Zero-dynamics design and its application to the stabilization of implicit systems

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Abstract

We present a formula that computes the output of an R-controllable, regular, single-input linear time-invariant implicit system in such a way that it has prescribed relative degree and zeros. The formula is inspired on different generalizations of Ackermann's formula.

A possible application is in the context of sliding-mode control of implicit systems where, as the first step, one can use the proposed formula to design a sliding surface with desired dynamic characteristics and, as the second step, apply a higher-order sliding-mode controller to enforce a sliding motion along the resulting sliding surface.

Keywords: Zero placement, Zero dynamics, Implicit systems, Minimum-phase systems.

1. Introduction

In order to derive a mathematical model of a given dynamical system one chooses first a set of descriptor variables (position, speed, acceleration, temperature, current, voltage etc) in an attempt to define the state. The relationship among the chosen variables gives rise to differential or algebraic equations, sometimes resulting in an implicit system. Implicit systems are also referred to as generalized, descriptor, differential-algebraic (DAE) or semi-state systems, and are mainly motivated by applications in electric circuits and electromechanical or mechanical systems such as constrained robots.

It is possible to bring a single-input–single-output explicit system with strictly positive relative degree into a normal form that clearly reveals its zero dynamics. If the system is minimum phase, that is, if the zero dynamics are stable, it is then possible to stabilize the system by means of a simple state feedback (it suffices to drive the system output to zero). The extension of such results to the case of implicit systems was reported, e.g., in [1, 2, 3], where the authors propose a normal form for implicit systems and analyze the stability of its zero dynamics.

The problem of choosing an output with desired zeros is referred to as *zero placement* [4]. Since the zeros of the transfer function of any linear time-invariant (LTI) system, explicit or implicit, coincide with the eigenvalues of its zero dynamics, the problem of zero placement can be assimilated to the problem of defining the eigenvalues of the zero dynamics.

There are several circumstances in which one might be interested in designing an output that induces specific zero dynamics. In sliding-mode control (SMC), for example, the strategy consists in two steps: the design of a so-called sliding surface and the design of the actual control law, whose goal is to bring the system state onto the sliding surface and constrain the state to slide along it thereafter [5]. In the SMC literature, the system behavior when sliding along the sliding surface is called the sliding dynamics. A closer look at the methodology reveals that the sliding dynamics are nothing else than the zero dynamics of a virtual output, called the sliding variable. A usual recipe to the design of the sliding surface is the application of a formula by Ackermann and Utkin [6]. The two-step approach results in a controlled system which is completely insensitive to a large class of

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external disturbances. From an application point of view, this robustness presents an advantage over the simpler strategy consisting on the application of Ackermann's formula directly (i.e., as opposed to Ackermann-Utkin's formula) in order to specify the eigenvalues of the dynamics on the entire state-space (i.e., as opposed to the lower-dimensional sliding surface).

The original formula by Ackermann and Utkin is restricted to sliding surfaces of co-dimension one, which implies that the sliding variable has relative degree one. This is natural in the context of conventional SMC, since step two requires the sliding variable to have relative degree precisely equal to one. However, modern (higher-order) SMC removes the restriction on the relative degree of the sliding surface in step two. It is then reasonable to adjust step one and aim at sliding surfaces with desired sliding dynamics and of co-dimension higher than one. This motivates the generalization of the formula by Ackermann and Utkin presented in [7]. The objective of this paper is to further extend the formula to the case of regular LTI implicit systems.

A formula to design a stabilizing state feedback for completely controllable (C-controllable) implicit systems, based on Ackermann's formula for explicit systems, can be found in [8]. Such formula does not require the implicit system to be in the so-called Weierstrass or quasi-Weierstrass form. Obviating the need to use Weierstrass' form, which can be thought of as a generalization of Jordan's form, represents an advantage in practical terms, since similarity transformations can sometimes induce large errors in the presence of parameter uncertainties [9]. The formula proposed here is more general, as it works for R-controllable systems (R-controllability is weaker than C-controllability) and serves to specify the zero dynamics instead of the system dynamics in the complete state space.

Other than the higher-order SMC application mentioned above, the main result can also be used to design an output such that the system is minimum phase and has relative degree one or zero. The closed loop is thus feedback equivalent to a passive system and any passivity-based techniques can be used to control it.

The paper structure is as follows: In Section 2 we introduce the basic theory for singular systems and state the problem formally. The main result is presented in Section 3. In Section 4 we analyze the implications of our main result in the stabilization problem of implicit systems and present a concrete example. Conclusions and future work are presented in Section 5.

2. Preliminaries

Consider the single-input LTI implicit system

$$E\dot{x} = Ax + Bu \tag{1a}$$
$$y = Cx \; , \tag{1b}$$

where $x \in \mathbb{R}^n$ and $u, y \in \mathbb{R}$ are the state, the control input and the output at time t, respectively (we omit the time arguments to ease the notation). The matrices $E, A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$ are constant and given. We have $\operatorname{rank} E = n_0 < n$. The output matrix $C \in \mathbb{R}^{1\times n}$, not given *a priori*, will be specified later.

Definition 1. [10] For any two matrices E, $A \in \mathbb{R}^{n\times n}$, the pencil $\lambda E - A$ is called *regular* if the determinant $\det(\lambda E - A)$ does not vanish identically.

If $\det(\lambda E - A) \equiv 0$ or if the matrices are non square, then the pencil is called singular [10]. System (1a) is called solvable if, for any admissible input and any given admissible initial condition, equation (1a) has a unique solution [11]. This happens when the pencil $\lambda E - A$ is regular [10]. In such a case, the implicit system (1a) is called a regular implicit system.

An implicit system is regular if, and only if, there exist nonsingular matrices L and R such that, by applying the coordinate transformation

$$\begin{bmatrix} x_s \\ x_f \end{bmatrix} = R^{-1}x \; , \quad x_s \in \mathbb{R}^{n_1} \; , \quad x_f \in \mathbb{R}^{n_2} \; ,$$

and multiplying (1a) on the left by L we obtain

$$\dot{x}_s = A_s x_s + B_s u \tag{2a}$$
$$N\dot{x}_f = x_f + B_f u \; , \tag{2b}$$

where N is nilpotent with index of nilpotence q (see [12] for details). System (1a) is called an implicit system with index q, for short. If the matrices A_s and N are in Jordan form, then system (2) is said to be in Weierstrass form [10], otherwise, system (2) is said to be in quasi-Weierstrass form [13]. Recall that $\deg\det(\lambda E - A) = n_1 < n$, where the function deg represents the degree of a polynomial [12]. The set of finite eigenvalues of a matrix pair (E, A) is denoted as $\Lambda(E, A) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n_1}\}$.

The solution of subsystem (2a) can be easily determined from well-known results on explicit systems [14]. The solution of Subsystem (2b) depends affinely on u and its first $q-1$ time derivatives [12, 15, 11]. Let U be the set of admissible input functions. In order to assure the continuity of x_f we require $j = \max\left\{i \in \mathbb{N} : \operatorname{Im} B_f \nsubseteq \ker N^i\right\}$ and $U = \mathcal{C}^j$, where $\ker N^i$ is the null space of the matrix N^i and $\operatorname{Im} B_f$ is the image of B_f. Notice that $j \leq q-1$.

Definition 2. [16] A regular pencil $\lambda \bar{E} - \bar{A}$ is in *standard form* if there exist scalars α and β such that $\alpha \bar{E} + \beta \bar{A} = I$, where I is the identity matrix.

By definition, for any regular pencil $\lambda E - A$ there always exists a scalar μ such that $\det(\mu E - A) \neq 0$. Taking any such μ and multiplying (1a) on the left by $L = (\mu E - A)^{-1}$ gives

$$\bar{E}\dot{x} = \bar{A}x + \bar{B}u \ . \tag{3}$$

It is not difficult to verify that the pencil $\lambda \bar{E} - \bar{A}$ is in standard form for $\alpha = \mu$ and $\beta = -1$. The representation (3) is called a standard form of the regular implicit system (1a) [16]. Thus, for regular systems the assumption of a standard form is always without loss of generality. Also, since (1), (2) and (3) are restricted equivalent systems [12], we have $\Lambda(E, A) = \Lambda(I, A_s) = \Lambda(\bar{E}, \bar{A})$.

Recall that a single-input–single-output LTI regular implicit system of the form (1) has the transfer function [12]

$$g(s) = C(sE - A)^{-1}B = \frac{\eta(s)}{\delta(s)} \ , \tag{4}$$

where the polynomials $\delta(s) = \det(sE - A)$ and $\eta(s)$ and are the denominator and the numerator after zero–pole cancellation. We define the relative degree of (1) as $r = \deg \delta(s) - \deg \eta(s)$.

Now, consider the rational function $\pi \in \mathbb{C}(s)$ given by [17, 12]

$$\pi(s) = \frac{1}{\mu - s} \ . \tag{5}$$

Strictly speaking, since π is not bijective, its inverse does not exist. However, we define $\pi^{-1} \in \mathbb{C}(s)$ as $\pi^{-1}(s) = \mu - 1/s$. Also, we agree that $\pi(\infty) = 0$.

Theorem 1. *[12] Consider a regular system* (1) *written in standard form. Let $g(s)$ be its transfer function. For $\tau = \pi(s)$ we have*

$$g(\pi^{-1}(\tau)) = C(\pi^{-1}(\tau)\bar{E} - \bar{A})^{-1}\bar{B} = \tau \bar{g}(\tau)$$

with $\bar{g}(\tau) = C(\tau I - \bar{E})^{-1}\bar{B}$.

Let us now turn to the questions of stability and controllability.

Theorem 2. *[12, 18] The regular implicit system* (1a) *is stable if and only if $\Lambda(E, A) \subset \mathbb{C}^-$, where $\mathbb{C}^-$ represents the open left-half complex plane.*

We shall now introduce the concept of reachable state and characterize the set of all possible states reachable from a zero initial condition. This turns out to be important when distinguishing the different notions of controllability in regular implicit systems.

Definition 3. For a regular implicit system of the form (2), a vector $x_r \in \mathbb{R}^n$ is said to be *reachable* if there exists an initial condition $x_s(0)$, an input $u(\cdot) \in \mathcal{C}^j$, and some $t_1 > 0$ such that $\begin{bmatrix} x_s^\top(t_1) & x_f^\top(t_1) \end{bmatrix} = x_r^\top$.

Let $X_t(x_{s0})$ be the set of reachable states at time t from the initial condition $x_s(0) = x_{s0}$. Denote by $X_t = \bigcup_{x_{s0} \in \mathbb{R}^{n_1}} X_t(x_{s0})$ the set of reachable states at time t from all admissible initial conditions.

Definition 4. [12, 18] The regular system (2) is called *R-controllable* if, for any prescribed $t_1 > 0$, $x_{s0} \in \mathbb{R}^{n_1}$ and $x_r \in X_{t_1}$, there exists an input $u(t) \in \mathcal{C}^j$ such that the state response of system (2) starting from the initial value x_{s0} satisfies $x(t_1) = x_r$.

The main results on controllability of regular systems using time-domain analysis are developed in [19, 11, 20, 21]. Let $\bar{\mathbb{C}}^+$ represent the closed right-half complex plane. We summarize some results on R-controllability in the following proposition.

Proposition 1. *The implicit system* (1a) *is*

1) R-controllable if, and only if,

$$\operatorname{rank}\begin{bmatrix} sE - A & B \end{bmatrix} = n \quad \textit{for all } s \in \mathbb{C} \ .$$

If the implicit system (1a) *is in standard form, then it is*

2) R-controllable if, and only if,

$$\operatorname{rank}\begin{bmatrix} \tau I - \bar{E} & \bar{B} \end{bmatrix} = n \quad \textit{for all } \tau \neq 0 \ .$$

Problem. *Consider a regular R-controllable system* (1a). *Let l be the rank of the controllability matrix of the pair (E, B). For a set of m prescribed zeros, $0 \leq m \leq l-1$, choose C such that the zeros of the transfer function $g(s) = C(sE - A)^{-1}B$ are the prescribed ones.*

As in the case of explicit systems, the stability of the zero dynamics of implicit systems is determined by the location of the zeros of $g(s)$. A stability criterion for the zero dynamics of implicit systems is established in [1, Thm. 7.12], from which the following corollary is derived.

Corollary 1. *Consider an implicit system of the form* (1). *The following statements are equivalent:*

(i) The zero dynamics of system (1) *are asymptotically stable.*

(ii) System (1) *is minimum phase, i.e., the transfer function* (4) *has no zeros in* $\bar{\mathbb{C}}^+$.

Thus, for stabilization purposes, it is reasonable to place the zeros of $g(s)$ in $\mathbb{C}^-$. The stability of the system can then be assured simply by steering the output to zero.

Let us close this section with the notion of external equivalence.

Definition 5. Two single-input–single-output systems are called externally equivalent if their transfer functions are equal.

3. Assignment of the zero-dynamics

In this section we propose a formula to solve the problem stated in the previous section. According to Proposition 1, item 2), its controllability is directly related to the controllability of an explicit representation with system and input matrices $\bar{E}$ and $\bar{B}$. The main idea is to work with such fictitious system on which well-known results can be applied. The desired properties are then recovered for the original implicit system by means of an inverse transformation. To this end, consider the system

$$\dot{\xi} = \bar{E}\xi + \bar{B}u \; . \tag{6}$$

Let P be the controllability matrix of the pair $(\bar{E}, \bar{B})$, $P = \begin{bmatrix}\bar{B} & \bar{E}\bar{B} & \cdots & \bar{E}^{n-1}\bar{B}\end{bmatrix}$, and let l be its rank. Let m be any integer satisfying $0 \le m \le l-1$. We will show that it is possible to assign m finite zeros to the transfer function. More precisely, let $\Gamma = \Gamma_f \cup \Gamma_i$, where

$$\Gamma_f = \{s_1, \ldots, s_m\} \quad \text{and} \quad \Gamma_i = \underbrace{\{\infty, \ldots, \infty\}}_{l-1-m}$$

are, respectively, the set of finite and infinite desired zeros of the transfer function, where the members are counted with multiplicity. We assume that $\mu \notin \Gamma_f$, which is not restrictive since the set of admissible μ's is dense. Define the polynomial

$$\eta^f(s) = \prod_{i=1}^{m}(s - s_i)$$

and note that its set of roots is Γ_f. Also, define the polynomials $\gamma^f(\tau) = \tau^m \cdot \big(\eta^f \circ \pi^{-1}(\tau)\big)$ (the symbol '$\circ$' denotes composition) and

$$\gamma(\tau) = \tau^{(l-1)-m}\gamma^f(\tau) \; . \tag{7}$$

Note that $\gamma^f(\tau)$ has degree m and that its set of roots is $\pi(\Gamma_f)$, with π defined in (5). The polynomial $\gamma(\tau)$ has degree $l-1$, and its set of roots is $\pi(\Gamma)$.

According to Kalman's decomposition theorem [14, 17], there exists a coordinate transformation $\hat{\xi} = Q^{-1}\xi$ such that system (6) is brought to the form

$$\dot{\hat{\xi}} = \hat{E}\hat{\xi} + \hat{B}u \tag{8}$$

with

$$\hat{\xi} = \begin{bmatrix}\xi_C \\ \xi_{\bar{C}}\end{bmatrix} \; , \quad \hat{E} = \begin{bmatrix}E_C & E_{12} \\ 0 & E_{\bar{C}}\end{bmatrix} \quad \text{and} \quad \hat{B} = \begin{bmatrix}B_C \\ 0\end{bmatrix} \; ,$$

where the pair (E_C, B_C) defines an l-dimensional controllable subsystem with state ξ_C and controllability matrix P_C (that is, $\operatorname{rank} P = \operatorname{rank} P_C = l \le n$). The pair $(E_{\bar{C}}, 0)$ is of course uncontrollable. The matrix Q is chosen as

$$Q = \begin{bmatrix}q_1 & \cdots & q_l & q_{l+1} & \cdots & q_n\end{bmatrix} \; , \tag{9}$$

where the vectors $\{q_1, \ldots, q_l\}$ correspond to the first l linearly independent columns of P, that is,

$$q_i = \bar{E}^{i-1}\bar{B} \quad \text{for } i = 1, \ldots, l \; , \tag{10}$$

and the remaining columns are chosen so that Q is nonsingular.

Lemma 3. *Consider a system of the form* (6) *and suppose that the transformation* $\hat{\xi} = Q^{-1}\xi$ *with* (10) *is used to bring it to the form* (8). *Then, the controllability matrix of the pair* (E_C, B_C) *is equal to the identity.*

Lemma 4. *Consider a system of the form* (6) *with controllability matrix of rank* l *and suppose that the transformation* $\hat{\xi} = Q^{-1}\xi$ *with* (10) *is used to bring it to the form* (8). *Let* $\bar{d}_i$ *be the* l*-dimensional row vector with the* ith *component equal to one and the rest of the components equal to zero. If*

$$\hat{C} = \begin{bmatrix}\bar{d}_l & 0_{n-l}\end{bmatrix}\gamma(\hat{E}) \; , \tag{11}$$

then the zeros of the transfer function

$$\hat{g}(\tau) = \hat{C}(\tau I - \hat{E})^{-1}\hat{B}$$

are the roots of the polynomial $\gamma(\tau)$.

Proof. From the block triangular structure of $\hat{E}$ and the block form of $\hat{B}$ we have

$$\hat{C} = \begin{bmatrix}\bar{d}_l & 0_{n-l}\end{bmatrix} \gamma\left(\begin{bmatrix}E_C & E_{12}\\ 0 & E_{\bar{C}}\end{bmatrix}\right) = \begin{bmatrix}C_C & *\end{bmatrix}$$

and $\hat{g}(\tau) = C_C(\tau I - E_C)^{-1}B_C$ with $C_C = \bar{d}_l\gamma(E_C)$. By the main result of [7] and the fact that $P_C = I$, we have

$$\hat{g}(\tau) = \frac{\gamma(\tau)}{\det(\tau I - E_C)} .$$

Theorem 5 (Main result)**.** *Consider a regular R-controllable system* (1a) *and suppose, without loss of generality, that it is written in standard form. Let l be the rank of the controllability matrix of the pair $(\bar{E}, \bar{B})$ and let*

$$C = \begin{bmatrix}\bar{d}_l & 0_{n-l}\end{bmatrix} Q^{-1}\gamma(\bar{E}) , \tag{12}$$

with Q as in (9), (10), *γ as in* (7) *and $\bar{d}_l$ as in Lemma 4. Then, the transfer function of* (1) *takes the form*

$$g(s) = \frac{\eta^f(s)}{\det(s\bar{E} - \bar{A})} . \tag{13}$$

Proof. Consider the fictitious system (6). The transformation $\hat{\xi} = Q^{-1}\xi$ induces the controllable–uncontrollable decomposition (8). We know from Lemma 4 that the output matrix (11) places the zeros of the transfer function $\hat{g}(\tau)$ at Γ. To recover the system representation in the original coordinates ξ, we use $C = \hat{C}Q^{-1}$ and $\gamma(\hat{E}) = Q^{-1}\gamma(\bar{E})Q$ to obtain (12). In other words, the numerator of $\bar{g}(\tau)$ is $\gamma(\tau)$.

Since the system is R-controllable, the uncontrollable poles can only be the ones at infinity, and such poles are mapped to zero by π. That is,

$$\det(\tau I - \bar{E}) = \tau^{n-l}\det(\tau I - E_C)$$

(see Proposition 1, point 2)). Thus, the transfer function of

$$\begin{aligned}\dot{\xi} &= \bar{E}\xi + \bar{B}u\\ \upsilon &= C\xi\end{aligned}$$

can be written as

$$\bar{g}(\tau) = \frac{\gamma(\tau)}{\det(\tau I - E_C)} = \frac{\tau^{n-l}\gamma(\tau)}{\det(\tau I - \bar{E})} .$$

It follows from Theorem 1 that the transfer function of the regular implicit system (1) is

$$g(\pi^{-1}(\tau)) = \tau\frac{\tau^{n-l}\left(\tau^{l-m-1}\gamma^f(\tau)\right)}{\det(\tau I - \bar{E})} = \frac{\tau^{n-m}\gamma^f(\tau)}{\det(\tau I - \bar{E})} .$$

By using the definition of γ^f we obtain

$$\begin{aligned}g(\pi^{-1}(\tau)) &= \frac{\tau^n\left(\eta^f\circ\pi^{-1}(\tau)\right)}{\det(\tau I - \bar{E})}\\ &= \frac{\tau^n\left(\eta^f\circ\pi^{-1}(\tau)\right)}{\tau^n\det\left((\mu - \frac{1}{\tau})\bar{E} - \bar{A}\right)} ,\end{aligned}$$

where the last equality follows from the fact that $I = \mu\bar{E} - \bar{A}$. Since $s = \pi^{-1}(\tau) = \mu - 1/\tau$, we have (13).

By using (12) it is possible to choose the output of a single-input implicit system such that its transfer function, $g(s)$, is improper (if $l - 1 \geq m > n_1$) or proper (if $n - 1 \geq m$). Furthermore, if $n_1 > m$, then $g(s)$ is strictly proper with relative degree $r = n_1 - m > 0$.

4. Stabilization of minimum-phase implicit systems

In this section we discuss the implications of Theorem (12) in the context of stabilization of implicit systems.

According to the following corollary, a regular implicit system with strictly positive relative degree can be expressed in a block triangular form in which one of the subsystems is completely accountable for the transfer function.

Corollary 2. *[22, Lem. 2, Thm. 1] Consider a regular system* (1) *with strictly positive relative degree. There exist two unitary matrices R and L such that the transformation*

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = R^{-1}x ,$$

together with premultiplication of (1) *by L, gives the externally equivalent system*

$$E_{11}\dot{x}_1 = A_{11}x_1 + B_1u \tag{14a}$$
$$E_{21}\dot{x}_1 + E_{22}\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u \tag{14b}$$
$$y = C_1x_1 , \tag{14c}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$ and E_{11} nonsingular. Furthermore, the transfer function is

$$g(s) = C(sE - A)^{-1}B = C_1(sE_{11} - A_{11})^{-1}B_1 .$$

Note that the invertibility of E_{11} enables the possibility to realize the transfer function using an explicit system. See [3] for related results in a behavioral context.

Recall that, if an explicit LTI system is minimum phase, then it can be easily stabilized by designing a state feedback that steers the output to zero [23, pp. 263]. The state of the closed-loop system, together with the input, then go asymptotically to zero.

It is thus clear from Corollary 2 that, if the output of (1) is chosen such that the system has a strictly positive relative degree and that, if the zeros of the transfer function (4) are contained in $\mathbb{C}^-$, then the zero dynamics of system (1) are asymptotically stable and the output and its derivatives $\dot{y}, \ddot{y}, \ldots, y^{(r-1)}$ can be expressed as functions of the substate x_1. The latter is useful for constructing an admissible input u capable of steering the output (1b) to zero.

Example: higher-order sliding-mode control

Consider the implicit system

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \dot{x} = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} (u + w) ,$$

where $w \in \mathbb{R}$ is the external perturbation at time t. We have $\det(sE - A)) = s^2 + 3s + 2$. Thus, the system is regular, $n_1 = 2$, $n_2 = 1$ and $q = 1$. Actually, this system is in standard form for $\mu = 0$. Clearly, $\operatorname{rank} P = \operatorname{rank} \begin{bmatrix} B & EB & E^2B \end{bmatrix} = 3$. Since $q = 1$, we have $j = 0$, so the system has well-defined solutions for $u \in \mathcal{C}^0$. Suppose we want to design an output of relative degree one. Let $s_1 = -1.5$ be the desired eigenvalue for the zero dynamics, we then have $\eta^f(s) = s + 1.5$. Application of (12) with the polynomial $\gamma(\tau) = \tau(1.5\tau - 1)$ gives $C = \begin{bmatrix} 0 & -2 & 3 \end{bmatrix}$. Thus, the system has relative degree $r = 1$ with respect to the output $y = Cx$, which can be verified with the transfer function

$$g(s) = C(sE - A)^{-1}B = \frac{2(s + 1.5)}{(s^2 + 3s + 2)} .$$

If we set

$$L = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} , \quad R = \begin{bmatrix} 0 & 0 & 1 \\ -0.52 & 0.85 & 0 \\ 0.85 & 0.52 & 0 \end{bmatrix}$$

in Corollary 2, we then obtain an implicit representation of the form (14) with

$$LER = \begin{bmatrix} 0.85 & 0.52 & 0 \\ 1.54 & 0.35 & 0 \\ 0.52 & -0.85 & 0 \end{bmatrix} ,$$
$$CR = \begin{bmatrix} 3.6 & -0.15 & 0 \end{bmatrix}$$

and

$$LAR = \begin{bmatrix} 0.52 & -0.85 & 0 \\ -0.85 & -0.52 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \quad LB = \begin{bmatrix} 0 \\ 0.5 \\ 0 \end{bmatrix} .$$

It is easy to see that

$$g(s) = C_1(sE_{11} - A_{11})^{-1}B_1 = \frac{2s + 3}{s^2 + 3s + 2} .$$

Note also that $LL^\top = RR^\top = I$, which confirms that it is possible to obtain block triangular forms by means of a unitary equivalent transformation (this represents an advantage in terms of structural robustness).

To robustly steer x to the origin (i.e., irrespective of the perturbation w), we propose a control strategy based on higher-order sliding-mode control — specifically, on the super twisting algorithm (STA) [24]. The STA provides and absolutely continuous control signal and exactly compensates perturbations which are Lipschitz in time, at least theoretically. The STA takes the form

$$u(t) = -k_1 \left| y(t) \right|^{\frac{1}{2}} \operatorname{sign}\left(y(t)\right) - \int_0^t k_2 \operatorname{sign}(y(\tau)) \mathrm{d}\tau .$$

Fig. 1 shows the system's response for the initial conditions $\begin{bmatrix} x_2(0) & x_3(0) \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$ and a perturbation $w(t) = \sin(t)$, which satisfies the Lipschitz condition in time. The gains were set as $k_1 = 7$, $k_2 = 2$. One can see that all state variables go to zero and the virtual output does it in finite time, in spite of the perturbations.

5. Conclusions and future work

The main result of this paper is a formula for designing a system output such that the system has the prescribed relative degree and zero dynamics with prescribed eigenvalues. The system is not required to be in Weierstrass form, which would require the use of Jordan forms that can only be obtained using numerically unstable algorithms [9]. A

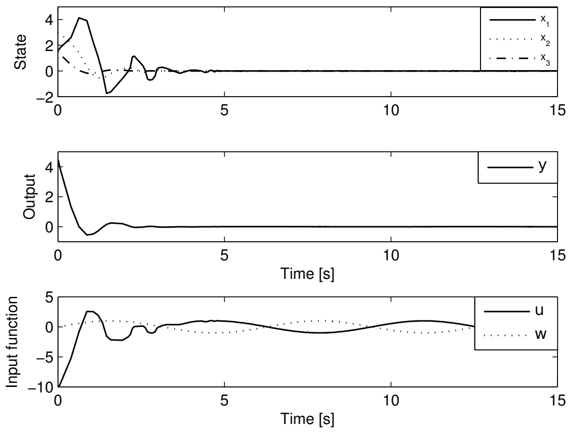

Figure 1: State, virtual output, control action and perturbations for the system described in the example.

transformation is proposed for computing the output derivatives. The transformation is numerically stable, as it is given by unitary matrices. The output derivatives can then be used to construct a control law that steers the output to zero, together with the state if the system is designed as minimum phase.

As a specific application, Theorem 5 can be used to design a sliding variable of arbitrary relative degree. On a second step, HOSM control techniques can then be used to construct the actual control law. It is worth noting that other techniques, such as passivity-based control, can also be used.

As future work we consider the use of other HOSM techniques, including continuous twisting and terminal algorithms.

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