INTEGRAL SLIDING SURFACE DESIGN USING AN $\mathcal{H}_\infty$ CRITERION FOR DECENTRALIZED CONTROL

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Abstract: The robustness properties of integral sliding mode and $\mathcal{H}_\infty$ control are exploited in the context of decentralized control. It is shown that integral sliding mode design successfully rejects the matched interconnections right from the initial time. It is demonstrated that by a proper selection of the sliding surface parameters, the effect of the unmatched interconnections, due to the discontinuous control, will be minimum; and that, at this minimum, the unmatched interconnections will not be amplified.

Keywords: Sliding-mode control, Variable structure systems, H-infinity control, Large-scale systems, Robust control.

1. INTRODUCTION

1.1 Motivation

Large-scale systems may require the use of decentralized control design when one, or several of the following difficulties occur:

(1) The system is widely distributed in space, so information transfer is too costly (e.g. power systems),
(2) implementation of a centralized feedback law is hard or impossible due to the system's decentralized structure (e.g. aerial and terrestrial traffic control),
(3) the complexity of analysis and design resulting from the system's order can be reduced by splitting the system into several subsystems (e.g. large flexible structures),
(4) the design criteria is robustness in the presence of structural perturbations where subsystems are disconnected and again connected during operation.

In general terms, the problem of decentralized control is that of finding a set of controllers satisfying an information constraint: the information available at each control station is only a subset of the state variables. The controllers are to be designed for stabilizing the set of interconnected subsystems that comprise the overall system. Clearly, such a decentralized feedback scheme would address difficulties 1 and 2.

As in the centralized case, several strategies have been proposed in order to solve the problem. For instance, eigenvalue assignment, or optimal control. The main disadvantage of these methods is that, at some stage of the design procedure, a solution to a set of simultaneous equations of at least the same order of the system needs to be found. Thus, difficulty 3 is left unsolved.

An alternative strategy is to consider each system independently and treat the interconnections as perturbations. This approach seeks to eliminate, or at least attenuate the perturbations using available robust techniques (Richter *et al.*, 1982; Akar and Özgüner, 2002; Yan *et al.*, 2004). A scheme like this one has the advantage of resolving difficulties 3 and 4. This is the approach taken in this paper.

Sliding mode control is a powerful and robust technique that fits well into this framework. The sliding mode controller drives the system's state into a "custom built" sliding (switching) surface and constraints the state thereafter. A system motion in a sliding surface, named *sliding mode*, is robust with respect to uncertainties an disturbances matched by a control but sensitive to unmatched ones. This control design strategy, although robust, has two main disadvantages:

- The classical sliding mode controllers are robust in the case of matched disturbances only.
- The trajectory of the designed solution is not robust even with respect to the matched disturbances during the time interval preceding the sliding motion (the reaching phase).

In order to address the unmatched disturbance issue, and to further improve the robustness properties of the system under control, the combination of sliding modes with other robust techniques has been investigated (Poznyak *et al.*, 2003; Choi, 2003; Cao and Xu, 2004).

Integral sliding mode control (Utkin and Shi, 1996) is a variation to conventional sliding mode with additional benefits such as elimination of the reaching phase and chattering reduction.

1.2 Main Contribution

An integral sliding surface using $\mathcal{H}_\infty$ control is proposed in order to solve the decentralized control problem, where the interactions among subsystems are viewed as perturbations and the matching assumption turns out to be too restrictive.

It is shown that by a proper selection of the sliding surface parameter, the effect of the unmatched perturbation, due to the discontinuous control, will be minimum; and that, at this minimum, the unmatched perturbation will not be amplified. This results are general and can be applied whenever an integral sliding mode surface is to be designed, whether in combination with $\mathcal{H}_\infty$ or other techniques.

2. PROBLEM STATEMENT

Consider a linear time invariant decentralized system with N control stations

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(x_i, t) + \sum_{j=1}^{N} A_{ij} x_j(t) \qquad i = 1, 2, \ldots, N \quad (1)$$

where $x_i(t) \in \mathbb{R}^{n_i}$ is the state vector and $u_i(x_i, t) \in \mathbb{R}^{m_i}$ is the control action of the ith station at time $t \in \mathbb{R}$. Note that u_i satisfies an information constraint, it depends on x_i only. A_i and B_i are matrices of appropriate dimensions. $\sum_{j=1}^{N} A_{ij} x_j$ represents the influence of the other stations, where the A_{ij}'s are, again, matrices of appropriate dimensions. In what follows, whenever the subscript i appears, it is assumed that the properties stated hold for all $i = 1, 2, \ldots, N$.

Assumption 1. The pairs $\{A_i, B_i\}$ are stabilizable.

Assumption 2. The matrices B_i have full rank m_i.

Assumption 3. The initial state $x(t_0)$ is bounded by a known constant q, $\|x(t_0)\| \leq q$.

The objective is to design each of the control laws u_i so that system (1) is semi–globally asymptotically stable.

2.1 Notation

The notation is fairly standard. I is the identity matrix of the corresponding dimension. B^T denotes the transpose of B. $B^\perp$ is a matrix whose columns span the orthogonal complement of span(B), i.e., $B^T B^\perp = 0$. B^+ is the pseudo–inverse of B, that is, $B^+ = (B^T B)^{-1} B^T$.

3. INTEGRAL SLIDING MODE CONTROL (ISMC)

From now on, consider the perturbed linear system

$$\dot{x}(t) = Ax(t) + Bu(x, t) + \phi, \qquad (2)$$

where the pair $\{A, B\}$ is controllable, matrix B has full rank and ϕ is a perturbation.

3.1 Design Principles

Conventional sliding mode control (SMC) is an effective technique with the ability to withstand disturbances of the matched type. The main idea is to use a discontinuous control action in order to force the system's state into a desired surface, regardless of the matched uncertainties.

Besides the robustness property just mentioned, conventional sliding mode control has another advantage: the dynamic equations are of lower order than the original system. This is due to the fact that the sliding surface, being embedded in the state space, is of lower dimension. In contrast, the motion equations resulting from an ISM controller are of the same order as the original system. This is a sacrifice made for the sake of robustness starting from the initial instant.

ISMC proposes a control law of the form

$$u(x,t) = u_0(x,t) + u_1(x,t) \in \mathbb{R}^m,$$

where u_1 is a discontinuous action designed to reject the matched disturbance by making the switching surface attractive, and u_0 is responsible for stabilizing the system in the presence of the unmatched perturbation.

The switching surface is chosen as

$$\begin{aligned} s(x,t) = G\left[x(t) - x(t_0)\right] - \\ - G\left[\int_{t_0}^{t} (Ax(\tau) + Bu_0(x,\tau)\,\mathrm{d}\tau\right] = 0 \end{aligned}$$

where G is an $m \times n$ matrix and GB is invertible. The integral term on the right hand side comes out as a dynamic equation of the same dimension as the state. A possible interpretation of s is as a penalizing factor of the difference between the desired trajectory

$$\int_{t_0}^{t} [Ax(\tau) + Bu_0(x,\tau)]\,\mathrm{d}\tau + x(t_0),$$

and the actual one, projected along G.

3.2 Design of the Discontinuous Component

The discontinuous control is defined as

$$u_1(x,t) = -\rho(x)\frac{(GB)^T s(x,t)}{\|(GB)^T s(x,t)\|}, \tag{3}$$

where $\rho(x)$ is a positive scalar function.

This control law guaranties the attractiveness of the sliding surface, that is, it makes the point $s(x,t) = 0$ stable. As a matter of fact, if the initial conditions are accurately known, $s = 0$ is guarantied right from the initial time t_0 (the reaching phase is eliminated), which in turn ensures the robustness property for all $t \geq t_0$.

A possible choice of $\rho(x)$ is

$$\rho(x) > \|(GB)^{-1}G\|\phi \tag{4}$$

By taking $V_s(s) = \|s\|^2/2$ as a Lyapunov function the stability of s is verified:

$$\dot{V} = s^T\dot{s}, \tag{5}$$

the derivative of s along time is

$$\begin{aligned} \dot{s} &= G\left[Ax + B(u_0 + u_1) + \phi - Ax - Bu_0\right] \\ &= -\rho GB\frac{(GB)^T s}{\|(GB)^T s\|} + G\phi \end{aligned} \tag{6}$$

Combining (5) and (6), one has

$$\begin{aligned} \dot{V}_s &= -\rho\|(GB)^T s\| + s^T G\phi \\ &\leq -\|(GB)^T s\|\left[\rho - \|(GB)^{-1}G\|\phi\right] \\ &\leq 0 \end{aligned}$$

To determine the motion equations, the equivalent control method is used,

$$\dot{s} = G\left[Bu_1 + \phi\right].$$

The perturbations can be represented as vectors consisting of two components, one that belongs to the space spanned by B, and another that belongs to the space orthogonal to B

$$\phi = BB^+\phi + \underbrace{B^\perp B^{\perp +}\phi}_{\eta}.$$

So $\dot{s}$ becomes

$$\dot{s} = GB\left[u_1 + B^+\phi + \eta\right].$$

Solving the equation for the equivalent control $u_{1\mathrm{eq}}$ yields

$$u_{1\mathrm{eq}} = -B^+\phi - (GB)^{-1}G\eta.$$

The new system's equation is obtained by substituting the equivalent control in the original equation

$$\dot{x} = Ax + Bu_0 + \left[I - B(GB)^{-1}G\right]\eta.$$

Note that the matched uncertainty has been eliminated and there is an extra degree of freedom: the choice of u_0 to ensure the stability of the system.

Although the matched uncertainty has been annihilated, there is a gain for the unmatched one. Essentially, the perturbation

$$\phi = BB^+\phi + \eta,$$

has been transformed into

$$\Gamma\eta, \quad \Gamma \triangleq [I - B(GB)^{-1}G].$$

Proposition 1. For any $B \in \mathbb{R}^{n\times m}$, $\mathrm{rank}(B) = m$, and any vector $\eta \in \mathbb{R}^n$, the norm

$$\left\|\left[I - B(GB)^{-1}G\right]\eta\right\|$$

attains it's minimum at $G = B^T$.

Proof: Consider first the problem of finding the vector $\eta_0 \in \mathrm{span}(B)$ which is closest to an arbitrary vector η, i.e., which minimises $\|\eta - \eta_0\|$. According to the projection theorem, η_0 is a unique minimising vector when $\eta - \eta_0$ is orthogonal to $\mathrm{span}(B)$.

To solve the later problem, set $\eta_0 = B\beta$. This ensures that $\eta_0 \in \mathrm{span}(B)$. Then, search for the vector $\hat{\beta}$ which makes $\eta - B\beta$ orthogonal to $\mathrm{span}(B)$, that is,

$$
\begin{aligned}
0 &= B^T\eta - B^T B\hat{\beta}\\
\hat{\beta} &= (B^T B)^{-1} B^T \eta\\
&= B^+\eta
\end{aligned}
$$

Realizing that

$$\left\| \left[I - B(B^T B)^{-1} B^T \right] \eta \right\|$$

is minimum, it only remains to set $G = B^T$ to complete the proof. It is important to notice that at $G = B^+$, the minimum is also attained. ■

Proposition 2. The unmatched perturbation η is not amplified, i.e. for $m < n$ the following identity holds:

$$\|\Gamma_B\| = \|I - BB^+\| = 1$$

Proof: Notice first that

$$
\begin{aligned}
\Gamma_B^T \Gamma_B &= \left[I - BB^+ \right] \left[I - BB^+ \right]\\
&= I - BB^+ = \Gamma_B,
\end{aligned}
$$

which means that Γ_B is a symmetric matrix and therefore all the eigen-values are real. Suppose that v is an eigen-vector associated to any eigen-value λ of Γ_B, that is,

$$
\begin{gathered}
\Gamma_B v = \lambda v\\
\Rightarrow\\
v^T \Gamma_B^T \Gamma_B v = \lambda^2 \|v\|^2.
\end{gathered}
\tag{7}
$$

But, since $\Gamma_B^T \Gamma_B = \Gamma_B$ we have

$$v^T \Gamma_B^T \Gamma_B v = v^T \Gamma_B v = \lambda \|v\|^2. \tag{8}$$

From (7) and (8), it is clear that the eigen-values of Γ_B must satisfy

$$\lambda^2 = \lambda.$$

The last equation has two solutions, $\lambda = 0$ and $\lambda = 1$.

Since $\mathrm{rank}(BB^+) < n$, the rank of $I - BB^+$ cannot be zero. This means that Γ_B must have at least one eigen-value different from zero, that is, the maximum eigen-value is one. The last statement implies that $\|\Gamma_B\| = 1$. ■

So, the gain Γ_B poses no problem. Furthermore, since by definition of $B^\perp$,

$$
\begin{gathered}
B^T B^\perp = 0\\
\left[I - B(B^T B)^{-1} B^T \right] B^\perp B^{\perp +} \phi = B^\perp B^{\perp +} \phi = \eta
\end{gathered}
$$

which means that the unmatched perturbation is left unchanged. Proposition 1 and 2 imply that it is not possible to attenuate the unmatched disturbance by the control component u_1; and to avoid amplification, only the projection of the difference between the actual and the desired trajectories along B should be penalized.

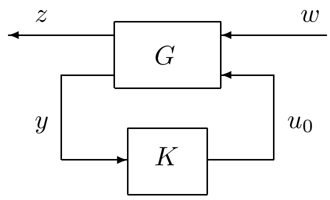

Fig. 1. General Block Diagram of a Control System.

Finally, the dynamics at the sliding surface are given by

$$\dot{x} = Ax + Bu_0 + \eta. \tag{9}$$

3.3 Design of the Continuous Component

There are several options to treat the unmatched term. The criteria for selecting a method is still robustness. In this paper the unmatched term is dealt with $\mathcal{H}_\infty$ control.

A general block diagram of a control system is depicted in figure 1. G is called the generalized plant and K is the controller. The output z is a penalty variable which may contain an error signal; y is composed of the available measurement variables; u_0 is the control input and w contains all external inputs, including disturbances, sensor noise and commands. The resulting closed–loop transfer matrix from w to z is denoted by T_{zw}.

The following is taken for granted:

Assumption 4. The generalized plant has the form

$$
\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u_0\\
z &= C_1 x + D_{12} u\\
y &= x.
\end{aligned}
$$

Assumption 5. (A, B_2) is stabilizable and (C_1, A) is detectable.

Assumption 6. $D_{12}^T \left[C_1 \; D_{12} \right] = \left[0 \; I \right]$.

Theorem 1. (Doyle *et al.*, 1989)
$\|T_{zw}\|_\infty < \gamma$ iff there is a positive semi–definite matrix X_∞ such that

$$A^T X_\infty + X_\infty A - X_\infty (B_2 B_2^T - \gamma^{-2} B_1 B_1^T) X_\infty + {} + C_1^T C_1 = 0. \tag{10}$$

Moreover, when this conditions hold, one such controller is

$$u_0 = -B^T X_\infty x.$$

Theorem 2. (Isidori and Astolfi, 1992)
If assumptions 4, 5, 6 are satisfied, and there is a solution to the Riccati equation (10) then

$$\int_0^T z^T(s) z(s)\,\mathrm{d}s \le \gamma^2 \int_0^T w^T(s) w(s)\,\mathrm{d}s,$$

for all $T > 0$. Moreover, the function

$$V(x) = x^T X_\infty x$$

satisfies

$$\dot{V} \leq -\|C_1 x\|^2 - \|u\|^2 + \gamma^2 \|w\|^2.$$

The equivalent dynamics (9) can be written using $\mathcal{H}_\infty$ notation as

$$\dot{x} = Ax + \underbrace{B^\perp B^{\perp +}}_{B_1} \underbrace{\phi}_{w} + \underbrace{B}_{B_2} u_0.$$

So the task is to find the smallest γ for which there is a positive semi–definite matrix X_∞ satisfying

$$A^T X_\infty + X_\infty A - X_\infty (BB^T - \gamma^{-2} B^\perp B^{\perp +}) X_\infty + \\ + C_1^T C_1 = 0.$$

And the integral sliding surface becomes

$$s(x,t) = B^T \left[x(t) - x(t_0) \right] - \\ - B^T \left[\int_{t_0}^{t} (A - B^T X_\infty) x(\tau)\, \mathrm{d}\tau \right].$$

4. APPLICATION TO DECENTRALIZED CONTROL

Consider again system (1). It can be thought as a set of perturbed systems

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(x_i, t) + \phi_i(x), \qquad (11)$$

$x = \mathrm{col}(x_1, \ldots, x_N)$ and the nominal systems are

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(x_i, t). \qquad (12)$$

The perturbations ϕ_i, resulting from the interconnections, are thus defined as

$$\phi_i(x) = \sum_{j=1}^{N} A_{ij} x_j(t) = A^i x. \qquad (13)$$

where $A^i = \sum_{j=1}^{N} A_{ij}$.

The problem can be restated as follows: *design N control laws for the nominal systems (12), so that perturbations (13) do not affect the stability of system (1).*

With the analysis developed so far, it it easy to derive sufficient conditions for stability of the interconnected system (1).

A set of controls

$$u_i = u_{i0} + u_{i1},$$

where the discontinuous components are given by (3) is proposed.

According to (9), each subsystem at the sliding surface is described by

$$\dot{x}_i = A_i x_i + \underbrace{B_i^\perp B_i^{\perp +} A^i}_{B_{i1}} \underbrace{x}_{w_i} + \underbrace{B_i}_{B_{i2}} u_{i0}.$$

Setting $C_{i1} = I$ and solving the Riccati equations for each γ_i ensures that the derivative along time of $V_i(x_i) = x_i^T X_{i\infty} x_i$ is

$$\dot{V}_i \leq -\|x_i\|^2 - \|u_{i0}\|^2 + \gamma_i^2 \|x\|^2$$

A straight forward approach to determine stability of an interconnected system, is to consider a composite Lyapunov function as the sum of the individual Lyapunov functions (Yan *et al.*, 2004)

$$V = \sum_{i=1}^{N} V_i(x_i)$$

$$\dot{V} \leq -\sum_{i=1}^{N} \left[\|x_i\|^2 + \|u_{i0}\|^2 - \gamma_i^2 \|x\|^2 \right] \\ = -\|x\|^2 (1 - \|\gamma\|^2) - \|u_0\|^2,$$

where

$$\gamma = \mathrm{col}(\gamma_1, \ldots, \gamma_N), \quad u_0 = \mathrm{col}(u_{01}, \ldots, u_{0N}).$$

If $\|\gamma\| < 1$, then it is possible to design an asymptotically stable system.

In order to determine the gains for the discontinuous controls u_{i1} (3), a bound on $\phi_i = A^i x$ is needed. Notice that

$$\lambda_{min}(X_\infty)\|x(t)\|^2 \leq x^T(t) X_\infty x(t) \leq \\ x^T(t_0) X_\infty x(t_0) \leq \lambda_{max}(X_\infty)\|x(t_0)\|^2, \quad (14)$$

which means that

$$||x(t)|| \leq \sqrt{\frac{\lambda_{max}(X_\infty)}{\lambda_{min}(X_\infty)}} ||x(t_0)|| = p||x(t_0)|| \leq p \times q$$

and the gain can be set as

$$\rho_i > \|A^i\| p \times q.$$

Naturally, we need to consider only the cases when X_∞ is strictly positive definite.

Since for any initial condition $x(t_0)$, a ρ_i can be found, and therefore a set of controllers stabilizing system (1), the semi-global stability of the closed-loop system can be concluded.

5. NUMERICAL EXAMPLE

To illustrate the algorithm developed we provide a simple example for a system to be controlled by two control stations.

Consider two identical systems of order two, with

$$A_1 = A_2 = \begin{bmatrix} 0 & 1 \\ -2 & -4 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$A_{1,2} = A_{2,1} = \begin{bmatrix} 0.2 & 0.2 \\ 2 & 2 \end{bmatrix}$$

Suppose a bound $\|x(t_0)\| \leq 5$ is known.

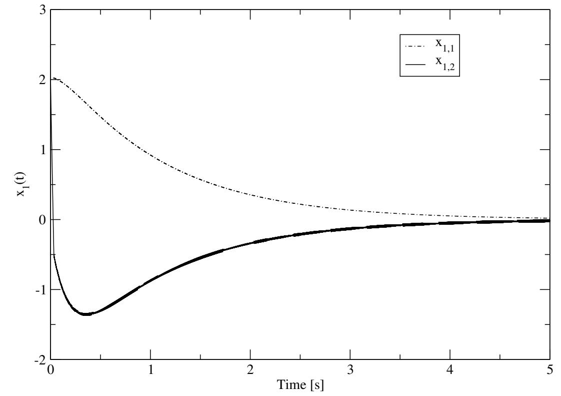

Fig. 2. Response of the nominal system $\dot{x}_1 = A_1 x_1 + B_1 u_1$.

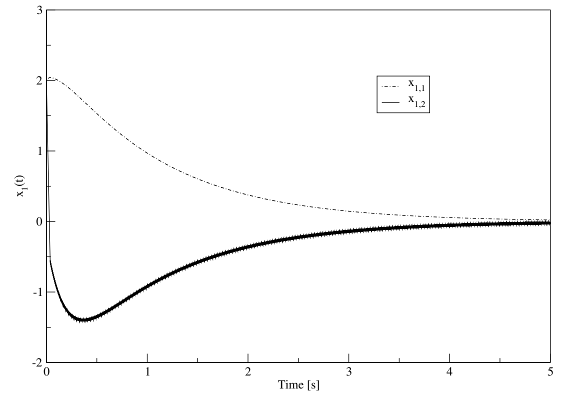

Fig. 3. Response of the interconnected system $\dot{x}_1 = A_1 x_1 + B_1 u_1 + A_{1,2} x_2$.

The unmatched perturbation is

$$\eta_i = B_i^{\perp} B_i^{\perp +} A_{i,j} x_j = \begin{bmatrix} 0.2 & 0.2 \\ 0 & 0 \end{bmatrix} x_j, \quad j \neq i.$$

$\gamma_i = 0.7$ is an approximate optimum for the $\mathcal{H}_\infty$ problem, where

$$X_{i\infty} = \begin{bmatrix} 1.791 & 0.350 \\ 0.350 & 0.210 \end{bmatrix}.$$

The condition $\|\gamma\| = \| \begin{bmatrix} 0.7 & 0.7 \end{bmatrix} \| < 1$ is satisfied, so a solution to the problem exists. The continuous component is set as

$$u_{0i} = -B_i^T X_i x_i = \begin{bmatrix} -0.350 & -0.210 \end{bmatrix} x_i.$$

Calculation of the gains ρ_i yields

$$\rho_i > \|A_{i,j}\| p_i q = 2.84 \sqrt{\frac{\lambda_{max}(X_{i\infty})}{\lambda_{min}(X_{i\infty})}} 5 = 52.74$$

So, $u_i = \begin{bmatrix} -0.350 & -0.210 \end{bmatrix} x_i - 55 \operatorname{sign}(s_i)$. The simulation of the system's response is showed in figures 2 and 3.

6. CONCLUSIONS

In this paper ISM and $\mathcal{H}_\infty$ control techniques were combined. A discontinuous action was used to eliminate the unmatched component of the perturbation. The selected surface ensures that the effect of the unmatched term, due to the discontinuous control is *minimal*; and that when the minimal is achieved, there is *no amplification* of such term. The effect of the unmatched term was then reduced using $\mathcal{H}_\infty$ control. This results are general and can be applied outside the context of decentralized control.

The techniques developed in this paper can be used to derive simple sufficient conditions for the stability of decentralized control systems. Such conditions have $\mathcal{H}_\infty$ flavor, as they are stated in terms of Riccati equations.

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REFERENCES

Akar, M. and Ü. Özgüner (2002). Decentralized sliding mode control design using overlapping decompositions. *Automatica* **28**, 1713–1718.

Cao, W. and J. Xu (2004). Nonlinear integral-type sliding surface for both matched and unmatched uncertain systems. *IEEE Trans. Automat. Contr.* **49**, 1355–1360.

Choi, H. (2003). An LMI-based switching surface design method for a class of mismatched uncertain systems. *IEEE Trans. Automat. Contr.* **48**, 1634–1638.

Doyle, J. C., P. P. Khargonekar and B. A. Francis (1989). State-space solutions to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. *IEEE Trans. Automat. Contr.* **34**, 831–847.

Isidori, A. and A. Astolfi (1992). Disturbance attenuation and $\mathcal{H}_\infty$-control via measurement feedback in nonlinear systems. *IEEE Trans. Automat. Contr.* **37**, 1283–1293.

Poznyak, A., Y. B. Shtessel and C. Jiménez (2003). Mini-max sliding mode control for multimodel linear time varing systems. *IEEE Trans. Automat. Contr.* **48**, 2141–2150.

Richter, S., S. Lefebvre and R. A. DeCarlo (1982). Control of a class of nonlinear systems by decentralized control. *IEEE Trans. Automat. Contr.* **27**, 492–494.

Utkin, V. and J. Shi (1996). Integral sliding mode in systems operating under uncertainty conditions. In: *Proc. Conference on Decision and Control.* Kobe, Japan. pp. 4591–4596.

Yan, X., C. Edwards and S. K. Suprgeon (2004). Decentralised robust sliding mode control for a class of nonlinear interconnected systems by static output feedback. *Automatica* **40**, 613–620.