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RESEARCH ARTICLE

Quantized and sampled output feedback for nonlinear systems

Manuel Mera, Fernando Castaños* and Alexander Poznyak

Departamento Control Automático, Cinvestav-IPN. México, D.F.

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In this paper we consider the analysis and design of an output feedback controller for a perturbed nonlinear system in which the output is sampled and quantized. Using the attractive ellipsoid method, which is based on Lyapunov analysis techniques, together with the relaxation of a nonlinear optimization problem, sufficient conditions for the design of a robust control law are obtained. Since the original conditions result in nonlinear matrix inequalities, a numerical algorithm to obtain the solution is presented. The obtained control ensures that the trajectories of the closed-loop system will converge to a minimal (in a sense to be made specific) ellipsoidal region. Finally, numerical examples are presented in order to illustrate the applicability of the proposed design method.

Keywords: Attractive Ellipsoid Method; Sampled-data Systems; Lyapunov-Krasovskii Functional

1 Introduction

Motivated by emerging applications in networked control systems (Peng *et al*, 2011, Peng and Tian 2007, Zhang and Yu 2007), the control community has witnessed a renewed interest in phenomena that is inherent to the digital implementation of continuous-time control systems, such as sampling and quantization. A major line of research in this area incorporates the information-theoretical aspects (such as channel capacity) of the networked control problem and aims at a theory that parallels the celebrated mathematical theory of communication (Shannon 1948). Interesting results have been obtained by following this direction; it is now possible, e.g., to relate the absolute value of the unstable eigenvalues of a system and the minimum channel capacity that is required in order to stabilize it (Nair and Evans 2003, Tatikonda and Mitter 2004, Matveev and Savkin 2007). While certainly of great theoretical interest, most of these results are, so far, limited to linear systems. The problem statement is cast in a stochastic framework and emphasis is given to the coding and decoding aspects of the communication channel (see Phat *et al*, (2004) for a coding scheme).

From a different point of view, quantization can be regarded either as a deterministic noise or as a deterministic perturbation, depending on whether quantization affects the control or the output signals. A robust-control approach, such as $\mathcal{H}_\infty$ (Gao and Chen 2008) or the sector bound (Fu and Xie 2005), can then be applied to cope with the quantization problem. Again, most of the results using this approach are limited to linear systems. In this paper we deal with the quantization problem by applying the attractive ellipsoid method (Glover and Schweppe 1971, Kurzhanski and Varaiya 2006, Polyak *et al*, 2004, Polyak and Topunov 2008, Davila and Poznyak 2011). This allows us to design dynamic feedback control laws for a class of *nonlinear* systems satisfying a quasi-Lipschitz condition (Azhmyakov *et al*, 2013a,b). The class of systems is fairly large, as it includes systems with hard or even discontinuous non linearities.

*Corresponding author. Email: castanos@ieee.org

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We consider static and time-invariant quantizers. Because of its time-invariance nature, the required quantizer has an infinite number of quantization levels and practical stability is obtained instead of asymptotic stability (see Brockett and Liberzon (2000) for a finite dynamic quantizer achieving asymptotic stability). The attractive ellipsoid method delivers an estimated region of convergence in the form of an ellipsoid. Using numerical methods, a controller is chosen with a clear performance criterion: to minimize the size of such ellipsoid.

To deal with the sampling problem, it is typically assumed that the system is already in discrete-time form. We do not make such assumption. In the spirit of Tian *et al*, (2008) and Fridman and Dambrine (2009), we consider continuous-time systems and approach the sampling problem from a time-delay systems perspective. To compute the aforementioned ellipsoid, we construct a Lyaunov-Krasovskii functional instead of the usual Lyapunov function. In this regard, the present work can be seen as an extension of the work presented in Mera *et al*, (2009) to the case when quantization phenomena are present.

Paper structure The problem is formally stated in the following section. Section 3 states conditions for a given ellipsoid to be attractive with respect to the closed-loop dynamics arising from the proposed controller. A sub-optimal algorithm for the minimization of such ellipsoid is given in Section 4. Numerical examples are given in Section 5. Conclusions can be found in Section 6.

2 Problem Formulation

Consider the nonlinear system

$$\dot{x}(t) = f(t, x(t)) + Bu(t) + \upsilon_x(t) , \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $\upsilon_x(t) \in \mathbb{R}^n$ are, respectively, the state vector, control input and perturbation at time $t \in \mathbb{R}_+$. We use the following model to describe a noisy, sampled and quantized output:

$$\bar{\bar{y}}(t) = Cx(t) + \omega_y(t) , \tag{2a}$$

$$\bar{y}(t) = \sum_{t_k} \bar{\bar{y}}(t_k)\chi_{[t_k,t_{k+1})}(t) , \tag{2b}$$

$$y(t) = \pi(\bar{y}(t)) . \tag{2c}$$

The vector $\omega_y(t) \in R^q$ in (2a) is the deterministic noise. The symbol $\chi_{[t_k,t_{k+1})}$ in (2b) denotes the characteristic function of the time interval $[t_k, t_{k+1})$, i.e.,

$$\chi_{[t_k,t_{k+1})}(t) := \begin{cases} 1 & \text{if } t \in [t_k, t_{k+1}) \\ 0 & \text{otherwise} \end{cases} , \quad k = 0, 1, 2, \ldots$$

Thus, $\bar{y} : \mathbb{R}_+ \to \mathbb{R}^q$ is the piecewise constant function which is obtained by sampling and holding $\bar{\bar{y}}$ at the discrete instants t_k. The actual system output at time t is $y(t) \in \mathbb{R}^q$, and is obtained by quantizing the sampled signal $\bar{y}$. Formally: Let $Y \subset \mathbb{R}^q$ be a countable set of possible output values. Then, $\pi : \mathbb{R}^q \to Y$ in (2c) is defined as a projection operator, i.e., as an operator that satisfies $\pi \circ \pi(\bar{y}) \equiv \pi(\bar{y})$. The image of π is a discrete subset of $\mathbb{R}^q$. The components of the measurable output $y(t)$ have the form as they are depicted in Fig. 1.

Let us now formulate our basic assumptions.

Assumption 2.1

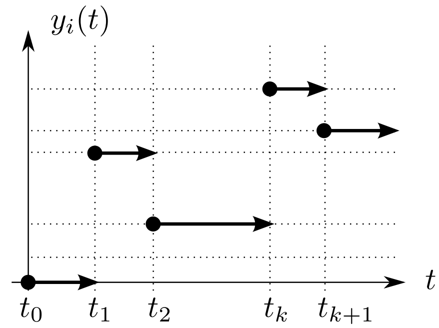

Figure 1. The components of the measurable output.

(i) The perturbation and noise are unknown but bounded. More precisely, there are known positive definite matrices $Q_x \in \mathbb{R}^n$ and $Q_y \in \mathbb{R}^q$ such that

$$\|\upsilon_x(t)\|^2_{Q_x} + \|\omega_y(t)\|^2_{Q_y} \leq 1 \quad \text{for all } t \in \mathbb{R}_+ \; . \tag{3}$$

Here, $\|\cdot\|_{Q_x}$ and $\|\cdot\|_{Q_y}$ are weighted norms given by Q_x and Q_y.

(ii) The function f is also unknown but satisfies the quasi-Lipschitz bound

$$\|f(t,x) - Ax(t)\|^2_{Q_x} \leq \delta + ||x(t)||^2_Q \quad \text{for all } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n \; , \tag{4}$$

where $\delta > 0$ is a scalar and $Q > 0$ and A are known $(n \times n)$-dimensional matrices.

(iii) The pair (A, B) is stabilizable and (A, C) is detectable.

(iv) The sampling intervals need not be regular, but there exists a maximum sampling interval

$$h := \max_k |t_{k+1} - t_k| \; .$$

(v) The quantization error is bounded, i.e., the positive scalar

$$c := \max_{\bar{y} \in \mathbb{R}^q} \|\pi(\bar{y}) - \bar{y}\|^2_{Q_y} \tag{5}$$

is finite.

Notice that (4) is not restrictive and comprises a large class of unknown nonlinear functions (Azhmyakov *et al*, 2013a,b). By defining the auxiliary function $\omega_x(t) := \upsilon_x(t) + f(t, x(t)) - Ax(t)$, we can rewrite (1) in the quasi-linear format

$$\dot{x}(t) = Ax(t) + Bu(t) + \omega_x(t) \; . \tag{6}$$

Condition (iii) then becomes natural. We assume (5) for simplicity but, as kindly pointed out by an anonymous reviewer, the sampling scenario can be extended to the case of mixed (linear/logarithmic) quantization: $\max_{\bar{y} \in \mathbb{R}^q} \|\pi(\bar{y}) - \bar{y}\|^2_{Q_y} \leq c + \|\bar{y}\|^2$.

We approach the partial-information problem using a conventional Luenberger observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)) \; , \tag{7}$$

where $L \in \mathbb{R}^{n \times q}$ is the observer gain. The control law is taken as

$$u(t) = K\hat{x}(t) \; , \tag{8}$$

with $K \in \mathbb{R}^{m \times n}$ the control gain.

Let us now introduce the estimation error $e(t) := x(t) - \hat{x}(t)$ and the auxiliary variable $\Delta y(t) := y(t) - \bar{\bar{y}}(t)$. It can be readily seen that $e(t)$ satisfies the dynamic equation

$$\begin{aligned}\dot{e}(t) &= Ax(t) + Bu(t) + \omega_x(t) - \big(A\hat{x}(t) + Bu(t) + L(\bar{\bar{y}} + \Delta y - C\hat{x}(t))\big)\\ \dot{e}(t) &= (A - LC)e(t) - L(\Delta y(t) + \omega_y(t)) + \omega_x(t)\ .\end{aligned} \tag{9}$$

It is possible to write the closed-loop equations (7) and (9) more compactly as

$$\dot{z}(t) = \tilde{A}z(t) + F\omega(t) + \psi(t)\ , \tag{10}$$

where we have defined the vectors

$$z(t) := \begin{pmatrix}\hat{x}(t)\\ e(t)\end{pmatrix}\ , \quad \omega(t) := \begin{pmatrix}\omega_x(t)\\ \omega_y(t)\end{pmatrix} \quad \text{and} \quad \psi(t) := \begin{pmatrix}L\\ -L\end{pmatrix}\Delta y(t)$$

and the matrices

$$\tilde{A} := \begin{pmatrix}A + BK & LC\\ 0 & A - LC\end{pmatrix} \quad \text{and} \quad F := \begin{pmatrix}0 & L\\ I & -L\end{pmatrix}\ .$$

Because of the presence of ω and ψ, it is not reasonable to expect $z(t)$ to converge to the origin as $t \to \infty$. On the other hand, if K and L are properly chosen, it reasonable to expect $z(t)$ to converge to a 'small' set containing the origin. Our problem is first to find an estimate of such set and then to find L and K that minimize (in a sense to be defined later) its 'size'.

3 Extended Attractive Ellipsoid Method

To estimate the region where the states of (10) converge, we use the ellipsoid method and propose an extension to deal with the sampling and the quantization of the output.

3.1 *Attractive sets*

Let us sketch the main idea first and let us recall a basic lemma about differential inequalities.

Lemma 3.1: *Let a function $V : \mathbb{R}_+ \to \mathbb{R}$ satisfy the differential inequality*

$$\dot{V}(t) \leq -\alpha V(t) + \beta\ . \tag{11}$$

Then, its solutions satisfy

$$V(t) \leq e^{-\alpha t}V(0) + \frac{\beta}{\alpha}(1 - e^{-\alpha t})\ . \tag{12}$$

Lemma 3.1 is a particular case of Theorem 4.1 (Hartman 2002, Ch. III). Now, suppose that

$$V(t) := \mathcal{V} \circ z(t)$$

with $\mathcal{V} : \mathbb{R}^{2n} \to \mathbb{R}_+$ differentiable, $z(t)$ a solution of (10) evaluated at time t and the operator $\circ$ denotes composition. Then, Eq. (12) with $\alpha > 0$ and $\beta \geq 0$ clearly implies that the sub-level set

$$\mathcal{V}_{\beta/\alpha} := \left\{ z \in \mathbb{R}^{2n}\ :\ \mathcal{V}(z) \leq \frac{\beta}{\alpha} \right\}$$

is attractive (i.e., $\limsup_{t\to\infty} V(t) \le \beta/\alpha$).

3.2 *A Lyapunov-Krasovskii functional*

Since sampling entails delays, instead of a regular function we suggest to use a Lyapunov-Krasovskii functional. More precisely, let $\mathcal{C}^0(\mathbb{R},\mathbb{R}^{2n})$ be the space of all continuous functions of $\mathbb{R}$ into $\mathbb{R}^{2n}$, differentiable almost everywhere; let $R > 0$ and $P > 0$ be $(2n \times 2n)$-dimensional matrices and let $\alpha > 0$ be a scalar. We propose the functional $V : \mathbb{R} \times \mathcal{C}^0(\mathbb{R},\mathbb{R}^{2n}) \to \mathbb{R}_+$, defined as[1]

$$V(t, z(\cdot)) := z^\top(t)P^{-1}z(t) + h\int_{\theta=-h}^{0}\int_{s=t+\theta}^{t} e^{\alpha(s-t)}\dot{z}^\top(s)R\dot{z}(s)\mathrm{d}s\mathrm{d}\theta\,. \tag{13}$$

Our primary goal is to derive sufficient conditions for $V(t, z(\cdot))$ to satisfy (11) with $\alpha > 0$ and $\beta \ge 0$ when z is a solution of (10). Let us begin with the case when z is arbitrary.

Theorem 3.2 : *For any given*

$$z(\cdot) \in \mathcal{C}^0(\mathbb{R},\mathbb{R}^{2n})\,, \quad h, \alpha, b \in \mathbb{R}\,, \quad P, R \in \mathbb{R}^{2n\times 2n}$$

such that $h > 0$, $\alpha > 0$, $P > 0$ *and* $R > 0$, *the time derivative of* $V(t, z(\cdot))$ *in* (13) *satisfies the bound*

$$\dot{V}(t, z(\cdot)) \le -\alpha V(t, z(\cdot)) + b\bar{\delta} + \eta(t, z(\cdot))^\top W \eta(t, z(\cdot))\,, \tag{14}$$

where

$$\eta(t, z(\cdot)) := \begin{pmatrix} z(t) \\ \dot{z}(t) \\ z(t) - z(t_k) \\ \omega(t) \end{pmatrix}, \quad W := \begin{pmatrix} \alpha P^{-1} + bQ_z & P^{-1} & 0 & 0 \\ * & h^2R & 0 & 0 \\ * & * & -he^{-\alpha h}R & 0 \\ * & * & * & -b\bar{Q} \end{pmatrix}, \tag{15a}$$

$$\bar{Q} := \begin{pmatrix} Q_x & 0 \\ 0 & Q_y \end{pmatrix}, \quad Q_z := \begin{pmatrix} I \\ I \end{pmatrix} Q \begin{pmatrix} I & I \end{pmatrix} \quad \text{and} \quad \bar{\delta} := \delta + 1\,. \tag{15b}$$

Before giving the proof of the theorem, let us state a pair of simple lemmas.

Lemma 3.3: *The perturbation* ω *satisfies the bound*

$$\|\omega(t)\|_{\bar{Q}}^2 \le \bar{\delta} + \|x(t)\|_Q^2\,. \tag{16}$$

Proof : Direct computation of the norm gives

$$\begin{aligned}\|\omega(t)\|_{\bar{Q}}^2 = \|\omega_x(t)\|_{Q_x}^2 + \|\omega_y(t)\|_{Q_y}^2 &= \|\upsilon_x(t) + f(t, x(t)) - Ax(t)\|_{Q_x}^2 + \|\omega_y(t)\|_{Q_y}^2 \\ &\le \|\upsilon_x(t)\|_{Q_x}^2 + \|f(t, x(t)) - Ax(t)\|_{Q_x}^2 + \|\omega_y(t)\|_{Q_y}^2\,. \end{aligned} \tag{17}$$

Substitution of (3) and (4) in (17) shows that

$$\|\omega(t)\|_{\bar{Q}}^2 \le 1 + \delta + \|x\|_Q^2\,.$$

[1]Recall that a delay system is infinite-dimensional and its response is defined uniquely by the initial condition $z_0 : [-h, 0] \to \mathbb{R}^{2n}$. Thus, the value of $z(\tau)$ in the interval $[-h, 0]$ is given by the initial condition that defines the specific solution of (10). Note that functional does not contain the single-integral term as there is in Fridman (2001) and Mera *et al*, (2009).

□

Lemma 3.4: *For any given $z(\cdot) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^{2n})$, $h > 0$, $\alpha > 0$, $R > 0$, we have*

$$-h \int_{t-h}^{t} e^{\alpha(s-t)} \dot{z}^\top(s) R \dot{z}(s) \mathrm{d}s \leq -h e^{-\alpha h} \int_{t_k}^{t} \dot{z}^\top(s) \mathrm{d}s R \int_{t_k}^{t} \dot{z}(s) \mathrm{d}s \,. \tag{18}$$

Proof : Since $e^{-\alpha h} \leq e^{\alpha(s-t)}$ for all $s \in [t-h, t]$ and R is positive definite, we have

$$-h \int_{t-h}^{t} e^{\alpha(s-t)} \dot{z}^\top(s) R \dot{z}(s) \mathrm{d}s \leq -h e^{-\alpha h} \int_{t-h}^{t} \dot{z}^\top(s) R \dot{z}(s) \mathrm{d}s \,. \tag{19}$$

By splitting the integration interval at the time $t_k \in [t-h, t)$, we obtain

$$\begin{aligned} -h e^{-\alpha h} \int_{t-h}^{t} \dot{z}^\top(s) R \dot{z}(s) \mathrm{d}s &= -h e^{-\alpha h} \int_{t-h}^{t_k} \dot{z}^\top(s) R \dot{z}(s) \mathrm{d}s - h e^{-\alpha h} \int_{t_k}^{t} \dot{z}^\top(s) R \dot{z}(s) \mathrm{d}s \\ &\leq -h e^{-\alpha h} \int_{t_k}^{t} \dot{z}^\top(s) R \dot{z}(s) \mathrm{d}s \leq -h e^{-\alpha h} \int_{t_k}^{t} \dot{z}^\top(s) \mathrm{d}s R \int_{t_k}^{t} \dot{z}(s) \mathrm{d}s \,, \end{aligned} \tag{20}$$

where the first inequality follows from the fact that h is positive the second one follows from Jensen's inequality (Poznyak 2008). Combining (19) and (20) yields (18). □

Proof (of Theorem 3.2): We begin by directly computing $\dot{V}$:

$$\begin{aligned} \dot{V}(t, z(\cdot)) = 2 z^\top(t) P^{-1} \dot{z}(t) &- \alpha h \int_{-h}^{0} \int_{t+\theta}^{t} e^{\alpha(s-t)} \dot{z}^\top(s) R \dot{z}(s) \mathrm{d}s \mathrm{d}\theta \\ &- h \int_{t-h}^{t} e^{\alpha(s-t)} \dot{z}^T(s) R \dot{z}(s) \mathrm{d}s + h^2 \dot{z}^T(t) R \dot{z}(t) \,. \end{aligned} \tag{21}$$

By adding and subtracting $\alpha V(t, z(\cdot))$ to the right-hand side of (21) we obtain

$$\begin{aligned} \dot{V}(t, z(\cdot)) = 2 z^\top(t) P^{-1} \dot{z}(t) + \alpha z^\top(t) P^{-1} z(t) &- h \int_{t-h}^{t} e^{\alpha(s-t)} \dot{z}^\top(s) R \dot{z}(s) \mathrm{d}s \\ &+ h^2 \dot{z}^\top(t) R \dot{z}(t) - \alpha V(t, z(\cdot)) \,. \end{aligned} \tag{22}$$

The following upper bound for $\dot{V}$ can be easily obtained from (22) and (18):

$$\dot{V}(t, z(\cdot)) \leq -\alpha V(t, z(\cdot)) + \eta_1(t, z(\cdot))^\top W_1 \eta_1(t, z(\cdot)) \,, \tag{23}$$

where

$$\eta_1(t, z(\cdot)) := \begin{pmatrix} z(t) \\ \dot{z}(t) \\ \int_{t_k}^{t} \dot{z}(s) \mathrm{d}s \end{pmatrix} = \begin{pmatrix} z(t) \\ \dot{z}(t) \\ z(t) - z(t_k) \end{pmatrix}$$

and W_1 is a symmetric matrix defined by

$$W_1 := \begin{pmatrix} \alpha P^{-1} & P^{-1} & 0 \\ * & h^2 R & 0 \\ * & * & -he^{-\alpha h} R \end{pmatrix} .$$

By adding and subtracting $b\|\omega(t)\|_{\bar{Q}}^2$ to the right-hand of (23), we can rewrite the upper bound as

$$\dot{V}(t, z(\cdot)) \leq -\alpha V(t, z(\cdot)) + b\|\omega(t)\|_{\bar{Q}}^2 + \eta(t, z(\cdot))^\top W_2 \eta(t, z(\cdot)) \, , \tag{24}$$

where

$$W_2 := \begin{pmatrix} \alpha P^{-1} & P^{-1} & 0 & 0 \\ * & h^2 R & 0 & 0 \\ * & * & -he^{-\alpha h} R & 0 \\ * & * & * & -b\bar{Q} \end{pmatrix} .$$

From (16), we have

$$\dot{V}(t, z(\cdot)) \leq -\alpha V(t, z(\cdot)) + b(\bar{\delta} + \|x(t)\|_Q^2) + \eta(t, z(\cdot))^\top W_2 \eta(t, z(\cdot)) \, . \tag{25}$$

Since

$$\|x(t)\|_Q^2 = \|\hat{x}(t) + e(t)\|_Q^2 = \left\| \begin{pmatrix} I & I \end{pmatrix} z(t) \right\|_Q^2 = z(t)^\top Q_z z(t) \, ,$$

we can finally rewrite (25) as (14). □

Now we will refine the bound given in Theorem 3.2 by restricting $z(\cdot)$ to the set of solutions of (10). In order to do so, we follow the idea presented in Fridman (2006) and Fridman and Niculescu (2008) which, originally devised for systems in descriptor form, consists in adding a term (the *descriptor term*) to the expression for $\dot{V}$. The descriptor term has to be zero for any solution z of the system. In our case, we will add the term

$$\mathcal{D}(t, z(\cdot)) := 2 \left(z(t)^\top \Pi_a + \dot{z}(t)^\top \Pi_b \right) \times \left(\tilde{A} z(t) + F\omega(t) + \psi(t) - \dot{z}(t) \right) \, ,$$

where Π_a and Π_b symmetric matrices in $\mathbb{R}^{2n \times 2n}$. Obviously, $\mathcal{D}$ is zero along the solutions of (10).

Theorem 3.5 : *Let ρ_1 be a positive scalar satisfying*

$$L^\top L \leq \rho_1 I \tag{26}$$

Then, for any

$$z(\cdot) \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^{2n}) \, , \quad h, \alpha, b, \varepsilon \in \mathbb{R} \, , \quad P, R, \Pi_a, \Pi_b \in \mathbb{R}^{2n \times 2n}$$

such that z is a solution of (10), $h > 0$, $\alpha > 0$, $P > 0$ *and* $R > 0$, *the time derivative of* $V(t, z(\cdot))$ *in* (13) *satisfies*

$$\dot{V}(t, z(\cdot)) \leq -\alpha V(t, z(\cdot)) + \beta + \xi(t, z(\cdot))^\top \Omega \xi(t, z(\cdot)) \, , \tag{27}$$

where

$$\Omega := \begin{pmatrix} \alpha P^{-1} + bQ_z + \Pi_a\tilde{A} + \tilde{A}^\top\Pi_a & P^{-1} - \Pi_a + \Pi_b\tilde{A} & 0 & \Pi_a F & \Pi_a \\ * & h^2R - 2\Pi_b & 0 & \Pi_b F & \Pi_b \\ * & * & -he^{-\alpha h}R + \varepsilon\rho Q_c & 0 & 0 \\ * & * & * & -b\bar{Q} & 0 \\ * & * & * & * & -\varepsilon I \end{pmatrix} \tag{28}$$

and

$$\xi(t, z(\cdot)) := \begin{pmatrix} z(t) \\ \dot{z}(t) \\ z(t) - z(t_k) \\ \omega(t) \\ \psi(t) \end{pmatrix}, \quad Q_c := \begin{pmatrix} I \\ I \end{pmatrix} C^\top Q_y C \begin{pmatrix} I & I \end{pmatrix}, \quad \beta := b\bar{\delta} + \varepsilon\rho(2 + c), \tag{29a}$$

$$\rho := 2\rho_1/\lambda_{\min}(Q_y). \tag{29b}$$

The following lemma will be needed before the proof of the theorem.

Lemma 3.6: *The uncertainty resulting from noise, sampling and quantization is bounded by*

$$\|\psi(t)\|^2 \le \rho\left((z(t) - z(t_k))^\top Q_c(z(t) - z(t_k)) + 2 + c\right). \tag{30}$$

Proof : We will begin by computing an upper bound for Δy (see p. 4). We have

$$\|\Delta y(t)\|^2_{Q_y} = \|y(t) - \bar{\bar{y}}(t)\|^2_{Q_y} \le \|y(t) - \bar{y}(t)\|^2_{Q_y} + \|\bar{y}(t) - \bar{\bar{y}}(t)\|^2_{Q_y}. \tag{31}$$

Notice that

$$\begin{aligned} \bar{\bar{y}}(t) - \bar{y}(t) &= C(x(t) - x(t_k)) + \omega_y(t) - \omega_y(t_k) \\ &= C\begin{pmatrix} I & I \end{pmatrix}(z(t) - z(t_k)) + \omega_y(t) - \omega_y(t_k), \end{aligned}$$

so

$$\|\bar{y}(t) - \bar{\bar{y}}(t)\|^2_{Q_y} \le (z(t) - z(t_k))^\top Q_c(z(t) - z(t_k)) + 2, \tag{32}$$

where we have used (3) to establish $\|\omega_y(t)\|^2_{Q_y} + \|\omega_y(t_k)\|^2_{Q_y} \le 2$. Substituting (32) and (5) in (31) gives

$$\|\Delta y(t)\|^2_{Q_y} \le (z(t) - z(t_k))^\top Q_c(z(t) - z(t_k)) + 2 + c. \tag{33}$$

The norm of ψ then satisfies

$$\|\psi(t)\|^2 = \left\|\begin{pmatrix} I \\ -I \end{pmatrix} L\Delta y(t)\right\|^2 = 2\Delta y(t)^\top L^\top L\Delta y(t) \le 2\rho_1\|\Delta y(t)\|^2 \le \frac{2\rho_1}{\lambda_{\min(Q_y)}}\|\Delta y(t)\|^2_{Q_y}. \tag{34}$$

From (34) and (33) we conclude (30). □

Proof (of Theorem 3.5): Adding the null term $\mathcal{D}(t,z(\cdot)) + \varepsilon\|\psi(t)\|^2 - \varepsilon\|\psi(t)\|^2$ to (14) gives

$$\dot{V}(t,z(\cdot)) \le -\alpha V(t,z(\cdot)) + b\bar{\delta} + \varepsilon\|\psi(t)\|^2 + \eta(t,z(\cdot))^\top W\eta(t,z(\cdot)) + \\ 2\left(z(t)^\top\Pi_a + \dot{z}(t)^\top\Pi_b\right) \times \left(\tilde{A}z(t) + F\omega(t) + \psi(t) - \dot{z}(t)\right) - \varepsilon\|\psi(t)\|^2 . \quad (35)$$

Substituting (30) in (35) establishes

$$\dot{V}(t,z(\cdot)) \le -\alpha V(t,z(\cdot)) + \beta + \varepsilon\rho(z(t)-z(t_k))^\top Q_c(z(t)-z(t_k)) + \eta(t,z(\cdot))^\top W\eta(t,z(\cdot)) + \\ 2\left(z(t)^\top\Pi_a + \dot{z}(t)^\top\Pi_b\right) \times \left(\tilde{A}z(t) + F\omega(t) + \psi(t) - \dot{z}(t)\right) - \varepsilon\|\psi(t)\|^2 . \quad (36)$$

Equation (27) is nothing but (36) written in compact form. □

3.3 *Main result*

The following corollary follows from Theorem 3.5 and Lemma 3.1.

Corollary 3.7: *Let*

$$\left\{\alpha > 0, b > 0, \varepsilon > 0, \rho_1 > 0, P^{-1} > 0, R > 0, \Pi_a, \Pi_b, L, K\right\} \quad (37)$$

be a set of control parameters such that

$$\Omega \le 0 \quad \textit{and} \quad L^\top L \le \rho_1 I\ , \quad (38)$$

with Ω defined by (28); Q_z, Q_c, $\bar{Q}$ and ρ given by (29) and (15). The ellipsoid

$$\mathcal{E} := \left\{z \in \mathbb{R}^{2n} \ : \ z^\top P^{-1} z \le \frac{\beta}{\alpha}\right\} ,$$

with β given by (29), (15) and $\alpha = a$ is an attractive set.

4 Numerical Aspects

Given Corollary 3.7, it is natural to look for a set of parameters (37) such that the attractive ellipsoid is minimal in some sense. An obvious objective function to minimize is trace P. Unfortunately, such problem is strongly nonlinear and difficult to solve, even numerically, so we will have to settle for a sub-optimal solution. Our goal here is to find a numerically tractable expression that ensures $\Omega \le 0$. More precisely, we seek an expression that is linear in the matrix parameters, so that the well-known convex tools for matrix inequalities can be applied.

The first step involves applying the Schur complement to several blocks of the original matrix Ω. In order to achieve this, we first simplify our parameter space by setting $\Pi_a = \Pi_b = P^{-1}$ and by restricting P^{-1} and R to the class of block diagonal matrices of the form

$$P^{-1} = \mathrm{diag}\left(P_1^{-1}, P_2^{-1}\right) \quad \text{and} \quad R = \mathrm{diag}\left(R_1, R_2\right) .$$

Let us define $\Omega_A := T\Omega T^\top$ with

$$T = \mathrm{diag}\left(P_2^{-1}P_1, I, P_2^{-1}P_1, I, \cdots, I\right) \in \mathbb{R}^{10n\times 10n} .$$

Since T is non-singular, $\Omega \leq 0$ is equivalent to $\Omega_A \leq 0$. To ease the notation, we will write Ω_A in terms of the block matrices $A_{ij} \in \mathbb{R}^{n\times n}$, with $i, j = 1, \ldots, 10$. Notice that the following elements of Ω_A are nonlinear with respect to P_2 and P_1:

$$\begin{aligned}
A_{11} &= P_2^{-1}\left(bP_1QP_1 + aP_1 + P_1(A^\top + K^\top B^\top) + (A+BK)P_1\right)P_2^{-1}\\
A_{12} &= A_{21}^\top = P_2^{-1}\left(LC + bP_1Q\right)\\
A_{13} &= A_{31}^\top = P_2^{-1}\left(A+BK\right)P_1P_2^{-1}\\
A_{33} &= P_2^{-1}\left(h^2P_1R_1P_1 - 2P_1\right)P_2^{-1}\,.
\end{aligned}$$

By defining the matrix $J := \left(P_1P_2^{-1}\; I\; 0 \cdots 0\right)$, it is possible to express Ω_A as

$$\Omega_A = \Omega_B + J^\top(bQ)J\,. \tag{39}$$

The matrix sub-blocks of Ω_B are the same as those of Ω_A: $B_{ij} = A_{ij}$ with the exception of

$$\begin{aligned}
B_{11} &= P_2^{-1}\left(aP_1 + P_1(A^\top + P_1K^\top B^\top) + (A+BK)P_1\right)P_2^{-1}\\
B_{12} &= B_{21}^\top = P_2^{-1}LC\,,
\end{aligned}$$

which are now simpler than A_{11} and A_{12}. Using a Schur complement argument and defining $\mathcal{O} = \left(0\; I\; 0 \cdots 0\right)$ it can be seen that Ω_A is negative semi-definite if and only if

$$\begin{pmatrix}\Omega_B & J^\top\\ J & -\frac{1}{b}Q^{-1}\end{pmatrix} = \begin{pmatrix}\Omega_B & \mathcal{O}^\top\\ \mathcal{O} & -\frac{1}{b}Q^{-1}\end{pmatrix} + \begin{pmatrix}0\\ \vdots\\ 0\\ P_1\end{pmatrix}\left(P_2^{-1}\; 0 \cdots 0\right) + \begin{pmatrix}P_2^{-1}\\ 0\\ \vdots\\ 0\end{pmatrix}\left(0 \cdots 0\; P_1\right) \leq 0\,. \tag{40}$$

According to the Λ-inequality (Poznyak 2008), $\mathcal{X}\mathcal{Y}^\top + \mathcal{Y}\mathcal{X}^\top \leq \mathcal{X}\Lambda\mathcal{X}^\top + \mathcal{Y}\Lambda^{-1}\mathcal{Y}^\top$ for any $\mathcal{X}$, $\mathcal{Y}$ and non-singular Λ with compatible dimensions. Now, setting $\mathcal{X} = \left(0 \cdots 0\; P_1\right)^\top$, $\mathcal{Y} = \left(P_2^{-1}\; 0 \cdots 0\right)^\top$ and $\Lambda = \Lambda_1^{-1}$ in (40) we obtain

$$\begin{pmatrix}\Omega_B & J^\top\\ J & -\frac{1}{b}Q^{-1}\end{pmatrix} \leq \begin{pmatrix}\Omega_C & \mathcal{O}^\top\\ \mathcal{O} & -\frac{1}{b}Q^{-1} + P_1\Lambda_1^{-1}P_1\end{pmatrix}\,, \tag{41}$$

where the sub-blocks of Ω_C are the same as those of Ω_B, this is, $C_{ij} = B_{ij}$, with the exception of C_{11} which is

$$C_{11} = B_{11} + P_2^{-1}\Lambda_1P_2^{-1} = P_2^{-1}\left(aP_1 + P_1(A^\top + P_1K^\top B^\top) + (A+BK)P_1 + \Lambda_1\right)P_2^{-1}\,.$$

Let us now introduce a new variable, G_f, which will serve as an upper bound for the nonlinear term $-\frac{1}{b}Q^{-1} + P_1\Lambda_1^{-1}P_1$ in (41), that is,

$$-\frac{1}{b}Q^{-1} + P_1\Lambda_1^{-1}P_1 < G_f\,. \tag{42}$$

This implies

$$\begin{pmatrix}\Omega_C & \mathcal{O}^\top\\ \mathcal{O} & -\frac{1}{b}Q^{-1} + P_1\Lambda_1^{-1}P_1\end{pmatrix} \leq \begin{pmatrix}\Omega_C & \mathcal{O}^\top\\ \mathcal{O} & G_f\end{pmatrix}\,. \tag{43}$$

Using Schur complements again, it can be readily shown that (42) is equivalent to

$$\begin{pmatrix} -G_f - \frac{1}{b}Q^{-1} & P_1 \\ P_1 & -\Lambda_1 \end{pmatrix} < 0 \, . \tag{44}$$

Now, we want to obtain a linear upper bound for Ω_C, i.e., a matrix Ω_D such that

$$\Omega_C \leq \Omega_D \, . \tag{45}$$

Notice that all the sub-blocks of Ω_C are linear in P_1 and P_2^{-1}, except for C_{11}, C_{13} and C_{33}, which are of the form $\mathcal{PMP}$, where $\mathcal{P}$ can take the value of P_1 or P_2^{-1} and $\mathcal{M}$ depends on P_1. These terms can be majored using the Λ-inequality again. Set $\mathcal{X} = \mathcal{P}$, $\mathcal{Y} = I$, $\Lambda = -\mathcal{M}$ and introduce a new term, $\mathcal{G}$, that will serve as an upper bound $\mathcal{PMP} < \mathcal{G}$. Then, according to the Λ-inequality, we have $2\mathcal{P} \leq -\mathcal{PMP} - \mathcal{M}^{-1}$, that is,

$$2\mathcal{P} + \mathcal{M}^{-1} \leq -\mathcal{PMP} \, . \tag{46}$$

Now, taking Schur complements, we know that

$$\begin{pmatrix} \mathcal{G} + 2\mathcal{P} & -I \\ -I & -\mathcal{M} \end{pmatrix} > 0$$

if and only if $-\mathcal{M} > 0$ and $\mathcal{G} + 2\mathcal{P} + \mathcal{M}^{-1} > 0$. Combining (46) with the last equation gives

$$\mathcal{G} - \mathcal{PMP} \geq \mathcal{G} + 2\mathcal{P} + \mathcal{M}^{-1} > 0 \, .$$

In summary, the inequality

$$\begin{pmatrix} -\mathcal{G} - 2\mathcal{P} & I \\ I & \mathcal{M} \end{pmatrix} < 0 \tag{47}$$

implies the desired result, $\mathcal{PMP} < \mathcal{G}$.

It is natural to propose an Ω_D with sub-blocks equal to those of Ω_C with the exception of D_{11}, $D_{13} = D_{31}^\top$ and D_{33}, which are matrix block variables that fulfill some additional restrictions. The first such restriction is that $D_{13} = D_{13}^\top$. To obtain the remaining constraints, let us write the upper-left $3n \times 3n$ sub-block matrix of the difference $\Omega_C - \Omega_D$,

$$\begin{aligned} \begin{pmatrix} C_{11} - D_{11} & 0 & C_{13} - D_{13} \\ 0 & 0 & 0 \\ C_{31} - D_{31} & 0 & C_{33} - D_{33} \end{pmatrix} &= \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} (C_{11} - D_{11} - C_{13} + D_{13}) \begin{pmatrix} I & 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} (C_{33} - D_{33} - C_{13} + D_{13}) \begin{pmatrix} 0 & 0 & I \end{pmatrix} + \begin{pmatrix} I \\ 0 \\ I \end{pmatrix} (C_{13} - D_{13}) \begin{pmatrix} I & 0 & I \end{pmatrix} \, . \end{aligned}$$

It is clear that $\Omega_C - \Omega_D \leq 0$ if

$$\begin{aligned} C_{11} - \frac{1}{2}(C_{13} + C_{13}^\top) &< D_{11} - D_{13} \\ C_{33} - \frac{1}{2}(C_{13} + C_{13}^\top) &< D_{33} - D_{13} \\ \frac{1}{2}(C_{13} + C_{13}^\top) &< D_{13} \end{aligned}$$

Application of (47) to the first inequality shows that

$$C_{11}-\frac{1}{2}(C_{13}+C_{13}^\top) = P_2^{-1}\left(aP_1 + \frac{P_1}{2}(A^\top + K^\top B^\top) + (A+BK)\frac{P_1}{2} + \Lambda_1\right)P_2^{-1} < D_{11}-D_{13}$$

is equivalent to

$$\begin{pmatrix} D_{13} - D_{11} - 2P_2^{-1} & I \\ I & aP_1 + \frac{P_1}{2}(A^\top + K^\top B^\top) + (A+BK)\frac{P_1}{2} + \Lambda_1 \end{pmatrix} < 0 \tag{48a}$$

(note that this inequality is linear with respect to P_1 and P_2^{-1}). Likewise, applying (47) to the second inequality,

$$\begin{aligned} C_{33} - \frac{1}{2}(C_{13} + C_{13}^\top) &= P_2^{-1}\left(h^2 P_1 R_1 P_1 - 2P_1 - (A+BK)\frac{P_1}{2} - \frac{P_1}{2}(A^\top + B^\top K^\top)\right)P_2^{-1} \\ &< P_2^{-1}\left(G_0 - (A+BK)\frac{P_1}{2} - \frac{P_1}{2}(A^\top + B^\top K^\top)\right)P_2^{-1} < D_{33} - D_{13} \end{aligned}$$

if

$$\begin{pmatrix} -2P_1 - G_0 & P_1 \\ P_1 & -h^2 R_1 \end{pmatrix} < 0 \tag{48b}$$

$$\begin{pmatrix} -2P_2^{-1} - D_{33} + D_{13} & I \\ I & G_0 - (A+BK)\frac{P_1}{2} - \frac{P_1}{2}(A^\top + B^\top K^\top) \end{pmatrix} < 0\,. \tag{48c}$$

Finally,

$$\frac{1}{2}(C_{13} + C_{13}^\top) = P_2^{-1}\left((A+BK)\frac{P_1}{2} + \frac{P_1}{2}(A^\top + B^\top K^\top)\right)P_2^{-1} < D_{13}$$

if

$$\begin{pmatrix} -2P_2^{-1} - D_{13} & I \\ I & (A+BK)\frac{P_1}{2} + \frac{P_1}{2}(A^\top + B^\top K^\top) \end{pmatrix} < 0\,. \tag{48d}$$

With these restrictions, inequality (45) holds so

$$\begin{pmatrix} \Omega_C & \mathcal{O}^\top \\ \mathcal{O} & G_f \end{pmatrix} \le \begin{pmatrix} \Omega_D & \mathcal{O}^\top \\ \mathcal{O} & G_f \end{pmatrix}\,.$$

Our constraint set is thus given by (44), (48) and

$$\begin{pmatrix} \Omega_D & \mathcal{O}^\top \\ \mathcal{O} & G_f \end{pmatrix} \le 0\,. \tag{49}$$

It is noteworthy that Ω_D is linear in P_1 and P_2^{-1}. However, there exist some bilinear matrix terms in P_1, P_2^{-1}, K and L all together. To deal with these and to obtain an LMI (in the matrix variables), we proceed to define

$$X_1 := P_1\,, \quad Y_1 := KP_1\,, \quad X_2 := P_2^{-1} \quad \text{and} \quad Y_2 := P_2^{-1}L\,,$$

So now all the inequalities are linear in the matrix arguments.

A natural objective for the controller is to minimize the volume of the ellipsoid, i.e., to minimize the trace of P, which amounts to minimizing the objective function $tr(X_1) + tr(X_2^{-1})$. This is still a nonlinear problem. By including the last linear constraint,

$$\begin{pmatrix} H & I \\ I & X_2 \end{pmatrix} > 0 \,, \tag{50}$$

we can now state the numerically tractable sub-optimal problem:

$$\text{minimize } tr(X_1) + tr(H) \,,$$

subject to the constraints $X_1, X_2, H, R_1, R_2 > 0$ and (44), (48), (49) and (50) with respect to the matrix variables $X_1, X_2, H, R_1, R_2, D_{11}, D_{13}, D_{33}, G_f, G_0 \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{n\times m}$, $Y_2 \in \mathbb{R}^{q\times n}$ and the scalar variables $a, b, \rho, \rho_q, \varepsilon$. The sub-optimal ellipsoid is defined by

$$P^{-1} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2^{-1} \end{pmatrix} .$$

The controller and observer gains can be obtained uniquely as $K = Y_1 X_1^{-1}$ and $L = X_2 Y_2$.

Notice that, still, this problem is bilinear in the matrix and scalar variables all together, so we propose the following algorithm:

```
fix ρ, ρ_q and ϵ
set a* to a very small value a_0
set b* to a very small value b_0
set T* to a very large value T_0
for j = 1 to m do
    for i = 1 to n do
        repeat
            try to solve semidefinite programming problem
            increase a* by STEP1
        until solution is feasible
        set T_i to trace(P^{-1}) evaluated in the solution
        if T_i < T* then
            set T* = T_i
        end if
        divide STEP1 by 2
    end for
    for i = 1 to n do
        repeat
            try to solve semidefinite programming problem
            increase b by STEP2
        until solution is feasible
        set T_i to trace(P^{-1}) evaluated in the solution
        if T_i < T* then
            set b* = b_i
        end if
        divide STEP2 by 2
    end for
end for
return a* and b*
```

(This algorithm can be readily implemented using off-the-shelf software and requires average programming skills only.)

5 Numerical Examples

The previous algorithm was implemented in Matlab to exemplify the applicability of our method. The objective was to design a robust controller based on the previously described method for two different systems. The first one is a two-dimensional nonlinear system, which dynamics are include a sign function and also has bounded state and output perturbations. While the second one is a four-dimensional linear system with bounded uncertainties and perturbations, its dynamics are modeled as a pair of double integrators.

The algorithm was implemented on a mid-end CPU. The time required to obtain a solution depends heavily on the selection of the starting values, if the solution exists but there is no information on the nature of the systems or an idea of the possible values of the solution, the average solving time will increase notably compared to the case when a set of feasible values is known. Most of the time the algorithm succeeded even when the starting values where selected far from a known solution. The algorithm was proposed with simplicity in mind. However, a faster and optimized version of this algorithm should be possible to device and implement.

5.1 *Example 1*

Consider the following discontinuous system:

$$\begin{aligned}
\dot{x}_1 &= \operatorname{sign}(x_2) + \upsilon_1 \\
\dot{x}_2 &= x_1 + 2u + \upsilon_2 \\
\bar{\bar{y}} &= x_1 + 2x_2 + \omega_y \ .
\end{aligned}$$

Let us assume that $|\upsilon_1|, |\upsilon_2| \leq 0.1$ and that $|\omega_y| \leq 0.2$. These bounds satisfy assumption (i) with $Q_x = Q_y = I_{2\times 2}$. Using the equivalent transformations discussed in Section 2, we can write the equivalent system (10) as

$$\begin{aligned}
\dot{x} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u + \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \\
\bar{\bar{y}} &= \begin{pmatrix} 1 & 2 \end{pmatrix} x + \omega_y \ .
\end{aligned}$$

By defining

$$Q_x := \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

it can be seen that

$$\|f(t,x) - Ax\|^2_{Q_x} = q_{11} \left(\operatorname{sign}(x_2) - x_2\right)^2 \leq q_{11} \left(x_2^2 + 1\right) \ .$$

Choosing $Q_x = Q = I_{2\times 2}$ and selecting $\delta = 1$, assumption (ii) is satisfied. Also, the resulting system is controllable and observable as needed in assumption (iii). The numerical treatment of the minimization problem was stated using the following parameters: the sample time interval is fixed at 0.01 seconds, so we can choose directly $h = 0.01$, the initial conditions for the dynamic system are $x_1(0) = x_2(0) = 10$ and the quantization constant selected was $c = 1$. Both, c and h constant, satisfy assumptions (iv) and (v). For the observer the initial conditions were chosen as the origin.

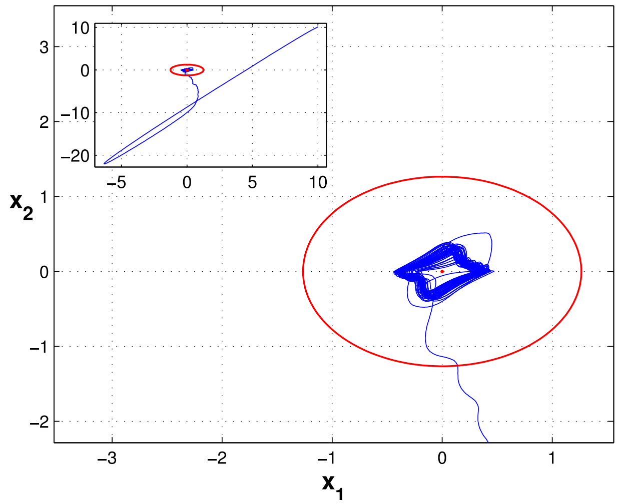

Figure 2. Estimated ellipsoid and system trajectories for Example 1.

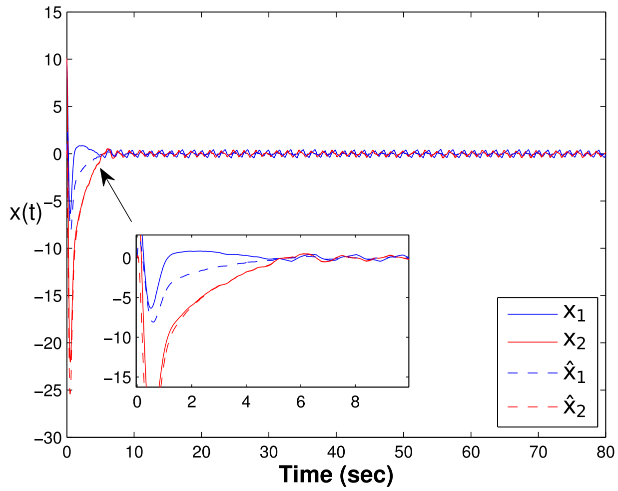

Figure 3. Actual and estimated states for Example 1.

Using the algorithm, the observer and the controller gains were obtained as

$$K = \begin{pmatrix}-16.7123 & 5.1423\end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix}1.9306 & 0.9996\end{pmatrix}^\top .$$

The simulated trajectories are shown in Figs. 2 and 3. The estimated ellipsoidal region is also shown in Fig. 2. Notice that the estimation is accurate enough since the ellipsoid encloses the trajectories tightly. An oscillatory effect can be appreciated. This is due to the fact that the trajectories are confined to a bounded region (because of (27)) while, due to uncertainty, the trajectories cannot converge to a fixed point. Fig. 3 shows how the estimated states converge to the actual ones. Finally, Fig. 4 shows a comparison between the control input u and the measurable output y. It is clear that this output is quantized. The effect of the quantization and the measurement noise can be appreciated.

5.2 *Example 2*

For the second example, an optical disk drive system is used. The dual-actuator disk drive system was modeled as a pair of double integrators (Phillips and Tomizuka 1995). The four states in this case were the relative position error and its derivative and the tracking error and its derivative.

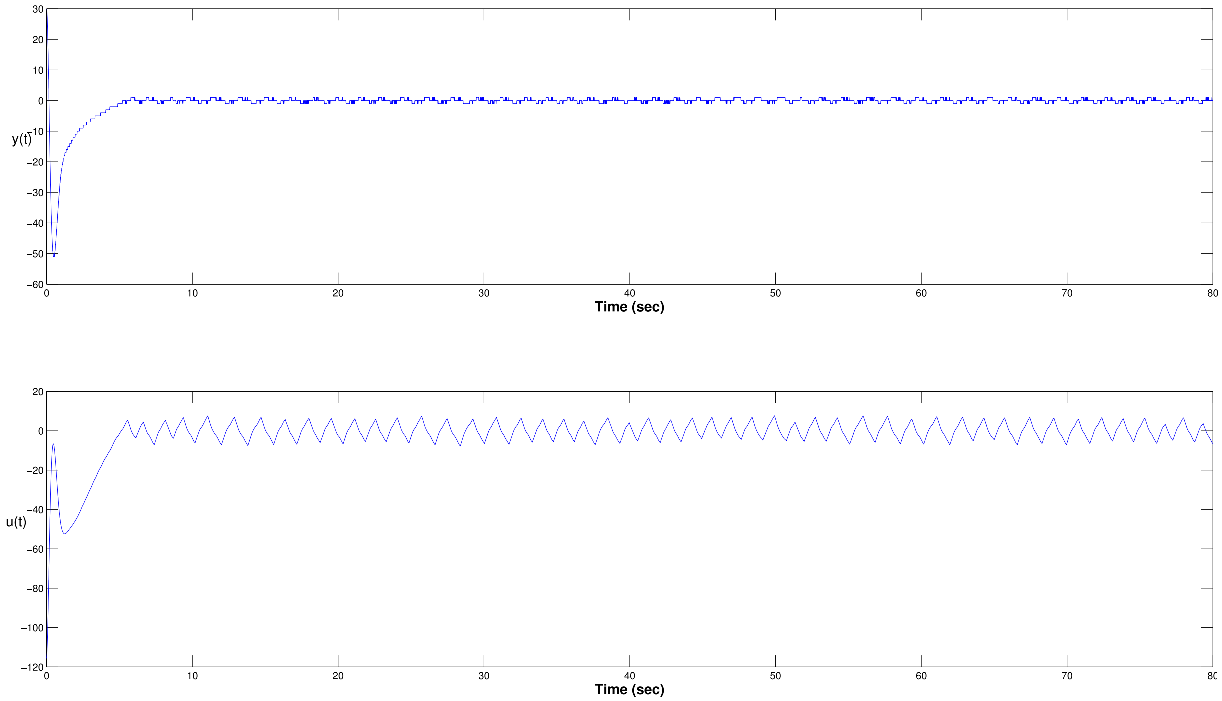

Figure 4. Input and output signals for Example 1.

The system is perturbed by $\|\omega_x\| \leq 0.1$ and $\|\omega_y\| \leq 0.2$

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 10 & 0 \\ 0 & 0 \\ -10 & -20 \end{pmatrix} u + \omega_x \\ \bar{\bar{y}} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x + \omega_y \; . \end{aligned} \tag{51}$$

The simulation was run using the same sample time $h = 0.01$, and $x_i(0) = 5$ with $i = 1, 2, 3, 4$ for initial conditions. Two different quantization constants were considered, the first one was $c = 1$ and the second was $c = 2$.

The observer and the controller gains were obtained as

$$K = \begin{pmatrix} -4.8202 & -3.6144 & -0.7786 & -1.5238 \\ 1.4946 & 1.1088 & 1.0644 & 1.1027 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 3.8594 & 3.7234 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 3.8594 & 3.7234 \end{pmatrix}^{\top} .$$

Fig. 5 shows the ellipsoid projection on the $x_1 - x_2$ plane with the respective trajectories when $c = 1$. The first three states and its estimations can be seen in Fig. 6. Both inputs and outputs are shown in Fig. 7. It is worth pointing out that the outputs in steady state only use the first level of quantization ($\pm c$).

The remaining figures were obtained using a quantization constant $c = 2$. Certain differences between the two choices of quantization constants are worth commenting. First, the estimated ellipsoidal region and the actual convergence region are obviously larger due to a bigger quantization constant effect, as can be seen in Fig. 8. Also, the estimation error is larger (Fig. 9). Finally, in Fig. 10, although the steady state output still only uses the first level of quantization $\pm c$, it is twice as large as in the first case.

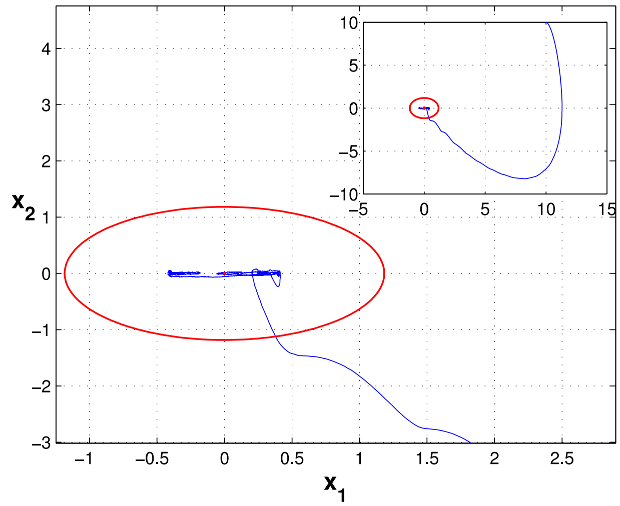

Figure 5. Ellipsoidal region and system trajectories for Example 2 with $c = 1$.

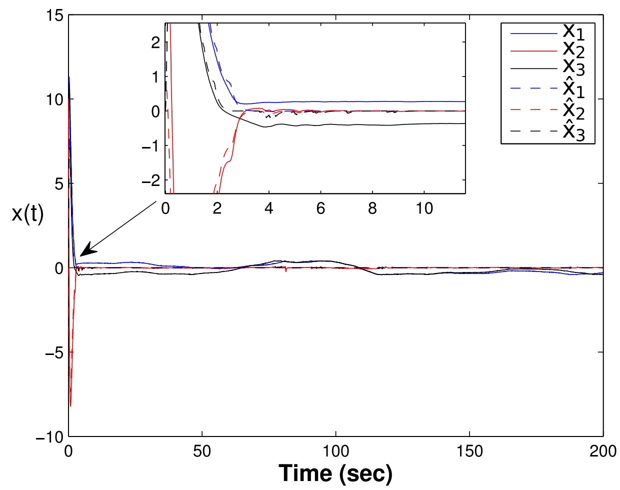

Figure 6. First two actual and estimated states for Example 2 with $c = 1$.

6 Conclusions

In this paper, a new analytical and numerical methodology for robust control design associated with nonlinear perturbed systems was developed. We also considered sampled-data and quantization at the output of these systems. In the model we used, the quantization error was bounded. The control design strategy proposed in this paper is an extension of the attractive ellipsoid method. This approach produces a control law such that the existence and an actual characterization of a minimal-size attractive ellipsoid for the closed-loop system can be guaranteed. The computational implementation of the aforementioned method led us to a complex nonlinear minimization problem with nonlinear matrix constraints. In this contribution we proposed an effective relaxation from this initial optimization problem to a semidefinite programming problem (linear in the matrix variables). The final product was an attractive ellipsoidal region with a minimal 'size'.

Finally, this approach can be generalized to systems with delays and networked control systems with relative ease. It also seems possible to apply the presented control design techniques in combination with some nonlinear feedback control strategies.

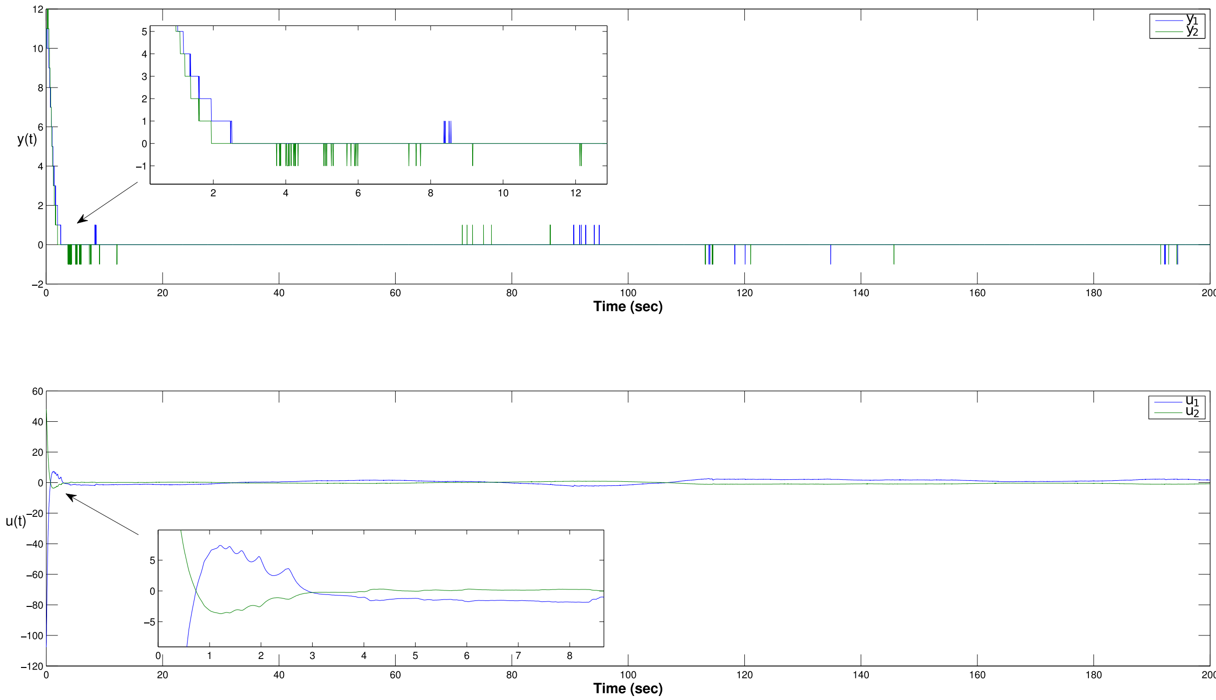

Figure 7. Input and output signals for Example 2 with $c = 1$.

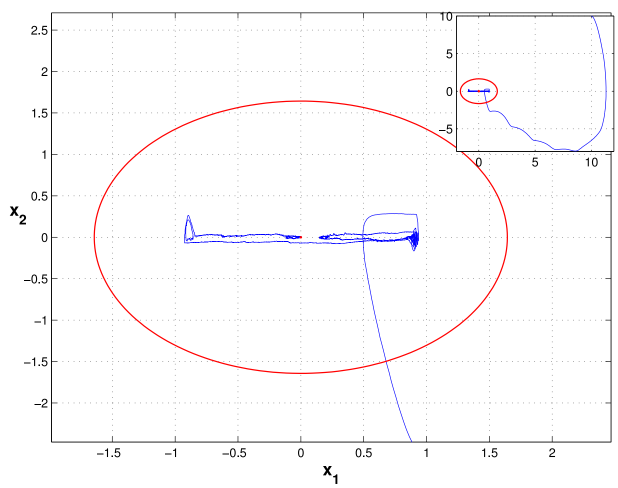

Figure 8. Ellipsoid and system trajectories for Example 2 with $c = 2$.

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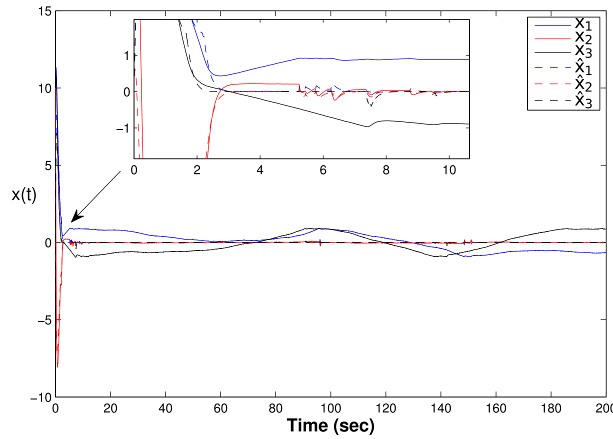

Figure 9. Actual states and estimated states for Example 2 with $c = 2$.

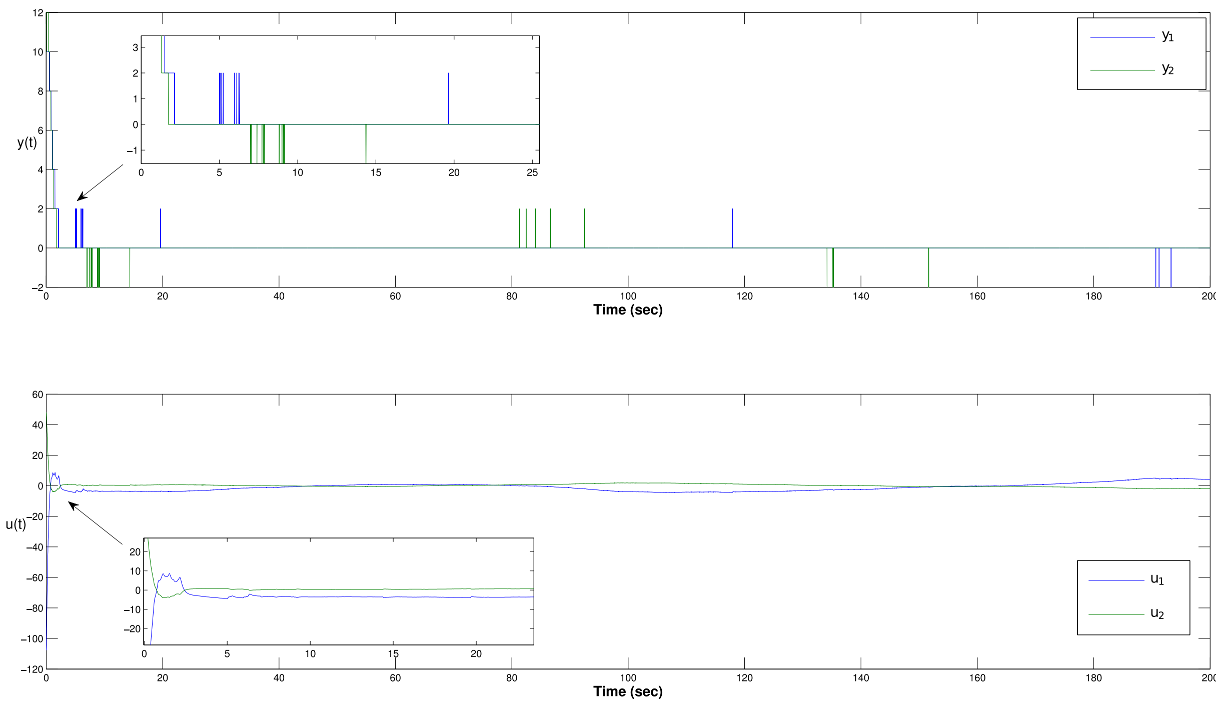

Figure 10. Input and output signals for Example 2 with $c = 2$.

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