Energy-Balancing Passivity-Based Control is Equivalent to Dissipation and Output Invariance

Fernando Castaños and Romeo Ortega

Abstract— **Passivity-based controllers (PBCs) achieve stabilization of nonlinear systems, rendering the closed-loop passive with a desired energy (storage) function. A natural question is under which conditions is it possible to make this function equal to the difference between the plant and controller energies—when the controller is said to be *energy-balancing*. In this paper we prove that a necessary and sufficient condition for energy-balancing is that the open and the closed-loop systems have the same dissipation functions and passive outputs. A second contribution of our work is the identification of a new passive output for port-Hamiltonian systems, which is invariant to the action of PBCs that modify only the energy function—so-called basic interconnection and damping assignment PBCs—proving that they are energy-balancing. To establish these results a new *algebraic* framework for analysis and design of PBCs, centered around the principles of output and dissipation invariance, is developed. Using this framework several PBC schemes reported in the literature are compared. Also, we present a systematic procedure to generate new passive outputs, this result is of interest on its own, since it allows to extend the applicability of PBC to systems that are non-minimum phase and/or have relative degree larger than one.**

Index Terms— **Passivity-based control, port-Hamiltonian systems, Energy-balance, Interconnection and Damping Assignment**

I. Introduction

In standard passivity-based control (PBC), the fundamental problem of feedback stabilization of nonlinear systems is reformulated in terms of feedback *passivation*. The objective is to find a state-feedback control law that renders the closed-loop system strictly output passive with a storage function having an isolated minimum at the given equilibrium and, to ensure asymptotic stabilization, a detectable passive output. Interested readers are referred to [1] for a tutorial account on this state-feedback approach to PBC, that is called "standard PBC", and to [2] for a historical review of PBC. A particular case of standard PBC is the so-called energy-shaping plus damping-injection technique, where the system is first rendered passive and then extra damping is introduced feeding back the passive output to ensure asymptotic stability.[1] An alternative, and far reaching, viewpoint of PBC as interconnection of dynamical systems, instead of a state-feedback action, may be found in [4], [5], [1]—see also [6], where standard PBCs are obtained as restrictions of these dynamic controllers.

The selection of the desired energy function in standard[2] PBC is, similarly to the selection of a Lyapunov function, a non-trivial task. In this paper it is assumed that the original system is cyclo-passive, see Assumption 1. This condition is a restatement of energy conservation, where the energy function is not required to be bounded from below. Hence, it is a rather weak assumption, verified by most physical systems, that does not imply any stability property whatsoever. Under the aforementioned assumption, the most natural desired storage function candidate is the difference between the energy of the plant and the energy of the controller. PBCs that verify this property are said to be *energy-balancing* (EB) [5]. A fundamental question that arises is then: Under which conditions a PBC is EB?

In [5] it is shown that, if the PBC ensures stability, a necessary condition for EB is that the dissipation function is equal to zero at the desired equilibrium, which consequently means that the system can be stabilized extracting a finite amount of energy from the controller. In this paper we prove that, even without the stability requirement, a *necessary and sufficient* condition for EB is that the open and the closed-loop systems have the same dissipation and output functions—hence providing a complete characterization of EB PBC.

Dissipation assignment has traditionally been regarded as an auxiliary, or even secondary, step to energy-shaping. The fundamental result mentioned above underscores the central role it plays in the understanding of PBC that motivates the development of a new *algebraic* framework for analysis and design of PBCs, centered around the principles of output and dissipation invariance. Using this framework several PBC schemes reported in the literature are compared in this paper, including the well-known Interconnection and Damping Assignment (IDA) PBC, in its basic and general formulations [7], [6], [8]. In Basic IDA (BIDA) it is assumed that the plant is described by a port-Hamiltonian model and the objective is to shape *only* the energy function—without modifying the interconnection and damping matrices. A second contribution of our work is the identification of a new passive output for port-Hamiltonian systems, which is

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Fernando Castaños and Romeo Ortega are with Laboratoire des Signaux et Systèmes, Supelec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France. castanos@ieee.org, ortega@lss.supelec.fr

[1]As shown in [3], the separation of the PBC design in two steps induces a loss of generality.

[2]For brevity, in the sequel the "standard" qualifier is omitted, in the understanding that we are dealing all the time with state-feedback PBC.

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invariant to the action of BIDA. Combining this result with the characterization of EB mentioned above shows that BIDA is EB. To establish these results a systematic procedure to generate new passive outputs is presented. The procedure is of interest on its own, since it allows to extend the applicability of PBC to systems that are non-minimum phase and/or have relative degree larger than one.

In the following section the PBC problem is formulated and an algebraic characterization, in terms of the added energy function and added dissipation, is given. In Section III the equivalence between dissipation and output invariance and the property of EB of PBC is established. In Section IV it is shown that, by suitably assigning the dissipation of the closed-loop system, it is possible to recover several existing PBCs—providing a framework to classify and compare them. In Section V, a procedure to generate zero-relative-degree passive outputs is proposed and the EB property of BIDA is established. Finally, we present the conclusions in Section VI.

Notation: The arguments of the functions are omitted once they are defined and there is no possibility of confusion. For a distinguished element $x^\star \in \mathbb{R}^n$ and a given function $f : \mathbb{R}^n \to \mathbb{R}^m$ we denote the constant vector $f^\star := f(x^\star)$.

II. Standard Passivity-Based Control

A. Definition of Passivity-Based Control

Consider a nonlinear system described by equations of the form

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x} &=& f(x) + g(x)u \\ y &=& h(x) \end{array} \right. , \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^m$ is the output. The remaining functions, f, g and h, are assumed to be smooth and of appropriate dimensions. The matrix g is assumed to be full rank—uniformly in x. We also impose the following.

Assumption 1: Σ is cyclo-passive. That is, there exists a $\mathcal{C}^1$ function $H : \mathbb{R}^n \to \mathbb{R}$, called the *storage* function, such that, for all $x_0 \in \mathbb{R}^n$, all $t \geq 0$ and all input functions $u(t)$

$$H(x(t)) - H(x(0)) \leq \int_0^t h^\top(x(s))u(s)\mathrm{d}s \;, \tag{2}$$

where $x(0) = x_0$ and $x(t)$ is the state of Σ at time t resulting from initial condition x_0 and input function $u(t)$.[3] Equivalently, if and only if

$$\dot{H} \leq y^\top u \;. \tag{3}$$

Recall that a system is passive if it is cyclo-passive and H *has a minimum* [1]. Clearly, every passive system is cyclo-passive but the converse is not true. In terms of energy exchange, cyclo-passive systems exhibit a net absorption of energy along *closed* trajectories [9], while passive systems absorb energy along *any* trajectory that starts from a state of minimal energy $x(0) = \arg\min H(x)$.

[3] Of course, we require that the integral in (2) is well defined.

The celebrated Hill-Moylan's Theorem [9] gives, in the spirit of Kalman-Yakubovich-Popov's Lemma, an *algebraic* characterization of cyclo-passive systems.[4]

Theorem 1: The system Σ (1) is cyclo-passive with storage function H if and only if there exists a function $d : \mathbb{R}^n \to \mathbb{R}_+$, called the *dissipation function*, such that,

$$\begin{array}{rcl} \nabla H^\top(x)f(x) &=& -d(x) \\ h(x) &=& g^\top(x)\nabla H(x) \;. \end{array} \begin{array}{r} (4a) \\ (4b) \end{array}$$

Using Hill-Moylan's Theorem one obtains the power balance equation for Σ

$$\dot{H} = y^\top u - d \;. \tag{5}$$

The objective in PBC is to "shape", via state-feedback (5). More precisely:

Definition 1 (The set PBC): The state-feedback $u_{\mathtt{SF}} : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a PBC (shorthand notation: $u_{\mathtt{SF}} \in$ PBC) if and only if there exist functions $H_\mathrm{d} : \mathbb{R}^n \to \mathbb{R}$ and $h_\mathrm{d} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$u = u_{\mathtt{SF}} + v \tag{6}$$

with v a new, virtual input, renders the closed-loop system

$$\Sigma_\mathrm{d} : \left\{ \begin{array}{rcl} \dot{x} &=& f_\mathrm{d}(x) + g(x)v \\ y_\mathrm{d} &=& h_\mathrm{d}(x) \end{array} \right. \tag{7}$$

where $f_\mathrm{d}(x) := f(x) + g(x)u_{\mathtt{SF}}(x)$, cyclo-passive with storage function $H_\mathrm{d}(x)$. That is, if it verifies

$$\dot{H}_\mathrm{d} \leq y_\mathrm{d}^\top v \;. \tag{8}$$

From Hill-Moylan's Theorem we have that the new power balance becomes

$$\dot{H}_\mathrm{d} = y_\mathrm{d}^\top v - d_\mathrm{d} \;. \tag{9}$$

where the new dissipation $d_\mathrm{d} : \mathbb{R}^n \to \mathbb{R}_+$ is given by

$$d_\mathrm{d}(x) = -\nabla H_\mathrm{d}^\top(x)(f(x) + g(x)u_{\mathtt{SF}}(x)) \tag{10}$$

Comparing the open-loop power balance (5) with the closed-loop power balance (9) we observe that, besides the energy and the dissipation, the output has also been modified. Since full-state-feedback is assumed, there is—a priori—no reason to maintain the original output y as the cyclo-passive output. We thus take the liberty to define the new output that, according to Proposition 1, should be of the form

$$y_\mathrm{d} = h_\mathrm{d} = g^\top \nabla H_\mathrm{d} \;. \tag{11}$$

This approach differs from the classic problem formulation of [10], where feedback passivation is defined (and sought) with respect to the original output. Changing the output is a natural way to satisfy the vector relative degree requirement and to overcome the minimal phase restriction on the plant. In terms of the usual analogy between passive systems and electrical ports, the change of the output corresponds to the addition of a current source $h - h_\mathrm{d}$ (see Figure 1).

[4] For ease of presentation a version of the theorem for systems with relative degree one is given first. In Section V the general version is stated.

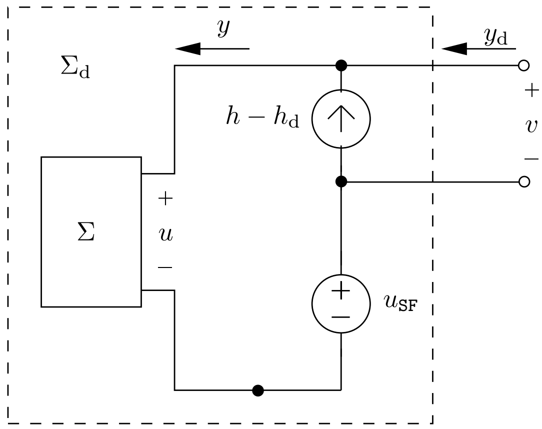

Fig. 1. Adding a current source to create a new passive output.

Remark 1: Definition 1 has been intentionally stated in a fairly general way. Notice, *e.g.*, that a null control $u_{\tt SF} = 0$ satisfies the requirements of the definition (take $H_{\rm d} = H$ and $d_{\rm d} = d$).

Remark 2: As announced above an algebraic framework to derive particular subsets of the set PBC will be proposed. To simplify notation we say $u_{\tt SF} \in \Omega$, where $\Omega \subset \mathrm{PBC}$ is either one of the sets $\{\mathrm{EB}, \mathrm{BIDA}, \mathrm{IDA}\}$, consisting of particular classes of PBCs—to be defined later.

B. Characterizing Passivity-Based Controllers

The following proposition, which constitutes the main thread of the paper, gives an algebraic characterization of the set PBC.

Proposition 1: $u_{\tt SF} \in PBC$ if and only if there exist functions $H_{\rm a} : \mathbb{R}^n \to \mathbb{R}$ and $d_{\rm a} : \mathbb{R}^n \to \mathbb{R}$, with

$$d_{\rm a}(x) \geq -d(x),$$

such that

$$h^\top(x)u_{\tt SF}(x) = -\nabla H_{\rm a}^\top(x)(f(x) + g(x)u_{\tt SF}(x)) - d_{\rm a}(x) \ . \tag{12}$$

Proof: To prove sufficiency, assume that (12) is satisfied and define

$$H_{\rm a} := H_{\rm d} - H \quad \text{and} \quad d_{\rm a} := d_{\rm d} - d \geq -d \ . \tag{13}$$

so that (12) can be rewritten as

$$h^\top u_{\tt SF} = -(\nabla H_{\rm d} - \nabla H)^\top (f + g u_{\tt SF}) + d - d_{\rm d}$$

or, equivalently, as

$$(h - g^\top \nabla H)^\top u_{\tt SF} = (d + \nabla H^\top f) - \nabla H_{\rm d} f_{\rm d} - d_{\rm d} \ . \tag{14}$$

From (4) and Assumption 1 we know that $h - g^\top \nabla H = 0$ and $d + \nabla H^\top f = 0$, so (14) becomes $\nabla H_{\rm d} f_{\rm d} = -d_{\rm d}$. Take $h_{\rm d}$ as in (11). According to Hill-Moylan's Theorem, the system $\Sigma_{\rm d}$ is cyclo-passive.

For necessity, assume that $\Sigma_{\rm d}$ is cyclo-passive with storage function $H_{\rm d}$ and output $h_{\rm d}$. Again, from Hill-Moylan's Theorem, we know that

$$\nabla H_{\rm d}^\top f_{\rm d} = -d_{\rm d} \ . \tag{15}$$

From (13) and $f_{\rm d} = f + g u_{\tt SF}$, equation (15) becomes

$$\begin{aligned} (\nabla H_{\rm a} + \nabla H)^\top (f + g u_{\tt SF}) &= -d_{\rm a} - d \\ \Leftrightarrow \quad \nabla H^\top g u_{\tt SF} &= -\nabla H_{\rm a}^\top (f + g u_{\tt SF}) - \\ &\quad - d_{\rm a} - (\nabla H^\top f + d) \ . \end{aligned}$$

Since $\nabla H^\top g = h$ and $\nabla H^\top f + d = 0$, we get

$$h^\top u_{\tt SF} = -\nabla H_{\rm a}^\top (f + g u_{\tt SF}) - d_{\rm a} \ .$$

This completes the proof. ■

III. Energy-Balancing PBC

As indicated in the Introduction the most natural desired storage function candidate is the difference between the energy of the plant and the energy of the controller, that is

$$H_{\rm d}(x(t)) = H(x(t)) - \int_0^t h^\top(x(s)) u_{\tt SF}(s) ds.$$

This motivates the definition of the following subset of PBC.

Definition 2 (Energy-Balancing): A PBC for the cyclo-passive system Σ (1) is said to be EB (*i.e.*, $u_{\tt SF} \in \mathrm{PBC} \cap \mathrm{EB}$) if and only if

$$-y^\top u_{\tt SF} = \dot{H}_{\rm a} \ , \tag{16}$$

where $H_{\rm a}$ is defined in (13).

Proposition 2: $u_{\tt SF} \in \mathrm{PBC} \cap \mathrm{EB}$ *if and only if*, the output and the dissipation remain invariant. That is, if and only if (9) holds with[5]

$$y_{\rm d} = y, \quad d_{\rm d} = d.$$

Proof: To prove sufficiency, assume $d_{\rm d} = d$ (*i.e.*, $d_{\rm a} = 0$) and $y_{\rm d} = y$. Since $y_{\rm d} = g^\top \nabla H_{\rm d}$ and $y = g^\top \nabla H$, $y_{\rm d} = y$ holds if and only if $g^\top \nabla H_{\rm d} = g^\top \nabla H$ or, equivalently, if and only if

$$g^\top \nabla H_{\rm a} = 0 \ . \tag{17}$$

Substituting (17) in (12) yields

$$h^\top u_{\tt SF} = -\nabla H_{\rm a}^\top f \ . \tag{18}$$

On the other hand, equation (17) implies that

$$\dot{H}_{\rm a} = \nabla H_{\rm a}^\top [f + g(u_{\tt SF} + v)] = \nabla H_{\rm a}^\top f \ . \tag{19}$$

Combining (18) and (19) one gets $-h^\top u_{\tt SF} = \dot{H}_{\rm a}$ (*i.e.*, $u_{\tt SF} \in$ EB).

For necessity, suppose that (16) holds. Then,

$$\begin{aligned} -h^\top u_{\tt SF} &= \nabla H_{\rm a}^\top [f + g(u_{\tt SF} + v)] \\ \Leftrightarrow \quad -\nabla H^\top g u_{\tt SF} &= [\nabla H_{\rm d} - \nabla H]^\top [f + g(u_{\tt SF} + v)] \\ \Leftrightarrow \quad -\nabla H^\top g u_{\tt SF} &= \nabla H_{\rm d}^\top f_{\rm d} + \nabla H_{\rm d}^\top g v - \\ &\quad - \nabla H^\top [f + gv] - \nabla H^\top g u_{\tt SF} \end{aligned}$$

or, equivalently,

$$\begin{aligned} \nabla H_{\rm d}^\top f_{\rm d} - \nabla H^\top f &= -\nabla H_{\rm d}^\top g v + \nabla H^\top g v \\ \Leftrightarrow \quad \nabla H_{\rm d}^\top f_{\rm d} - \nabla H^\top f &= -\nabla H_{\rm a}^\top g v \ . \end{aligned} \tag{20}$$

[5] A PBC that satisfies these conditions is said to be output- and dissipation-preserving, respectively.

Equation (20) must hold for all v, in particular, for $v = 0$. This implies that $\nabla H_{\mathrm{d}}^\top f_{\mathrm{d}} = \nabla H^\top f$, which is equivalent to $d_{\mathrm{d}} = d$. Thus, equation (20) becomes

$$\nabla H_{\mathrm{a}}^\top g v = 0 \quad \forall\, v \, ,$$

which implies $\nabla H_{\mathrm{a}}^\top g = 0$. As stated before, this is equivalent to $y_{\mathrm{d}} = y$. ∎

IV. Overcoming the Dissipation Obstacle

A. Stabilization and the Dissipation Obstacle

When PBC is used for stabilization of an equilibrium, $x^\star \in \mathbb{R}^n$, the storage function is typically used as a Lyapunov function, so it is required that

$$x^\star = \arg\min H_{\mathrm{d}} \, . \tag{21}$$

Since $\nabla H_{\mathrm{d}}^\star = 0$ is a necessary condition for (21) it is clear from (11), that the output y_{d} must be zero at the equilibrium (*i.e.*, $y_{\mathrm{d}}^\star = 0$). Likewise, from equation (10), we also have that the dissipation at the equilibrium must be zero (*i.e.*, $d_{\mathrm{d}}^\star = 0$). EB PBCs, that preserve output and dissipation, impose then to the *open-loop* system that

$$d^\star = -(\nabla H^\star)^\top f^\star = 0, \quad y^\star = 0.$$

This is the so-called *dissipation obstacle* [5].

B. IDA PBC

It is clear that dissipation should be modified to stabilize, with PBCs, systems that dissipate energy at the equilibrium. A candidate dissipation function d_{d}, which is compatible with the requirement $d_{\mathrm{d}}^\star = 0$ and overcomes the dissipation obstacle, is given in the following proposition, where the well-known IDA PBC is re-derived.

Proposition 3: Fix

$$d_{\mathrm{d}}(x) = \nabla H_{\mathrm{d}}^\top(x) R_{\mathrm{d}}(x) \nabla H_{\mathrm{d}}(x) \tag{22}$$

with $R_{\mathrm{d}} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $R_{\mathrm{d}} = R_{\mathrm{d}}^\top \geq 0$.

(i) $u_{\mathtt{SF}} \in \mathrm{PBC}$ if and only if

$$g(x) u_{\mathtt{SF}}(x) = -f(x) - R_{\mathrm{d}}(x)\nabla H_{\mathrm{d}}(x) + \alpha(x) \tag{23}$$

for some function $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ such that $\alpha^\top \nabla H_{\mathrm{d}}$ is identically zero. Then:

(ii) If $x^\star$ is an equilibrium of the closed-loop that satisfies (21) then $\alpha^\star = 0$

(iii) For any $J_{\mathrm{d}} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $J_{\mathrm{d}} = -J_{\mathrm{d}}^\top$, the function

$$\alpha(x) = J_{\mathrm{d}}(x) \nabla H_{\mathrm{d}}(x),$$

satisfies both restrictions: $\alpha^\star = 0$ and $\alpha^\top \nabla H_{\mathrm{d}} = 0$. Furthermore, the closed-loop system, Σ_{d}, takes the port-Hamiltonian (PH) [1] form[6]

$$\Sigma_{\mathrm{d}} : \left\{ \begin{array}{rcl} \dot{x} & = & F_{\mathrm{d}}(x)\nabla H_{\mathrm{d}}(x) + g(x) v \\ y_{\mathrm{d}} & = & g^\top(x) \nabla H_{\mathrm{d}}(x) \end{array} \right. , \tag{24}$$

where $F_{\mathrm{d}}(x) := J_{\mathrm{d}}(x) - R_{\mathrm{d}}(x)$.

[6]In the literature of PH systems, J_{d} is called the interconnection and R_{d} the damping. For obvious reasons, this control strategy is known as IDA [7].

Proof: For sufficiency of (i), assume (23) and premultiply by $\nabla H_{\mathrm{d}}^\top$:

$$\begin{array}{rcl} \nabla H_{\mathrm{d}}^\top g u_{\mathtt{SF}} & = & -\nabla H_{\mathrm{d}}^\top f - \nabla H_{\mathrm{d}}^\top R_{\mathrm{d}} \nabla H_{\mathrm{d}} \\ & \Leftrightarrow & \\ (\nabla H^\top + \nabla H_{\mathrm{a}}^\top) g u_{\mathtt{SF}} & = & -\nabla H_{\mathrm{a}}^\top f - \nabla H^\top f - \\ & & -\, \nabla H_{\mathrm{d}}^\top R_{\mathrm{d}} \nabla H_{\mathrm{d}} \, . \end{array}$$

By reordering terms we get

$$\nabla H^\top g u_{\mathtt{SF}} = -\nabla H_{\mathrm{a}}^\top (f + g u_{\mathtt{SF}}) - \nabla H^\top f - \nabla H_{\mathrm{d}}^\top R_{\mathrm{d}} \nabla H_{\mathrm{d}} \, . \tag{25}$$

Notice that the aggregated dissipation is

$$d_{\mathrm{a}} = \nabla H_{\mathrm{d}}^\top R_{\mathrm{d}} \nabla H_{\mathrm{d}} + \nabla H f \, , \tag{26}$$

So (25) can be expressed as

$$\begin{array}{rcl} \nabla H^\top g u_{\mathtt{SF}} & = & -\nabla H_{\mathrm{a}}^\top (f + g u_{\mathtt{SF}}) - d_{\mathrm{a}} \\ & \Leftrightarrow & \\ h^\top u_{\mathtt{SF}} & = & -\nabla H_{\mathrm{a}}^\top (f + g u_{\mathtt{SF}}) - d_{\mathrm{a}} \, . \end{array}$$

Hence, according to Proposition 1, $u_{\mathtt{SF}} \in \mathrm{PBC}$.

For necessity, assume that $u_{\mathtt{SF}} \in \mathrm{PBC}$, *i.e.*, that (12) holds. Then, from (26),

$$\begin{array}{rcl} \nabla H^\top g u_{\mathtt{SF}} & = & -\nabla H_{\mathrm{a}}^\top (f + g u_{\mathtt{SF}}) - \nabla H^\top f - \\ & & -\, \nabla H_{\mathrm{d}}^\top R_{\mathrm{d}} \nabla H_{\mathrm{d}} \\ \Leftrightarrow \quad 0 & = & \nabla H_{\mathrm{d}}^\top (g u_{\mathtt{SF}} + f + R_{\mathrm{d}} \nabla H_{\mathrm{d}}) \, . \end{array}$$

The latter implies the existence of a vector field α, satisfying (23) and

$$\alpha^\top \nabla H_{\mathrm{d}} = 0 \, .$$

Regarding (ii), notice that for a control that satisfies (23), the drift $f_{\mathrm{d}} = f + g u_{\mathtt{SF}}$ of the controlled system is

$$f_{\mathrm{d}} = f - f - R_{\mathrm{d}} \nabla H_{\mathrm{d}} + \alpha = -R_{\mathrm{d}} \nabla H_{\mathrm{d}} + \alpha \, . \tag{27}$$

If $x^\star$ is an equilibrium of the closed-loop, then $f_{\mathrm{d}}^\star = 0$ and $\nabla H_{\mathrm{d}}^\star = 0$. These equations, together with (27) imply that $\alpha^\star = 0$.

The first assertion of (iii) is proved by noting that $\nabla H_{\mathrm{d}}^\star = 0$ implies $\alpha^\star = J_{\mathrm{d}}^\star \nabla H_{\mathrm{d}}^\star = 0$. Orthogonality follows from the fact that $\nabla H_{\mathrm{d}}^\top J_{\mathrm{d}} \nabla H_{\mathrm{d}} = 0$ for any skew-symmetric matrix J_{d}.

The second assertion of (iii) can be verified replacing α in (27) to get:

$$\begin{array}{rcl} f_{\mathrm{d}} & = & -R_{\mathrm{d}} \nabla H_{\mathrm{d}} + J_{\mathrm{d}} \nabla H_{\mathrm{d}} \\ & = & F_{\mathrm{d}} \nabla H_{\mathrm{d}} \, . \end{array}$$

∎

C. Basic IDA PBC

Although in some cases the choice of the matrices J_{d} and R_{d} in IDA may be motivated by physical considerations, besides the requirement of the solvability of the matching equations, there are no general guidelines. If the original system already has the PH form

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x} & = & F(x) \nabla H(x) + g(x) u \\ y & = & g^\top(x) \nabla H(x) \end{array} \right. , \tag{28}$$

with $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ $F + F^\top \leq 0$, one natural first choice of F_d is simply

$$F_\mathrm{d} = F\ .$$

In this case the controller is called Basic IDA (BIDA) and the equation to solve is, according to (23),

$$\begin{aligned} gu_{\mathtt{SF}} &= -F\nabla H - R_\mathrm{d}\nabla H_\mathrm{d} + J_\mathrm{d}\nabla H_\mathrm{d} \\ &= F\nabla H_\mathrm{a}\ . \end{aligned} \tag{29}$$

Notice that, in general, in BIDA the dissipation is modified from

$$d = -\nabla H^\top f = -\nabla H^\top F\nabla H = \nabla H^\top R\nabla H$$

to

$$d_\mathrm{d} = -\nabla H_\mathrm{d}^\top f_\mathrm{d} = -\nabla H_\mathrm{d}^\top F\nabla H_\mathrm{d} = \nabla H_\mathrm{d}^\top R\nabla H_\mathrm{d}\ .$$

We close this section with an interesting property of BIDA controllers.

Proposition 4: A BIDA controller that is output-preserving is necessarily dissipation-preserving, consequently, it is EB.

Proof: Premultiply (29) by $\nabla H_\mathrm{a}^\top$ to obtain

$$\nabla H_\mathrm{a}^\top gu_{\mathtt{SF}} = \nabla H_\mathrm{a}^\top F\nabla H_\mathrm{a} = -\nabla H_\mathrm{a}^\top R\nabla H_\mathrm{a}\ ,$$

where $R(x) := -\frac{1}{2}(F(x) + F^\top(x))$. Under the assumption of output preservation, *i.e.*, $\nabla H_\mathrm{a}^\top g = 0$, equation (17) shows that $\nabla H_\mathrm{a}^\top R\nabla H_\mathrm{a} = 0$. Since R is symmetric and positive semidefinite,

$$R\nabla H_\mathrm{a} = 0\ .$$

This means that dissipation is preserved:

$$d_\mathrm{d} = (\nabla H + \nabla H_\mathrm{a}^\top)R(\nabla H + \nabla H_\mathrm{a}) = \nabla HR\nabla H = d\ .$$

■

V. Basic IDA-PBC is Energy-Balancing

In the preceding sections we used a particular version of Hill-Moylan's Theorem for systems without feedthrough terms (*cf.* Theorem 1). In this section we show that the incorporation of a feedthrough component allows to generate new cyclo-passive outputs. In particular, to identify one which is invariant to the action of BIDA. It turns out that dissipation associated to the new output is also invariant under BIDA. Output and dissipation invariance then establish that BIDA is EB (with respect to the definition of the new output).

A. Passivity-Based Control for Systems with Feedthrough

Let us start by recalling the general version of Hill-Moylan's Theorem [9].

Theorem 2: Consider a system with feedthrough described by

$$\Sigma^j : \left\{ \begin{array}{rcl} \dot{x} &=& f(x) + g(x)u \\ y^j &=& h^j(x) + j(x)u \end{array} \right. ,\quad j \in \mathbb{R}^{m\times m}$$

where $j : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ and $h^j : \mathbb{R}^n \to \mathbb{R}^m$. Σ^j is cyclo-passive with storage function H if and only if, for some $q \in \mathbb{N}$, there exist functions $l : \mathbb{R}^n \to \mathbb{R}^q$ and $w : \mathbb{R}^n \to \mathbb{R}^{q\times m}$ such that

$$\nabla H^\top(x)f(x) = -|l(x)|^2 \tag{30a}$$

$$h^j(x) = g^\top(x)\nabla H(x) + 2w^\top(x)l(x), \tag{30b}$$

$$w^\top(x)w(x) = \frac{1}{2}(j^\top(x) + j(x))\ , \tag{30c}$$

with $|\cdot|$ the Euclidean norm.

The power balance equation for Σ^j is

$$\dot{H} = (y^j)^\top u - d^j \tag{31}$$

with the dissipation given by

$$d^j(x) = |l(x) + w(x)u|^2\ . \tag{32}$$

Theorem 2 can be used to *construct* new cyclo-passive outputs. Indeed, it provides a means to parameterize the output function h^j and the dissipation function d^j in terms of the free square matrix j (hence the notation). If we set $j = 0$, then $\Sigma^j = \Sigma$ and, according to Assumption 1, equation (30a) must hold for some l—hence, l is fixed. Now, for all j, whose symmetric part is positive semidefinite, there always exist w satisfying (30c). w can then be used to define, via (30b) and (32), h^j and d^j, respectively.

Considering relative degree zero systems allows for an extension, provided by the free matrix j, of the set PBC given in Definition 1.[7]

Definition 3 (The extended set PBC): The state-feedback $u_{\mathtt{SF}} \in$ PBC if and only if there exists functions $H_d : \mathbb{R}^n \to \mathbb{R}$ and $h_\mathrm{d}^j : \mathbb{R}^n \to \mathbb{R}^m$ such that the system

$$\Sigma_\mathrm{d}^j : \left\{ \begin{array}{rcl} \dot{x} &=& f_\mathrm{d}(x) + g(x)v \\ y_\mathrm{d}^j &=& h_\mathrm{d}^j(x) + j(x)v \end{array} \right. \tag{33}$$

with $f_\mathrm{d}(x) := f(x) + g(x)u_{\mathtt{SF}}(x)$, is cyclo-passive with storage function H_d, *i.e.*, it satisfies the dissipation inequality

$$\dot{H}_\mathrm{d} \leq (y_\mathrm{d}^j)^\top v\ .$$

Again, from Hill-Moylan's Theorem we get the power balance equation for Σ^j

$$\dot{H}_\mathrm{d} = (y_\mathrm{d}^j)^\top v - d_\mathrm{d}^j\ , \tag{34}$$

with dissipation

$$d_\mathrm{d}^j(x) = |l_d(x) + w(x)v|^2$$

and

$$h_\mathrm{d}^j = g^\top\nabla H_\mathrm{d} + 2w^\top l_\mathrm{d}, \tag{35}$$

where $l_\mathrm{d} : \mathbb{R}^n \to \mathbb{R}^q$ verifies

$$\nabla H_\mathrm{d}^\top(x)f_\mathrm{d}(x) = -|l_\mathrm{d}(x)|^2 \tag{36}$$

and w satisfies (30c).

[7]To avoid cluttering the notation this new set is still called PBC.

B. Generation of Cyclo-Passive Outputs for PH Systems

Although Proposition 2 is applicable to general nonlinear systems, our interest in this paper is restricted to the case when Σ is a PH system described by (28). For this class of systems a new cyclo-passive output, which is an extension of the power-shaping output introduced in [11] to the case when F is not full rank, is constructed.

To present the main result, which is contained in Proposition 5, the notion of *generalized inverse* of a matrix, a technical assumption and two lemmata, are needed.

Definition 4: [12] Let A be an $n \times o$ matrix of arbitrary rank. A generalized inverse of A is an $o \times n$ matrix A^- such that

$$AA^-A = A\ .$$

It should be pointed out that, in general, A^- is not unique; but it always exists [12, Lemma 2.2.3].

Assumption 2: Σ is a PH system described by (28) and satisfies

$$F^\top (F^-)^\top F = F \tag{37}$$

and

$$\text{span } g \subseteq \text{span } F\ . \tag{38}$$

It is important to underscore that equation (37) does not depend on the particular choice of F^- (see [12]). Furthermore, if F is nonsingular, then (37) and (38) are immediately satisfied.

Lemma 1: the equation

$$F^\top(x)Z(x)F(x) = -F(x)\ , \tag{39}$$

with unknown $Z : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, is consistent (*i.e.*, at least one such Z exists) if and only if (37) is satisfied.

Proof: Equation (39) is a special case of the linear matrix equation

$$AXB = C\ , \tag{40}$$

where X is the unknown. According to [12, Theorem 2.3.2], equation (40) is consistent if and only if

$$AA^-CB^-B = C\ . \tag{41}$$

By matching the terms in (39) and (40) we get

$$A = F^\top\ , \quad X = Z\ , \quad B = F \quad \text{and} \quad C = -F\ .$$

By substituting these in (41) we obtain

$$\begin{array}{rcl} -F^\top (F^\top)^- F F^- F &=& -F \\ \Leftrightarrow \quad F^\top (F^-)^\top F &=& F \end{array}$$

(recall that $FF^-F = F$ and that a possible generalized inverse of $F^\top$ is $(F^-)^\top$). ■

Lemma 2: Equations (39) and (38) imply that

$$F^\top Z g = -g\ . \tag{42}$$

Proof: Equation (38) implies the existence of a mapping $\beta : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ such that

$$g(x) = F(x)\beta(x)\ .$$

On the other hand, equation (39) implies that

$$F^\top Z F \beta = -F\beta$$

for any β. Combining the last two equations yields (42). ■

Proposition 5: Consider a system Σ satisfying Assumption 2 and define

$$Z(x) := -(F^-)^\top(x) F(x) F^-(x)\ . \tag{43}$$

The system

$$\Sigma^j : \left\{ \begin{array}{rcl} \dot{x} &=& F(x)\nabla H(x) + g(x)u \\ y^j &=& g^\top(x)Z(x)F(x)\nabla H(x) + \\ && + g^\top(x)Z(x)g(x)u \end{array} \right. \tag{44}$$

is cyclo-passive with storage function H.

Proof: The proof is established verifying the conditions of Theorem 2. Notice that for system (44) we have

$$j = g^\top Z g \tag{45}$$

and

$$h^j = g^\top Z F \nabla H\ . \tag{46}$$

We will show that there exists functions l and w such that (30) is satisfied. Because of (37) and Lemma 1, equation (39) is consistent. Under Assumption 2, (43) is a particular positive semidefinite solution. Equation (39) implies that

$$\nabla H^\top F^\top Z F \nabla H = -\nabla H^\top F \nabla H\ .$$

Given Z compute $Y : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ as

$$Y^\top Y = \frac{Z^\top + Z}{2}, \tag{47}$$

which can always be obtained since $(Z + Z^\top) \geq 0$. It is then easy to see that

$$l := Y F \nabla H, \tag{48}$$

satisfies (30a). Furthermore,

$$w = Yg\ . \tag{49}$$

satisfies

$$w^\top w = g^\top \frac{Z^\top + Z}{2} g = \frac{1}{2}(j^\top + j)\ .$$

Substituting l and w into (30b) one obtains

$$\begin{array}{rcl} h^j &=& g^\top \nabla H + 2 g^\top Y^\top Y F \nabla H \\ &=& g^\top \nabla H + g^\top (Z^\top + Z) F \nabla H \\ &=& g^\top \nabla H - g^\top \nabla H + g^\top Z F \nabla H \\ &=& g^\top Z F \nabla H\ , \end{array}$$

where (47) is used to obtain the second identity, while (39) and Lemma 2 are invoked in the third one. ■

Remark 3: When F is nonsingular, the new cyclo-passive output y^j coincides with the power-shaping output of [11]. It is shown in [13] that the generation of the new output, for a class of electrical and electromechanical systems, is tantamount to the application of the classical Thevenin-Norton transformation of electrical circuits. Additional connections with power-shaping may be found in these two papers.

C. Basic IDA PBC is Energy-Balancing

As one might expect, in the zero-relative-degree case there is also a connection between energy-balancing and output and dissipation invariance.

Proposition 6: $u_{\mathtt{SF}} \in \mathrm{PBC}\ \cap\ \mathrm{EB}$ if the output and the dissipation remain invariant. That is, if (34) holds with

$$y^j = y^j_{\mathrm{d}} \quad \text{and} \quad d^j = d^j_{\mathrm{d}} \; . \tag{50}$$

Proof: By substracting (31) from (34) it is readily seen that

$$\dot{H}_{\mathrm{a}} = (y^j_{\mathrm{d}})^\top v - (y^j)^\top u + d^j_{\mathrm{d}} - d^j \; , \tag{51}$$

with H_{a} defined as in (13). Substitution of the hypothesis (50) into (51) yields

$$\dot{H}_{\mathrm{a}} = (y^j)^\top (v - u) \; .$$

Since $u = u_{\mathtt{SF}} + v$,

$$\dot{H}_{\mathrm{a}} = -(y^j)^\top u_{\mathtt{SF}} \; . \tag{52}$$

■

The next proposition shows that y^j and d^j are invariant under BIDA control.[8]

Proposition 7: Consider the cyclo-passive system (43), (44) and suppose that Assumption 2 holds. The BIDA control given by (29) is a PBC with

$$h^j_{\mathrm{d}} = g^\top Z F \nabla H_{\mathrm{d}} \; . \tag{53}$$

Moreover, the controller is output and dissipation preserving. Therefore, it is EB.

Proof: We will show first that the closed-loop

$$\Sigma^j_{\mathrm{d}} : \left\{ \begin{array}{rcl} \dot{x} &=& F\nabla H_{\mathrm{d}} + gv \\ y^j_{\mathrm{d}} &=& g^\top Z(F\nabla H_{\mathrm{d}} + gv) \end{array} \right. \tag{54}$$

is cyclo-passive with storage function H_{d}. To this effect, we will prove that there exists an l_{d} such that (36) and (35) are valid. Indeed, equation (39) implies that

$$\nabla H_{\mathrm{d}}^\top F^\top Z F \nabla H_{\mathrm{d}} = -\nabla H_{\mathrm{d}}^\top F \nabla H_{\mathrm{d}} \; ,$$

so

$$l_{\mathrm{d}} = Y F \nabla H_{\mathrm{d}} \; ,$$

with Y defined as in (47), satisfies (36). Selecting w as (49) and substituting into (35) gives

$$\begin{aligned} h^j_{\mathrm{d}} &= g^\top \nabla H_{\mathrm{d}} + g^\top (Z^\top + Z) F \nabla H_{\mathrm{d}} \\ &= g^\top \nabla H_{\mathrm{d}} - g^\top \nabla H_{\mathrm{d}} + g^\top Z F \nabla H_{\mathrm{d}} \\ &= g^\top Z F \nabla H_{\mathrm{d}} \; . \end{aligned} \tag{55}$$

This proves our cyclo-passivity claim.

For output preservation, we will prove that

$$\begin{aligned} y^j &= y^j_{\mathrm{d}} \\ \Leftrightarrow \quad h^j + j u_{\mathtt{SF}} &= h^j_{\mathrm{d}} \; . \end{aligned} \tag{56}$$

Equation (45) and (29) imply that

$$j u_{\mathtt{SF}} = g^\top Z g u_{\mathtt{SF}} = g^\top Z F \nabla H_{\mathrm{a}}.$$

Replacing in (55) yields

$$h^j + j u_{\mathtt{SF}} = g^\top Z F \nabla H + g^\top Z F \nabla H_{\mathrm{a}} = g^\top Z F \nabla H_{\mathrm{d}} \; . \tag{57}$$

From (57) and (53) one obtains (56).

Regarding dissipation, we will prove that $d^j = d^j_{\mathrm{d}}$, that is,

$$|l + w u_{\mathtt{SF}} + wv|^2 = |l_{\mathrm{d}} + wv|^2 \; . \tag{58}$$

Direct substitution of the expressions of l and w gives

$$\begin{aligned} l + w u_{\mathtt{SF}} &= Y F \nabla H + Y g u_{\mathtt{SF}} \\ &= Y F \nabla H + Y F \nabla H_{\mathrm{a}} \\ &= Y F \nabla H_{\mathrm{d}} \; . \end{aligned}$$

Since l_{d} is equal to $YF\nabla H_{\mathrm{d}}$, we conclude (58). ■

Remark 4: Notice that the property of energy-balancing for BIDA is established with respect to the definition of the new passive output (*cf.* (52)), which is obviously different from (16).

VI. Conclusions

A framework for analysis and design of PBC, based on the principles of dissipation and output preservation, has been derived. This framework allows to classify various PBCs according to Table I, where the key algebraic equations[9] that define the sets are given in parenthesis.

		Dissipation		**Output**
$u_{\mathtt{SF}} \in$ EB ($y^\top u_{\mathtt{SF}} = -\nabla H_{\mathrm{a}}^\top f$)	$\Longleftrightarrow$	Preserved	&	Preserved
$u_{\mathtt{SF}} \in$ BIDA ($g u_{\mathtt{SF}} = (J - R)\nabla H_{\mathrm{a}}$)	$\Longrightarrow$	$\nabla H_{\mathrm{d}}^\top R \nabla H_{\mathrm{d}}$	&	$g^\top \nabla H_{\mathrm{d}}$
if, in addition $g^\top \nabla H_{\mathrm{a}} = 0$ or $j = g^\top Z g$	$\Longrightarrow$	Preserved	&	Preserved
$u_{\mathtt{SF}} \in$ IDA ($g u_{\mathtt{SF}} =$ $-f + (J_{\mathrm{d}} - R_{\mathrm{d}})\nabla H_{\mathrm{d}}$)	$\Longrightarrow$	$\nabla H_{\mathrm{d}}^\top R_{\mathrm{d}} \nabla H_{\mathrm{d}}$	&	$g^\top \nabla H_{\mathrm{d}}$

TABLE I

Classifying different PBCs according to their dissipation and output preservation properties.

The equivalence between output and dissipation preservation and the important property of EB has been established. In this regard, we identified zero-relative-degree outputs that are invariant under BIDA control, rendering it EB.

The properties of output and dissipation preservation are also important in dynamic PBC, such as Control by Interconnection (CbI). CbI is output and dissipation preserving by construction (see Fig. 2 and [6] for details). We hope then that the results presented here will provide a means to extend the work done in [6], where the relationships among CbI and different PBCs are studied (see also [14]).

[8] A similar result (using different arguments) was obtained in [13] for the case when F is nonsingular.

[9] If the energy functions are seen as unknown these equations are, of course, partial differential equations.

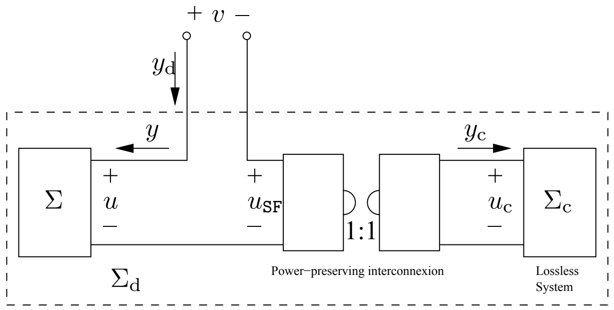

Fig. 2. The CbI scheme.

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