Preprints of the 20th World Congress
The International Federation of Automatic Control
Toulouse, France, July 9-14, 2017

Set-valued discrete-time sliding-mode control of uncertain linear systems

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Abstract: This paper focuses on the discrete-time sliding-mode control problem, that is, given an uncertain linear system under the effect of external matched perturbations, to design a set-valued control law that achieves the robust regulation of the plant and at the same time reduces substantially the chattering effect in both the input and the sliding variables. The cornerstone is the implicit Euler discretization technique together with a differential inclusion framework which allow us to make a suitable selection of the control values that will compensate for the disturbances. Numerical examples confirm the effectiveness of the proposed methodology.

Keywords: Differential inclusions, robust control, sliding-mode control, discrete-time systems, linear uncertain systems, Lyapunov stability.

1. INTRODUCTION

There exists an extensive literature on discrete-time sliding-mode control which, at this point, can be divided into two groups. In one group we have the works that rely on discontinuous control actions, as for example Bartoswewicz [1998], Galias and Yu [2007], Gao et al. [1995], Kaynak and Denker [1993], Spurgeon [1991]. The sliding-mode control law is discretized using an explicit Euler technique and is limited by the condition that the ideal sliding-mode is never reached, leading to concepts such as *quasi sliding*, a term that refers to the fact that the system trajectories will ultimately belong to a boundary layer of the sliding manifold even in the absence of disturbances. The main problem with the discontinuous control approach is the susceptibility to the appearance of chattering. Indeed, at a point of discontinuity the control law cannot take values lying between its different limits, so a high frequency switching becomes necessary for maintaining the system in the sliding phase [Utkin 1992]. It is thus not surprising to see considerably high levels of chattering in these schemes.

The central idea among the second group of controllers is that, similar to the differential inclusions described in the work of Filippov and Arscott [1988], the discrete-time system should be governed by a difference inclusion, not a difference equation [Acary and Brogliato 2010, Acary et al. 2012, Huber et al. 2016b,c]. These works are based on the use of set-valued control laws for which a selection compensating the matched disturbances is possible.

In practical terms, the difference between both approaches lays on the type of discretization used. Whereas the former group employs an explicit Euler discretization, the second one employs an implicit one. In the latter case, the resulting controller turns out to be Lipschitz continuous, which results in a substantial reduction of chattering, Huber et al. [2016b,c], Wang et al. [2015].

The present work falls into the second group and is dedicated to the study of uncertain systems, i.e., we consider the case where the system matrices are uncertain. The class of uncertainty considered is large enough to embrace parametric uncertainty as well as nonlinear unmodeled dynamics and external perturbations. It is also worth remarking that the works by Acary and Brogliato [2010], Acary et al. [2012], Huber et al. [2016b,c] do not consider uncertainty in the system parameters.

The paper is organized as follows: Section 2 sets the notation and recalls some concepts from convex analysis. Section 3 presents, very shortly, the design of continuous-time sliding-mode controllers for systems with model uncertainty and external matched disturbances. Section 4 constitutes the main body of the paper. Here, the methodology design of discrete-time sliding mode controllers is presented together with well-posedness and stability results. Finally, Section 5 shows the effectiveness of the proposed controller and its superior performance when compared against explicit Euler discretization techniques.

2. PRELIMINARIES AND NOTATION

Let $\mathbb{R}^n$ be a n-dimensional linear space, given with the classical Euclidean inner product denoted as $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$.

Definition 1. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function. The subdifferential of f at $x \in \operatorname{Dom} f$ is given by the set

$$\partial f(x) := \{\zeta \in \mathbb{R}^n \mid \langle \zeta, \eta - x\rangle \leq f(\eta) - f(x), \text{ for all } \eta \in \mathbb{R}^n\}.$$

Definition 2. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function. The proximal map $\operatorname{Prox}_f : \mathbb{R}^n \to \mathbb{R}^n$ is the unique minimizer of $f(w) + \frac{1}{2}\|x - w\|^2$, that is,

$$f(\operatorname{Prox}_f(x)) + \frac{1}{2}\|x - \operatorname{Prox}_f(x)\|^2 = \min_{w\in\mathbb{R}^n}\left\{f(w) + \frac{1}{2}\|x - w\|^2\right\}.$$

Note that for $\Psi_{\mathcal{C}}$, the indicator function of the set $\mathcal{C}$, the proximal map corresponds to the well-know projection operator, see Hiriart-Urruty and Lemaréchal [1993]. The following result, extracted from [Bauschke and Combettes 2011, Proposition 12.26], establishes a link between the two former concepts.

Proposition 3. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function. Then, $p = \operatorname{Prox}_f(x)$ if, and only if, $x - p \in \partial f(p)$.

Remark 4. It follows from Proposition 3 that the map $(I + \alpha\partial f)^{-1}$ is singled valued. More specifically, $\operatorname{Prox}_{\alpha f} = (I + \alpha\partial f)^{-1}$. Indeed, assume that y_i, $i = 1, 2$ are such that $y_i \in (I + \alpha\partial f)^{-1}(x)$. We have, $x - y_i \in \alpha\partial f(y_i)$, $i = 1, 2$. Hence, Proposition 3 gives $y_1 = y_2 = \operatorname{Prox}_{\alpha f}(x)$.

In the upcoming discussion the conjugate function $f^\star$ of a proper function will play an important role. Here we recall its definition.

Definition 5. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. The conjugate of f is,

$$f^\star(z) := \sup_{x\in\mathbb{R}^n}\{\langle z, x\rangle - f(x)\}.$$

Theorem 6. (Moreau's decomposition). Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, lower semicontinuous function and let $\alpha \in \mathbb{R}$ be strictly positive. Then, for any $x \in \mathbb{R}^n$, the following identity holds:

$$x = \operatorname{Prox}_{\alpha f}(x) + \alpha \operatorname{Prox}_{f^\star/\alpha}(x/\alpha).$$

Along this paper we denote the identity matrix in $\mathbb{R}^{n\times n}$ as I_n. The set $\mathcal{B}_n := \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ represents the unit open ball with center at the origin in $\mathbb{R}^n$ with the Euclidean norm. The interior, closure, and boundary of a set $\mathcal{S} \subset \mathbb{R}^n$ are denoted as $\operatorname{int}\mathcal{S}$, $\operatorname{cl}\mathcal{S}$, and $\operatorname{bd}\mathcal{S}$ respectively.

3. A QUICK REVIEW OF CONTINUOUS-TIME SLIDING-MODE CONTROL

We begin with a quick look at the continuous-time sliding-mode control problem. To this end, let us consider the uncertain plant

$$\dot{x} = (A+\Delta_A(t, x))x(t)+B(u(t)+w(t, x)),\ x(0) = x_0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ represents the state of the system, $u(t) \in \mathbb{R}$ is the scalar control input and $w(t, x) \in \mathbb{R}$ accounts for external disturbances and unmodeled dynamics. The matrices A, Δ_A and B are of the appropriate dimensions. It is assumed that the matrix $\Delta_A(t, x)$ is unknown but is uniformly upper-bounded by

$$\Delta_A(t, x)\Lambda\Delta_A^\top(t, x) < I_n \quad (2)$$

with $\Lambda = \Lambda^\top > 0$ a known matrix. We also make the following standard assumptions.

Assumption 7. The pair (A, B) is stabilizable.

Assumption 8. The disturbance term $w(t, x)$ is uniformly bounded in the $\mathcal{L}^\infty$ sense, that is, there exists $W > 0$ such that $\sup_{t\geq0}\|w(t, x)\| \leq W < +\infty$.

The first step in the design of sliding-mode controllers consists in fixing the sliding surface $\sigma(x) = 0$ in such a way that the behaviour of the system constrained to the sliding surface satisfies the performance requirements. The second step consists in the design of the control law that will steer the state towards the sliding surface and will maintain it there, even in the presence of model uncertainties and external disturbances. An assumption concerning the sliding surface is the following.

Assumption 9. The matrix $C \in \mathbb{R}^{1\times n}$ is such that the product CB is nonsingular.

The previous assumption ensures the uniqueness of the equivalent control (see, e.g. Utkin et al. [2009]). Namely, by considering the sliding surface as the hyperplane $\sigma = Cx$, the equivalent control is computed from the invariance condition $\dot\sigma = 0$ as

$$C(Ax^{\mathrm{eq}} + B(u^{\mathrm{eq}} + w)) + \Delta_A(t, x^{\mathrm{eq}})x^{\mathrm{eq}}) = 0 \Rightarrow$$
$$u^{\mathrm{eq}} = -(CB)^{-1}C\left(Ax^{\mathrm{eq}} + \Delta_A(t, x^{\mathrm{eq}})x^{\mathrm{eq}}\right) - w.$$

Substitution of the equivalent control into (1) leads to the expression of the dynamics in sliding phase,

$$\dot{x}^{\mathrm{eq}} = \left(I_n - B(CB)^{-1}C\right)(A + \Delta_A(t, x^{\mathrm{eq}}))x^{\mathrm{eq}}, \quad (3)$$

from which it becomes clear that the matrix characterizing the sliding hyperplane plays a role in the reduced system dynamics. There exists many methods for the design of the sliding surface, e.g., LQR design [Utkin 1992, Chapter 9], eigenvalue placement [Utkin et al. 2009, Chapter 7], $\mathcal{H}_\infty$ control [Castaños and Fridman 2006], linear matrix inequalities [Polyakov and Poznyak 2011], see also [Shtessel et al. 2014, Section 2.4.2], among others. Here we relegate the design of the sliding surface in continuous time to the background and focus instead on the discrete-time setting. As mentioned above, the second step consists in designing the set-valued control law that will bring the system into the sliding regime. The design procedure is divided into two steps. Namely, first we compute a control law for the nominal version of (1) (i.e., $\Delta_A \equiv 0$ and $w \equiv 0$) and then the set-valued controller that will provide the necessary robustness. Thus, the control law is set as

$$u = u^{\mathrm{nom}} - \gamma_1(x)\operatorname{Sgn}(\sigma), \quad (4)$$

where u^{nom} is a control input for the nominal system and $\gamma_1 : \mathbb{R}^n \to \mathbb{R}_+$ is a control gain. It is worth remarking that the trajectories of the closed-loop (1), (4) will reach the sliding surface $\sigma = Cx$ in finite time, from where the reduced system will go asymptotically to the origin whenever the matrix C is well-designed.

In conclusion, the common methodology design for sliding-mode controllers in continuous time relies on the appropriate design of the matrix C that will make the reduced system asymptotically stable, whereas the set-valued controller will compensate for all the matched disturbances.

In the upcoming section we show in detail the analogous methodology for the implementation of discrete-time sliding mode controllers.

4. DISCRETE-TIME SET-VALUED SLIDING-MODE CONTROL

The first step consists in obtaining a discrete-time model of (1) by using Euler's method, i.e., we take a constant sampling time $h = t_{k+1} - t_k > 0$ for all $k \geq 0$ and we obtain

$$x_{k+1} = (I_n + hA)x_k + hB(u_k + w(k, x_k)) + h\Delta_A(k, x_k)x_k. \tag{5}$$

In this work we do not study in detail the effect of the discretization scheme applied to the plant (1). However, we do make emphasis in the discretization of the set-valued component of the controller, that is, the discretization of the signum multifunction. Along all this section we also consider Assumptions 7 through 8, together with the bound on $\Delta_A(t, x)$ expressed by (2). Henceforth, because of space limitations, we will omit the arguments of the uncertainties/disturbances $\Delta_A(k, x_k)$ and $w(k, x_k)$. The proof of the incoming proposition can be found in Miranda-Villatoro et al. [2016].

Proposition 10. Assumption 7 implies that, for some $a > 0$ such that $0 < 2ha < 1$, there exists a symmetric positive definite matrix $X \in \mathbb{R}^{n\times n}$ satisfying the matrix inequality:

$$B_\perp^\top \left(AX + XA^\top + 2aX\right) B_\perp + hB_\perp^\top \left(XA^\top B_\perp \left(B_\perp^\top X B_\perp\right)^{-1} B_\perp^\top AX\right) B_\perp < 0, \tag{6}$$

where $B_\perp \in \mathbb{R}^{n\times(n-1)}$ is a full rank orthogonal complement of the matrix B, that is, $B^\top B_\perp = 0$.

The following result establishes a bound for Δ_A that will be useful in the forthcoming sections.

Proposition 11. Let $X = X^\top > 0$ be such that

$$X - I_n > 0. \tag{7}$$

Then,

$$\Lambda^{-1} - \Delta_A^\top B_\perp (B_\perp^\top X B_\perp)^{-1} B_\perp^\top \Delta_A > 0. \tag{8}$$

Proof. The proof follows from Assumption 2 and the Schur's complement formula.

In the sequel we will assume that X satisfies (6) together with (7) and consequently (8) also holds.

4.1 Design of the sliding surface

It is common to consider the sliding surface for the discrete-time design as the same as in its continuous-time counterpart. However, as is pointed out in Spurgeon [1991], it is more suitable to make a *redesign* of such surface which guarantees, in the presence of uncertainties, the desired performance of the closed-loop discrete-time system. Thus, letting $\left\{x_k \in \mathbb{R}^n | Sx_k = 0, S \in \mathbb{R}^{1\times n}\right\}$ be such linear surface, we make the following assumption.

Assumption 12. The product SB is nonsingular.

Analogous to the continuous-time context, we start computing the equivalent control in order to see how the disturbance affects the sliding regime. In the discrete-time case, the necessary sliding condition $\dot{\sigma} = 0$ is transformed into the fixed-point condition $\sigma_{k+1} = \sigma_k$, from which we obtain the equivalent control as

$$u_k^{\text{eq}} = \frac{1}{h}(SB)^{-1}\left(\sigma_k - S(I_n + hA)x_k - hS\Delta_A x_k\right) - w_k \tag{9}$$

Hence, the equivalent closed-loop dynamics in *ideal* sliding motion results in:

$$x_{k+1}^{\text{eq}} = \left(I_n - B(SB)^{-1}S\right)(I_n + hA + h\Delta_A)x_k^{\text{eq}} + B(SB)^{-1}\sigma_k. \tag{10}$$

From (10) it becomes clear that the structure of the sliding surface will be similar to that in the continuous-time framework, c.f. (3). Throughout this section we set

$$S = (B^\top X^{-1} B)^{-1} B^\top X^{-1}, \tag{11}$$

where $X = X^\top > 0$ is an $n \times n$ matrix that satisfies (6) and (7). See Miranda-Villatoro et al. [2016] for a detailed account about this selection. It is noteworthy that the sliding hyperplane depends implicitly on the sampling time $h > 0$, thus different samplings will lead to different sliding surfaces, a property that is not obtained by direct discretization of the continuous-time sliding surface $\sigma = Cx$.

4.2 Controller design

In this subsection we formulate the discrete version of the two-step design methodology mentioned in the previous section. Namely, we compute a control law of the form $u_k = u_k^{\text{nom}} + u_k^{\text{sv}}$, where u_k^{nom} is a control law for the nominal plant, i.e., (5) with $\Delta_A(k, x_k) \equiv w(k, x_k) \equiv 0$, whereas u_k^{sv}, designed as a set-valued map, is responsible for compensating the matched disturbances. We will show that the implicit Euler discretization applied to u_k^{sv} inherits the robustness of the signum multifunction.

The first step consists in computing the nominal control using the fixed-point condition $\sigma_{k+1} = \sigma_k$, which leads to

$$u_k^{\text{nom}} = \frac{1}{h}(SB)^{-1}\left(\sigma_k - S(I_n + hA)x_k\right). \tag{12}$$

Substitution of (12) into the discrete-time dynamics (5) yields

$$x_{k+1} = \left(I_n - B(SB)^{-1}S\right)(I_n + hA)x_k + B(SB)^{-1}\sigma_k + hB(u_k^{\text{sv}} + w_k) + h\Delta_A x_k. \tag{13}$$

The next step consists of the decoupling of the matched and mismatched parts of the disturbances. To this end, consider the coordinate transformation $z_k = Tx_k$ with T given as

$$T = \begin{bmatrix} B_\perp^\top \\ (B^\top X^{-1} B)^{-1} B^\top X^{-1} \end{bmatrix}. \tag{14}$$

After simple computations we obtain a closed-loop system in the so-called regular form,

$$z_{k+1}^1 = B_\perp^\top (I_n + hA + h\Delta_A) X B_\perp \left(B_\perp^\top X B_\perp\right)^{-1} z_k^1 + B_\perp^\top (I_n + hA + h\Delta_A) B\sigma_k \tag{15a}$$

$$\sigma_{k+1} = \sigma_k + h(u_k^{\text{sv}} + w_k + \eta_k^{\text{m}}), \tag{15b}$$

where the term η_k^{m} refers to the matched part of the disturbance $\Delta_A(k, x_k)x_k$, i.e.,

$$\eta_k^{\text{m}} = S\Delta_A T^{-1} z_k = (B^\top X^{-1} B)^{-1} B^\top X^{-1} \Delta_A T^{-1} z_k.$$

It is clear that the disturbance term η_k^{m} satisfies a linear growth condition (since the term $\Delta_A(t,x)$ is uniformly bounded, c.f. (2)). Thus, the following holds.

Proposition 13. The disturbance term η_k^{m} satisfies the linear growth condition $|\eta_k^{\mathrm{m}}| \leq \sqrt{\bar{\kappa}}\|z_k\|$ for some $\bar{\kappa} > 0$ and finite.

The set-valued controller We continue with the design of the controller's multivalued part. This is where we depart from the explicit Euler discretization scheme. Because of the implicit discretization method employed, it is possible to make a selection for the values of the controller that will compensate the disturbances that affect the closed-loop system. Now we introduce the implementable controller u_k^{sv} using the implicit discretization approach studied in Acary and Brogliato [2010], Acary et al. [2012], Huber et al. [2016a] and tested experimentally in Huber et al. [2016b,c], Wang et al. [2015]. The approach has proved to be very efficient in terms of the chattering alleviation. Roughly speaking, we consider a discrete-time scheme by creating a virtual nominal system from where the selection process is achieved. Next, the controller computed from the virtual nominal system is applied to the original discrete-time plant. Formally, we consider the extended system

$$\begin{aligned} z_{k+1}^1 &= B_\perp^\top(I_n + hA + h\Delta_A)XB_\perp\left(B_\perp^\top XB_\perp\right)^{-1} z_k^1 \\ &\quad + B_\perp^\top(I_n + hA + h\Delta_A)B\sigma_k && \text{(16a)} \\ \sigma_{k+1} &= \tilde{\sigma}_{k+1} + h(w_k + \eta_k^{\mathrm{m}}) && \text{(16b)} \\ \tilde{\sigma}_{k+1} &= \sigma_k + hu_k^{\mathrm{sv}} && \text{(16c)} \\ -u_k^{\mathrm{sv}} &\in \gamma_1\tilde{\sigma}_{k+1} + \gamma_2 \operatorname{Sgn}(\tilde{\sigma}_{k+1}), && \text{(16d)} \end{aligned}$$

where $\gamma_i \in \mathbb{R}$, $i = 1, 2$, are positive gains specified below. System (16) represents the implementable discrete-time dynamics. The variable $\tilde{\sigma}_{k+1}$ may be seen as the state of a nominal, undisturbed system, or as a dummy variable used to compute u_k^{sv}. In this approach, the selection of the values of the controller is made by using the virtual undisturbed system (16c)-(16d), where the perturbation term is implicitly taken into account through the use of the real state σ_k in the computation of (16c). Considering the subsystem (16c)-(16d), we have

$$\begin{aligned} &\sigma_k - \tilde{\sigma}_{k+1} \in h\gamma_1\tilde{\sigma}_{k+1} + h\gamma_2 \operatorname{Sgn}(\tilde{\sigma}_{k+1}) \\ \Leftrightarrow\ &\frac{\sigma_k}{1+h\gamma_1} \in \left(I + \frac{h\gamma_2}{1+h\gamma_1}\operatorname{Sgn}\right)(\tilde{\sigma}_{k+1}) && \text{(17)} \\ \Leftrightarrow\ &\tilde{\sigma}_{k+1} = \operatorname{Prox}_{\frac{h\gamma_2}{1+h\gamma_1}f}\left(\frac{\sigma_k}{1+h\gamma_1}\right). && \text{(18)} \end{aligned}$$

It follows from (16c) that the input selection applied to the system is explicitly given by

$$\begin{aligned} u_k^{\mathrm{sv}} &= -\frac{1}{h}\left(\sigma_k - \operatorname{Prox}_{\frac{h\gamma_2}{1+h\gamma_1}f}\left(\frac{\sigma_k}{1+h\gamma_1}\right)\right) \\ &= -\frac{1}{1+h\gamma_1}\left(\gamma_1\sigma_k + \gamma_2 \operatorname{Proj}_{[-1,1]}\left(\frac{\sigma_k}{h\gamma_2}\right)\right), && \text{(19)} \end{aligned}$$

where, in the last inequality, we have used Theorem 6 once again. Equation (19) shows the non-anticipation and the uniqueness of the control law (16d). Hence, the discrete-time closed-loop subsystem (16b)-(16d) is equivalent to

$$\begin{aligned} \sigma_{k+1} &= \tilde{\sigma}_{k+1} + h(w_k + \eta_k^{\mathrm{m}}) \\ \tilde{\sigma}_{k+1} &= \operatorname{Prox}_{\frac{h\gamma_2}{1+h\gamma_1}f}\left(\frac{\sigma_k}{1+h\gamma_1}\right). \end{aligned} \qquad (20)$$

In this context the variable $\tilde{\sigma}_k$ is called the discrete sliding variable and, when $\tilde{\sigma}_{k^*+n} = 0$ for all $n \geq 1$ and some $k^* < +\infty$, we say that the system is in the *discrete-time sliding phase* [Huber et al. 2016a].

Remark 14. The concept of discrete-time sliding phase used in this work stands in contrast with the concept of ideal sliding phase. The ideal sliding phase condition depends on the original variable σ_k and it is never reached in a real application, whereas the discrete-time sliding phase condition, defined for $\tilde{\sigma}_k$, is attained after a finite number of steps (see Corollary 19).

Remark 15. Even though it may seem cumbersome at first sight, the implicit discretization provides an efficient methodology to regularize the signum multifunction. Note that the controller (19) is the sum of two functions, a linear one and a saturation (see Figure 1). Note also that the regularization is not arbitrary, as it depends on the sampling time $h > 0$ and has interesting features. Namely, it yields the discrete-time sliding regime after a finite number of steps. Moreover, when the discrete-time sliding phase is reached, the control input becomes independent of the gains γ_1 and γ_2 (this is in perfect analogy with the continuous-time scenario) and it is indirectly estimating $w_k + \eta_k^{\mathrm{m}}$ with a one-step delay (see Lemma 16, Corollary 17, Theorem 18 and Corollary 19 below).

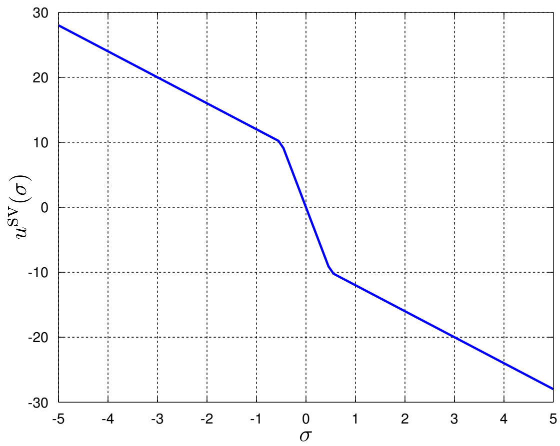

Fig. 1. Control law (19) obtained from the use of the implicit Euler discretization algorithm with $h = 50$ ms, $\gamma_1 = 5$, $\gamma_2 = 10$ and σ ranging from -5 up to 5.

4.3 Stability of the closed-loop system

In this section we prove the stability of the entire closed-loop system (16). With this end in mind, we start with a result characterizing the discrete-time sliding-phase condition ($\tilde{\sigma}_{k+1} = \tilde{\sigma}_k = 0$ for all $k \geq k^*$ and some $0 < k^* < +\infty$) in terms of σ_k.

Lemma 16. Consider the subsystem (20). The following two statements are equivalent:

1) $\sigma_k \in h\gamma_2 \operatorname{Sgn}(0)$ for some $k \in \mathbb{N}$.
2) $\tilde{\sigma}_{k+1} = 0$.

In addition, if for some $k^* \in \mathbb{N}$, $\tilde{\sigma}_{k^*+1} = 0$, then $\tilde{\sigma}_{k^*+p} = 0$ for all $p \geq 1$, whenever $w_k + \eta_k^{\mathrm{m}} \in \gamma_2 \operatorname{Sgn}(0)$ for all $k \geq k^*$.

Proof. It follows from (20) and the characterization of the proximal map given in Proposition 3.

The following result supports the use of the scheme proposed in Acary and Brogliato [2010], Acary et al. [2012].

Corollary 17. Let the matched disturbance $w_k + \eta_k^{\mathrm{m}} \in \gamma_2 \operatorname{Sgn}(0)$ for all $k \geq k^*$ and some $0 < k^* < +\infty$. Then, in the discrete-time sliding phase the control input u_k^{sv} satisfies

$$u_k^{\mathrm{sv}} = -w_{k-1} - \eta_{k-1}^{\mathrm{m}}.$$

In words, the input obtained from the implicit scheme (16) compensates for the disturbance with a delay of one step once the discrete-time sliding phase has been reached. Moreover, during such phase u_k^{sv} is independent of the gains γ_i, $i = 1, 2$, a crucial fact that was experimentally verified in Huber et al. [2016b,c]. This last property turns out to be fundamental in the application of the control scheme (16) since it helps to drastically reduce the chattering effect of the closed-loop system.

Practical stability of the difference equation (16) is proved in the following theorem whose complete proof can be consulted in Miranda-Villatoro et al. [2016].

Theorem 18. Let Assumptions 7-12 hold and consider the closed-loop system (16), were $X = X^\top > 0$ and $\gamma_1 > 0$ are such that

$$\bar{Q} := \begin{bmatrix} \bar{Q}_{11} & -\frac{1}{2}B_\perp^\top AB - \bar{Q}_{12}^\top \\ -\frac{1}{2}B^\top A^\top B_\perp - \bar{Q}_{12} & \bar{Q}_{22} \end{bmatrix} > 0, \quad (21)$$

where

$$\bar{Q}_{11} := B_\perp^\top \Big(aX - I_n - \frac{1}{2}X\Lambda^{-1}X - h\left(2X\Lambda^{-1}X + XA^\top B_\perp G^{-1} B_\perp^\top AX\right) \Big) B_\perp,$$

$\bar{Q}_{12} := \frac{h}{2} B^\top A^\top B_\perp G^{-1} B_\perp^\top AXB_\perp$, and

$$\bar{Q}_{22} := \gamma_1 - \frac{1}{2}B^\top \Lambda^{-1} B - hB^\top \left(2\Lambda^{-1} + \frac{3}{2}A^\top B_\perp G^{-1} B_\perp^\top A\right) B.$$

hold, where $G = B_\perp^\top X B_\perp$. In addition, let $L_c \subset \mathbb{R}^n$ be the compact set

$$L_c := \left\{ \begin{bmatrix} z^1 \\ \sigma \end{bmatrix} \in \mathbb{R}^n \;\middle|\; \frac{1}{2} z^{1\top} \left(B_\perp^\top X B_\perp\right)^{-1} z^1 + \frac{1}{2}\sigma^2 \leq c^2 \right\}. \quad (22)$$

For any initial condition $z_0 = \begin{bmatrix} z_0^{1\top} & \sigma_0 \end{bmatrix}^\top$ choose $c > 0$ such that $z_0 \in L_c$. Choose any neighborhood of the origin. Then, there exists $h > 0$ small enough and fixed such that, for all $\gamma_2 > 0$ satisfying

$$\gamma_2 \geq \beta + W + (\sqrt{\bar{\kappa}} + 2h\gamma_1^2)\bar{z} \quad (23)$$

with $\bar{z} := \max\{\|z\|, z \in L_c\}$, the state of the discrete-time closed-loop system (16a)-(16d) will be ultimately contained in such neighborhood. In other words, the system is semi-globally practically asymptotically stable.

From Theorem 18 we conclude that the trajectories of the closed-loop system (16) are uniformly bounded (since they belong to the compact set L_c). From this property we derive the following result.

Corollary 19. Let all conditions and assumptions of Theorem 18 hold. Let also the gain $\gamma_2 > 0$ satisfy

$$\gamma_2 \geq \beta + (1+\alpha)(r + W + \sqrt{\bar{\kappa}}\bar{z}) + \max\left\{ 2h\gamma_1\bar{z}, \frac{(W + \sqrt{\bar{\kappa}}\bar{z})^2}{r} \right\} \quad (24)$$

for some constants $\beta, r > 0$. Then, there exists $k_0 = k_0(\alpha, r) > 0$, which is finite and such that $\tilde{\sigma}_{k_0} = 0$. Moreover, $\tilde{\sigma}_k = 0$ for all $k \geq k_0$, that is, the discrete-time sliding phase is reached after a finite number of steps.

In conclusion, we have shown how the implicit discretization approach can deal with uncertainties in the system in the same way that a continuous-time sliding-mode controller does. Moreover, the resulting discrete-time closed-loop system inherits the robustness property of the continuous-time setting and, as will be revealed in the upcoming section, the implicit approach reduces substantially the chattering in both the input and sliding variables when compared with the explicit discretization approach.

5. NUMERICAL EXAMPLE

This section is devoted to show the performance obtained when the control law described in the previous section is implemented. Let us start considering the benchmark system

$$\begin{aligned} \dot{x}_1 &= bx_2 \\ \dot{x}_2 &= u + w, \end{aligned} \quad (25)$$

where $x \in \mathbb{R}^2$ is the state of the system, the term $w \in \mathbb{R}$ accounts for external disturbances and the parameter $b \in \mathbb{R}$ is assumed constant but unknown.

Following the design methodology exposed in the previous section, we set the sampling time $h = 10$ ms and, for simulation purposes, we set $x(0) = [-3, 3]^\top$, $b = 1 + \delta_b$ with $\delta_b = 0.5$ and $w(t) = 2\sin(t)\sin(\pi t)$. Using the software tool cvx [Grant and Boyd 2014] together with the solver SeDuMi [Sturm 1999] we solve (6), (7) and (21). We obtain

$$\gamma_1 = 4.7890, \; X = \begin{bmatrix} 2.0546 & -4.2843 \\ -4.2843 & 21.1234 \end{bmatrix}, \; S = [2.0852 \;\; 1].$$

Now, setting $\gamma_2 = 10$, we apply the discretized control input given by (19) to the continuous.-time plant (25) by using a zero-order-hold mechanism for the coupling between the continuous-time and discrete-time signals. The results are depicted in Figure 2 (the sampled values u_k are linearly interpolated in the plot).

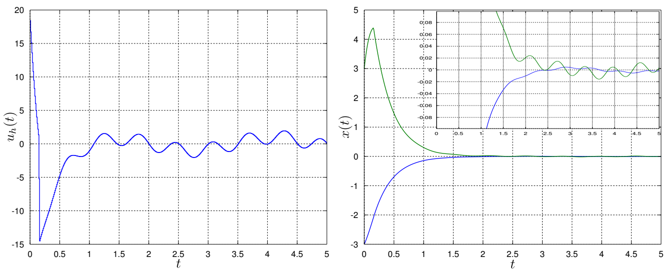

Fig. 2. Time trajectory of the piecewise linear control input and the state of the closed-loop system (25), (19) with sampling time $h = 10$ ms and gains $\gamma_1 = 4.789$, $\gamma_2 = 10$.

In order to compare the results obtained, we also consider the discrete-time control law obtained form the application of the explicit Euler discretization. Namely, we consider the control law

$$u_k^* = -\gamma_1 \sigma_k - \gamma_2 \frac{\sigma_k}{|\sigma_k| + 0.001}. \tag{26}$$

It is noteworthy that in this case we replace the set-valued signum map with a single-valued function that is a regularization of the former. The simulation results, depicted in Figure 3, reveal that the controller proposed in this work has a superior performance than the explicit discretization with an arbitrary regularization.

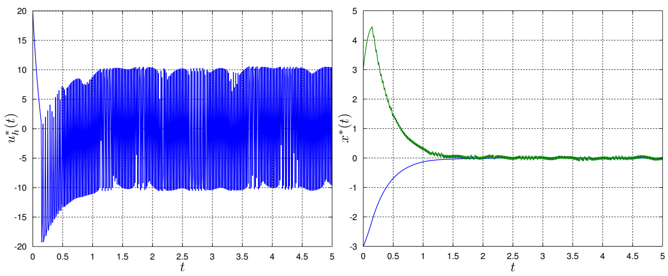

Fig. 3. Time trajectories of the closed-loop system (25), (26) with sampling time $h = 10$ ms and gains $\gamma_1 = 4.789$, $\gamma_2 = 10$.

6. CONCLUDING REMARKS

We presented a methodology for designing discrete-time sliding-mode controllers with complex uncertainty in the system. We dealt with parametric and matched external disturbances. It was shown that the use of the implicit discretization for the set-valued part of the controller is well-posed and enables us to uniquely choose a value from the image that will compensate the disturbances. The advantage of making a selection rather than switching is translated into the suppression of the chattering effect, confirming previous analytic and experimental results obtained in a less general framework not encompassing parametric uncertainties.

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