Min–max piecewise constant optimal control for multi-model linear systems

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This work addresses a finite-horizon linear-quadratic optimal control problem for uncertain systems driven by piecewise constant controls. The precise values of the system parameters are unknown, but assumed to belong to a finite set, i.e., there exist only finitely many possible models for the plant. Uncertainty is dealt within a min–max context, seeking the *best* control for the *worst* possible plant, in the sense of some specified cost functional. The optimal control is derived using a multi-model version of the method of Lagrange multipliers, which specifies the control in terms of a discrete-time Riccati equation and an optimization problem over a simplex. A numerical algorithm for computing the optimal control is proposed and tested by simulation.

Keywords: multi-model optimal control; min–max optimization; piecewise constant control; linear-quadratic regulator (LQR).

1. Introduction

Multi-model dynamical systems arise in several areas of control, sometimes as a result of the uncertainty in the system parameters and others when the original model is largely complicated and needs to be divided into subsystems, each of which characterizes an important feature in some region of the parameter or the state space. For this class of models the optimal control problem can be formulated in such a way that an operation of maximization is taken over the set of uncertainties and an operation of minimization is taken over the control strategies. This is known as a min–max optimal control problem (Boltyanski & Poznyak, 2011). In this approach, the original system model is replaced by a finite set of dynamic models such that each model describes a particular uncertain case.

The multi-model min–max optimal control problem has been considered by several authors such as Varga (1996), Poznyak *et al.* (2002), Azhmyakov *et al.* (2010), Besselmann *et al.* (2012), Ramirez & Camacho (2002), Bemporad *et al.* (2003), to name a few. The purpose in these references is to obtain a control signal $u(\cdot)$ which guarantees that the cost does not exceed the 'best cost' incurred by the plant realizing the 'worst parameters'.

For the continuous-time case, the min–max problem was solved by Poznyak *et al.* (2002) using the so-called robust maximum principle. In addition, the multi-model control problem was studied from a dynamic programming perspective, and a natural relationship between dynamic programming and the robust maximum principle was established for a class of linear-quadratic (LQ) problems (Azhmyakov *et al.*, 2010).

In the discrete-time case, there exists an abundant series of works dealing with the min–max problem. These works can be mainly divided by the kind of disturbance that are taken into consideration. Namely, additive or parametric. Among the authors that consider an additive disturbance we can mention Bemporad *et al.* (2001), Sakizlis *et al.* (2004), Kerrigan & Maciejowski (2004), Alamo *et al.* (2005),

Olaru & Ayerbe (2006), Alexis *et al.* (2014) and Rubagotti *et al.* (2014). On the other hand, the parametric disturbance case is treated in the works of Mu noz de la Pe na *et al.* (2005), Pistikopoulos *et al.* (2009), Falugi *et al.* (2008), Kouramas *et al.* (2013) and Chang *et al.* (2014). In both cases (additive or parametric disturbance), the cost functional is considered quadratic using a given p-norm (where p usually takes the values 1 or ∞). In addition, all of them consider restrictions in both the control input and the system state. All the aforementioned works are based on model predictive control (MPC) techniques, where multi-parametric programming is the key tool, and most of them focus on reducing the computational effort, leaving aside the analysis of necessary and/or sufficient conditions of optimal trajectories. The reader is kindly directed to the above-mentioned works and references for a detailed account about MPC.

The multi-model control problem is also relevant when the designer is not only concerned about optimizing control effort, but it is interested in optimizing communication bandwidth as well. Bandwidth optimization is common in applications belonging to the field of networked control systems, where the controller is not completely dedicated to the plant or it does not have full access to the network resources at every time (see, e.g., Yang, 2006; Hespanha *et al.*, 2007). The communication constraint leads to the consideration of a particular set of admissible controls, given by piecewise constant functions on the interval $[t_0, t_N]$, which, in general, are non-uniformly spaced in time. Non-uniformity is motivated by the fact that, in terms of bandwidth optimization, a uniform sampling is not necessarily a 'good' choice (Bini & Buttazzo, 2014). Non-uniform switching sequences also appear in the field of compressed sampling (Candes & Wakin, 2008; Nagahara *et al.*, 2012a; Bryan & Leise, 2013), where there is a given number m of samples, which, in general, are also non-uniformly spaced in time. The objective is to recover the full signal in the receptor side. This technique has shown to be promising in band-limited networked control applications (Nagahara *et al.*, 2012b).

As another motivating example, consider the case when the objective is to control a set consisting of different instances or realizations of a generic plant (i.e., all models have the same mathematical structure, but each realization has possibly different parameters). The target is to control all the systems with only one control signal (a simultaneous control problem Saadatjoo *et al.*, 2009). In this context, the multi-model setting arises inherently and the min–max approach provides a reasonable criterion for optimality.

Motivated by the applications described above, we restrict the control actions to the class of piecewise constant functions of time. We also assume that the sequence of switching times is known *a priori*, inasmuch as the choice of the switching times can be carried out independently from the choice of the control levels when performing bandwidth optimization (Skafidas & Nerode, 1998; Xu & Antsaklis, 2004).

The purpose of this paper is to derive an optimal min–max control strategy in an *analytic* fashion. We depart from the numerically oriented MPC perspective and instead develop necessary conditions for optimal trajectories in the form of an extended Lagrangian multiplier rule for the case when parametric uncertainty belongs to a finite set, that is, when there is a finite set of possible models (see Remark 2.2). The LQ optimal control problem is stated formally in Section 2. Motivated by the piecewise constant nature of the control laws, the problem will be reformulated in a discrete-time context.

1.1 *Main contribution*

Theorem 3.1 in Section 3.2 states the solution to the LQ problem. Using convex analysis and generalized gradients, a discrete-time extended Riccati equation is obtained. The equation is parametrized by μ, a vector whose elements are convex multipliers for the costs incurred by the individual plants.

To compute the optimal control, a maximization problem constrained to a finite-dimensional simplex has to be solved for μ. Additionally, it is shown that a complementary slackness condition holds between μ and the individual costs.

In Section 3.3, we propose a numerical algorithm to find μ and the corresponding optimal control. This algorithm can be used in conjunction with MPC. It is based on the classical gradient with projection approach, but the gradient is approximated using Kiefer–Wolfowitz algorithm (Poznyak, 2009). The complementary slackness condition turns out to be critical in deciding a practical stopping criterion for the numerical algorithm.

Finally, we present numerical examples illustrating the usefulness of Theorem 3.1, the feasibility of the numerical algorithm and the soundness of the min–max paradigm when confronted to multi-models.

2. Problem formulation

Consider the following general continuous-time linear system with parametric uncertainty:

$$\dot{x}(t) = A^\alpha(t)x(t) + B^\alpha(t)u(t), \quad x(t_0) = x_0, \quad \alpha \in \mathscr{A}, \quad t \in [t_0, t_N], \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector and $u(t) \in \mathbb{R}^m$ is the control input. The actual realization of the system matrix and input matrix, $A^\alpha(t)$ and $B^\alpha(t)$, is unknown, but these are known to belong to finite set which is indexed by α. In other words, equation (2.1) describes one realization taken out from a finite set of possible systems. The index α is taken to be constant, *but unknown*, through all the process lifetime, i.e., from t_0 until t_N. However, it is possible to compute the control iteratively over a receding horizon, in an MPC fashion. This approach allows one to face problems where the parameters can change slowly over the time. For a fixed $\alpha \in \mathscr{A}$, we will denote the solution of (2.1) by $x^\alpha(t)$.

2.1 *Admissible controls*

The family of admissible controls takes a stepwise form. This is motivated by some applications in networked control systems. For example, when passing through a digital band-limited communication channel, the control input u may only take a finite number of changes (switches) on its levels over the whole interval $[t_0, t_N]$. Let the times at which these switches occur be given by a monotonically increasing sequence[1]

$$\delta = \{t_0, t_1, \ldots, t_{N-1}\}, \tag{2.2}$$

where δ is bounded by t_N. The set of admissible controls is given by

$$\mathscr{U}_{\text{ad}}^\delta = \left\{ u : [t_0, t_N] \to \mathbb{R}^m \mid u(t) = \sum_{k=0}^{N-1} \chi_{[t_k, t_{k+1})}(t)v_k, t_k \in \delta, v_k \in \mathbb{R}^m, k = 0, \ldots, N-1 \right\} \tag{2.3}$$

with

$$\chi_{[t_k, t_{k+1})}(t) = \begin{cases} 1 & \text{if } t \in [t_k, t_{k+1}), \\ 0 & \text{otherwise} \end{cases}$$

the characteristic function of the interval $[t_k, t_{k+1})$ and v_k the value of $u(t)$ when $t \in [t_k, t_{k+1})$. It is worth mentioning that the switching sequences δ that are taken under consideration are non-uniform in general.

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[1] Note that, since the controller is updated at most $N - 1$ times, a reduction on N directly translates into a reduction of the required bandwidth of the communication channel.

The reason is that, regarding minimal bandwidth consumption, a uniform sampling is not always the best choice (see, e.g., Bini & Buttazzo, 2014 and Remark 2.1).

2.2 *Problem statement*

Associated to (2.1), we consider the quadratic cost functional

$$J^{\alpha}(u) = \frac{1}{2} x^{\alpha}(t_N)^{\top} G x^{\alpha}(t_N) + \frac{1}{2} \int_{t=0}^{t_N} (x^{\alpha}(t)^{\top} Q(t) x^{\alpha}(t) + u(t)^{\top} R(t) u(t)) \, \mathrm{d}t, \tag{2.4}$$

where the usual positivity conditions are assumed

$$G = G^{\top} \geqslant 0, \quad Q(t) = Q(t)^{\top} \geqslant 0 \quad \text{and} \quad R(t) = R(t)^{\top} \geqslant \varepsilon I \tag{2.5}$$

for all $t \in [t_0, t_N]$ and some $\varepsilon > 0$.

The optimization problem consists in finding a control action $u^*(\cdot) \in \mathscr{U}_{\mathrm{ad}}^{\delta}$ that provides a 'good' behaviour for all systems from the given collection of models (2.1), including the 'worst' case. The resulting control strategy is applied to (2.1), regardless of the actual α-realization. We state this formally as the min–max optimization problem

$$\begin{aligned} \text{minimize} \quad & J(u), \\ \text{subject to} \quad & u(\cdot) \in \mathscr{U}_{\mathrm{ad}}^{\delta}, \end{aligned} \tag{2.6}$$

where $J(u) = \max_{\alpha \in \mathscr{A}} J^{\alpha}(u)$ subject to (2.1) (see, e.g., Azhmyakov *et al.*, 2010; Boltyanski & Poznyak, 2011).

REMARK 2.1 All admissible controls $u(\cdot)$ clearly depend on the switching sequence δ but, as it is noted in Skafidas & Nerode (1998) and Xu & Antsaklis (2004), it is possible to perform optimization in two stages: First, seek an optimal switching sequence δ and then look for an optimal control input for the given switching sequence. We focus on the second stage, so it is assumed that the switching sequence is given *a priori*.

REMARK 2.2 It is worth mentioning that, for the case when the set of considered uncertainties is a compact polyhedron (i.e., we have a infinite number of possible realizations of (2.1)), and the map $\alpha \mapsto J^{\alpha}(u)$ is convex, the problem is reduced to study only those uncertainties that belongs to the set of extreme points of the polyhedron, i.e., its vertices (Bazaraa *et al.*, 2006, Theorem 3.4.7).

REMARK 2.3 As mentioned above, there exist mainly two approaches of the min–max control problem, depending on the kind of disturbance. The first approach considers an additive disturbance of the form

$$\dot{x} = Ax(t) + Bu(t) + w(t),$$

whereas the second one deals with parametric disturbances of the form

$$\dot{x} = A^{\alpha}(t)x(t) + B^{\alpha}(t)u(t).$$

Although both problems have been dealt with before, the parametric disturbed case is still a research focus, being more difficult to solve than the additive parametric case (Kouramas *et al.*, 2013).

The stepwise nature of the control motivates the use of a discrete-time approach, using δ as the sampling time-sequence. Indeed, the problem is finite-dimensional, since it is only necessary to find N values of the control signal. Now, let

$$v = [v_0^\top \quad \cdots \quad v_{N-1}^\top]^\top \in \mathbb{R}^{mN} \tag{2.7}$$

be the decision vector and allow us construct the vector x^α, obtained by appending each point of the resulting discrete-time orbit,

$$x^\alpha = [x_1^{\alpha\top} \quad x_2^{\alpha\top} \quad \cdots \quad x_N^{\alpha\top}]^\top \in \mathbb{R}^{nN}.$$

The discrete-time representation of problem (2.6) is thus

$$\text{minimize} \quad \bar{J}(v, x, x_0), \tag{2.8}$$

where the discrete costs are given by

$$\bar{J}(v, x, x_0) = \max_{\alpha \in \mathscr{A}} \bar{J}^\alpha(v, x^\alpha, x_0)$$

subject to

$$x_{k+1}^\alpha = \Phi_k^\alpha x_k^\alpha + \Gamma_k^\alpha v_k, \quad x_0^\alpha = x_0, \quad \alpha \in \mathscr{A}, \ k = 0, \ldots, N-1$$

with

$$J^\alpha(v, x^\alpha, x_0) = \frac{1}{2} x_N^{\alpha\top} G x_N^\alpha + \frac{1}{2} \sum_{k=0}^{N-1} [x_k^{\alpha\top} \Pi_k^\alpha x_k^\alpha + 2 x_k^{\alpha\top} \Theta_k^{\alpha\top} v_k + v_k^\top \Psi_k^\alpha v_k]. \tag{2.9}$$

To alleviate the notation, we will denote the cost functional $\bar{J}^\alpha(v, x^\alpha, x_0)$ simply by $\bar{J}^\alpha(v, x^\alpha)$, bearing in mind the dependence of the initial condition in the cost.[2]

It is straightforward to verify that the weighting matrices in (2.9) are

$$\Gamma^\alpha(t, t_k) := \int_{t_k}^t \Phi^\alpha(t, \tau) B^\alpha(\tau) \, d\tau \in \mathbb{R}^{n \times m},$$

$$\Pi_k^\alpha := \int_{t_k}^{t_{k+1}} \Phi^\alpha(t, t_k)^\top Q(t) \Phi^\alpha(t, t_k) \, dt \in \mathbb{R}^{n \times n},$$

$$\Theta_k^\alpha := \int_{t_k}^{t_{k+1}} \Gamma^\alpha(t, t_k)^\top Q(t) \Phi^\alpha(t, t_k) \, dt \in \mathbb{R}^{m \times n},$$

$$\Psi_k^\alpha := \int_{t_k}^{t_{k+1}} (\Gamma^\alpha(t, t_k)^\top Q(t) \Gamma^\alpha(t, t_k) + R(t)) \, dt \in \mathbb{R}^{m \times m},$$

$$\Gamma_k^\alpha := \Gamma^\alpha(t_{k+1}, t_k)$$

and $\Phi_k^\alpha := \Phi^\alpha(t_{k+1}, t_k)$ with $\Phi^\alpha(t, t_0)$ the state transition matrix for the α-system (2.1). Notice that $\Pi_k^\alpha = \Pi_k^{\alpha\top} \geqslant 0$ and $\Psi_k^\alpha = \Psi_k^{\alpha\top} > 0$ for all $k = 0, \ldots, N-1$.

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[2] Moreover, we will write in (3.3) the cost as a function of v only, to reflect the fact that x^α depends on v through the constraint equations.

3. Solution to the min–max problem

3.1 *Multi-model method of Lagrange multipliers*

Recall that, in the classical discrete-time LQ problem (i.e., when $\mathscr{A}$ is a singleton, $\mathscr{A} = \{1\}$), the optimal control can be obtained using the Lagrange multipliers framework (Ogata, 1995, ch. 8). The following lemma states that the framework is still valid, *mutatis mutandis*, for the discrete-time multi-model case.

LEMMA 3.1 Let (v^*, x^*) be an optimal pair solution of (2.8), then the following conditions are satisfied:

$$0 \in \text{conv}\{\nabla_v L^\alpha(v^*, x^*, \lambda^\alpha) : \alpha \in I(v^*)\}, \tag{3.1a}$$

$$0 = \nabla_x L^\alpha(v^*, x^*, \lambda^\alpha) \quad \text{for all } \alpha \in I(v^*), \tag{3.1b}$$

$$0 = \nabla_\lambda L^\alpha(v^*, x^*, \lambda^\alpha) \quad \text{for all } \alpha \in I(v^*), \tag{3.1c}$$

where conv denotes convex closure and $L^\alpha : \mathbb{R}^{mN} \times \mathbb{R}^{nN} \times \mathbb{R}^{nN} \to \mathbb{R}$ denotes the Lagrangian of the problem, that is,

$$L^\alpha(v, x^\alpha, \lambda^\alpha) = \bar{J}^\alpha(v, x^\alpha) + \lambda^{\alpha\top} g^\alpha(v, x^\alpha) \tag{3.2}$$

with

$$g^\alpha(v, x^\alpha) = [g_0^{\alpha\top}(v, x^\alpha), \dots, g_{N-1}^{\alpha\top}(v, x^\alpha)]^\top \in \mathbb{R}^{nN},$$

$$g_k^\alpha(v, x^\alpha) = -x_{k+1}^\alpha + \Phi_k^\alpha x_k^\alpha + \Gamma_k^\alpha v_k.$$

$I(v)$ denotes the set of indices where $\bar{J}^\alpha$ reaches the maximum value (with v fixed), i.e.,

$$I(v) = \{\alpha \in \mathscr{A} : \bar{J}^\alpha(v, x^\alpha) = \bar{J}(v, x)\}.$$

Proof. In order to make the proof easier to follow, we introduce the function

$$W^\alpha(v) := \bar{J}^\alpha(v, x^\alpha(v)), \tag{3.3}$$

where we are exploiting the fact that x^α is a function of v through the restriction equations. Indeed, is easy to see that for each v, x^α is uniquely defined by the constraint $g^\alpha(v, x^\alpha) = 0$. Moreover, $W^\alpha(\cdot)$ is again convex since it is the composition of an affine and a convex function.

Our immediate goal is to derive necessary conditions for optimality of the problem

$$\text{minimize:} \quad \bar{J}(v, x),$$

which is equivalent to

$$\text{minimize} \quad W(v) \tag{3.4}$$

with $W(v) = \max_{\alpha \in \mathscr{A}} W^\alpha(v)$. Note that W is obtained by choosing the maximum among a finite set of functions, so it is in general a non-smooth function, even in the case where each $W^\alpha(v)$ is smooth.

Is well know (Mäkelä & Neittaanmäki, 1992, p. 70) that a necessary condition for optimality of (3.4) is $0 \in \partial_c W(v^*)$, where $\partial_c W$ denotes the convex subdifferential of W at v^*.[3] The subdifferential of the max function has been reported in the literature (see, e.g., Mäkelä & Neittaanmäki, 1992, p. 49) and it is known to satisfy $\partial_c W(v) = \text{conv}\{\nabla W^\alpha(v) : \alpha \in I(v)\}$, where $I(v) = \{\alpha \in \mathscr{A} : W^\alpha(v) = W(v)\}$ (i.e., $I(v)$

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[3] Note that W is the maximum of a finite set of convex functions, which is again convex.

is the set of indices where the maximum is reached). Thus, we have the following necessary condition for optimality

$$0 \in \mathrm{conv}\{\nabla W^\alpha(v^*) : \alpha \in I(v^*)\}.$$

It is worth stressing the fact that the right-hand side does not involve all the possible α-realizations of the uncertain plant. Only those plants for which the maximum is reached play a role in the optimization procedure. In the following section, we propose a method for finding these *extreme* plants (see also Remark 3.2).

From the definition of W^α, we have

$$\nabla W^\alpha(v) = \nabla_v \bar{J}^\alpha(v, x^\alpha) + \nabla_x \bar{J}(v, x^\alpha(v)) \nabla x^\alpha(v),$$

where we defined the gradient of a map $F : \mathbb{R}^n \to \mathbb{R}^m$ as

$$\nabla F(x) = \begin{bmatrix} \dfrac{\partial F_1(x)}{\partial x_1} & \cdots & \dfrac{\partial F_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m(x)}{\partial x_1} & \cdots & \dfrac{\partial F_m(x)}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

(i.e., we adhere to the convention that the gradient of a scalar function is as a row vector).

Since the plant is a well-defined system, there exist at least one pair $(\bar{v}, \bar{x}^\alpha)$ for which $g^\alpha(\bar{v}, \bar{x}^\alpha) = 0$. Furthermore,

$$\nabla_x g^\alpha(v, x^\alpha) = \begin{bmatrix} -I_n & 0 & 0 & \cdots & 0 & 0 \\ \Phi_1^\alpha & -I_n & 0 & \cdots & 0 & 0 \\ 0 & \Phi_2^\alpha & -I_n & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -I_n & 0 \\ 0 & 0 & 0 & \cdots & \Phi_{N-1}^\alpha & -I_n \end{bmatrix}$$

is an invertible matrix, so it is possible to apply the implicit function theorem, write x^α as a function of v and write the gradient

$$\nabla x^\alpha(v) = -[\nabla_x g^\alpha(v, x^\alpha(v))]^{-1} \nabla_v g^\alpha(v, x^\alpha(v)).$$

This results in

$$\nabla W^\alpha(v) = \nabla_v \bar{J}^\alpha(v, x^\alpha(v)) - \nabla_x \bar{J}^\alpha(v, x^\alpha(v))[\nabla_x g^\alpha(v, x^\alpha(v))]^{-1} \nabla_v g^\alpha(v, x^\alpha(v)).$$

Now, define $\lambda^{\alpha\top} := -\nabla_x \bar{J}^\alpha(v, x^\alpha(v))[\nabla_x g^\alpha(v, x^\alpha(v))]^{-1} \in \mathbb{R}^{1 \times nN}$, write the Lagrangian as in (3.2) and immediately obtain (3.1a). From the definition of λ^α, we have

$$0 = \nabla_x \bar{J}^\alpha(v, x^\alpha(v)) + \lambda^{\alpha\top} \nabla_v g^\alpha(v, x^\alpha(v)),$$

which is the same as (3.1b). Finally, note that (3.1c) is a restatement of the constraints. □

3.2 *Extended Riccati equation, complementary slackness*

Throughout the rest of this section, we will work simultaneously with all α-realizations. In order to make the presentation clearer, we introduce the following extended matrices:

$$\mathbf{G}(\mu) = \mathrm{diag}\{\mu_1 G, \ldots, \mu_{|\mathscr{A}|} G\} \in \mathbb{R}^{n|\mathscr{A}| \times n|\mathscr{A}|},$$

$$\boldsymbol{\Pi}_k(\mu) = \mathrm{diag}\{\mu_1 \Pi_k^1, \ldots, \mu_{|\mathscr{A}|} \Pi_k^{|\mathscr{A}|}\} \in \mathbb{R}^{n|\mathscr{A}| \times n|\mathscr{A}|},$$

$$\boldsymbol{\Theta}_k(\mu) = [\mu_1 \Theta_k^1 \quad \cdots \quad \mu_{|\mathscr{A}|} \Theta_k^{|\mathscr{A}|}] \in \mathbb{R}^{m \times n|\mathscr{A}|},$$

$$\boldsymbol{\Psi}_k(\mu) = \sum_{\alpha=1}^{|\mathscr{A}|} \mu_\alpha \Psi_k^\alpha \in \mathbb{R}^{m \times m},$$

$$\boldsymbol{\Phi}_k = \mathrm{diag}\{\Phi_k^1, \ldots, \Phi_k^{|\mathscr{A}|}\} \in \mathbb{R}^{n|\mathscr{A}| \times n|\mathscr{A}|},$$

$$\boldsymbol{\Gamma}_k = [\Gamma_k^{1\top} \quad \Gamma_k^{2\top} \quad \cdots \quad \Gamma_k^{|\mathscr{A}|\top}]^\top \in \mathbb{R}^{n|\mathscr{A}| \times m},$$

$$\mathbf{M}(\mu^*) = \mathrm{diag}(\mu_1^* I_n, \ldots, \mu_{|\mathscr{A}|}^* I_n),$$

where $|\mathscr{A}|$ is the cardinality of $\mathscr{A}$. Note that the vector μ (formally defined below as an element of a simplex) only intervenes in the matrices that are related to the cost. Also, note that the symmetry and positive (semi) definiteness of G, Π_α^k and Ψ_α^k are inherited by $\mathbf{G}(\mu)$, $\boldsymbol{\Pi}_k(\mu)$ and $\boldsymbol{\Psi}_k(\mu)$, for all $k \in 0, \ldots, N-1$ and for all μ in the simplex defined below.

DEFINITION 3.1 The simplex of dimension $|\mathscr{A}| - 1$ is the set

$$\mathscr{S}^{|\mathscr{A}|} = \left\{ \mu \in \mathbb{R}^{|\mathscr{A}|} : \sum_{\alpha \in \mathscr{A}} \mu_\alpha = 1, \ \mu_\alpha \geqslant 0, \ \text{for all } \alpha \in \mathscr{A} \right\}.$$

Let us now define the extended vectors

$$\mathbf{x}_k = \begin{bmatrix} x_k^1 \\ \vdots \\ x_k^{|\mathscr{A}|} \end{bmatrix} \in \mathbb{R}^{n|\mathscr{A}|} \quad \text{and} \quad \boldsymbol{\Lambda}_k(\mu^*) = \begin{bmatrix} \mu_1^* \lambda_k^1 \\ \vdots \\ \mu_{|\mathscr{A}|}^* \lambda_k^{|\mathscr{A}|} \end{bmatrix} \in \mathbb{R}^{n|\mathscr{A}|},$$

so that we can formulate the main contribution of this work. The result can be interpreted as a discrete-time version of the robust maximum principle developed by one of the authors in Boltyanski & Poznyak (2011) and applied to the LQ problem.

THEOREM 3.1 Consider the multi-model linear system (2.1) and the cost functional (2.4) subject to the usual positivity assumptions (2.5). Let δ be a switching sequence given *a priori* and of the form (2.2) and let $\mathscr{U}_{\mathrm{ad}}^\delta$ be the set of admissible controllers defined by (2.3). The control $u^{**}(\cdot) \in \mathscr{U}_{\mathrm{ad}}^\delta$ that solves the min–max problem (2.6) is given by

$$u^{**}(t) := u^*(t, \mu^*) = \sum_{k=0}^{N-1} \chi_{[t_k, t_{k+1})}(t) v_k^*(\mu^*), \quad t_k \in \delta, \tag{3.5}$$

$$v_k^*(\mu) = -(\boldsymbol{\Psi}_k(\mu) + \boldsymbol{\Gamma}_k^\top \mathbf{P}_{k+1}(\mu) \boldsymbol{\Gamma}_k)^{-1}(\boldsymbol{\Theta}_k(\mu) + \boldsymbol{\Gamma}_k^\top \mathbf{P}_{k+1}(\mu) \boldsymbol{\Phi}_k) \mathbf{x}_k,$$

where the boldface matrices were defined previously, except for $\mathbf{P}_k(\mu)$, which is defined implicitly as the positive-definite solution of the discrete-time Riccati equation

$$\mathbf{P}_k(\mu) = \boldsymbol{\Pi}_k(\mu) + \boldsymbol{\Phi}_k^\top \mathbf{P}_{k+1}(\mu)\boldsymbol{\Phi}_k - (\boldsymbol{\Theta}_k(\mu) + \boldsymbol{\Gamma}_k^\top \mathbf{P}_{k+1}(\mu)\boldsymbol{\Phi}_k)^\top$$
$$\times (\boldsymbol{\Psi}_k(\mu) + \boldsymbol{\Gamma}_k^\top \mathbf{P}_{k+1}(\mu)\boldsymbol{\Gamma}_k)^{-1}(\boldsymbol{\Theta}_k(\mu) + \boldsymbol{\Gamma}_k^\top \mathbf{P}_{k+1}(\mu)\boldsymbol{\Phi}_k) \tag{3.6a}$$

with boundary condition

$$\mathbf{P}_N(\mu) = \mathbf{G}(\mu). \tag{3.6b}$$

The optimal vector μ^* is the solution of

$$\max_{\mu \in \mathscr{S}^{|\mathscr{A}|}} \frac{1}{2}\mathbf{x}_0^\top \mathbf{P}_0(\mu)\mathbf{x}_0. \tag{3.7}$$

Moreover, the complementary slackness condition

$$\mu_\alpha^*[J^\alpha(u^{**}) - J(u^{**})] = 0 \tag{3.8}$$

is satisfied for every $\alpha \in \mathscr{A}$.

REMARK 3.1 Equations (3.5) and (3.6) define the solution of a classical discrete-time LQ problem for the extended system $\mathbf{x}_{k+1} = \boldsymbol{\Phi}_k\mathbf{x}_k + \boldsymbol{\Gamma}_k v_k$, the only difference being the dependence that the matrices have on μ. In view of this observation and the structure of the extended cost matrices, one concludes that the cost for the extended system is a convex combination of the individual costs. In symbols,

$$\frac{1}{2}\mathbf{x}_N^\top \mathbf{G}(\mu)\mathbf{x}_N + \frac{1}{2}\sum_{k=0}^{N-1}(\mathbf{x}_k^\top \boldsymbol{\Pi}_k(\mu)\mathbf{x}_k + 2\mathbf{x}_k\boldsymbol{\Theta}_k(\mu)^\top v_k + v_k^\top \boldsymbol{\Psi}_k(\mu)v_k) = \sum_{\alpha \in \mathscr{A}} \mu_\alpha \bar{J}^\alpha(v, x^\alpha).$$

REMARK 3.2 The complementarity slackness condition reveals that only the *extreme* plants (those for which the maximum is attained) play a role in the computation of the optimal control. Indeed, the optimal vector μ^* acts as an indicator for the set of extreme plants (having non-zero elements in the α-coordinates if, and only if, the corresponding plant is extreme).

Proof of Theorem 3.1. It has already been established that the continuous-time problem (2.6) is equivalent to the discrete-time problem (2.8). Thus, our problem consists in minimizing $\bar{J}(v, x)$. Lemma 3.1 states that the optimal control pair must satisfy (3.1), which translates to

$$0 = \sum_{\alpha \in I(v^*)} \{\mu_\alpha^*(\nabla_v \bar{J}^\alpha(v^*, x^*) + \lambda^{\alpha\top}\nabla_v g^\alpha(v^*, x^*))\} \tag{3.9a}$$

for some $\mu^* \in \mathscr{S}^{|I(v^*)|}$ (i.e., such that $\sum_{\alpha \in I(v^*)} \mu_\alpha^* = 1$ and $\mu_\alpha^* \geqslant 0$) and

$$0 = \nabla_x \bar{J}^\alpha(v^*, x^*) + \lambda^{\alpha\top}\nabla_x g^\alpha(v^*, x^*), \tag{3.9b}$$
$$0 = g^\alpha(v^*, x^*) \tag{3.9c}$$

for all $\alpha \in I(v^*)$. Allow us to embed μ^* in the larger simplex $\mathscr{S}^{|\mathscr{A}|}$ by putting zeros in the $\alpha \notin I(v^*)$-entries, so the new vector $\mu^* \in \mathscr{S}^{|\mathscr{A}|}$ acts an indicator for the *extreme* plants.

Computing (3.9) explicitly gives

$$
\begin{aligned}
0 &= \sum_{\alpha \in \mathscr{A}} \mu_\alpha^* (\Psi_k^\alpha v_k^* + \Theta_k^\alpha x_k^{\alpha*} + \Gamma_k^{\alpha\top} \lambda_{k+1}^\alpha), \\
0 &= \mu_\alpha^* [\Pi_k^\alpha x_k^{\alpha*} + \Theta_k^{\alpha\top} v_k^* + \Phi_k^\top \lambda_{k+1}^\alpha - \lambda_k^\alpha], \\
0 &= \mu_\alpha^* [\lambda_N - G x_N^{\alpha*}], \\
0 &= \mu_\alpha^* [-x_{k+1}^{\alpha*} + \Phi_k^\alpha x_k^{\alpha*} + \Gamma_k^\alpha v_k^*]
\end{aligned}
\tag{3.10}
$$

for all $k = 0, \ldots, N-1$. In order to make the notation more compact, we will make use of the block matrices defined above. The optimality conditions (3.10) can now be rewritten as

$$
0 = \boldsymbol{\Psi}_k(\mu^*) v_k^* + \boldsymbol{\Theta}_k(\mu^*) \mathbf{x}_k^* + \boldsymbol{\Gamma}_k^\top \boldsymbol{\Lambda}_{k+1}(\mu^*), \tag{3.11a}
$$

$$
0 = \boldsymbol{\Pi}_k(\mu^*) \mathbf{x}_k^* + \boldsymbol{\Theta}_k(\mu^*)^\top v_k^* + \boldsymbol{\Phi}_k^\top \boldsymbol{\Lambda}_{k+1}(\mu^*) - \boldsymbol{\Lambda}_k(\mu^*), \tag{3.11b}
$$

$$
0 = \mathbf{G}(\mu^*) \mathbf{x}_N^* - \boldsymbol{\Lambda}_N(\mu^*), \tag{3.11c}
$$

$$
0 = \mathbf{M}(\mu^*)[-\mathbf{x}_{k+1}^* + \boldsymbol{\Phi}_k \mathbf{x}_k^* + \boldsymbol{\Gamma}_k v_k^*] \tag{3.11d}
$$

for all $k = 0, \ldots, N-1$. Note that these equations are similar to the classical discrete-time LQ optimality equations of an $n|\mathscr{A}|$-dimensional linear system parametrized by μ^* (Ogata, 1995, p. 582). To obtain the optimal control v^*, we simply follow the classical discrete-time approach (only the main steps are reported since the approach is well known).

Recall that it is always possible to write the adjoint variable $\boldsymbol{\Lambda}(\mu^*)$ as a linear function of the (extended) state $\mathbf{x}^*$, i.e., as $\boldsymbol{\Lambda}_k(\mu^*) = \mathbf{P}_k(\mu^*) \mathbf{x}_k^*$. It is straightforward to verify that, for $\boldsymbol{\Lambda}_k(\mu^*)$ to satisfy (3.11), $\mathbf{P}_k(\mu^*)$ must be the (positive semi-definite) solution of the μ^*-parametrized Riccati equation (3.6). The optimal control then takes the form

$$
v_k^*(\mu^*) = -(\boldsymbol{\Psi}_k(\mu^*) + \boldsymbol{\Gamma}_k^\top \mathbf{P}_{k+1}(\mu^*) \boldsymbol{\Gamma}_k)^{-1} (\boldsymbol{\Theta}_k(\mu^*) + \boldsymbol{\Gamma}_k^\top \mathbf{P}_{k+1}(\mu^*) \boldsymbol{\Phi}_k) \mathbf{x}_k^*
$$

(see Ogata, 1995, Chapter 8 for a detailed development of the classical discrete-time LQ problem).

It is not difficult to prove that the optimal cost of the extended system is equal to $\frac{1}{2} \mathbf{x}_0^\top \mathbf{P}_k(\mu^*) \mathbf{x}_0$ (see Ogata, 1995, p. 575) so, according to Remark 3.1,

$$
\sum_{\alpha \in \mathscr{A}} \mu_\alpha^* \bar{J}^\alpha(v^*(\mu^*), x^{\alpha*}) = \frac{1}{2} \mathbf{x}_0^\top \mathbf{P}_k(\mu^*) \mathbf{x}_0. \tag{3.12}
$$

On the other hand, according to Lemma A.1 in the Appendix, this vector also satisfies

$$
\mu^* = \arg \max_{\mu \in \mathscr{S}^{|\mathscr{A}|}} \sum_{\alpha \in \mathscr{A}} \mu_\alpha \bar{J}^\alpha(v^*(\mu), x^{\alpha*}). \tag{3.13}
$$

Statement (3.7) now follows directly from (3.12) and (3.13).

Lemma A.1 also implies that the components of μ^* will be different from zero at the α-positions where $J^\alpha(u^{**}) = J(u^{**})$ (i.e., where the maximum is attained) and will be zero otherwise. This fact is equivalent to the complementary slackness condition (3.8). □

REMARK 3.3 Note from (3.7) that μ^* depends on the initial conditions and the system parameters only (not on the whole state trajectory). This allow us to separate the optimization problem in two simpler

subproblems. Namely, the first part consists in solving the μ-parametrized Riccati equation (3.6). The second part consists in finding the solution μ^* of (3.7). Both stages can be accomplished off-line.

3.3 *Numerical algorithm*

Theorem 3.1 provides the feedback control equations in terms of the parameters of every α-model and μ^*. Thus, in order to determine the control law completely, it is necessary to solve (3.7). Gradient-based algorithms are widely used for numerical optimization. These are methods where the search directions are defined by the gradient of a target function at the current iteration point. Computing the gradient of the performance index $\frac{1}{2}\mathbf{x}_0^\top \mathbf{P}_0(\mu)\mathbf{x}_0$ directly is a challenging task, mainly because of the recursion in the Riccati equation (3.6). To circumvent this problem, we propose to use an approximation of the gradient in combination with a projection algorithm that guarantees that μ belongs to $\mathscr{S}^{|\mathscr{A}|}$.

Let $f(\mu) \in \mathbb{R}$ be a convex cost function to be minimized. The classical gradient with projection algorithm for finding the minimum is given by the recursion (Levitin & Polyak, 1965)

$$\mu^j = \text{Proj}[\mu^{j-1} - \gamma_j \nabla f(\mu^{j-1})]_{\mathscr{S}^{|\mathscr{A}|}}, \tag{3.14}$$

where $\text{Proj}[\cdot]_{\mathscr{S}^{|\mathscr{A}|}}$ refers to the projection of a point in $\mathbb{R}^{|\mathscr{A}|}$ into the set $\mathscr{S}^{|\mathscr{A}|}$, i.e.,

$$x^* = \text{Proj}[y]_{\mathscr{B}} \quad \text{if and only if } x^* = \arg\min_{x \in \mathscr{B}} \|y - x\|$$

and γ_j is a positive and small number called the step-size,[4] The convexity of $f(\cdot)$ implies that $f(\mu^j) \to f(\mu^*)$, as $j \to \infty$ where μ^* is the minimum of f constrained to $\mu^* \in \mathscr{S}^{|\mathscr{A}|}$ (see Levitin & Polyak, 1965). Equivalently, for all $\varepsilon_1 > 0$, there exists an $N_{\varepsilon_1} > 0$ such that

$$|f(\mu^j) - f(\mu^*)| < \varepsilon_1$$

for all $j \geqslant N_{\varepsilon_1}$. As mentioned above, we will approximate $\nabla f(\mu^{j-1})$ in (3.14). For simplicity, we will use the classical Kiefer–Wolfowitz approximation,

$$Y_m(\mu^{j-1}) = \frac{1}{2\beta_m} \sum_{i=1}^{|\mathscr{A}|} [f(\mu^{j-1} + \beta_m e_i) - f(\mu^{j-1} - \beta_m e_i)]e_i \in \mathbb{R}^{|\mathscr{A}|}, \tag{3.15}$$

where the sequence β_m vanishes as $m \to \infty$ and the vector e_i represents the ith element of the canonical basis in $\mathbb{R}^{|\mathscr{A}|}$ (see Chin, 1997 for a comparative study of gradient approximations).

LEMMA 3.2 Consider the approximation (3.15). If the gradient $f(\cdot)$ exists at μ^{j-1} and $\lim_{m\to\infty} \beta_m = 0$, then the limit of $Y_m(\mu^{j-1})$ as $m \to \infty$ is equal to the gradient of $f(\cdot)$ at μ^{j-1}.

Proof. The proof is in Poznyak (2009) but we briefly repeat it here for completeness. Since f is differentiable at μ^{j-1}, we can write the first order approximation of $f(\mu^{j-1} + \beta_m e_i)$ at μ^{j-1} as

$$f(\mu^{j-1} + \beta_m e_i) = f(\mu^{j-1}) + \langle \nabla f(\mu^{j-1}), \beta_m e_i \rangle + o(\beta_m),$$

where $o(\cdot)$ satisfies

$$\lim_{x\to 0} \left| \frac{o(x)}{x} \right| = 0.$$

---

[4] Actually, there is a large variety of possibilities for choosing the step-size γ_j. See, e.g. Bertsekas (1999).

Representing $f(\mu^{j-1} - \beta_m e_i)$ in the corresponding fashion gives

$$
\begin{aligned}
Y_m(\mu^{j-1}) &= \frac{1}{\beta_m} \sum_{i=1}^{|\mathscr{A}|} [\langle \nabla f(\mu^{j-1}), \beta_m e_i \rangle + o(\beta_m)] e_i \\
&= \sum_{i=1}^{|\mathscr{A}|} \left[\frac{\partial f(\mu^{j-1})}{\partial \mu_i^{j-1}} + \frac{o(\beta_m)}{\beta_m} \right] e_i \\
&= \nabla f(\mu^{j-1}) + \frac{o(\beta_m)}{\beta_m} \sum_{i=1}^{|\mathscr{A}|} e_i.
\end{aligned}
$$

Taking the limit as $m \to \infty$ gives the desired result. $\qquad\square$

With the convergence of the recursion (3.15) assured, it seems natural to adjust the classical gradient with projection algorithm as follows.

PROPOSITION 3.1 Consider the recursion

$$
\mu^j = \text{Proj}[\mu^{j-1} - \gamma_j Y_j(\mu^{j-1})]_{\mathscr{S}^{|\mathscr{A}|}} \tag{3.16}
$$

with $f(\cdot)$ a convex (and therefore continuous) function (Rockafellar, 1970, Corollary 10.1.1). Then, $\lim_{j\to\infty} f(\mu^j) = f(\mu^*)$.

Proof. It follows from Lemma 3.2 that $\nabla f(\mu^{j-1}) = \lim_{m\to\infty} Y_m(\mu^{j-1})$. By substituting this expression in (3.14), we obtain

$$
\mu^j = \text{Proj} \left[\mu^{j-1} - \gamma_j \lim_{m\to\infty} Y_m(\mu^{j-1}) \right]_{\mathscr{S}^{|\mathscr{A}|}} = \lim_{m\to\infty} \text{Proj}[\mu^{j-1} - \gamma_j Y_m(\mu^{j-1})]_{\mathscr{S}^{|\mathscr{A}|}}, \tag{3.17}
$$

where the second equation follows from the continuity of $\text{Proj}[\cdot]_{\mathscr{S}^{|\mathscr{A}|}}$. Let us define the term

$$
\mu^{j,m} := \text{Proj}[\mu^{j-1} - \gamma_j Y_m(\mu^{j-1})]_{\mathscr{S}^{|\mathscr{A}|}}.
$$

Then we can write (3.17) as $\mu^j = \lim_{m\to\infty} \mu^{j,m}$ and $f(\mu^j) = \lim_{m\to\infty} f(\mu^{j,m})$, where we have used again a continuity argument to compute the second limit. Thus, for all $\varepsilon_2 > 0$, there exists an $M_{\varepsilon_2} > 0$ such that

$$
|f(\mu^{j,m}) - f(\mu^j)| < \varepsilon_2
$$

for all $m \geqslant M_{\varepsilon_2}$. Finally, for every $\varepsilon > 0$, we can choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $\varepsilon = \varepsilon_1 + \varepsilon_2$. It is then straightforward to verify that

$$
\begin{aligned}
|f(\mu^{j,m}) - f(\mu^*)| &= |f(\mu^{j,m}) - f(\mu^*) \pm f(\mu^j)| \\
&\leqslant |f(\mu^{j,m}) - f(\mu^j)| + |f(\mu^j) - f(\mu^*)| \\
&< \varepsilon_1 + \varepsilon_2 = \varepsilon
\end{aligned}
$$

for all $j, m \geqslant \max\{N_{\varepsilon_1}, M_{\varepsilon_2}\}$. Consequently, for the sequence $\mu^{j,j} = \mu^j$ we have

$$
|f(\mu^{j,j}) - f(\mu^*)| < \varepsilon \quad \text{for all } j \geqslant \max\{N_{\varepsilon_1}, M_{\varepsilon_2}\}.
$$

The convergence of the recursion (3.16) has been proved. □

In order to apply the recursion (3.16) to our optimization problem, it suffices to set $f(\mu)$ as $f(\mu) = -\mathbf{x}_0^\top \mathbf{P}_0(\mu)\mathbf{x}_0$ and specify a stop criterion. The latter can be obtained from the complementary slackness condition (3.8) in Theorem 3.1. More precisely, the fact that, at the optimal value μ^*, the components μ_α^* that are different from zero must have equal associated costs can be used to determine whether or not an optimal μ^* has been found. In a real implementation, however, the slackness condition is impossible to achieve exactly, due to finite machine precision. For this reason, we introduce a threshold value $\varepsilon \ll 1$ for which the stop condition is

$$|\mu_\alpha^j[J^\alpha(u^*(\mu^{j-1})) - J(u^*(\mu^{j-1}))]| < \varepsilon, \tag{3.18}$$

for all $\alpha \in \mathscr{A}$ and for some $j \in \mathbb{N}$ sufficiently large.

The procedure for computing the optimal control law for problem (2.6) is summarized in the following algorithm:

1. Fix $\mu^0 \in \mathscr{S}^{|\mathscr{A}|}$ and set $j = 1$.

2. Compute the matrix $\mathbf{P}_0(\mu^{j-1})$ as the positive-definite solution of (3.6).

3. Make one step through the recursion (3.16) by computing the term $\mu^{j-1} - \gamma_j Y_j(\mu^{j-1})$ and projecting on the simplex $\mathscr{S}^{|\mathscr{A}|}$.

4. Compute the product $\mu_\alpha^j[J^\alpha(u^*(\mu^j)) - J(u^*(\mu^j))]$ for every α in $\mathscr{A}$ and verify the stop condition (3.18)[5]. If True, go to step 5; else increase j by one and go to step 2.

5. Compute the optimal control law as in (3.5).

4. Numerical examples

EXAMPLE 4.1 Consider the following multi-model system

$$\dot{x} = A^\alpha x + B^\alpha u, \quad x_0 = [3 \quad -2]^\top \tag{4.1}$$

with $\alpha \in \mathscr{A} = \{1, 2\}$,

$$A^1 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A^2 = 10A^1, \quad B^2 = B^1.$$

Also, consider the cost functional

$$J^\alpha(u) = \frac{1}{2}x^{\alpha\top}(10)Gx^\alpha(10) + \int_0^{10}(x^{\alpha\top}Qx^\alpha + u^\top Ru)\,dt,$$
$$G = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 50 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = 10. \tag{4.2}$$

---

[5] At the expense of a large computational burden, the term $J(u^*(\mu^j))$ can be obtained by integrating the system equations numerically. Alternatively, $J(u^*(\mu^j))$ can be approximated by $\frac{1}{2}\mathbf{x}_0^\top \mathbf{P}_0(\mu^j)\mathbf{x}_0$ (notice that $\lim_{j\to\infty}(J(u^*(\mu^j)) - \frac{1}{2}\mathbf{x}_0^\top \mathbf{P}_0(\mu^j)\mathbf{x}_0) = 0$).

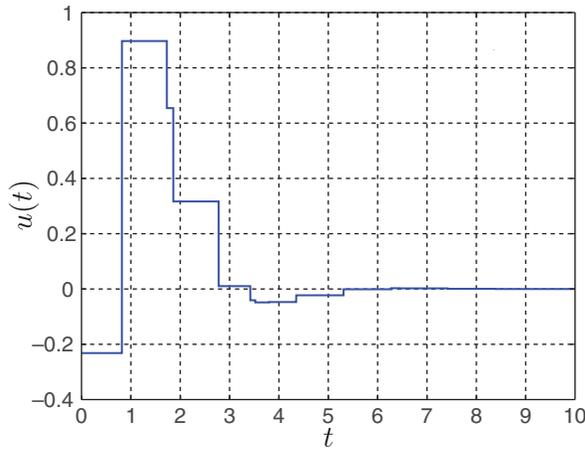Fig. 1. Optimal min–max control.

Note that the worst dominant plant (i.e., the plant whose individual cost J^{α_w} is always greater than the other costs J^{α}, for all $\alpha \in \mathscr{A} \setminus \{\alpha_w\}$ and for all admissible controls) is given by $\alpha = 1$ (the slowest plant). We expected the proposed algorithm to be able to identify it. In this case, we consider a random switching sequence

$$\delta = \delta_1 \cup \delta_2,$$
$$\delta_1 = [0\ 0.82\ 1.73\ 1.86\ 2.78\ 3.42\ 3.52\ 3.80\ 4.35],$$
$$\delta_2 = [5.31\ 6.28\ 6.44\ 7.42\ 8.38\ 8.87\ 9.68\ 9.83\ 10].$$

Setting $\varepsilon = 2 \times 10^{-5}$ and applying the algorithm described in Section 3.3, one obtains the optimal vector

$$\mu^* = [1\quad 0]^\top.$$

After substituting the optimal parameter in (3.5), the optimal control u^* is obtained. Such control is plotted in Fig. 1. The optimal min–max cost $J(u^{**})$ is equal to 139.1381. The individual costs that result from applying the optimal min–max control $u^{**}(\cdot)$ to every α-system independently are $J^1(u^{**}) = 139.1381$ and $J^2(u^{**}) = 20.7546$, which confirms that the worst plant was identified correctly. Finally, the state trajectories for each system are presented in Fig. 2(a and b).

EXAMPLE 4.2 Consider now the linear multi-model system

$$\dot{x}^\alpha = A^\alpha x^\alpha + B^\alpha u,$$
$$A^\alpha = \begin{bmatrix} 0 & 1 \\ (\alpha - 0.9)\,\mathrm{sign}(1.1 - \alpha) & -(4 - \alpha)^2 \end{bmatrix}, \quad B^\alpha = \begin{bmatrix} 0 \\ \sqrt{\alpha} \end{bmatrix}, \quad x_0 = \begin{bmatrix} -5 \\ 3 \end{bmatrix} \tag{4.3}$$
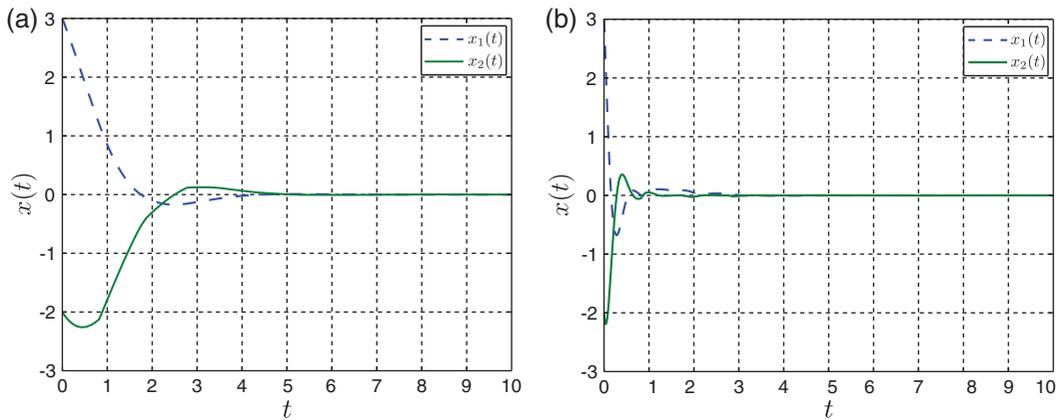
FIG. 2. Time trajectories for the system (4.1) with $\alpha = 1, 2$. (a) $\alpha = 1$ and (b) $\alpha = 2$.

with $\alpha \in \mathscr{A} = \{1, 2, 3, 4\}$ and the cost functional

$$J(u) = \frac{1}{2} x^{\alpha\top} G x^{\alpha} + \int_0^{20} (x^{\alpha\top} Q x^{\alpha} + u^\top R u)\, \mathrm{d}t,$$

$$G = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 50 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = 10.$$

(4.4)

In this case, we consider a random switching sequence δ given by

$$\delta_1 = [0\ 0.82\ 1.73\ 1.86\ 2.78\ 3.42\ 3.52\ 3.80\ 4.35],$$
$$\delta_2 = [5.31\ 6.28\ 6.44\ 7.42\ 8.38\ 8.87\ 9.68\ 9.83\ 10.26],$$
$$\delta_3 = [11.18\ 11.98\ 12.94\ 13.60\ 13.64\ 14.49\ 15.43\ 16.11\ 16.87],$$
$$\delta_4 = [17.62\ 18.02\ 18.68\ 20.00\ 20.52\ 22.36\ 23.96\ 25.88\ 27.20],$$
$$\delta_5 = [27.28\ 28.98\ 30.86\ 32.22\ 33.74\ 35.24\ 36.04\ 37.36\ 40.00],$$
$$\delta = \cup_{i=1,\ldots,5}\delta_i.$$

Applying the algorithm described with $\varepsilon = 5 \times 10^{-3}$ gives

$$\mu^* = [0.4842\quad 0.1842\quad 0.1432\quad 0.1884]^\top$$

(note that the model is unstable and marginally stable for $\alpha = 1$ and $\alpha = 4$, respectively). Figure 3 shows the optimal control (3.5) after plugging in μ^*. The optimal min–max cost is equal to $J(u^{**}) = 3688.1$ and the individual costs are

$$J^1(u^{**}) = 3688.1, \quad J^2(u^{**}) = 3688.1, \quad J^3(u^{**}) = 3688.1 \quad \text{and} \quad J^4(u^{**}) = 3688.1. \tag{4.5}$$

In this case, all the plants play a role in the optimal control and can be viewed as extreme plants in the min–max sense. The min–max control strategy is well suited for the multi-model case in the sense that, *for every plant*, an appropriate upper bound on the cost is ensured. For comparison purposes, we
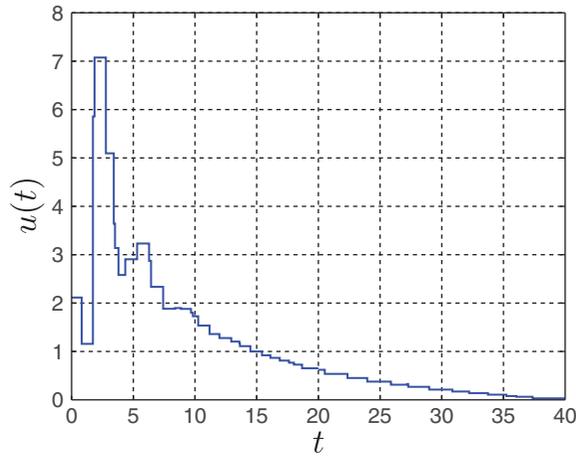
Fig. 3. Optimal min–max control, u^{**}.

Table 1 *The costs that each plant incurs when subject to $u^{\alpha*}$, the optimal control for the α-system. Compare with* (4.5)

	$J^1(u)$	$J^2(u)$	$J^3(u)$	$J^4(u)$
$u = u^{1*}$	2384.4	4900.0	7649.7	1.22×10^5
$u = u^{2*}$	2.462×10^4	570.77	1526.7	6.465×10^4
$u = u^{3*}$	3.889×10^4	1194.2	381.16	1.269×10^4
$u = u^{4*}$	4.454×10^4	1749.6	691.35	485.76

have computed each control $u^{\alpha*}$, where $u^{\alpha*}$ is defined as the optimal control for the plant α. Next, we have computed the cost that each plant incurs when subject to $u^{\alpha*}$. All costs are shown in Table 1. It can be seen that, for all $u^{\alpha*}$, there is at least one plant that incurs a cost which is larger than $J(u^{**})$. This supports the claim that the min–max is a reasonable criterion. Finally, the state trajectories for each system are presented in Fig. 4.

REMARK 4.1 From equation (3.16) it is easy to see that the computation time for μ^* strongly depends on the value of the parameter γ_j. In previous examples, a simple updating rule for γ_j was used. Namely, we consider $\gamma_j = 0.35/j$ and $\gamma_j = 0.5/j$ for Examples 4.1 and 4.2, respectively. With these data, it took about 50 iteration steps (1.708 s) to converge to μ_* for Example 4.1, whereas it took about 184 iteration steps (56.99 s) for Example 4.2. All simulations were carried out using Matlab R2011b over a PC with an Intel $i7 - 4770$ processor at 3.4 GHz.

It is worth noting that previous time measurements give only a rough idea about the speed of the algorithm, but in applications it would be better to look into adaptation techniques for the parameter γ_j, such as Armijo's rule (Bertsekas, 1976) or some other non-linear programming technique for the programme (3.13).
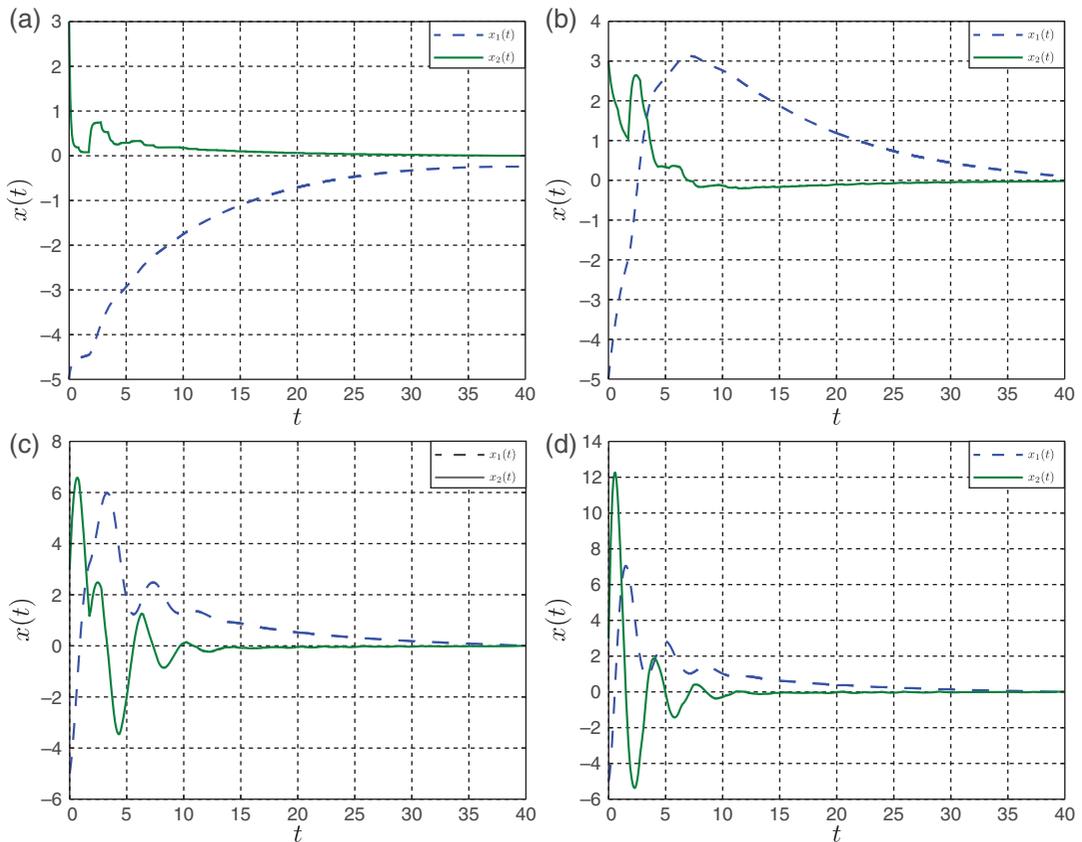
FIG. 4. Trajectories of (4.3) subject to u^{**}. (a) $\alpha = 1$, (b) $\alpha = 2$, (c) $\alpha = 3$ and (d) $\alpha = 4$.

5. Conclusions

By employing generalized gradients we have formulated a multi-model method of Lagrange multipliers. When applied to the discrete-time min–max optimal control problem, the method leads to a Riccati equation for an extended plant with a state vector obtained by aggregating the state of each individual plant. This is in perfect analogy with the continuous-time solution (Boltyanski & Poznyak, 2011). The Riccati equation is parametrized by μ, a member of a simplex whose elements determine the weight assigned to the cost of each plant. Non-smooth analysis specifies μ as the solution of a maximization problem over a simplex, thus completely characterizing the optimal solution. Non-smooth analysis also leads to a complementary slackness condition on μ, which turns out to be useful when computing the solution numerically.

Numerical experiments show the effectiveness of the min–max approach in the context of a multi-model setting, in the sense that the cost of each plant is kept at a reasonable level. It is worth mentioning, however, that the resulting optimal control is essentially open-loop, since it depends on the state trajectories of all the models, whether they are actually realized or not. The problem of obtaining a state-feedback control thus remains open, but points the direction for continuing this line of research.

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Appendix

Lemma A.1 Consider the problem of finding the maximum element among a finite set indexed by $\mathscr{A}$, i.e., $\max_{\alpha \in \mathscr{A}}\{z^1, \ldots, z^{|\mathscr{A}|}\}$. This problem is equivalent to the following linear programme:

$$\underset{\mu \in \mathscr{S}^{|\mathscr{A}|}}{\text{maximize}} \sum_{\alpha \in \mathscr{A}} \mu_\alpha z^\alpha \, .$$

Moreover, the solution μ^* of the linear programme satisfies $\mu_k^* = 0$ for all $k \notin I := \{\alpha \in \mathscr{A} : z^\alpha = z^0\}$ with $z^0 := \max\{z^1, \ldots, z^{|\mathscr{A}|}\}$.

Proof. Note that

$$z^0 = \sum_{\alpha \in \mathscr{A}} \mu_\alpha z^0 \geqslant \sum_{\alpha \in \mathscr{A}} \mu_\alpha z^\alpha \quad \text{for all } \mu \in \mathscr{S}^{|\mathscr{A}|} \, ,$$

so that $z^0 \geqslant \zeta^* := \max_{\mu \in \mathscr{S}^{|\mathscr{A}|}} \sum_{\alpha \in \mathscr{A}} \mu_\alpha z^\alpha$. On the other hand, we have $\zeta^* \geqslant z^\alpha$ for all $\alpha \in \mathscr{A}$. Therefore, $z^0 = \zeta^*$. The proof is established by contradiction: suppose that there exists some indices $b \in \bar{I} := \{\alpha \in \mathscr{A} : z^\alpha < z^0\}$ such that $\mu_b^* \neq 0$. We have

$$
\begin{aligned}
z^0 &= \sum_{\alpha \in \mathscr{A}} \mu_\alpha^* z^\alpha \\
&= \sum_{\alpha \in I} \mu_\alpha^* z^0 + \sum_{\alpha \in \bar{I}} \mu_\alpha^* z^\alpha \\
&\leqslant \sum_{\alpha \in I} \mu_\alpha^* z^0 + \sum_{\alpha \in \bar{I}} \mu_\alpha^* \tilde{z},
\end{aligned}
$$

where $\tilde{z} = \max_{\alpha \in \bar{I}} z^\alpha$. The last inequality implies that

$$
\sum_{\alpha \in \bar{I}} \mu_\alpha^* z^0 = \left(1 - \sum_{\alpha \in I} \mu_\alpha^*\right) z^0 \leqslant \sum_{\alpha \in \bar{I}} \mu_\alpha^* \tilde{z}.
$$

In other words, $z^0 \leqslant \tilde{z}$, the desired contradiction. $\qquad\square$