}\|z\|\right)\|\sigma\|,\end{aligned}$$

where we have used (24) and the fact that $K > 0$. Hence, if (26) holds, then $\dot{V} < -\rho\|\sigma\|$. Finally, after integration of both sides of the last inequality an upper-bound for the time t^* such that $\sigma(t) = 0$ for all $t \geq t^*$ is obtained as $t^* \leq \sqrt{2V(0)}/\rho$. □

It is worth to mention that Theorem 24 does not make mention of the uniqueness of solutions, but we have proved instead that all the solutions converge to the sliding surface. The next step consists in showing the asymptotic stability of the whole system (13), (22).

THEOREM 25. *Let the assumptions of Theorem 22 hold. Then, the origin of the closed-loop system* (13)*,* (22) *is globally asymptotically stable.*

Proof. Consider the Lyapunov-function candidate

$$V(z_1, \sigma) := \frac{1}{2} z_1^\top (B_\perp^\top P B_\perp)^{-1} z_1 + \frac{1}{2}\sigma^\top \sigma. \tag{27}$$

Let ζ be an element in $\mathbf{M}(\sigma)$, differentiating (27) along the system trajectories yields

$$\begin{aligned} \dot{V} &\leq -\lambda_{\min}(\tilde{Q})\|z\|^2 + \sigma^\top \left(-\gamma(z)\zeta + \hat{w}(t,z) + \hat{\phi}_m(t,z)\right) \\ &\leq -\lambda_{\min}(\tilde{Q})\|z\|^2 - \left(\varepsilon\gamma(z) - (W + \sqrt{\kappa}\|z\|)\right)\|\sigma\| \\ &< -\alpha\|z\|^2, \end{aligned} \tag{28}$$

where $\alpha = \lambda_{\min}(\tilde{Q}) > 0$, the matrix $\tilde{Q} = \tilde{Q}^\top > 0$ is defined in (79) and we made use of (24). This concludes the proof. □

According to Theorem 25 the stability of the origin is in fact exponential. However, notice that at the light of Theorem 24 the sliding variable σ converges to the origin of $\mathbb{R}^m$ in finite time, whereas z_1 decays exponentially to zero.

An important case arises when we ask for a constant gain $\gamma > 0$. In this case the existence of solutions has been deeply studied (see, e.g., [9], [15], [20]) and from the practical point of view, we sacrifice the global stability for semi-global stability and the uniqueness of solutions is retrieved.

COROLLARY 26. *Let the Assumptions 7-19 hold, let $\alpha > 0$, $\delta > 0$ and $P = P^\top$ be such that* (5)*,* (20) *hold, and let $L_c \subset \mathbb{R}^n$ be a compact set specified below in the proof. Then, for each initial condition that satisfies $(z_1(0), \sigma(0)) \in L_c$, for some $c > 0$, the origin of the closed-loop system* (13) *with set-valued controller*

$$-u^{\mathrm{sv}} \in K\sigma + \gamma \mathbf{M}(\sigma), \tag{29}$$

where $K = K^\top > 0$ satisfies (19)*, is semi-globally asymptotically stable whenever*

$$\varepsilon\gamma = \rho + W + \sqrt{\kappa} \max_{z \in L_c} \{\|z\|\}, \tag{30}$$

where $z = [z_1^\top, \sigma^\top]^\top$, κ is given in (21)*, and $\rho > 0$ is an arbitrary constant.*

Proof. Consider the positive definite function $V(z_1, \sigma)$ as in (27) and let

$$L_c := \{(z_1, \sigma) \in \mathbb{R}^n \mid V(z_1, \sigma) \leq c\}$$

be the level sets of V. As first step we prove the positive invariance of the set L_c. To this end we take the time derivative of V along the system trajectories, yielding again (28) with $\gamma(z)$ replaced by γ. In the light of (30), we can conclude that $\dot{V} < 0$

for all $\sigma \in \mathrm{bd}(L_c)$ and the positive invariance follows. Now, let $(z_1(0), \sigma(0)) \in L_c$ for some $c > 0$, then from (28) and the fact that the maximum in (30) is attained in the boundary of L_c it follows that $\dot{V} < -\alpha\|z\|^2$ for all $t \geq 0$ and we arrive at the conclusion. ∎

From Corollary 26 it follows that the multivalued controller (29) drives the system (13) into the sliding surface $\{x \in \mathbb{R}^n \mid \sigma(x) = 0\}$ in finite time. Moreover, as a consequence of the maximal monotonicity of the set-valued map $\gamma\mathbf{M}(\cdot)$ we have uniqueness of solutions of the closed-loop system (13), (29). Indeed, consider the following differential inclusion

$$\dot{z} \in f(t,z) - \gamma\mathbf{N}(z), \tag{31}$$

where

$$f(t,z) = \begin{bmatrix} B_\perp^\top \left(A + \hat{\Delta}_A(t,z)\right) PB_\perp \left(B_\perp^\top PB_\perp\right)^{-1} & B_\perp^\top \left(A + \hat{\Delta}_A(t,z)\right) B \\ 0 & -K \end{bmatrix} \begin{bmatrix} z_1 \\ \sigma \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{w}(t,z) + \hat{\phi}_m(t,z) \end{bmatrix}$$

is a locally Lipschitz function in its second argument and $\mathbf{N} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a maximal monotone set-valued map described by $z \mapsto [0, \zeta^\top]^\top$ and $\zeta \in \mathbf{M}(\sigma)$. Thus, a direct application of Proposition 3.13 in [9] leads us to the uniqueness of solutions.

It is a well known fact that in the continuous-time setting the selection of the values that maintain the sliding regime depends explicitly on the values of the disturbances $\hat{w}$ and $\hat{\phi}_m$, which are by definition unknown. For that reason, in practical applications it is common to use a regularized version of the controller (22), which leads to the concept of boundary layer control [46]. In general, the regularization is made in an arbitrary way. In our context the regularization is well defined by means of the Yosida regularization and, as was shown in the proof of Theorem 22, this approach leads to trajectories that are in a neighbourhood of one solution of the differential inclusion (13). In the sequel we present an example for the case of the unit vector approach.

Consider the set-valued map $\mathbf{M}$ as in Example 17 and a constant gain $\gamma > 0$. From the proof of Theorem 22, it follows that our regularized control is given by the maximal monotone single-valued map $\mathcal{M}^\mu$, which in this case is given by

$$\mathcal{M}^\mu(\sigma) = \nabla f^\mu(\sigma) = \frac{1}{\mu}\left(\sigma - \mathrm{Prox}_{\mu f}(\sigma)\right) = \begin{cases} \frac{\sigma}{\|\sigma\|}, & \text{if } \|\sigma\| > \mu, \\ \frac{1}{\mu}\sigma, & \text{otherwise.} \end{cases} \tag{32}$$

It is worth to mention that (32) differs from the commonly used regularization $\frac{\sigma}{\|\sigma\|+\rho}$ with $\rho > 0$ sufficiently small. Therefore, in the maximal monotone approach we have a unique way of computing the regularized controller coming from a set-valued maximal monotone map leading to a closed-loop system whose trajectories converge into a neighborhood of the origin. In the next section we shall study the design of this kind of maximal monotone controllers in the discrete-time setting.

4. Design of discrete-time sliding-mode controllers by using maximal monotone maps. In this section we present a methodology for the digital implementation of discrete-time sliding-mode controllers using maximal monotone maps. The design process is revisited step-by-step in order to show how the implicit discrete-time scheme proposed in [1, 2] allows us to make a proper selection of the values of the control input at each sampling instant, and consequently reduces drastically the chattering effect at high sampling rates.

4.1. The plant representation. We start considering the discrete-time model of (4) through the use of the Euler's method, i.e., we take a constant sampling time $t_{k+1} - t_k = h > 0$ for all $k \geq 0$ and obtain

$$x_{k+1} = (I_n + hA)x_k + hB(u_k + w(k, x_k)) + h\Delta_A(k, x_k)x_k. \tag{33}$$

It is worth to mention that in the absence of the parametric disturbances, $\Delta_A(k, x_k) \equiv 0$, the system (33) becomes linear and the ZOH (Zero-Order Hold) method can be applied in order to obtain the equations of the dynamics in discrete time. Nevertheless, that is not the general case analyzed in this paper. Note that, because of the presence of the nonlinear term $\Delta_A(k, x_k)$, it is not possible to compute, in general, the equations of the ZOH discretization in a closed-form, which requires the knowledge of the solution of the nonlinear system, as well as the exact value of the parameters. Instead, the first order approximation described by the explicit Euler algorithm is used in this work for the discretization of the plant dynamics. In addition, just as stated in [28, Theorem 2], under the assumption that the sampling time is *small enough*, the property of stability is independent of the number of terms considered in the exact ZOH of the nonlinear system. That is, the property of stability for the discrete-time closed-loop system (47) is the same as the stability of an exact ZOH method whenever the sampling time $h > 0$ is sufficiently small.

Along all this section we also consider that Assumptions 7 through 19 hold. In the discrete-time context the counterpart of Proposition 11 is given as:

PROPOSITION 27. *Assumption 7 implies that for some $a > 0$ such that $0 < 2ha < 1$, there exists a symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$ satisfying the matrix inequality:*

$$B_\perp^\top \left(AX + XA^\top + 2aX\right) B_\perp + hB_\perp^\top \left(XA^\top B_\perp \left(B_\perp^\top X B_\perp\right)^{-1} B_\perp^\top AX\right) B_\perp < 0. \tag{34}$$

Proof. Stabilizability of the system (33) is equivalent to the existence of a matrix $K \in \mathbb{R}^{m\times n}$ such that for any $2ha \in (0, 1)$, there exists a matrix, $D_1 \in \mathbb{R}^{n\times n}$, $D_1 = D_1^\top > 0$ satisfying the discrete-time Lyapunov equation

$$(1 - 2ha)D_1 - (I + hA - hBK)^\top D_1 (I + hA - hBK) > 0.$$

Pre and post multiplying by D_1^{-1} and setting $D_2 = KD_1^{-1}$ yields,

$$\begin{aligned} &-h(2aD_1^{-1} + AD_1^{-1} + D_1^{-1}A^\top - BD_2 - D_2^\top B^\top) \\ &\qquad - h^2 \left(AD_1^{-1} - BD_2\right)^\top D_1 \left(AD_1^{-1} - BD_2\right) > 0. \end{aligned}$$

Hence, applying Schur's complement formula we obtain the LMI

$$\begin{bmatrix} -h(2aD_1^{-1} + AD_1^{-1} + D_1^{-1}A^\top - BD_2 - D_2^\top B^\top) & h(D_1^{-1}A^\top - D_2^\top B) \\ h(AD_1^{-1} - BD_2) & D_1^{-1} \end{bmatrix} > 0.$$

Recalling that $B_\perp \in \mathbb{R}^{n\times(n-m)}$ has full column rank, it follows that the previous inequality implies

$$\begin{bmatrix} -hB_\perp^\top(2aD_1^{-1} + AD_1^{-1} + D_1^{-1}A^\top)B_\perp & hB_\perp^\top D_1^{-1}A^\top B_\perp \\ hB_\perp^\top AD_1^{-1}B_\perp & B_\perp^\top D_1^{-1}B_\perp \end{bmatrix} > 0, \tag{35}$$

where we have applied the full row rank congruence transformation

$$\begin{bmatrix} B_\perp^\top & 0_{n-m\times n} \\ 0_{n\times n-m} & B_\perp^\top \end{bmatrix} \in \mathbb{R}^{2(n-m)\times 2n}.$$

Finally, applying once again the Schur's complement formula to (35) and setting $X = D_1^{-1}$ we obtain the desired result. ∎

Notice that any solution of (34) is also a solution of (5) for any $h > 0$, and when $h = 0$ the left-hand sides of (34) and (5) coincide.

To finish this subsection we compute a bound for $\Delta_A(k, x_k)$ that will be useful in the forthcoming sections.

Proposition 28. *Let $X = X^\top > 0$ be such that*

$$X - I_n > 0, \tag{36}$$

then,

$$\Lambda^{-1} - \Delta_A(k, x_k)^\top B_\perp (B_\perp^\top X B_\perp)^{-1} B_\perp^\top \Delta_A(k, x_k) > 0. \tag{37}$$

Proof. From Assumption 9 together with the bound on X imposed by (36) it follows that

$$\Delta_A(k, x_k)\Lambda\Delta_A(k, x_k)^\top < X.$$

Since $B_\perp$ has full column rank, it follows that

$$B_\perp^\top X B_\perp - B_\perp^\top \Delta_A(k, x_k)\Lambda\Delta_A(k, x_k)^\top B_\perp > 0.$$

Using the Schur's complement formula we obtain,

$$\begin{bmatrix} B_\perp^\top X B_\perp & B_\perp^\top \Delta_A(k, x_k) \\ \Delta_A(k, x_k)^\top B_\perp & \Lambda^{-1} \end{bmatrix} > 0,$$

and applying once again the Schur's complement formula we obtain the desired result. ∎

In the sequel we will assume that X satisfies (34) together with (36) and consequently (37) also holds.

4.2. Design of the sliding surface. In this subsection the methodology for the design of the sliding surface mimics its continuous counterpart. First, we start with a sliding manifold of the form $\tilde{H} := \{x \in \mathbb{R}^n \mid Sx = 0\}$ and conditions on the matrix S are derived. In fact, it is shown that the resulting hyperplane has the same structure as its continuous-time analog H. We make the following assumption,

Assumption 29. The product SB is nonsingular.

Analogous to the continuous-time context, we start computing the equivalent control in order to see how the disturbance affects the sliding regime. In the discrete-time case, the sliding variable is given as $\sigma_k := Sx_k$ and the necessary sliding condition $\dot{\sigma} = 0$ is transformed into the fixed-point condition $\sigma_{k+1} = \sigma_k$, from which we obtain the equivalent control as[1]

$$u_k^{\mathrm{eq}} = \frac{1}{h}(SB)^{-1}\left(\sigma_k - S(I_n + hA)x_k - hS\Delta_A(k, x_k)x_k\right) - w(k, x_k) \tag{38}$$

[1] As alluded above, what we call the equivalent control here is not the same as what is called the equivalent control in [25].

Notice that the fixed-point condition $\sigma_{k+1} = \sigma_k$ is usually neglected and changed for the condition $\sigma_{k+1} = 0$. We will see that the fixed-point condition is well fitted for the estimation of the control law that will achieve the sliding motion. The equivalent closed-loop dynamics in sliding mode results in

$$\begin{aligned} x_{k+1}^{\mathrm{eq}} = \left(I_n - B(SB)^{-1}S\right)(I_n + hA)x_k^{\mathrm{eq}} + B(SB)^{-1}\sigma_k \\ + h\left(I_n - B(SB)^{-1}S\right)\Delta_A(k, x_k)x_k. \end{aligned} \tag{39}$$

From (39) it becomes clear that the structure of the sliding surface will be the same as in the continuous-time framework, i.e., throughout this section we set $S = (B^\top X^{-1}B)^{-1}B^\top X^{-1}$. Notice that both surfaces (C and S) are not exactly the same since P satisfies (5) and X satisfies (34) instead, but S tends to C as h decreases to zero.

4.3. Controller design. In this subsection we follow the discrete version of the two-steps design methodology used in the previous section. The main difference with the continuous part relies on the discretization scheme used for the control u^{sv}. It is shown that the implicit discretization approach inherits the robustness provided by the maximal monotone operators presented in Section 3. The first step consists in computing the nominal control using the fixed-point condition $\sigma_{k+1} = \sigma_k$, which leads to

$$u_k^{\mathrm{nom}} = \frac{1}{h}(SB)^{-1}\left(\sigma_k - S(I_n + hA)x_k\right). \tag{40}$$

Substitution of (40) into the discrete-time dynamics (33) yields

$$x_{k+1} = \left(I_n - B(SB)^{-1}S\right)(I_n + hA)\,x_k + B(SB)^{-1}\sigma_k + hB\,(u_k^{\mathrm{sv}} + w_k) + h\Delta_A(k, x_k)x_k.$$

Consider the coordinates transformation $z_k = Tx_k$ with T given in (11) but changing the matrix P by its discrete-time counterpart X. Hence, after simple computations we get the closed-loop system in regular form,

$$\begin{aligned} z_{k+1}^1 &= B_\perp^\top(I_n + hA + h\hat{\Delta}_A(k, z_k))XB_\perp\left(B_\perp^\top XB_\perp\right)^{-1} z_k^1 \\ &\quad + B_\perp^\top(I_n + hA + h\hat{\Delta}_A(k, z_k))B\sigma_k \end{aligned} \tag{41a}$$

$$\sigma_{k+1} = \sigma_k + h(u_k^{\mathrm{sv}} + \hat{w}(k, z_k) + \eta_k^m), \tag{41b}$$

where $\hat{\Delta}_A(k, z_k) := \Delta_A(k, T^{-1}z_k)$, $\hat{w}(k, z_k) := w(k, T^{-1}z_k)$, and the term η_k^m refers to the matched part of the disturbance $\hat{\Delta}_A(k, z_k)T^{-1}z_k$, that is, $\eta_k^m = S\hat{\Delta}_A(k, z_k)T^{-1}z_k$ with $S = (B^\top X^{-1}B)^{-1}B^\top X^{-1}$, see Remark 14. It is noteworthy that system (41) is the discrete-time counterpart of (13). It is clear that the disturbance term η_k^m satisfies a linear growth condition similar to that associated with the term ϕ_m, as stated in following.

PROPOSITION 30. *The disturbance term η_k^m satisfies the linear growth condition $\|\eta_k^m\| \le \sqrt{\bar{\kappa}}\|z_k\|$, where*

$$\bar{\kappa} := \frac{\lambda_{\max}(X)\lambda_{\max}(\Lambda^{-1})}{\lambda_{\min}(B^\top X^{-1}B)\lambda_{\min}(X)} \max\left\{\frac{1}{\lambda_{\min}(B_\perp^\top XB_\perp)}, \lambda_{\max}(B^\top X^{-1}B)\right\}. \tag{42}$$

4.3.1. The set-valued controller. We continue with the design of the multi-valued part of the controller. The main difference with the continuous-time part is contained here where, because of the discretization method employed, it is possible to make a selection for the values of the controller that will compensate for the disturbances that affect the resulting closed-loop system. Specifically, we use the implicit Euler's method and we show how the system automatically makes the selection of the values that will compensate for the disturbance. As a motivation of the implicit scheme used, we study first the following *equivalent* controller,

$$-u_k^{\mathrm{sv}} \in \gamma\mathbf{M}(\sigma_{k+1}), \tag{43}$$

where $\gamma > 0$ is considered constant.

Remark 31. Note that unlike the continuous-time case, the operator $\gamma\mathcal{M}$ is maximal monotone. The main reason why we are considering a constant gain $\gamma > 0$ is that, whereas the lack of the maximal monotonicity was not a problem in the continuous-time setting, it becomes a critical issue in the discrete-time case since it implies the well-posedness of the resolvent and Yosida approximations, both of which, as is revealed below, are used for the computation of the explicit values of the feedback control.

At this point two important questions arise: is the proposed set-valued controller (43) non-anticipative? and why is it called 'equivalent'? The label 'equivalent' corresponds to the fact that, during the sliding phase, u_k^{sv} is equal to $u_k^{\mathrm{eq}} - u_k^{\mathrm{nom}}$. In other words, the control action $u_k = u_k^{\mathrm{nom}} + u_k^{\mathrm{sv}}$, with u_k^{sv} satisfying (43), coincides with the equivalent control (38). Indeed, consider the closed-loop system (41b), (43). It follows that,

$$\begin{aligned} \sigma_k - \sigma_{k+1} + h(\hat{w}(k, z_k) + \eta_k) \in h\gamma\mathbf{M}(\sigma_{k+1}) \iff \\ \sigma_{k+1} = J_{\gamma\mathbf{M}}^h(\sigma_k + h(\hat{w}(k, z_k) + \eta_k)), \end{aligned} \tag{44}$$

where $J_{\gamma\mathbf{M}}^h$ refers to the resolvent of the maximal monotone map $\gamma\mathbf{M}$ of index h. Hence, the discrete-time closed-loop dynamics of the sliding variable results in the difference equation (44). An explicit expression for the controller is obtained after substitution of (44) into (41b) as

$$u_k^{\mathrm{sv}} = -\frac{1}{h}(I - J_{\gamma\mathbf{M}}^h)(\sigma_k + h(\hat{w}(k, z_k) + \eta_k^m)) = -\mathcal{M}_\gamma^h\left(\sigma_k + h(\hat{w}(k, z_k) + \eta_k^m)\right). \tag{45}$$

where the map $\mathcal{M}_\gamma^h$ refers to the Yosida approximation of the set-valued map $\gamma\mathbf{M}$ of index h. At this point it is worth to mention that the selection process was done automatically by the system, i.e., the closed-loop system selects one and only one input from the maximal monotone map $\mathbf{M}$ in order to compensate for the disturbance term $\hat{w}(k, z_k) + \eta_k^m$. Thus, in ideal sliding mode $\sigma_{k+1} = \sigma_k = 0$ implies $u_k^{\mathrm{sv}} = -\frac{1}{h}(I - J_{\gamma\mathbf{M}}^h)(h(\hat{w}(k, z_k) + \eta_k^m))$. Now, assuming that $\hat{w}(k, z_k) + \eta_k^m \in \gamma\mathbf{M}(0)$ it follows that $u_k^{\mathrm{sv}} = -\hat{w}(k, z_k) - \eta_k^m$ (since $J_{\gamma\mathbf{M}}^h(w) = 0$ for all $w \in \gamma\mathbf{M}(0)$). Therefore, $u_k = u_k^{\mathrm{nom}} + u_k^{\mathrm{sv}} = u_k^{\mathrm{eq}}$. The previous development reveals that the implicit controller (43) makes sense.

Now we introduce the missing term u_k^{sv} using an implicit approach, which has been studied theoretically in [1, 2, 25] and tested experimentally in [26, 27, 48], showing to be a very efficient way to deal with the chattering effect on both the input and the output signals. It is clear that in a real implementation setting the selection procedure cannot be achieved directly, because if we try to mimic the same steps presented in the previous situation, we will have to impose the unreal assumption that we know perfectly the disturbance term $\hat{w}_k + \eta_k^m$, see (45). Therefore, some modification to the discrete-time controller (43) must be done. Roughly speaking, we consider the discrete-time scheme proposed in [1, 2, 25] in which a virtual nominal system is created and from which the selection process is achieved. Next, the controller computed from the virtual nominal system is applied to the original discrete-time plant. Formally, instead of (41), (43), we consider the extended system,

$$
\begin{aligned}
&(46a) && z_{k+1}^1 = B_\perp^\top (I_n + hA + h\hat{\Delta}_A(k, z_k)) X B_\perp \left(B_\perp^\top X B_\perp\right)^{-1} z_k^1 \\
& && \qquad + B_\perp^\top (I_n + hA + h\hat{\Delta}_A(k, z_k)) B \sigma_k \\
&(46b) && \sigma_{k+1} = \tilde{\sigma}_{k+1} + h(\hat{w}(k, z_k) + \eta_k^m) \\
&(46c) && \tilde{\sigma}_{k+1} = \sigma_k + h u_k^{\mathrm{sv}} \\
&(46d) && -u_k^{\mathrm{sv}} \in K\tilde{\sigma}_{k+1} + \gamma \mathbf{M}(\tilde{\sigma}_{k+1}),
\end{aligned}
$$

where $K \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix specified below. System (46) represents the implementable discrete-time dynamics associated with the real continuous-time system (13). The variable $\tilde{\sigma}_{k+1}$ may be seen as the state of a nominal, undisturbed system, or as a dumb variable allowing to calculate the controller u_k^{sv}. In this approach, the control selection is made using the virtual undisturbed system (46c)-(46d), and the perturbation term is implicitly taken into account through the use of the real state σ_k in (46c). Following the same steps as in (44), we have

$$
\begin{aligned}
\sigma_k - \tilde{\sigma}_{k+1} \in hK\tilde{\sigma}_{k+1} + h\gamma\mathbf{M}(\tilde{\sigma}_{k+1}) &\iff \sigma_k \in \left(I + h(K + \gamma\mathbf{M})\right)(\tilde{\sigma}_{k+1}) \\
&\iff \tilde{\sigma}_{k+1} = \left(I + h(K + \gamma\mathbf{M})\right)^{-1}(\sigma_k) \\
&\iff \tilde{\sigma}_{k+1} = J_{\mathbf{N}}^h(\sigma_k), \qquad (47)
\end{aligned}
$$

where $K = K^\top > 0$ is an $m \times m$ matrix and the set-valued map $\mathbf{N} := K + \gamma\mathbf{M}$ that maps $p \mapsto \{q \in \mathbb{R}^m \mid q = Kp + \gamma\zeta, \zeta \in \mathbf{M}(p)\}$ is also maximal monotone [41, Exercise 12.4]. It follows from (46c) that the input selection applied to the system is explicitly given by

$$
u_k^{\mathrm{sv}} = -\frac{1}{h}\left(I - J_{\mathbf{N}}^h\right)(\sigma_k) =: -\mathcal{N}^h(\sigma_k), \qquad (48)
$$

where $\mathcal{N}^h$ refers to the Yosida approximation of $\mathbf{N}$ of index h. Equation (48) shows the non-anticipation and the uniqueness of the control (46d) (since $\mathcal{N}^h$ is single valued). Hence, the discrete-time closed-loop subsystem (46b)-(46d) is equivalent to

$$
\begin{cases}
\sigma_{k+1} = \tilde{\sigma}_{k+1} + h(\hat{w}(k, z_k) + \eta_k^m), \\
\tilde{\sigma}_{k+1} = J_{\mathbf{N}}^h(\sigma_k).
\end{cases} \qquad (49)
$$

In this context the variable $\tilde{\sigma}_k$ is called the discrete sliding variable and, when $\tilde{\sigma}_{k+n} = 0$ for all $n \geq 1$ and some $k < +\infty$, we say that the system is in the *discrete-time sliding phase* [25].

Remark 32. Note that we have shown that the implicit discretization scheme (46) is well-posed and implementable. Indeed, the values of the controller were obtained explicitly from the *unique* solution of (46c)-(46d), that is, (48). It is also worth to mention that, under the proposed scheme, u_k^{sv} is a function of the current state σ_k

and design parameters, i.e., K, γ and $\mathbf{M}$. Hence, the controller is implementable and the closed-loop system reduces to,

$$z^1_{k+1} = B_\perp^\top (I_n + hA + h\hat{\Delta}_A(k,z_k))XB_\perp \left(B_\perp^\top X B_\perp\right)^{-1} z^1_k + B_\perp^\top (I_n + hA + h\hat{\Delta}_A(k,z_k))B\sigma_k \tag{50a}$$

$$\sigma_{k+1} = \sigma_k - \mathcal{N}^h(\sigma_k) + h(\hat{w}(k,z_k) + \eta^m_k). \tag{50b}$$

In the next section the stability properties of the closed-loop (50), equivalently (46), are studied in detail.

4.4. Stability of the closed-loop. In this section the stability of system (46) is proved. We start by computing the necessary conditions that the matrices X and K must satisfy under the assumption of ideal sliding phase, that is, $\sigma_k = 0$. This step allows us to compare the discrete-time and the continuous-time approaches showing their similarities, and also providing some convergence results. To this end, we start considering the following discrete-time reduced order system

$$z^1_{k+1} = B_\perp^\top (I_n + hA + h\hat{\Delta}_A(k,z_k))XB_\perp \left(B_\perp^\top X B_\perp\right)^{-1} z^1_k \tag{51}$$

together with the Lyapunov-function candidate $V(z^1_k) = \frac{1}{2} z^{1\top}_k \left(B_\perp^\top X B_\perp\right)^{-1} z^1_k$. Computing the difference $\Delta V := V(z^1_{k+1}) - V(z^1_k)$ along the trajectories of (51) and setting $G := B_\perp^\top X B_\perp$ and $s_k := G^{-1} z^1_k$ yields

$$\begin{aligned}
\Delta V &= \frac{1}{2} z^{1\top}_{k+1} \left(B_\perp^\top X B_\perp\right)^{-1} z^1_{k+1} - \frac{1}{2} z^{1\top}_k \left(B_\perp^\top X B_\perp\right)^{-1} z^1_k \\
&= \frac{h}{2} s_k^\top B_\perp^\top \left(AX + XA^\top + hXA^\top B_\perp G^{-1} B_\perp^\top AX\right) B_\perp s_k \\
&\quad + h s_k^\top B_\perp^\top \hat{\Delta}_A(k,z_k) X B_\perp s_k + h^2 s_k^\top B_\perp^\top X A^\top B_\perp G^{-1} B_\perp^\top \hat{\Delta}_A(k,z_k) X B_\perp s_k \\
&\quad + \frac{h^2}{2} s_k^\top B_\perp^\top X \hat{\Delta}_A(k,z_k)^\top B_\perp G^{-1} B_\perp^\top \hat{\Delta}_A(k,z_k) X B_\perp s_k.
\end{aligned} \tag{52}$$

Making use of the inequality $2p^\top Z^\top Y q \leq p^\top Z^\top \Psi Z p + q^\top Y^\top \Psi^{-1} Y q$, where $\Psi = \Psi^\top > 0$, gives the bounds

$$s_k^\top B_\perp^\top \hat{\Delta}_A(k,z_k) X B_\perp s_k \leq \frac{1}{2} s_k^\top B_\perp^\top \hat{\Delta}_A(k,z_k) \Psi_1 \hat{\Delta}_A(k,z_k)^\top B_\perp s_k + \frac{1}{2} s_k^\top B_\perp^\top X \Psi_1^{-1} X B_\perp s_k, \tag{53}$$

$$s_k^\top E^\top G^{-1} B_\perp^\top \hat{\Delta}_A(k,z_k) X B_\perp s_k \leq \frac{1}{2} s_k^\top E^\top G^{-1} \Psi_2 G^{-1} E s_k + \frac{1}{2} s_k^\top B_\perp^\top X \hat{\Delta}_A(k,z_k)^\top B_\perp \Psi_2^{-1} B_\perp^\top \hat{\Delta}_A(k,z_k) X B_\perp s_k, \tag{54}$$

where $E = B_\perp^\top X A^\top B_\perp$. Setting $\Psi_1 = \Lambda$ and $\Psi_2 = G$, where Λ is any positive definite matrix that satisfies Assumption 9, and then applying the results from Propositions 27 and 28 transforms (52) into

$$\Delta V \leq -h s_k^\top B_\perp^\top \left(aX - \frac{1}{2} I_n - \frac{1}{2} X\Lambda^{-1}X - hX\Lambda^{-1}X - \frac{h}{2} XA^\top B_\perp \left(B_\perp^\top X B_\perp\right)^{-1} B_\perp^\top AX \right) B_\perp s_k. \tag{55}$$

Therefore, $\Delta V < 0$ if

$$B_\perp^\top \left(aX - \frac{1}{2} I_n - \frac{1}{2} X \Lambda^{-1} X - h X \Lambda^{-1} X \right. \\ \left. - \frac{h}{2} X A^\top B_\perp \left(B_\perp^\top X B_\perp \right)^{-1} B_\perp^\top A X \right) B_\perp > 0. \tag{56}$$

Notice the resemblance of (56) with (18). In fact, once again we have that any solution of (56) is a solution of (18) and in the special case when $h = 0$ the right-hand sides of both matrix inequalities coincide. Similarly to the continuous-time case, we will ask for a stronger version of (56). Namely,

$$\bar{Q} := \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12}^\top & \bar{Q}_{22} \end{bmatrix} > 0, \tag{57}$$

where

$$\bar{Q}_{11} := B_\perp^\top \left(aX - I_n - \frac{1}{2} X \Lambda^{-1} X - h \left(2 X \Lambda^{-1} X + X A^\top B_\perp G^{-1} B_\perp^\top A X \right) \right) B_\perp,$$
$$\bar{Q}_{12} := -\frac{1}{2} B_\perp^\top A B - \frac{h}{2} B_\perp^\top X A^\top B_\perp G^{-1} B_\perp^\top A B,$$
$$\bar{Q}_{22} := K - \frac{1}{2} B^\top \Lambda^{-1} B - h B^\top \left(2\Lambda^{-1} + \frac{3}{2} A^\top B_\perp G^{-1} B_\perp^\top A \right) B.$$

It is also worth to notice that for any $h > 0$, a solution (X, K) of the matrix inequality (57) is also a solution of the matrix inequality (19). Additionally, in analogy with the continuous-time context, repeated application of Schur's complement formula gives us the equivalence between the matrix inequality (57) and the LMI

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^\top & R_{22} \end{bmatrix} > 0, \tag{58}$$

where,

$$R_{11} := \begin{bmatrix} B_\perp^\top (aX - I_n) B_\perp & -\frac{1}{2} B_\perp^\top A B & -h B_\perp^\top X A^\top B_\perp \\ -\frac{1}{2} B^\top A^\top B_\perp & K & -h B^\top A^\top B_\perp \\ -h B_\perp^\top A X B_\perp & -h B_\perp^\top A B & 2h B_\perp^\top X B_\perp \end{bmatrix}$$
$$R_{12} := \begin{bmatrix} -h B_\perp^\top X A^\top B_\perp & 0 & B_\perp^\top X & 0 \\ 0 & -h B^\top A^\top B_\perp & 0 & B^\top \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$R_{22} := \begin{bmatrix} 2h B_\perp^\top X B_\perp & 0 & 0 & 0 \\ 0 & h B_\perp^\top X B_\perp & 0 & 0 \\ 0 & 0 & \frac{2}{1+2h}\Lambda & 0 \\ 0 & 0 & 0 & \frac{2}{1+2h}\Lambda \end{bmatrix}.$$

Assumption 33. Along all this section we will assume that X and K are such that (34), (36) and (58) hold.

The following result gives conditions for achieving the discrete-time sliding phase ($\tilde{\sigma}_{k+1} = \tilde{\sigma}_k = 0$ for all $k \geq k^*$ for some $0 < k^* < +\infty$).

LEMMA 34. *Let Assumption 19 hold. The following two statements are equivalent:*

1) $\sigma_k \in h\gamma\mathbf{M}(0)$ *for some* $k \in \mathbb{N}$.
2) $\tilde{\sigma}_{k+1} = 0$.

In addition, if for some $k_0 \in \mathbb{N}$, $\tilde{\sigma}_{k_0+1} = 0$, *then* $\tilde{\sigma}_{k_0+n} = 0$ *for all* $n \geq 1$, *whenever* $\hat{w}(k, z_k) + \eta_k^m \in \gamma\mathbf{M}(0)$ *for all* $k \geq k_0$.

Proof. The equivalence between 1) and 2) is clear from (49). Namely, $\tilde{\sigma}_{k+1} = 0$ is equivalent to $J_{\mathbf{N}}^h(\sigma_k) = 0$, which in fact is the same as $\sigma_k \in (I + h(K + \gamma\mathbf{M}))(0)$. For the second part of the proof we start from the assumption that, for some $k_0 \in \mathbb{N}$, $\tilde{\sigma}_{k_0+1} = 0$. Hence, again from (49) it follows that

$$\sigma_{k_0+1} = \tilde{\sigma}_{k_0+1} + h(w_{k_0} + \eta_{k_0}^m) = h(w_{k_0} + \eta_{k_0}^m) \in h\gamma\mathbf{M}(0). \tag{59}$$

Therefore, applying the first part of the lemma we obtain $\tilde{\sigma}_{k_0+2} = 0$. The results follows by induction. ∎

The following result supports the use of the scheme proposed in [1, 2].

COROLLARY 35. *Let the matched disturbance* $\hat{w}(k, z_k) + \eta_k^m \in \gamma\mathbf{M}(0)$ *for all* $k \geq k^*$ *for some* $0 < k^* < +\infty$. *Then, in the discrete-time sliding phase the control input* u_k^{sv} *satisfies*

$$u_k^{\mathrm{sv}} = \hat{w}_{k-1} + \eta_{k-1}^m.$$

Proof. Since in sliding phase $\tilde{\sigma}_{k+1} = \tilde{\sigma}_k = 0$ it follows from (48) that $u_k^{\mathrm{sv}} = -\frac{\sigma_k}{h}$ and from (49) we have that $\sigma_k = h(\hat{w}_{k-1} + \eta_{k-1}^m)$ and the result follows. ∎

In words, the input obtained from the implicit scheme (46) compensates for the disturbance with a delay of one step once the discrete-time sliding phase has been reached. Moreover, it is worth to notice that in the discrete-time sliding phase the input u_k^{sv} is independent of the gain γ, a crucial fact that is experimentally verified in [26, 27]. This last property becomes fundamental in the application of the control scheme (46) since it helps to drastically reduce the chattering effect of the closed-loop system.

Remark 36. It is worth to mention that the scheme proposed in [1], [2] and stated in (46) for the computation of the control input seems to be connected to the approach of integral sliding modes for the estimation of the disturbance [47]. Indeed, we can see that equation (46c) represents some sort of nominal system from which the control input is obtained instead of using the perturbed system (46b). Moreover, Corollary 35 confirms that, as a consequence of taking the implicit discretization, the obtained controller is *automatically* compensating the matched disturbance terms with a one-step delay.

Practical stability of the difference equation (46) is proved by the following theorem.

THEOREM 37. *Let Assumptions 7-29 hold. Consider the closed-loop system* (46) *where* $X = X^\top > 0$ *and* $K = K^\top > 0$ *are such that Assumption 33 holds. In addition, let* $L_c \subset \mathbb{R}^n$ *be the compact set*

$$L_c := \left\{ \begin{bmatrix} z^1 \\ \sigma \end{bmatrix} \in \mathbb{R}^n \middle| \frac{1}{2} z^{1\top} \left(B_\perp^\top X B_\perp\right)^{-1} z^1 + \frac{1}{2}\sigma^\top\sigma \leq c^2 \right\}. \tag{60}$$

Then, for any initial condition $z_0 = \begin{bmatrix} z_0^{1\top} & \sigma_0^\top \end{bmatrix}^\top$ *which lies in* L_c *for some* $c > 0$, *there exists* $h > 0$ *small enough and fixed such that for* $\gamma > 0$ *satisfying*

$$\gamma\varepsilon = \rho + W + (\sqrt{\bar{\kappa}} + 2h\|K\|^2)\bar{z}, \tag{61}$$

where $\bar{z} := \max\{\|z\|, z \in L_c\}$ *and* $\rho > 0$ *is an arbitrary constant, the origin of the discrete-time closed-loop system* (46a)-(46d) *is semi-globally practically stable. In fact, for any initial condition* $z_0 \in L_c$ *the trajectories converge to a ball* $c_h^* \mathbb{B}_n$ *where* $c_h^* < c$ *is specified below in the proof and* $\lim_{h \to 0} c_h^* = 0$.

Proof. See the Appendix. ∎

Remark 38. Roughly speaking, semiglobal practical stability of the origin refers to the stability of a set (containing the origin) in which, the *size* of the set can be made arbitrary small and the region of attraction can be made arbitrary large by suitably adjusting a set of parameters (in our case the parameters are the sampling time $h > 0$ and the controller gain $\gamma > 0$). The reader is addressed to [16] for a detailed exposition of the concept and related results.

Remark 39. Practical stability fits within the boundary layer approach [45]. In our case we add the prefix semi-global because the disturbance is not uniformly bounded, so the gain γ would have to depend on the state for global stability.

COROLLARY 40. *Let all conditions and assumptions of Theorem 37 hold. Also, let the gain* $\gamma > 0$ *satisfy*

$$\gamma\varepsilon = \rho + (1+\alpha)(r + W + \sqrt{\bar{\kappa}}\bar{z}) + \max\left\{2h\|K\|^2\bar{z}, \frac{(W + \sqrt{\bar{\kappa}}\bar{z})^2}{r}\right\} \tag{62}$$

for some constants $\rho, r > 0$ *and* $\varepsilon > 0$ *such that* $\varepsilon\mathbb{B}_m \subset \mathbf{M}(0)$. *Then, there exists* $k_0 > 0$, $k_0 = k_0(\alpha, r)$, *which is finite and such that the variable* $\tilde{\sigma}_{k_0} = 0$. *Moreover,* $\tilde{\sigma}_k = 0$ *for all* $k \geq k_0$, *that is, the discrete-time sliding phase is reached in a finite number of steps.*

Proof. From Theorem 37 it follows that for all $k > 0$ the state z_k is uniformly bounded (since $z_k \in L_c$ for all $k \geq 0$). This boundedness property allows us to analyze the subsystem (49) and to take the disturbance term $\hat{w}(k, z_k) + \eta_k^m$ as uniformly bounded. Let us consider first the case where $\|\sigma_{k+1}\| > h\left(r + W + \sqrt{\bar{\kappa}}\bar{z}\right)$ for some $k \in \mathbb{N}$ and some $r > 0$ as in (62). Notice that this implies $\|\tilde{\sigma}_{k+1}\| \geq hr$. Consider the Lyapunov-function candidate $V_\sigma = \frac{1}{2}\sigma_k^\top \sigma_k$. From (87) we have that

$$\begin{aligned} \Delta V_\sigma &\leq -h\left(\gamma\varepsilon - \|\hat{w}(k, z_k) + \eta_k^m\|\right)\|\tilde{\sigma}_{k+1}\| + h^2\|\hat{w}(k, z_k) + \eta_k^m\|^2 \\ &\leq -h\left(\gamma\varepsilon - \left(W + \sqrt{\bar{\kappa}}\bar{z}\right) - \frac{\left(W + \sqrt{\bar{\kappa}}\bar{z}\right)^2}{r}\right)\|\tilde{\sigma}_{k+1}\| \end{aligned} \tag{63}$$

Thus, $\Delta V_\sigma < 0$ whenever $\|\sigma_{k+1}\| > h\left(r + W + \sqrt{\bar{\kappa}}\bar{z}\right)$. It follows that $\mathrm{dist}(\sigma_k, h(r + W + \sqrt{\bar{\kappa}}\bar{z})\mathbb{B}_m) \to 0$ as $k \to \infty$. Hence, there exists a finite $k_0(\alpha, r) > 0$ such that $\|\sigma_k\| \leq (1+\alpha)h(r + W + \sqrt{\bar{\kappa}}\bar{z})$ for all $k \geq k_0$, and

$$\frac{\|\sigma_k\|}{h} \leq (1+\alpha)(r + W + \sqrt{\bar{\kappa}}\bar{z}) \leq \gamma\varepsilon. \tag{64}$$

Since by assumption $\varepsilon\mathbb{B}_m \subset \mathbf{M}(0)$ a direct application of Lemma 34 gives us the desired result. On the other hand, if $\|\sigma_{k+1}\| < h(r + W + \sqrt{\bar{\kappa}}\bar{z})$ we have that

$$\frac{\|\sigma_{k+1}\|}{h} \leq r + W + \sqrt{\bar{\kappa}}\bar{z} \leq \gamma\varepsilon,$$

and the proof is complete. ∎

4.5. Convergence of the discrete-time solutions. Here we prove that the trajectories of the closed-loop discrete-time system (46) converge to trajectories of the closed-loop continuous-time system (13) as the sampling rate $h > 0$ decreases to zero. To this end consider the following piecewise continuous functions:

$$z_h^1(t) := z_k^1 + \frac{t - t_k}{h}\left(z_{k+1}^1 - z_k^1\right) \quad \text{for all } t \in [t_k, t_{k+1}] \tag{65a}$$

$$\sigma_h(t) := \sigma_k + \frac{t - t_k}{h}\left(\sigma_{k+1} - \sigma_k\right) \quad \text{for all } t \in [t_k, t_{k+1}], \tag{65b}$$

together with the step functions

$$\tilde{\sigma}_h^*(t) := \tilde{\sigma}_{k+1} \quad \text{for all } t \in (t_k, t_{k+1}] \tag{66a}$$

$$\sigma_h^*(t) := \sigma_k \quad \text{for all } t \in (t_k, t_{k+1}] \tag{66b}$$

$$z_h^{1*}(t) := z_k \quad \text{for all } t \in (t_k, t_{k+1}]. \tag{66c}$$

From Theorem 37 it follows that for a given initial condition $[z_h^1(0)^\top, \sigma_h(0)^\top]^\top \in \mathbb{R}^n$ the trajectories z_h^1 and σ_h are maintained for all times $t > 0$ inside a compact set L_c for some $c > 0$. Hence, they are uniformly bounded. Moreover, we have that the derivatives of z_h^1 and σ_h exist for almost all $t > 0$, and satisfy

$$\dot{z}_h^1(t) = \frac{z_{k+1}^1 - z_k^1}{h}, \text{ for all } t \in (t_k, t_{k+1}) \tag{67a}$$

$$\dot{\sigma}_h(t) = \frac{\sigma_{k+1} - \sigma_k}{h}, \text{ for all } t \in (t_k, t_{k+1}). \tag{67b}$$

It follows from (46a) and the continuity of $\hat{\Delta}_A(k, z_k)$ that $\dot{z}_h^1$ is uniformly bounded. On the other hand, by (49) we have that

$$\begin{aligned}\dot{\sigma}_h &= \frac{\tilde{\sigma}_{k+1} + h(\hat{w}(k, z_k) + \eta_k^m) - \sigma_k}{h} = \frac{J_{\mathbf{N}}^h(\sigma_k) - \sigma_k}{h} + \hat{w}(k, z_k) + \eta_k^m \\ &= -\mathcal{N}^h(\sigma_k) + \hat{w}(k, z_k) + \eta_k^m, \end{aligned} \tag{68}$$

where $\mathcal{N}^h$ is defined in (48). Thus, from the fact that $\|\mathcal{N}^h(\sigma_k)\| \leq \|\operatorname{Proj}_{\mathbf{N}(\sigma_k)}(0)\|$ [4, Theorem 2 p. 144] and recalling that $\eta_k^m = S\hat{\Delta}_A(k, z_k)T^{-1}z_k$ together with the uniform boundedness of $\hat{\Delta}_A(k, z_k)$ and $\hat{w}(k, z_k)$ (Assumptions 9 and 10 respectively), it follows that $\dot{\sigma}_h$ is uniformly bounded too. Hence, we have a pair of equicontinuous sequences of functions $\{z_h\}_{h>0}$ and $\{\sigma_h\}_{h>0}$ and using a similar argument as the one used in the proof of Theorem 22, we get the existence of continuous functions z^1 and σ such that $[z_h, \sigma_h] \to [z, \sigma]$, strongly in $\mathcal{L}_2([0, T]; \mathbb{R}^n)$ and $[\dot{z}_h, \dot{\sigma}_h] \to [\dot{z}, \dot{\sigma}]$ weakly in $\mathcal{L}_2([0, T]; \mathbb{R}^n)$ for any $T > 0$. Additionally, we have

$$\begin{aligned}\|\sigma_h - \sigma_h^*\|^2_{\mathcal{L}_2([0,T];\mathbb{R}^m)} &= \sum_{k=0}^{N-1} \int_{t_k}^{t_{k+1}} (t - t_k)^2 \|\dot{\sigma}_h(t)\|^2 dt \\ &\leq C_1^2 \sum_{k=0}^{N-1} \left.\frac{(t - t_k)^3}{3}\right|_{t_k}^{t_{k+1}} \\ &\leq \frac{C_1^2 T h^2}{3},\end{aligned}$$

where $C_1 > 0$ is an upperbound of $\|\dot{\sigma}_h\|$. Hence $\sigma_h^* \to \sigma$ as $h \downarrow 0$. In a similar fashion, we also have $z_h^* \to z$ as $h \downarrow 0$. Moreover, as was pointed out above, any solution X of the matrix inequalities (34), (57) converges to a matrix P, solution of (5) and (19), as h decreases to zero. Therefore, from (67) and (46) we get

$$\begin{aligned}\dot{z}_h^1 &= B_\perp^\top \left(A + \hat{\Delta}_A(k, z_k)\right) XB_\perp \left(B_\perp^\top XB_\perp\right)^{-1} z_h^{1*} + B_\perp^\top \left(A + \hat{\Delta}_A(k, z_k)\right) B\sigma_h^*,\\ &\to B_\perp^\top \left(A + \hat{\Delta}_A(k, z_k)\right) PB_\perp \left(B_\perp^\top PB_\perp\right)^{-1} z^1 + B_\perp^\top \left(A + \hat{\Delta}_A(k, z_k)\right) B\sigma = \dot{z}^1\end{aligned}$$

and

$$\dot{\sigma}_h - w_h^* - \eta_h^{m*} \to \dot{\sigma} - w - \phi_m \quad \text{as} \quad h \downarrow 0,$$

both weakly in $\mathcal{L}_2([0,T];\mathbb{R}^{n-m})$ and $\mathcal{L}_2([0,T];\mathbb{R}^m)$, respectively. Finally, from (68) we have that $-\dot{\sigma}_h + w_h^* + \eta_h^{m*} = \mathcal{N}^h(\sigma_h^*)$ and $J_{\mathbf{N}}^h(\sigma_h^*) \to \sigma$ strongly in $\mathcal{L}_2([0,T];\mathbb{R}^m)$. Indeed,

$$\begin{aligned}\|\sigma - J_{\mathbf{N}}^h(\sigma_h^*)\| &\le \|\sigma - J_{\mathbf{N}}^h(\sigma)\| + \|J_{\mathbf{N}}^h(\sigma) - J_{\mathbf{N}}^h(\sigma_h^*)\|\\ &\le h\|\mathcal{N}^h(\sigma)\| + \|\sigma - \sigma_h^*\|\\ &\le h\|\operatorname{Proj}_{\mathbf{N}(\sigma)}(0)\| + \|\sigma - \sigma_h^*\|,\end{aligned}$$

where we used the non-expansivity of the resolvent. It follows that $J_{\mathbf{N}}^h(\sigma_h^*) \to \sigma$ uniformly in $C([0,T];\mathbb{R}^m)$ as $h \downarrow 0$ (and consequently, strongly in $\mathcal{L}_2([0,T];\mathbb{R}^m)$). Consequently, using the fact that $\mathcal{N}(\sigma_h^*) \in \mathbf{N}(J_{\mathbf{N}}^h(\sigma_h^*))$, where $\mathbf{N} = K + \gamma\mathbf{M}$ [4, Theorem 2 p.144], after the application of Proposition 2 in Section 2 we conclude that the pair (z^1, σ) is a solution of the differential inclusion (13).

Remark 41. Previous developments reveal that the implicit discretization scheme for the set-valued part of the controller u_k^{sv} makes sense and at the same time allows us to inherit the robustness of the continuous-time closed-loop system.

In the next section we present some numerical examples, showing the robustness of the implemented discrete-time controller as well as the suppression of the chattering effect.

5. Numerical example. Consider the following benchmark dynamical system

$$(69) \qquad \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\\ -1 & -2 & 3 & 1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\\ 1 & 1 \end{bmatrix} u,$$

$x \in \mathbb{R}^5$, $u \in \mathbb{R}^2$, with the parametric uncertainty

$$(70) \qquad \Delta_A(t,x) = \begin{bmatrix} 0.1\cos x_1 & 0.1 & -0.1 & -0.1 & 0\\ 0 & 0.1\sin x_2 & 0.2 & 0.3 & -0.4\\ 0.33 & 0.1 & 0 & 0 & -0.1\sin x_3\\ 0 & 0 & 0.14\cos t & 0.2 & 0\\ 1 & 0.4 & 0.1\sin x_4 & 0 & 0.1 \end{bmatrix}.$$

In addition, we take into account the effects of a matched and bounded external disturbance $w(t) = \begin{bmatrix} 2\sin(t) & 5\sin(0.63t) \end{bmatrix}^\top$. First, we show the continuous-time case with the regularized control law provided by the Yosida approximation of the set-valued map $\mathbf{M}$ and, after that, the discrete-time case is exposed. In this example we consider the set-valued map $\mathbf{M}$ as the subdifferential of the infinity norm, i.e., let $f(\sigma) = \|\sigma\|_\infty = \max_i |\sigma_i|$. Hence,

$$\begin{aligned} \mathbf{M}(\sigma) = \partial f(\sigma) &:= \{\zeta \in \mathbb{R}^m \mid f(\eta) - f(\sigma) \geq \langle \zeta, \eta - \sigma \rangle, \text{ for all } \eta \in \mathbb{R}^m\} \\ &= \text{conv}\{\partial f^i(\sigma) | i \in I(\sigma)\}, \end{aligned} \tag{71}$$

where $f^i(\sigma) := |\sigma_i|$ and $I(\sigma) := \{i \in \{1, \ldots, n\} | f^i(\sigma) = f(\sigma)\}$ is the set of indices where the maximum is achieved [41, Exercise 8.31]. For the continuous-time case we use the regularized controller given by the Yosida approximation to the maximal monotone operator $\mathbf{M}$. Notice that, in the continuous-time case, the selection of the values for reaching the sliding phase will depend of the disturbance terms and therefore there is no suitable selection process. Invoking [7, Example 23.3] we have that $J^\mu_{\partial f} = \text{Prox}_{\mu f}$, where $\text{Prox}_{\mu f}$ refers to the proximal map of the function μf defined in Section 2. In order to compute the Yosida approximation first notice that the Moreau's decomposition Theorem [7, Theorem 14.3] gives

$$\mathcal{M}^\mu(\sigma) = \frac{1}{\mu}\left(I - J^\mu_{\mathbf{M}}\right)(\sigma) = \text{Prox}_{f^\star/\mu}\left(\frac{\sigma}{\mu}\right).$$

So we proceed to compute the conjugate function $f^\star(\sigma) := \sup_{x \in \mathbb{R}^m}\{\langle x, \sigma \rangle - f(x)\}$. Let us first consider the case when σ is such that $\sum_i |\sigma_i| \leq 1$. Then we have

$$\begin{aligned} 0 = \langle 0, \sigma \rangle - f(0) \leq f^\star(\sigma) &= \sup_{x \in \mathbb{R}^m} \{\langle x, \sigma \rangle - \|x\|_\infty\} \\ &\leq \sup_{x \in \mathbb{R}^m} \left\{\sum_{i=1}^m |\sigma_i||x_i| - \|x\|_\infty\right\} \\ &\leq \sup_{x \in \mathbb{R}^m} \left\{\|x\|_\infty \left(\sum_{i=1}^m |\sigma_i| - 1\right)\right\} = 0. \end{aligned}$$

Hence, $f^\star(\sigma) = 0$ whenever $\|\sigma\|_1 \leq 1$. On the other hand, consider the case where $\sum_i |\sigma_i| > 1$. In this case we have

$$\begin{aligned} f^\star(\sigma) &= \sup_{x \in \mathbb{R}^m} \{\langle x, \sigma \rangle - \|x\|_\infty\} \\ &\geq \sup_{b \in \mathbb{R}_+} \left\{\sum_{i=1}^m \sigma_i b \,\text{sign}(\sigma_i)\|\sigma\|_\infty - b \left\| \begin{bmatrix} \text{sign}(\sigma_1)\|\sigma\|_\infty \\ \vdots \\ \text{sign}(\sigma_m)\|\sigma\|_\infty \end{bmatrix} \right\|_\infty \right\} \\ &= \sup_{b \in \mathbb{R}_+} \left\{ b\|\sigma\|_\infty \left(\sum_{i=1}^m |\sigma_i| - 1\right)\right\} = +\infty. \end{aligned}$$

It follows that $f^\star(\sigma) = \Psi_{\mathbb{B}^1_m}(\sigma)$, where $\mathbb{B}^1_m := \{x \in \mathbb{R}^m | \|x\|_1 \leq 1\}$ and the function Ψ_C denotes the indicator function of the set C. Therefore,

$$\mathcal{M}^\mu(\sigma) = \text{Prox}_{\Psi_{\mathbb{B}^1_m}}\left(\frac{\sigma}{\mu}\right) = \text{Proj}_{\mathbb{B}^1_m}\left(\frac{\sigma}{\mu}\right).$$

The next step consists in the computation of C. Following the steps described in Section 3 we have that $C = (B^\top P^{-1} B) B^\top P^{-1}$ where $P = P^\top > 0$ is a solution of (5), (20). Using the software package CVX [24] together with the solver SeDuMi [43] to solve the LMIs (5) and (20) we obtain

$$P = \begin{bmatrix} 2.3075 & -3.3999 & -1.4020 & 2.5063 & -2.0431 \\ -3.3999 & 18.3866 & 1.4443 & -9.7181 & 9.8744 \\ -1.4020 & 1.4443 & 13.8392 & -19.8470 & -9.7614 \\ 2.5063 & -9.7181 & -19.8470 & 70.0849 & 38.7141 \\ -2.0431 & 9.8744 & -9.7614 & 38.7141 & 38.7003 \end{bmatrix},$$

together with

$$K = \begin{bmatrix} 14.6386 & -2.411 \\ -2.4111 & 14.2337 \end{bmatrix}.$$

It follows that

$$C = \begin{bmatrix} 1.5052 & 0.9790 & 0.0350 & -0.0210 & 0.0210 \\ -0.0019 & -1.7935 & 0.3140 & -0.7935 & 1.7935 \end{bmatrix}.$$

Figure 1 shows the trajectories, the sliding variable and the control input of the closed-loop system (69) with regularized control input $u = u^{\text{nom}} - K\sigma - \gamma(z)\mathcal{M}^{\mu}(\sigma)$, taking $\mu = 0.001$, $a = 1.4$, whereas the gain $\gamma(z)$ is as given in (25), with values $\gamma(z) = 7 + 29.28\|z\|$ and the initial condition $x(0) = \begin{bmatrix} 1 & -1 & 1 & 0 & -1 \end{bmatrix}^{\top}$. The simulations were carried up in Matlab using a Dormand-Prince solver (`ode45`) with variable time-step and relative tolerance of 10^{-6}. Also it is worth to mention that there is no chattering present neither in the input nor in the output σ, since the control input is Lipschitz continuous, see (48), and well-posed over all $\mathbb{R}^m$, see Figure 1.

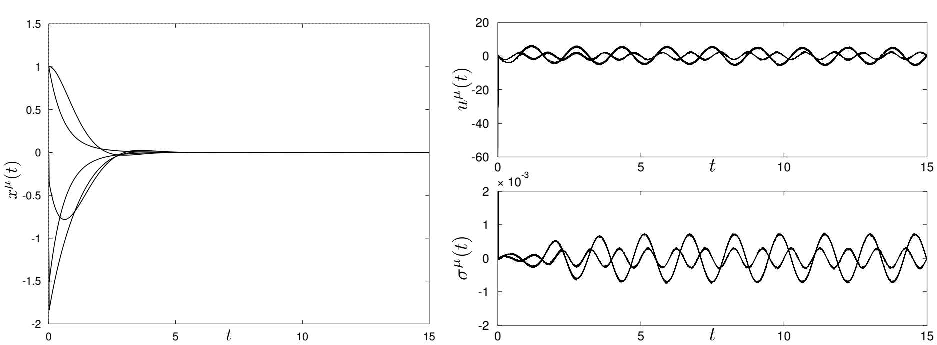

Fig. 1: Time evolution of the control input $u = u^{\text{nom}} - K\sigma - \gamma(z)\mathcal{M}^{\mu}(\sigma)$ and the corresponding system trajectories and sliding variable with $\mu = 0.001$.

For the discrete-time setting, we simulate the continuous-time plant with a ZOH sampling mechanism and we implement the discrete-time controller described in Section 4.3. We use the set-valued maximal monotone map $\mathbf{M}$ defined in (71). In this context, instead of computing the Yosida approximation of $\mathbf{N} = K + \gamma\mathbf{M}$, we introduce another way of computing the control input u^{sv} from the Yosida approximation of the set-valued map $\mathbf{M}$. From (46c)-(46d) it follows that $(I_n + hK)\tilde{\sigma}_{k+1} - \sigma_k \in -h\gamma\mathbf{M}(\tilde{\sigma}_{k+1})$

or, equivalently,

$$\theta\sigma_k - \theta(I_n + hK)\tilde{\sigma}_{k+1} \in \theta h\gamma \mathbf{M}(\tilde{\sigma}_{k+1})$$
$$\Updownarrow$$
$$\theta\sigma_k + (I_n - \theta I_n - \theta hK)\,\tilde{\sigma}_{k+1} \in (I + \theta h\gamma\mathbf{M})\,(\tilde{\sigma}_{k+1})$$
$$\Updownarrow$$
$$\tilde{\sigma}_{k+1} = J_{\mathbf{M}}^{\theta h\gamma}\left(\theta\sigma_k + (I_n - \theta(I_n + hK))\tilde{\sigma}_{k+1}\right). \tag{72}$$

We claim that the right-hand side of (72) is a contraction for $\theta > 0$ sufficiently small. Indeed, recalling that the resolvent $J_{\mathbf{M}}^{\mu}$ is non-expansive for any $\mu > 0$ it follows that

$$\begin{aligned}\left\| J_{\mathbf{M}}^{\theta h\gamma}\left(\theta\sigma_k + (I_n - \theta(I_n + hK))\tilde{\sigma}^1_{k+1}\right) - J_{\mathbf{M}}^{\theta h\gamma}\left(\theta\sigma_k + (I_n - \theta(I_n + hK))\tilde{\sigma}^2_{k+1}\right)\right\| \\ \leq \|I_n - \theta(I_n + hK)\|\|\tilde{\sigma}^1_{k+1} - \tilde{\sigma}^2_{k+1}\|.\end{aligned}$$

Hence, taking $\theta > 0$ small enough we have that $\|I_n - \theta(I_n + hK)\| < 1$ and then $J_{\mathbf{M}}^{\theta h\gamma}$ is a contraction. Consequently, the method of successive approximations can be applied in order to find the fixed point $\tilde{\sigma}_{k+1}$ of (72) and the control input u_k^{sv} at each sampling instant. We set three different sampling periods, $h \in \{50\,\mathrm{ms}, 5\,\mathrm{ms}, 0.5\,\mathrm{ms}\}$, $a = 1.4$, whereas γ was computed from (61) as $\gamma = 237.77$ for $h = 50$ ms, $\gamma = 51.17$ for $h = 5$ ms and $\gamma = 49.63$ for $h = 0.5$ ms, and $x_0 = \begin{bmatrix}1 & -1 & 1 & 0 & -1\end{bmatrix}^\top$ as before. In the three cases we solve (34), (36) and (58) and we obtain the following sliding surfaces $H_h := \{x \in \mathbb{R}^n \mid S_h x = 0\}$:

$$S_{h_1} = \begin{bmatrix} 1.4759 & 0.9867 & 0.0042 & -0.0133 & 0.0133 \\ 0.1065 & -1.6527 & 0.6364 & -0.6527 & 1.6527 \end{bmatrix}$$
$$S_{h_2} = \begin{bmatrix} 1.4733 & 0.9912 & 0.0266 & -0.0088 & 0.0088 \\ 0.0317 & -1.7821 & 0.3248 & -0.7821 & 1.7821 \end{bmatrix}$$
$$S_{h_3} = \begin{bmatrix} 1.4701 & 0.9977 & 0.0332 & -0.0023 & 0.0023 \\ 0.0280 & -1.7837 & 0.3083 & -0.7837 & 1.7837 \end{bmatrix}.$$

For the simulation of the system, we use the same Matlab configuration setting as in the previous case. Figures 2-3 show the evolution of the trajectories of the closed-loop system (69) with a control scheme dictated by (46), as well as the evolution in time of the sliding variable and the control input. The subindices in the labels of the plots indicate the sampling time h for the current variable. Notice that in all the three cases there is no chattering at all, neither in the input nor in the output, c.f. Figure 4. It is noteworthy that the control compensates for the disturbance as stated in Corollary 35.

Finally, Figure 4 shows the plots of the control input, sliding variable and system trajectories of the closed-loop system (69) when the conventional unit vector control is applied using an explicit discretization for the set-valued part of the controller, that is, $u(t_k) = u^{\mathrm{nom}}(t_k) - K\sigma(t_k) - \gamma\frac{\sigma(t_k)}{\|\sigma(t_k)\|+0.001}$ on $[t_k, t_{k+1})$ with sampling time $h = 5$ ms. Notice that, when we regularize the control input in the conventional way there is no selection procedure, which in the end results in the appearance of chattering in the system. Numerical chattering (i.e., the chattering due to the time-discretization) is known to be intrinsic to explicit discretizations [22, 23, 27].

6. Concluding remarks. In this work we present a family of set-valued sliding-mode controllers making use of the so-called maximal monotone operators. The proposed methodology has the advantage of embracing the two main approaches which

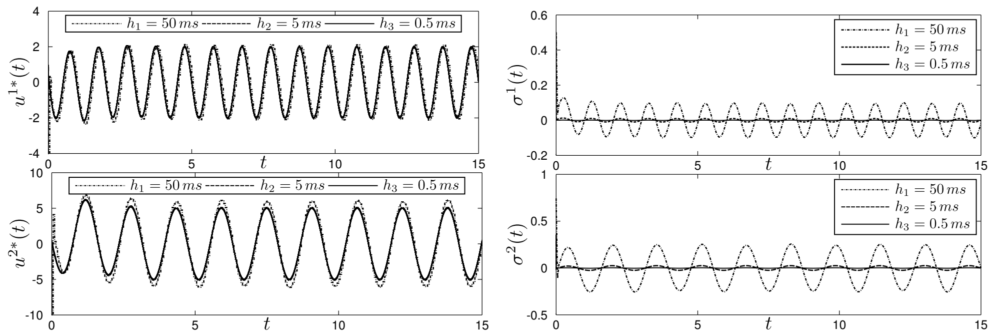

Fig. 2: Time evolution of the control input $u_k = u_k^{\mathrm{nom}} + u_k^{\mathrm{sv}}$ (left) and the associated sliding variable (right), for the sampling times $h \in \{50\,\mathrm{ms}, 5\,\mathrm{ms}, 0.5\,\mathrm{ms}\}$.

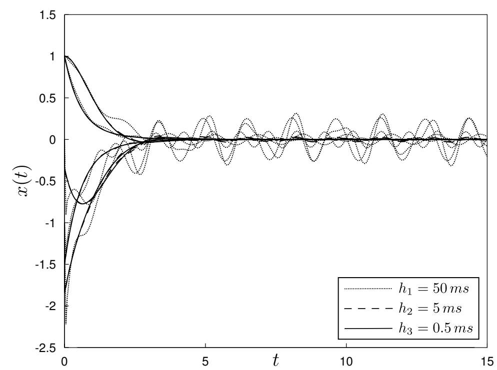

Fig. 3: Time evolution of the piecewise linear trajectories $x(t)$ of the discrete-time system (46) for the sampling times $h \in \{50\,\mathrm{ms}, 5\,\mathrm{ms}, 0.5\,\mathrm{ms}\}$.

exist in the literature of sliding-mode control, namely, the unit vector control and the componentwise control, among others. Additionally, the proposed scheme allows us to deal with the multivariable case without any modification and provides a unique and well-posed way of regularization of the set-valued controller through the use of the Yosida approximation.

All along the article we deal with parametric and matched external disturbances. A study for both the continuous and discrete-time cases was presented. In the continuous-time case it was shown that the proposed set-valued controller is well-posed even in the case when the right-hand side is not maximal monotone. Moreover, the convergence of the trajectories as the Yosida approximation converges to the set-valued control was established. On the other hand, the implementation of the controllers obtained from the continuous-time setting was analyzed. It was shown that the use of the implicit discretization for the set-valued part of the controller is well-posed, and allows us to make a selection for the values of the controller that will compensate for the disturbances in a unique fashion. The advantage of making a selection rather than switching is translated into the suppression of the chattering effect, confirming previous analytical and experimental results obtained in a less

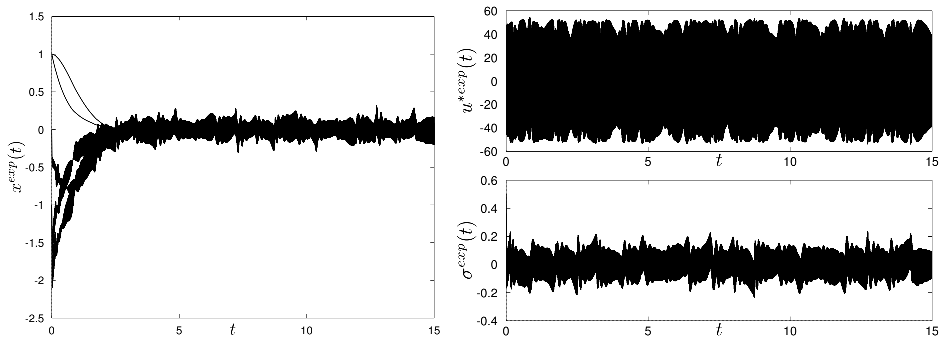

Fig. 4: Time evolution of the control input $u = u^{\mathrm{nom}} - K\sigma - \gamma\sigma/(\|\sigma\| + 0.001)$ and the corresponding system trajectories and sliding variable with a sampling step $h = 5$ ms.

general framework not encompassing parametric uncertainties.

Appendix A. Appendix.

A.1. Proof of Theorem 22.

Proof. The proof follows a classical approach. Namely, first we approximate the solutions of the differential inclusion (13),(22) by using differential equations. After that, the boundedness of the solutions of the differential equation for all times $t \in [0, +\infty)$ is proved. Finally, the application of the Arzelà-Ascoli [31, Theorem 1.3.8] and the Banach-Alaoglu [31, Theorem 2.4.3] theorems gives us the convergence of the sequence formed from the solutions of the differential equation to one solution of the differential inclusion (13),(22), see e.g., [3]. We start with the proof as follows. Consider first the differential equation

$$\left\{\begin{array}{ll} (73a) & \dot{z}_1^{\mu} = B_\perp^\top \left(A + \hat{\Delta}_A(t, z^\mu)\right) PB_\perp \left(B_\perp^\top PB_\perp\right)^{-1} z_1^\mu + B_\perp^\top \left(A + \hat{\Delta}_A(t, z^\mu)\right) B\sigma^\mu \\ (73b) & \dot{\sigma}^\mu = -K\sigma^\mu + \hat{w}(t, z^\mu) + \hat{\phi}_m(t, z^\mu) - \gamma(z^\mu)\mathcal{M}^\mu(\sigma^\mu), \end{array}\right.$$

where $z^\mu = [z_1^{\mu\top} \sigma^{\mu\top}]^\top$ and the map $\mathcal{M}^\mu : \mathbb{R}^m \to \mathbb{R}^m$ refers to the Yosida approximation of index $\mu > 0$ of the map $\mathbf{M}$ (see Definition 1). It is a well known fact that the Yosida approximation is a Lipschitz continuous function with constant $1/\mu$. Hence, it follows that there exists one solution to (73) in $[0, T)$ for some $T > 0$. Next, using a Lyapunov analysis we show that the solution of (73) exists for all times $t > 0$. To this end, consider the positive definite function

$$(74) \qquad V(z_1^\mu, \sigma^\mu) := \frac{1}{2} z_1^{\mu\top} (B_\perp^\top PB_\perp)^{-1} z_1^\mu + \frac{1}{2}\sigma^{\mu\top}\sigma^\mu,$$

where we recall that $B_\perp$ is full column rank and hence $B_\perp^\top PB_\perp > 0$. Deriving V

along the trajectories of (73) leads to

$$
\begin{aligned}
\dot{V} &= z_1^{\mu\top}(B_\perp^\top P B_\perp)^{-1}\dot{z}_1^\mu + \sigma^{\mu\top}\dot{\sigma}^\mu \\
&= z_1^{\mu\top}(B_\perp^\top P B_\perp)^{-1}B_\perp^\top\left(A+\hat{\Delta}_A(t,z^\mu)\right)PB_\perp(B_\perp^\top P B_\perp)^{-1}z_1^\mu \\
&\quad + z_1^{\mu\top}(B_\perp^\top P B_\perp)^{-1}B_\perp^\top\left(A+\hat{\Delta}_A(t,z^\mu)\right)B\sigma^\mu - \sigma^{\mu\top}K\sigma^\mu \\
&\quad + \sigma^{\mu\top}\left(-\gamma(z^\mu)\mathcal{M}^\mu(\sigma^\mu) + \hat{w}(t,z^\mu) + \hat{\phi}_m(t,z^\mu)\right) \\
&\le \frac{1}{2}\bar{z}_1^{\mu\top}B_\perp^\top(AP+PA^\top)B_\perp\bar{z}_1^\mu + \bar{z}_1^{\mu\top}B_\perp^\top AB\sigma^\mu + \bar{z}_1^{\mu\top}B_\perp^\top\hat{\Delta}_A(t,z^\mu)PB_\perp\bar{z}_1^\mu \\
&\quad + \bar{z}_1^{\mu\top}B_\perp^\top\hat{\Delta}_A(t,z^\mu)B\sigma^\mu - \sigma^{\mu\top}K\sigma^\mu \\
&\quad + \sigma^{\mu\top}\left(-\gamma(z^\mu)\mathcal{M}^\mu(\sigma^\mu) + \hat{w}(t,z^\mu) + \hat{\phi}_m(t,z^\mu)\right),
\end{aligned}
\tag{75}
$$

where, $\bar{z}_1^\mu = (B_\perp^\top P B_\perp)^{-1}z_1^\mu$. The next step consists in finding bounds for the terms that involve the unknown matrix $\hat{\Delta}_A$. Using the inequality $2p^\top X^\top Yq \le p^\top X^\top \Psi X p + q^\top Y^\top \Psi^{-1} Y q$, where $\Psi = \Psi^\top > 0$, gives us the bounds

$$
\bar{z}_1^{\mu\top}B_\perp^\top\hat{\Delta}_A P B_\perp\bar{z}_1^\mu \le \frac{1}{2}\bar{z}_1^{\mu\top}B_\perp^\top\hat{\Delta}_A\Psi\hat{\Delta}_A^\top B_\perp\bar{z}_1^\mu + \frac{1}{2}\bar{z}_1^{\mu\top}B_\perp^\top P\Psi^{-1}PB_\perp\bar{z}_1^\mu \tag{76}
$$

$$
\bar{z}_1^{\mu\top}B_\perp^\top\hat{\Delta}_A B\sigma^\mu \le \frac{1}{2}\bar{z}_1^{\mu\top}B_\perp^\top\hat{\Delta}_A\Psi\hat{\Delta}_A^\top B_\perp\bar{z}_1^\mu + \frac{1}{2}\sigma^{\mu\top}B^\top\Psi^{-1}B\sigma^\mu. \tag{77}
$$

Taking $\Psi = \Lambda$ where $\Lambda = \Lambda^\top > 0$ satisfies Assumption 9, the substitution of (76)-(77) into (75) yields

$$
\begin{aligned}
\dot{V} &\le -\bar{z}_1^{\mu\top}B_\perp^\top\left(aP - I_n - \frac{1}{2}P\Lambda^{-1}P\right)B_\perp\bar{z}_1^\mu + \bar{z}_1^\top B_\perp^\top AB\sigma^\mu \\
&\quad - \sigma^{\mu\top}\left(K - \frac{1}{2}B^\top\Lambda^{-1}B\right)\sigma^\mu + \sigma^{\mu\top}\left(-\gamma(z^\mu)\mathcal{M}^\mu(\sigma^\mu) + \hat{w}(t,z^\mu) + \hat{\phi}_m(t,z^\mu)\right) \\
&\le -\lambda_{\min}(\tilde{Q})\|z^\mu\|^2 - \gamma(z^\mu)\sigma^{\mu\top}\mathcal{M}^\mu(\sigma^\mu) + \left(W + \sqrt{\kappa}\|z^\mu\|\right)\|\sigma^\mu\|,
\end{aligned}
\tag{78}
$$

where $\tilde{Q} \in \mathbb{R}^{n\times n}$ is given as

$$
\tilde{Q} := \begin{bmatrix} \left(B_\perp^\top P B_\perp\right)^{-1} & 0 \\ 0 & I_m \end{bmatrix} Q \begin{bmatrix} \left(B_\perp^\top P B_\perp\right)^{-1} & 0 \\ 0 & I_m \end{bmatrix} > 0 \tag{79}
$$

and Q is defined in (19). We proceed to analyze the term $\langle\sigma^\mu, \mathcal{M}^\mu(\sigma^\mu)\rangle$ as follows. From the definition of the Yosida approximation (Definition 1 in Section 2) we have that $\sigma^\mu = \mu\mathcal{M}^\mu(\sigma^\mu) + J_{\mathbf{M}}^\mu(\sigma^\mu)$, and $(J_{\mathbf{M}}^\mu(\sigma^\mu), \mathcal{M}^\mu(\sigma^\mu)) \in \operatorname{Graph}\mathbf{M}$. Hence, making use of both previous facts together with (24) in Proposition 21 gives

$$
\begin{aligned}
\langle\sigma^\mu, \mathcal{M}^\mu(\sigma^\mu)\rangle &= \mu\|\mathcal{M}(\sigma^\mu)\|^2 + \langle J_{\mathbf{M}}^\mu(\sigma^\mu), \mathcal{M}^\mu(\sigma^\mu)\rangle \\
&\ge \mu\|\mathcal{M}(\sigma^\mu)\|^2 + \varepsilon\|J_{\mathbf{M}}^\mu(\sigma^\mu)\| \\
&= \mu\|\mathcal{M}(\sigma^\mu)\|^2 + \varepsilon\|\sigma^\mu - \mu\mathcal{M}^\mu(\sigma^\mu)\|.
\end{aligned}
$$

Now, recalling that $\|\sigma^\mu - \mu\mathcal{M}^\mu(\sigma^\mu)\| \ge \|\sigma^\mu\| - \mu\|\mathcal{M}^\mu(\sigma^\mu)\|$, we have

$$
\langle\sigma^\mu, \mathcal{M}^\mu(\sigma^\mu)\rangle \ge \varepsilon\|\sigma^\mu\| + \mu\|\mathcal{M}^\mu(\sigma^\mu)\|\left(\|\mathcal{M}^\mu(\sigma^\mu)\| - \varepsilon\right). \tag{80}
$$

Substitution of (80) into (78) results in

$$
\begin{aligned}
\dot{V} &\leq -\lambda_{\min}(\tilde{Q})\|z^\mu\|^2 + \|\sigma^\mu\|(W + \sqrt{\kappa}\|z^\mu\|) - \gamma(z^\mu)\big(\varepsilon\|\sigma^\mu\| \\
&\quad + \mu\|\mathcal{M}^\mu(\sigma^\mu)\|\left(\|\mathcal{M}^\mu(\sigma^\mu)\| - \varepsilon\right)\big) \\
&\leq -\lambda_{\min}(\tilde{Q})\|z^\mu\|^2 - \left(\varepsilon\gamma(z^\mu) - W - \sqrt{\kappa}\|z^\mu\|\right)\|\sigma^\mu\| \\
&\quad - \gamma(z^\mu)\mu\|\mathcal{M}^\mu(\sigma^\mu)\|\left(\|\mathcal{M}^\mu(\sigma^\mu)\| - \varepsilon\right).
\end{aligned} \tag{81}
$$

Now we continue with the proof showing that for all $\sigma^\mu \notin \mu\varepsilon\mathbb{B}_m$ the term $\|\mathcal{M}^\mu(\sigma^\mu)\| - \varepsilon$ is nonnegative. To this end, first notice that for any $v \in \mu\varepsilon\mathbb{B}_m \subset \mu\mathbf{M}(0)$, the resolvent $J^\mu_{\mathbf{M}}$ at the point v is zero. Indeed, let $\varepsilon > 0$ be such that $\varepsilon\mathbb{B}_m \subset \mathbf{M}(0)$. Then, it follows that for any $v \in \mu\varepsilon\mathbb{B}_m$, $v \in \mu\mathbf{M}(0) = (I + \mu\mathbf{M})(0)$. Therefore, $J^\mu_{\mathbf{M}}(v) = 0$. From the non-expansiveness property of the resolvent it follows that $\|J^\mu_{\mathbf{M}}(\sigma^\mu)\| \leq \|\sigma^\mu - v\|$, for all $v \in \mu\varepsilon\mathbb{B}_m$. So, from the definition of the Yosida approximation, taking $v = \mu\varepsilon\frac{\sigma^\mu}{\|\sigma^\mu\|}$, and recalling that we are analyzing the case where $\|\sigma^\mu\| \geq \mu\varepsilon$, we have

$$
\begin{aligned}
\|\mathcal{M}^\mu(\sigma^\mu)\| &= \frac{1}{\mu}\|\sigma^\mu - J^\mu_{\mathbf{M}}(\sigma^\mu)\| \geq \frac{1}{\mu}\left(\|\sigma^\mu\| - \|J^\mu_{\mathbf{M}}(\sigma^\mu)\|\right) \\
&\geq \frac{1}{\mu}\left(\|\sigma^\mu\| - \left\|\sigma^\mu - \mu\varepsilon\frac{\sigma^\mu}{\|\sigma^\mu\|}\right\|\right) \\
&= \frac{1}{\mu}\left(\|\sigma^\mu\| - \left(1 - \frac{\mu\varepsilon}{\|\sigma^\mu\|}\right)\|\sigma^\mu\|\right) = \varepsilon.
\end{aligned}
$$

Previous developments show that it is sufficient to consider only the case when the sliding variable $\sigma^\mu \in \varepsilon\mu\mathbb{B}_m$ (since for the case $\sigma^\mu \notin \varepsilon\mu\mathbb{B}_m$ we have already shown that (81) is strictly negative). Hence, letting $\|\sigma^\mu\| \leq \mu\varepsilon$ and recalling that in this case $J^\mu_{\mathbf{M}}(\sigma^\mu) = 0$, it follows that $\mathcal{M}^\mu(\sigma^\mu) = \frac{1}{\mu}\sigma^\mu$ and (81) transforms into

$$
\begin{aligned}
\dot{V} &\leq -\lambda_{\min}(\tilde{Q})\|z^\mu\|^2 - \left(\varepsilon\gamma(z^\mu) - W - \sqrt{\kappa}\|z^\mu\|\right)\|\sigma^\mu\| - \gamma(z^\mu)\|\sigma^\mu\|\left(\frac{\|\sigma^\mu\|}{\mu} - \varepsilon\right) \\
&\leq -\lambda_{\min}(\tilde{Q})\|z^\mu\|^2 - \left(\varepsilon\gamma(z^\mu) - W - \sqrt{\kappa}\|z^\mu\|\right)\|\sigma^\mu\| - \gamma(z^\mu)\frac{\|\sigma^\mu\|^2}{\mu} + \gamma(z^\mu)\varepsilon^2\mu.
\end{aligned}
$$

Let $L_c = \{z^\mu \in \mathbb{R}^n \mid V(z^\mu) \leq c,\}$ be the level sets of the function V and let $c > 0$ be such that the initial condition $z_0 \in L_c$ and $r\mathbb{B}_n \subset L_c$ for some $r > 0$. Then $\gamma(\cdot)$ is uniformly bounded in L_c by some $\bar{\gamma} > 0$, and for any $z \in L_c \setminus r\mathbb{B}_n$ we have that

$$
\dot{V} \leq -\left(\lambda_{\min}(\tilde{Q}) - \frac{\bar{\gamma}\varepsilon^2\mu}{r^2}\right)\|z^\mu\|^2 - \left(\varepsilon\gamma(z^\mu) - W - \sqrt{\kappa}\|z^\mu\|\right)\|\sigma^\mu\| - \gamma(z^\mu)\frac{\|\sigma^\mu\|^2}{\mu}. \tag{82}
$$

From (82) we conclude that, for all $\mu > 0$ small enough such that

$$
\mu < \frac{r^2\lambda_{\min}(\tilde{Q})}{\varepsilon^2\bar{\gamma}} =: \mu^*, \tag{83}
$$

the set L_c is positively invariant (since $\dot{V} < 0$ in $\operatorname{bd} L_c$) and boundedness of the trajectories on the time interval $[0, T]$ follows. A classical argument by contradiction proves the existence of solutions of (73) for all $T > 0$. It remains to show that for any $z^\mu(0) = z(0) = z_0 \in \mathbb{R}^n$ the sequences $\{z^\mu\}_{\mu>0}$ formed by the solutions of (73) converge to a solution of (13),(22) as $\mu \downarrow 0$. Continuing with the proof, let $z_0^\mu \in \mathbb{R}^n$ be fixed, then there exists a $c > 0$ such that $z^\mu(0) \in L_c$, and we have that any solution of (73) satisfies $z^\mu \in C([0,T];\mathbb{R}^n)$ for any $T > 0$, where $C([0,T];\mathbb{R}^n)$ refers to the Banach space of continuous functions from $[0,T]$ to $\mathbb{R}^n$ with norm $\|y\| = \sup_{t\in[0,T]} \|y(t)\|$. Further, the sequence of trajectories $\{z^\mu\}_{\mu>0}$ is uniformly bounded for all $0 < \mu < \mu^*$, where μ^* satisfies (83) (recall that the set L_c is positively invariant). On the other hand, from the assumption that the domain of $\mathbf{M}$ is all $\mathbb{R}^m$ it follows that $\mathcal{M}^\mu(\sigma^\mu(t))$ is uniformly bounded. Actually, from the fact that the set L_c is a compact subset of $\mathbb{R}^n$, it follows that there exist a compact subset $\tilde{L}_c \subset \mathbb{R}^m$ such that $\sigma^\mu(t) \in \tilde{L}_c$ for all $t \geq 0$ and all $0 < \mu < \mu^*$, and a finite collection of open sets $\{O_i\} \subset \mathbb{R}^m$ such that:

1. $\tilde{L}_c \subset \cup_{i=1}^r O_i$,
2. For each $i \in \{1,\dots,r\}$, $\mathbf{M}(O_i) \subset b_i\mathbb{B}_m$ for some $0 < b_i < +\infty$.

Consequently, $\mathbf{M}(\sigma^\mu(t)) \subset \cup_{i=1}^r \mathbf{M}(O_i) \subset \max_{i\in\{1,\dots,r\}} b_i\mathbb{B}_m$. Hence, invoking (2) it follows that $\|\mathcal{M}^\mu(\sigma^\mu(t))\| \leq \|\operatorname{Proj}_{\mathbf{M}(\sigma^\mu(t))}(0)\| \leq \max_{i\in\{1,\dots,r\}} b_i$. Therefore, from Assumption 9, together with (73) and the conclusion about the boundedness of its solutions it follows that, for any $0 < \mu < \mu^*$, $\dot{z}^\mu \in \mathcal{L}_\infty([0,T];\mathbb{R}^n)$ is uniformly bounded. Hence, we have that the sequence $\{z^\mu\}_{\mu>0}$ is equicontinuous. By a direct application of the Arzelà-Ascoli Theorem [31, Theorem 1.3.8] we get that there exists a subsequence $\{z^\mu\}_{\mu>0}$ such that $z^\mu \to z$ for some $z \in C([0,T];\mathbb{R}^n)$ uniformly in $[0,T]$. On the other hand, because $\dot{z}^\mu \in \mathcal{L}_\infty([0,T];\mathbb{R}^n)$, an application of the Banach-Alaoglu Theorem [31, Theorem 2.4.3] shows that there exists a function $q \in \mathcal{L}_\infty([0,T];\mathbb{R}^n)$ such that $\dot{z}^\mu \to q$ in the weak* topology, i.e.,

$$\lim_{\mu\downarrow 0} \int_0^T \langle \dot{z}^\mu(t) - q(t), s(t)\rangle dt = 0 \quad \text{for all } s \in \mathcal{L}_1([0,T];\mathbb{R}^n).$$

Moreover, from the fact that $z(t) = z(0) + \int_0^T q(t)dt$ we infer that $q = \dot{z}$ almost everywhere. Notice that, since the considered time domain is bounded, we have that $\mathcal{L}_2([0,T];\mathbb{R}^n) \subset \mathcal{L}_1([0,T];\mathbb{R}^n)$ [30, Corollary 1, Chapter VIII]. Hence, $\dot{z}^\mu$ converges weakly in $\mathcal{L}_2([0,T];\mathbb{R}^n)$. From the continuity assumption of $\hat{\Delta}_A$ and the convergence of z^μ and $\dot{z}^\mu$ to z and $\dot{z}$ respectively, it becomes clear that z satisfies (13a). In fact,

$$\begin{aligned}\dot{z}_1^\mu = B_\perp^\top(A + \hat{\Delta}_A(t,z^\mu))PB_\perp\left(B_\perp^\top PB_\perp\right)^{-1} z_1^\mu + B_\perp^\top\left(A + \Delta_A(t,z^\mu)\right)B\sigma^\mu \to \\ B_\perp^\top(A + \hat{\Delta}_A(t,z))PB_\perp\left(B_\perp^\top PB_\perp\right)^{-1} z + B_\perp^\top\left(A + \Delta_A(t,z)\right)B\sigma = \dot{z}_1.\end{aligned}$$

Additionally, setting $\theta^\mu := \dot{\sigma}^\mu + K\sigma^\mu - \hat{w} - \hat{\phi}_m$ we have that, for any $\varphi \in \mathcal{L}_2([0,T];\mathbb{R}^m)$,

$$\begin{aligned}&\int_0^T \left\langle \frac{\theta^\mu(t)}{\gamma(z^\mu(t))} - \frac{\theta(t)}{\gamma(z(t))}, \varphi(t)\right\rangle dt \\ &\quad = \int_0^T \left(\frac{1}{\gamma(z^\mu(t))} - \frac{1}{\gamma(z(t))}\right)\langle\theta^\mu(t),\varphi(t)\rangle\, dt + \int_0^T \left\langle \frac{\theta^\mu(t)-\theta(t)}{\gamma(z(t))}, \varphi(t)\right\rangle dt\end{aligned}$$

From (25) if follows that $\gamma(z) > \frac{\varrho}{\varepsilon}$ for any $z \in \mathbb{R}^n$. Thus, there exists a $\bar{\mu} > 0$ such that, for all $\bar{\mu} \leq \mu^*$ we have

$$\begin{aligned}(84)\quad &\int_0^T \left\langle \frac{\theta^{\bar{\mu}}(t)}{\gamma(z^{\bar{\mu}}(t))} - \frac{\theta(t)}{\gamma(z(t))}, \varphi(t)\right\rangle dt \\ &\quad \leq \int_0^T \frac{\varepsilon^2}{\rho^2} L_\gamma \|z^{\bar{\mu}}(t) - z(t)\|\|\theta^{\bar{\mu}}(t)\|\|\varphi(t)\| dt + \int_0^T \frac{\varepsilon}{\rho}\left\langle \theta^{\bar{\mu}}(t) - \theta(t), \varphi(t)\right\rangle dt,\end{aligned}$$

where $L_\gamma > 0$ refers to the Lipschitz constant of the function γ. Hence,

$$\zeta^{\bar{\mu}} := \frac{\dot{\sigma}^{\bar{\mu}} + K\sigma^{\bar{\mu}} - \hat{w}(t, z^{\bar{\mu}}) - \hat{\phi}_m(t, z^{\bar{\mu}})}{\gamma(z^{\bar{\mu}})} \to \frac{\dot{\sigma} + K\sigma - \hat{w}(t, z) - \hat{\phi}_m(t, z)}{\gamma(z)} =: \zeta \quad \text{as } \bar{\mu} \downarrow 0 \tag{85}$$

weakly in $\mathcal{L}_2([0, T]; \mathbb{R}^m)$ for any $T > 0$. Finally, from [4, p. 146] it follows that the set-valued map $\mathbf{M}$ seen as a set-valued map from $\mathcal{L}_2([0, T], \mathbb{R}^m)$ to the subsets of $\mathcal{L}_2([0, T], \mathbb{R}^m)$ is also maximal monotone. Since $J^{\bar{\mu}}_{\mathbf{M}}(\sigma^{\bar{\mu}}) \to \sigma$ uniformly in $C([0, T], \mathbb{R}^m)$ [4, p.144], and consequently strongly in $\mathcal{L}_2([0, T]; \mathbb{R}^m)$, the left-hand side of (85) is equal to $\zeta^{\bar{\mu}} = \mathcal{M}^{\bar{\mu}}(\sigma^{\bar{\mu}})$ and $\mathcal{M}^{\bar{\mu}}(\sigma^{\bar{\mu}}) \in \mathbf{M}(J^{\bar{\mu}}_{\mathbf{M}}(\sigma^{\bar{\mu}}))$ [4, p. 144]. Invoking Proposition 2 in Section 2 allows us to conclude that $\zeta \in \mathbf{M}(\sigma)$, that is, the differential inclusion (13),(22) is satisfied. This finishes the proof. ∎

A.2. Proof of Theorem 37.

Proof. Mimicking (27), let us consider the Lyapunov function candidate $V^k(z) = V^k_{z^1} + V^k_\sigma$, where $V^k_{z^1} := \frac{1}{2} z^{1\top}_k (B_\perp^\top X B_\perp)^{-1} z^1_k$ and $V^k_\sigma := \frac{1}{2}\sigma_k^\top \sigma_k$. Let $\Delta V = \Delta V_{z^1} + \Delta V_\sigma$ where $\Delta V_\sigma := V^{k+1}_\sigma - V^k_\sigma$ and $\Delta V_{z^1} := V^{k+1}_{z^1} - V^k_{z^1}$. We split the proof into two parts. The first part consists in finding a proper upper-bound for the difference ΔV_σ. After this, we continue analyzing the term ΔV_{z^1}. Finally we put all terms together and the practical stability follows. Consider the positive definite function $V^k_{\tilde{\sigma}} = \frac{1}{2}\tilde{\sigma}_k^\top \tilde{\sigma}_k$ and its respective difference $\Delta V_{\tilde{\sigma}} = V^{k+1}_{\tilde{\sigma}} - V^k_{\tilde{\sigma}}$. Then, making use of (46c) and (46d) it follows that

$$\begin{aligned}
\Delta V_{\tilde{\sigma}} &= \frac{1}{2}\tilde{\sigma}_{k+1}^\top \tilde{\sigma}_{k+1} - \frac{1}{2}\tilde{\sigma}_k^\top \tilde{\sigma}_k \\
&= \frac{1}{2}\tilde{\sigma}_{k+1}^\top (\tilde{\sigma}_{k+1} - \sigma_k) - \frac{1}{2}\tilde{\sigma}_k^\top \tilde{\sigma}_k + \frac{1}{2}\tilde{\sigma}_{k+1}^\top \sigma_k \\
&= \tilde{\sigma}_{k+1}^\top (\tilde{\sigma}_{k+1} - \sigma_k) - \frac{1}{2}\tilde{\sigma}_k^\top \tilde{\sigma}_k + \tilde{\sigma}_{k+1}^\top \sigma_k - \frac{1}{2}\tilde{\sigma}_{k+1}^\top \tilde{\sigma}_{k+1} \\
&\le -h\tilde{\sigma}_{k+1}^\top (K\tilde{\sigma}_{k+1} + \gamma\zeta_{k+1}) + V^k_\sigma - V^k_{\tilde{\sigma}},
\end{aligned} \tag{86}$$

where $\zeta_{k+1} \in \mathbf{M}(\tilde{\sigma}_{k+1})$ and we have used the inequality $2\tilde{\sigma}_{k+1}^\top \sigma_k \le \tilde{\sigma}_{k+1}^\top \tilde{\sigma}_{k+1} + \sigma_k^\top \sigma_k$ in the last step. Adding and subtracting the term $V^{k+1}_\sigma + V^{k+1}_{\tilde{\sigma}}$ in (86) yields

$$\Delta V_{\tilde{\sigma}} \le -h\tilde{\sigma}_{k+1}^\top K\tilde{\sigma}_{k+1} - h\gamma\tilde{\sigma}_{k+1}^\top \zeta_{k+1} + \frac{1}{2}\sigma_{k+1}^\top \sigma_{k+1} - \frac{1}{2}\tilde{\sigma}_{k+1}^\top \tilde{\sigma}_{k+1} + \Delta V_{\tilde{\sigma}} - \Delta V_\sigma$$

which, after substitution of (46c) into (46b), leads to

$$\begin{aligned}
\Delta V_\sigma \le{} & -h\tilde{\sigma}_{k+1}^\top K\tilde{\sigma}_{k+1} - h\gamma\tilde{\sigma}_{k+1}^\top \zeta_{k+1} - \frac{1}{2}\tilde{\sigma}_{k+1}^\top \tilde{\sigma}_{k+1} \\
& + \frac{1}{2}\left(\tilde{\sigma}_{k+1} + h\left(\hat{w}(k, z_k) + \eta^m_k\right)\right)^\top \left(\tilde{\sigma}_{k+1} + h\left(\hat{w}(k, z_k) + \eta^m_k\right)\right) \\
={} & -h\tilde{\sigma}_{k+1}^\top K\tilde{\sigma}_{k+1} - h\gamma\tilde{\sigma}_{k+1}^\top \zeta_{k+1} + h\tilde{\sigma}_{k+1}^\top \left(\hat{w}(k, z_k) + \eta^m_k\right) + h^2\|\hat{w}(k, z_k) + \eta^m_k\|^2.
\end{aligned} \tag{87}$$

From (46c) and (46d) it follows that $\tilde{\sigma}_{k+1} = \sigma_k - hK\tilde{\sigma}_{k+1} - h\gamma\zeta_{k+1}$ with $\zeta_{k+1} \in$

$\mathbf{M}(\tilde{\sigma}_{k+1})$. Then (87) transforms into

$$\begin{aligned}
\Delta V_\sigma &\le -h\left(\sigma_k - hK\tilde{\sigma}_{k+1} - h\gamma\zeta_{k+1}\right)^\top K\left(\sigma_k - hK\tilde{\sigma}_{k+1} - h\gamma\zeta_{k+1}\right) - h\gamma\tilde{\sigma}_{k+1}^\top\zeta_{k+1} \\
&\quad + h\tilde{\sigma}_{k+1}^\top\left(\hat{w}(k,z_k) + \eta_k^m\right) + h^2\|\hat{w}(k,z_k) + \eta_k^m\|^2 \\
&\le -h\sigma_k^\top K\sigma_k + 2h^2\sigma_k^\top K\left(K\tilde{\sigma}_{k+1} + \gamma\zeta_{k+1}\right) \\
&\quad - h\gamma\tilde{\sigma}_{k+1}^\top\zeta_{k+1} + h\tilde{\sigma}_{k+1}^\top\left(\hat{w}(k,z_k) + \eta_k^m\right) + h^2\|\hat{w}(k,z_k) + \eta_k^m\|^2 \\
&\le -h\sigma_k^\top K\sigma_k - h\left(\gamma\varepsilon - \|\hat{w}(k,z_k) + \eta_k^m\| - 2h\|K\|^2\|\sigma_k\|\right)\|\tilde{\sigma}_{k+1}\| \\
&\quad + 2h^2\gamma\|K\|\|\zeta_{k+1}\|\|\sigma_k\| + \frac{h^2}{2}\|\hat{w}(k,z_k) + \eta_k^m\|^2,
\end{aligned} \tag{88}$$

where we made use of Proposition 21 in the last step. On the other hand, let us recall that $G = (B_\perp^\top XB_\perp)$ and let us set $s_k := G^{-1}z_k^1$. Substitution of (46a) into ΔV_{z^1}, after some simple algebra, leads to

$$\begin{aligned}
\Delta V_{z^1} &= \frac{1}{2}z_{k+1}^{1\top}G^{-1}z_{k+1}^1 - \frac{1}{2}z_k^{1\top}G^{-1}z_k^1 \\
&= \frac{1}{2}\Bigg(B_\perp^\top(I_n + hA + h\hat{\Delta}_A(k,z_k))XB_\perp s_k \\
&\qquad + B_\perp^\top(I_n + hA + h\hat{\Delta}_A(k,z_k))B\sigma_k\Bigg)^\top G^{-1}\Bigg(B_\perp^\top(I_n + hA \\
&\qquad + h\hat{\Delta}_A(k,z_k))XB_\perp s_k + B_\perp^\top(I_n + hA + h\hat{\Delta}_A(k,z_k))B\sigma_k\Bigg) - \frac{1}{2}s_k^\top Gs_k \\
&= \frac{1}{2}s_k^\top B_\perp^\top X\left(I_n + hA + h\hat{\Delta}_A(k,z_k)\right)^\top B_\perp G^{-1}B_\perp^\top(I_n + hA \\
&\qquad + h\hat{\Delta}_A(k,z_k))XB_\perp s_k - \frac{1}{2}s_k^\top Gs_k \\
&\qquad + s_k^\top B_\perp^\top X\left(I_n + hA + h\hat{\Delta}_A(k,z_k)\right)^\top B_\perp G^{-1}B_\perp^\top(hA + h\hat{\Delta}_A(k,z_k))B\sigma_k \\
&\qquad + \frac{h^2}{2}\sigma_k^\top B^\top\left(A + \hat{\Delta}_A(k,z_k)\right)^\top B_\perp G^{-1}B_\perp^\top(A + \hat{\Delta}_A(k,z_k))B\sigma_k.
\end{aligned} \tag{89}$$

Notice that the first two terms in (89) are equal to (52). Then, from (55) it follows that

$$\begin{aligned}
\Delta V_{z^1} &\le -hs_k^\top B_\perp^\top\left(aX - \frac{1}{2}I_n - \left(\frac{1}{2} + h\right)X\Lambda^{-1}X - \frac{h}{2}XA^\top B_\perp G^{-1}B_\perp^\top AX\right)B_\perp s_k \\
&\quad + hs_k^\top B_\perp^\top AB\sigma_k + hs_k^\top B_\perp^\top\hat{\Delta}_A(k,z_k)B\sigma_k + h^2s_k^\top B_\perp^\top XA^\top B_\perp G^{-1}B_\perp^\top AB\sigma_k \\
&\quad + h^2s_k^\top B_\perp^\top X\hat{\Delta}_A(k,z_k)^\top B_\perp G^{-1}B_\perp^\top\hat{\Delta}_A(k,z_k)B\sigma_k \\
&\quad + h^2s_k^\top B_\perp^\top XA^\top B_\perp G^{-1}B_\perp^\top\hat{\Delta}_A(k,z_k)B\sigma_k \\
&\quad + h^2s_k^\top B_\perp^\top X\hat{\Delta}_A(k,z_k)^\top B_\perp G^{-1}B_\perp^\top AB\sigma_k + \frac{h^2}{2}\sigma_k^\top B^\top A^\top B_\perp G^{-1}B_\perp^\top AB\sigma_k \\
&\quad + \frac{h^2}{2}\sigma_k^\top B^\top\hat{\Delta}_A(k,z_k)^\top B_\perp G^{-1}B_\perp^\top\hat{\Delta}_A(k,z_k)B\sigma_k \\
&\quad + h^2\sigma_k^\top B^\top A^\top B_\perp G^{-1}B_\perp^\top\hat{\Delta}_A(k,z_k)B\sigma_k.
\end{aligned} \tag{90}$$

Applying the inequality $2p^\top U^\top \Psi V q \le p^\top U^\top \Psi U p + q^\top V^\top \Psi^{-1} V q$, where $\Psi = \Psi^\top > 0$, to every cross term in which $\hat{\Delta}_A(k, z_k)$ appears in (89), yields the following bounds

$$
\begin{aligned}
&s_k^\top B_\perp^\top \hat{\Delta}_A(k, z_k) B\sigma_k \le \\
&\qquad \frac{1}{2} s_k^\top B_\perp^\top \hat{\Delta}_A(k, z_k) \Psi_1 \hat{\Delta}_A(k, z_k)^\top B_\perp s_k + \frac{1}{2}\sigma_k^\top B^\top \Psi_1^{-1} B\sigma_k,
\end{aligned}
$$

$$
\begin{aligned}
&s_k^\top B_\perp^\top X \Pi_1^\top B_\perp G^{-1} B_\perp^\top \Pi_2 B\sigma_k \le \\
&\qquad \frac{1}{2} s_k^\top B_\perp^\top X \Pi_1^\top B_\perp G^{-1} \Psi_2 G^{-1} B_\perp^\top \Pi_1 X B_\perp s_k \\
&\qquad\qquad + \frac{1}{2}\sigma_k^\top B^\top \Pi_2^\top B_\perp \Psi_2^{-1} B_\perp^\top \Pi_2 B\sigma_k,
\end{aligned}
$$

$$
\begin{aligned}
&\sigma_k^\top B^\top A^\top B_\perp G^{-1} B_\perp^\top \hat{\Delta}_A(k, z_k) B\sigma_k \le \\
&\qquad \frac{1}{2}\sigma_k^\top B^\top A^\top B_\perp G^{-1} \Psi_2 G^{-1} B_\perp^\top A B\sigma_k \\
&\qquad\qquad + \frac{1}{2}\sigma_k^\top B^\top \hat{\Delta}_A(k, z_k)^\top B_\perp \Psi_2^{-1} B_\perp^\top \hat{\Delta}_A(k, z_k) B\sigma_k,
\end{aligned}
$$

where we set $\Pi_1 = A$ or $\Pi_1 = \hat{\Delta}_A(k, z_k)$ according to the term in question and similarly for Π_2. Setting $\Psi_1 = \Lambda$ and $\Psi_2 = G$, the substitution of previous bounds into (90) gives

$$
\begin{aligned}
\Delta V_{z^1} \le{}& -h s_k^\top B_\perp^\top \left(aX - \frac{1}{2} I_n - \left(\frac{1}{2} + h\right) X\Lambda^{-1} X - \frac{h}{2} X A^\top B_\perp G^{-1} B_\perp^\top A X \right) B_\perp s_k \\
&+ h s_k^\top B_\perp^\top A B\sigma_k + \frac{h}{2} s_k^\top B_\perp^\top \hat{\Delta}_A(k, z_k) \Lambda \hat{\Delta}_A(k, z_k)^\top B_\perp s_k + \frac{h}{2}\sigma_k B^\top \Lambda^{-1} B\sigma_k \\
&+ h^2 s_k^\top B_\perp^\top X A^\top B_\perp G^{-1} B_\perp^\top A B\sigma_k \\
&+ h^2 s_k^\top B_\perp^\top X \hat{\Delta}_A(k, z_k)^\top B_\perp G^{-1} B_\perp^\top \hat{\Delta}_A(k, z_k) X B_\perp s_k \\
&+ \frac{h^2}{2} s_k^\top B_\perp^\top X A^\top B_\perp G^{-1} B_\perp^\top A X B_\perp s_k + \frac{3h^2}{2}\sigma_k^\top B^\top A^\top B_\perp G^{-1} B_\perp^\top A B\sigma_k \\
&+ 2h^2 \sigma_k^\top B^\top \hat{\Delta}_A(k, z_k)^\top B_\perp G^{-1} B_\perp^\top \hat{\Delta}_A(k, z_k) B\sigma_k.
\end{aligned}
\tag{91}
$$

Taking into account (37) together with Assumption 9 reduces (91) into

$$
\begin{aligned}
\Delta V_{z^1} \le{}& -h s_k^\top B_\perp^\top \left(aX - I_n - \left(\frac{1}{2} + 2h\right) X\Lambda^{-1} X - h X A^\top B_\perp G^{-1} B_\perp^\top A X \right) B_\perp s_k \\
&+ h s_k^\top B_\perp^\top A B\sigma_k + h^2 s_k^\top B_\perp^\top X A^\top B_\perp G^{-1} B_\perp^\top A B\sigma_k \\
&+ h\sigma_k B^\top \left(\left(\frac{1}{2} + 2h\right)\Lambda^{-1} + \frac{3}{2} h A^\top B_\perp \left(B_\perp^\top X B_\perp\right)^{-1} B_\perp^\top A \right) B\sigma_k
\end{aligned}
\tag{92}
$$

Addition of (87) and (92) leads to

$$
\begin{aligned}
\Delta V \le{}& -h z_k^\top \hat{Q} z_k - h\left(\gamma\varepsilon - \|\hat{w}(k, z_k) + \eta_k^m\| - 2h\|K\|^2\|\sigma_k\|\right)\|\tilde{\sigma}_{k+1}\| \\
&+ 2h^2\gamma\|K\|\|\zeta_{k+1}\|\|\sigma_k\| + \frac{h^2}{2}\|\hat{w}(k, z_k) + \eta_k^m\|^2,
\end{aligned}
\tag{93}
$$

where $\hat{Q} = \hat{Q}^\top \in \mathbb{R}^{n\times n}$ is given as

$$\hat{Q} := \begin{bmatrix} \left(B_\perp^\top X B_\perp\right)^{-1} & 0 \\ 0 & I_m \end{bmatrix} \bar{Q} \begin{bmatrix} \left(B_\perp^\top X B_\perp\right)^{-1} & 0 \\ 0 & I_m \end{bmatrix} > 0, \tag{94}$$

and $\bar{Q}$ is defined in (57). Now, let $L_c := \{(z_k^1, \sigma_k) \in \mathbb{R}^n \mid V(z_k^1, \sigma_k) \leq c^2\}$ be such that $(z_0^1, \sigma_0) \in L_c$ and $\|z_k\| > r$ in the boundary of L_c for some fixed $r > 0$. We proceed to show that L_c is invariant. To this end, first notice that $\zeta_{k+1} \in \mathbf{M}(\tilde{\sigma}_{k+1})$ is bounded in L_c. Indeed, from (49) and the non-expansiveness property of the resolvent, it follows that $\tilde{\sigma}_{k+1}$ is bounded in L_c. Additionally, recalling that $\mathbf{M}$ is defined over all $\mathbb{R}^m$, it follows that $\mathbf{M}$ is bounded on bounded sets [41, Corollary 12.38] and consequently $\zeta_{k+1} \in \mathbf{M}(\tilde{\sigma}_{k+1})$ is bounded in L_c by some $\bar{\zeta} > 0$. Moreover, it follows from Proposition 30 that, in L_c, $\|\hat{w}(k, z_k) + \eta_k^m\| \leq W + \sqrt{\bar{\kappa}}\bar{z}$, where $\bar{z} := \max\{\|z\|, z \in L_c\}$. Consequently, for any $(z_k^1, \sigma_k) \in \mathrm{bd}(L_c)$ we have that

$$\begin{aligned} \Delta V &\leq -h\lambda_{\min}(\hat{Q})\|z_k\|^2 - h\left(\gamma\varepsilon - W - \sqrt{\bar{\kappa}}\bar{z} - 2h\|K\|^2\bar{z}\right)\|\tilde{\sigma}_{k+1}\| \\ &\quad + 2h^2\gamma\|K\|\|\bar{\zeta}\|\|z_k\| + \frac{h^2}{2}\left(W + \sqrt{\bar{\kappa}}\|z_k\|\right)^2 \\ &\leq -h\lambda_{\min}(\hat{Q})\|z_k\|^2 - h\left(\gamma\varepsilon - W - \sqrt{\bar{\kappa}}\bar{z} - 2h\|K\|^2\bar{z}\right)\|\tilde{\sigma}_{k+1}\| + h^2 l_c, \end{aligned} \tag{95}$$

where $l_c := 2\gamma\|K\|\|\bar{\zeta}\|\bar{z} + \frac{1}{2}\left(W + \sqrt{\bar{\kappa}}\bar{z}\right)^2$. Two cases arise:

Case 1, $\left(\|z_k\|^2 > \frac{h}{\lambda_{\min}(\hat{Q})} l_c\right)$.. From (61) and (95) it follows that the difference ΔV^k is strictly negative. Hence, if $z_k \in L_c$ it follows that $z_{k+1} \in L_c$.

Case 2, $\left(\|z_k\|^2 \leq \frac{h}{\lambda_{\min}(\hat{Q})} l_c\right)$.. In this case (95) lead us to,

$$V^{k+1} \leq V^k + h^2 l_c. \tag{96}$$

Roughly speaking, in this case the Lyapunov function may fail to be decreasing. However, if it increases, it will be in *small* quantities in such a way that the system's state stays inside L_c. Formally, letting $h > 0$ be such that

$$c^2 > \max_{\|z\|^2 \leq \frac{h}{\lambda_{\min}(\hat{Q})} l_c} V(z) + h^2 l_c \tag{97}$$

will imply $V^{k+1} \leq c^2$, that is, $z_{k+1} \in L_c$. Hence, selecting $c > 0$ big enough and $h > 0$ small enough, it follows that $z_0 \in L_c \setminus \sqrt{\frac{h}{\lambda_{\min}(\hat{Q})} l_c}\mathbb{B}_n$. Thus, we fall in Case 1 and $z_1 \in L_c$. Let $k^* \in \mathbb{N}$ be such that $z_{k^*} \in \sqrt{\frac{h}{\lambda_{\min}(\hat{Q})} l_c}\mathbb{B}_n$ (if that k^* does not exists, then we are always in Case 1 and the state will converge asymptotically to the ball $\sqrt{\frac{h}{\lambda_{\min}(\hat{Q})} l_c}\mathbb{B}_n$ and we are done). So, we fall in Case 2 and condition (97) will assure $z_{k^*+1} \in L_c$. Indeed, from (96) it follows that the state z_{k^*+1} remains inside the ball $c_h^*\mathbb{B}_n$ with c_h^* given as

$$c_h^{*2} = \left(\max\left(\frac{1}{\lambda_{\min}(B_\perp^\top X B_\perp)}, 1\right)\frac{1}{\lambda_{\min}(\hat{Q})} + h\right) h l_c, \tag{98}$$

from where practical stability follows. This concludes the proof. ∎

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