Robust output regulation of linear passive systems using maximally monotone controls

Félix A. Miranda and Fernando Castaños

Abstract— **The output regulation problem is addressed. We start with a linear passive system without feed-forward term and we look for a static set-valued output feedback control law which guarantees the regulation of the output to a desired fixed value, even in the presence of external matched and bounded perturbations. The multivalued control is taken as the subdifferential of some proper, convex, lower semicontinuous function with effective domain restricted to a closed convex set S, yielding to a maximally monotone set-valued map. The existence and convergence of system trajectories to an equilibrium set are established. By using the so-called Yosida approximation, some implementation issues regarding the multivalued control law are addressed. Finally, some examples are presented to illustrate the robustness features in the output of the closed-loop.**

I. Introduction

The study of systems with multivalued right-hand sides is an active area of research in the control community, its study for modeling and analysis of processes is extensive, see e.g. [1], [2], [3], [4], [5], [6], to name a few. Among all of this works, maximal monotone set-valued maps yield to a well-posed problems enjoying important features. Namely, existence and uniqueness of solutions, continuity of solutions with respect to initial conditions and approximation of solutions through maximally monotone Lipschitz single-valued maps (Yosida approximation); see, e.g., [2], [7], [8], [9].

On the other hand, the problem of designing a multivalued control in order to achieve a desired response is less explored, except for the case of sliding mode control [10] and recently some works using maximally monotone operators such as [9], [11], [12], [13].

The output regulation problem with a feed-forward term Du, was treated in previous works [12] where, under some conditions (which strongly depend on the inverse of the matrix D) it is possible to achieve perfect output regulation independently of the initial conditions of the system, even in the presence of parametric and external disturbances. Recently [13] the output regulation problem was treated using an internal model approach. The authors consider the design of a state feedback control law for systems of Lur'e type with multivalued right-hand side and developed a static and a dynamic control law which depend on both the system parameters and the system state.

In [11] the robustness and stability issues for Lagrangian systems with a maximally monotone set-valued control law were studied. The authors analyzed a control that only depends on the generalized velocity of the system. It is a well-known fact that Euler-Lagrange systems represent passive maps with the output taken as the generalized velocity [14, Chapter 4]. Taking advantage of this fact the authors obtain a static output feedback for which finite-time stability of the origin is established.

In this note, we propose the use of a static set-valued control law (more general than the conventional signum multifunction used in sliding-mode control) defined as the convex subdifferential of some proper, convex and lower semicontinuous function φ restricted to a closed and convex effective domain S. The closed-loop system shows useful and attractive features. Namely, the output is confined to S for all time t, the set of equilibrium points is globally stable (and globally asymptotically stable in the strict passivity case) and the output converges in finite time.

This note is organized as follows: The following subsection introduces mathematical preliminaries and notation are introduced. Section II establishes the robust output regulation problem, together with the static set-valued control law considered along this note. Some facts about existence and uniqueness of solutions, as well as existence, uniqueness, and stability of equilibrium points are established. In Section III the regulation problem is solved for the case of matched disturbances together with comments about the implementation. Some examples are presented. Finally, conclusions and further research directions are suggested in Section IV.

A. Mathematical preliminaries and notation

A set-valued function or multifunction $\mathbf{F}:\mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a map that associates with any $w \in \mathbb{R}^n$ a subset $\mathbf{F}(w) \subset \mathbb{R}^n$. The domain of $\mathbf{F}$ is given by $\operatorname{Dom}\mathbf{F} = \{w \in \mathbb{R}^n : \mathbf{F}(w) \neq \emptyset\}$. Related with the definition of a multifunction is the concept of its graph,

$$\operatorname{Graph}\mathbf{F} = \{(w,z) \in \mathbb{R}^n \times \mathbb{R}^n : z \in \mathbf{F}(w)\}.$$

The graph is used to define the concept of monotonicity of a multifunction in the following way: A set-valued function $\mathbf{F}$ is said to be monotone if for all $(w,z) \in \operatorname{Graph}\mathbf{F}$ and all $(w',z') \in \operatorname{Graph}\mathbf{F}$, the relation $\langle z-z', w-w'\rangle \geq 0$ is preserved with $\langle \cdot,\cdot \rangle$ the usual scalar product on $\mathbb{R}^n$. Moreover, $\mathbf{F}$ is called strongly monotone if $\langle z-z', w-w'\rangle \geq \alpha\|w-w'\|^2$ for some $\alpha > 0$. A monotone map $\mathbf{F}$ is called maximally monotone if, for every pair $(\hat{w},\hat{z}) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \operatorname{Graph}\mathbf{F}$, there exits $(w,z) \in \operatorname{Graph}\mathbf{F}$ with $\langle z-\hat{z}, w-\hat{w}\rangle < 0$ or, in other words, if no enlargement of its graph is possible in $\mathbb{R}^n \times \mathbb{R}^n$ without destroying monotonicity.

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Departamento de Control Automático, Cinvestav del IPN, Av. Instituto Politécnico Nacional 2508, México D.F., México. {fmiranda,fcastanos}@ctrl.cinvestav.mx

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous (lsc) function. The subdifferential $\partial f(w)$ of $f(\cdot)$ at $w \in \mathbb{R}^n$ is defined by

$$\partial f(w) = \{\zeta \in \mathbb{R}^n : f(\sigma) - f(w) \geq \langle \zeta, \sigma - w\rangle \quad \text{for all } \sigma \in \mathbb{R}^n\} .$$

It is a well known result that the subdifferential of a proper, convex, lsc function is a maximally monotone operator (see e.g., [1, Prop. 3.4.1]). An important convex function used through this note is the indicator function of a convex set S, defined by $\Psi_S(w) = 0$ if $w \in S$ and $\Psi(w) = +\infty$ otherwise.

It is easy to see that when $f(\cdot)$ is equal to the indicator function of a closed convex set S the subdifferential coincides with the normal cone of the set S at the point $w \in S$, i.e.,

$$\partial \Psi_S(w) = N_S(w) = \{\xi \in \mathbb{R}^n : 0 \geq \langle \xi, \sigma - w\rangle \quad \text{for all } \sigma \in S\} .$$

Notice that if w is in the interior of S then $N_S(w) = \{0\}$. If $w \notin S$, then we adopt the convention $N_S(w) = \emptyset$.

Let $\mathbf{F}$ be a maximally monotone set-valued map. The Yosida approximation of $\mathbf{F}$, denoted by F_λ, is a single-valued, Lipschitz continuous, maximally monotone operator given by

$$F_\lambda = \frac{1}{\lambda}\left(I - J_\lambda\right), \quad J_\lambda = \left(1 + \lambda F\right)^{-1},$$

where J_λ is the so-called resolvent of F. See, e.g. [1, Chapter 3], [15, Chapter 12] for a detailed account of properties about Yosida approximation.

Remark 1: As it is pointed out in [1] the Yosida approximation of a subdifferential can be obtained in the following way. Let $\partial f(w)$ be the subdifferential of the function $f : \mathbb{R}^n \to \mathbb{R}$ at the point w. Consider the Moreau-Yosida approximation of f,

$$f_\varepsilon(w) = \inf_{\zeta \in \mathbb{R}^n} \left\{ f(\zeta) + \frac{1}{2\varepsilon}\|\zeta - w\|^2 \right\},$$

where $\varepsilon > 0$. Then, the gradient $\nabla f_\varepsilon(w)$ is the Yosida approximation of $\partial f(w)$ [1, Th. 3.4].

Along this note, we denote the unitary open ball of $\mathbb{R}^n$ as $\mathcal{B}_n$.

We prove an elementary fact about solid convex cones.

Lemma 1: Let $\mathcal{K} \subset \mathbb{R}^m$ be a convex cone with non-empty interior, and let $\mathcal{V} \subset \mathbb{R}^m$ be a bounded set. Then, there exists $\vartheta \in \mathcal{K}$ such that $\{\vartheta\} + \mathcal{V} \subset \mathcal{K}$.

Proof: Since $\mathcal{K}$ is solid, there exists a base $\{\xi_1, \ldots, \xi_m\}$ contained in $\mathcal{K}$. Thus, every element of $\mathcal{V}$ can be written as $v = \sum_i^m \xi_i \nu_i$. Due to the boundedness of $\mathcal{V}$, there exists $\bar{\nu}$ such that $|\nu_i| \leq \bar{\nu}$, $i = 1, \ldots, m$. By setting $\vartheta = \sum_i^m \xi_i \bar{\nu} \in \mathcal{K}$, it is clear that $\vartheta + v \in \mathcal{K}$, since $\vartheta + v = \sum_i^m \xi_i(\nu_i + \bar{\nu})$ and $(\nu_i + \bar{\nu}) \geq 0$. ■

II. Problem statement

Consider the following linear system

$$\Sigma \begin{cases} \dot{x}(t) = Ax(t) + B_1 u(t) + B_2(v + \nu(t)) \\ y(t) = Cx(t) \\ x(0) = x_0, \end{cases}$$

where, $x(t) \in \mathbb{R}^n$ represents the system state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^m$ is the system output, the constant term $v \in \mathbb{R}^m$ is a system parameter and A, B_1, B_2 and C are constant matrices of suitable dimensions. The term $\nu(t) \in \mathbb{R}^m$ accounts for an exogenous perturbation signal which is considered bounded.

A special case to consider is when $B_1 = B_2$ (i.e., the matched perturbation case). In this case the parameter v can be assimilated by the control: $u(t) + v$ can be taken as the new control.

Some techniques, such as conventional sliding mode control, can achieve robust output regulation against parametric and external disturbances, by using the signum multifunction (which is the subdifferential of the absolute value function), but we do not restrict ourselves to that case. Instead, we consider a set-valued control law represented as the subdifferential of some proper, convex, lsc function with effective domain restricted to a convex set S.

The robust output regulation problem can be announced as follows.

Problem: The output regulation problem consists in regulating the output y to a desired fixed value $y_{\rm d}$, even in the presence of an external perturbation signal ν and (possibly unmatched) parametric uncertainty.

The class of linear systems that we consider is characterized by the following assumption.

Assumption 1: System Σ is passive, i.e., there exists a symmetric positive definite matrix P such that [17]

$$A^\top P + PA \leq 0 \tag{1}$$
$$PB_1 = C^\top. \tag{2}$$

Special cases of passive systems to consider are the so-called strictly passive systems, which satisfy the Lyapunov inequality strictly, i.e.

$$A^\top P + PA + \varepsilon P \leq 0 \tag{3}$$
$$PB_1 = C^\top, \tag{4}$$

for some $\varepsilon > 0$.

A. The set-valued controller

We propose the use of the following set-valued static control

$$-u(t) \in \partial\Phi(y(t)) \tag{5}$$

where $\Phi = \Psi_S + \varphi$, with Ψ_S the indicator function of a closed convex set S and $\varphi : \mathbb{R}^m \to \mathbb{R}$ is a proper, convex and lsc function. Observe that Φ is the restriction of φ to S and is proper, convex and lsc.

It is a well known result that, under the passivity assumption, system Σ admits a unique solution $x \in C^0(\mathbb{R}_+; \mathbb{R}^n)$

such that $\dot{x} \in L^{\infty}_{\text{loc}}(\mathbb{R}_+;\mathbb{R}^n)$ and $y(t) = Cx(t) \in S = \text{Dom}(\partial\Phi)$ for all time $t \geq 0$, whenever $y_0 = Cx_0 \in S$. See [9] and [7] for a detailed account about this fact.

With the multivalued control (5), the closed-loop system results in

$$\dot{x}(t) - Ax(t) - B_2(v + \nu(t)) \in -B_1\partial\Phi(y(t)) \quad \text{(6a)}$$
$$y(t) = Cx(t) \quad \text{(6b)}$$
$$Cx(0) = y_0 \in S. \quad \text{(6c)}$$

The closed-loop system can also be written as a *hemivariational inequality of evolution* [18], by using the subdifferential's definition. Namely, multiplying both sides of (6a) by P and applying the chain rule for subdifferentials [16, Ch. VI, Th. 4.2.1], we have

$$\langle -P\dot{x}(t) + PAx(t) + B_2(v + \nu(t)), \sigma - x(t)\rangle \leq \hat{\Phi}(\sigma) - \hat{\Phi}(x(t)),\ \forall\, \sigma \in \mathbb{R}^n.$$

Moreover, system (6) is related to complementarity systems [19] and projected dynamical systems [20] as stated in [8].

III. Robust output regulation

In this section the robust output regulation problem is dealt with. We start introducing Lemma 2 about reachability of the desired output, then we prove that it is always possible to achieve output regulation of admissible outputs even in the presence of external matched disturbances.

A. Perfect output regulation

We starting writing the equilibrium points associated to the nominal closed-loop system (6) as

$$Ax_* + B_2v \in B_1\partial\Phi(Cx_*). \quad (7)$$

Multiplying both sides of (7) by P and using (2) together with the chain rule for subdifferentials, we have that solutions of (7) are also solutions of the inclusion

$$P\,(Ax_* + B_2v) \in \partial\hat{\Phi}(x_*), \quad \hat{\Phi} = \Phi \circ C, \quad (8)$$

and vice-versa. Now, notice that the linear map $-PA$ is monotone (in fact it is maximally monotone [1, Prop. 3.3]) then a direct use of Theorem 3.11.2 of [21] gives the existence result for any $v \in \mathbb{R}^m$.

Once the existence of equilibria has been established, we introduce the set

$$\mathcal{E}(v,S) = \Big\{x \in \mathbb{R}^n : P(Ax + B_2v) \in \partial\hat{\Phi}(x)\Big\}$$

as well as the set of points which produce the desired output y_{d},

$$\mathcal{C} = \{x \in \mathbb{R}^n : Cx = y_{\text{d}},\ y_{\text{d}} \in S\} \subset \text{Dom}\,\partial\hat{\Phi}.$$

We will need an extra assumption concerning the convex set $S \subset \mathbb{R}^m$.

Assumption 2: The normal cone to $S \subset \mathbb{R}^m$ at y_{d} is *solid*, i.e., it has non empty interior.

The last assumption is very mild. It is clear that if we take the set S as a polyhedron with y_{d} as one of its vertex then $\text{int}\, N_S(y_{\text{d}}) \neq \emptyset$. As we shall see later (in the proof of Theorem 1), the condition about non emptiness of $N_S(y_{\text{d}})$ is crucial in order to remove the effect of the bounded disturbance ν.

Conditions under which we obtain $x \in \mathcal{E}(v,S) \cap \mathcal{C}$ are established in the following Lemma.

Lemma 2: If for some $x_* \in \mathcal{E}(v,S)$, the inclusion

$$P(Ax_* + B_2v) \in \text{rint}\,\partial\hat{\Phi}(\xi)$$

is satisfied for some $\xi \in \mathcal{C}$, where $\text{rint}\,\mathcal{A}$ refers to the interior of the set $\mathcal{A}$ relative to its *affine hull* (see e.g. [22, Sec. 6]), then $\mathcal{E}(v,S) \subset \mathcal{C}$.

In words, every equilibrium is associated with the desired output.

Proof: Let $x_* = x_1 \in \mathcal{E}(v,S)$ and define $\eta_i := P(Ax_i + B_2v)$ with $x_i \in \mathcal{E}(v,S)$, $i = 1,2$. It is easy to see that, by the definition of η_i, the following equation is fulfilled

$$\sup_{\sigma \in \text{Dom}\,\hat{\Phi}} \Big\{\langle \eta_1, \sigma\rangle - \hat{\Phi}(\sigma)\Big\} = \langle \eta_1, x_1\rangle - \hat{\Phi}(x_1). \quad (9)$$

Since by assumption $\eta_1 \in \text{rint}\,\partial\hat{\Phi}(\xi)$, there exists $\varepsilon > 0$ such that

$$\{\eta_1\} + \Big(\text{aff}\,\partial\hat{\Phi}(\xi) \cap \varepsilon\mathcal{B}_n\Big) \subset \partial\hat{\Phi}(\xi),$$

for some $\xi \in \mathcal{C}$. Notice that, due to $\partial\hat{\Phi}(\xi) = C^\top\partial\Phi(y_{\text{d}})$ and considering that $\text{int}\,\partial\Phi(y_{\text{d}})$ is not empty in $\mathbb{R}^m$ (see Assumption 2), we have $\text{aff}\,\partial\hat{\Phi} = \text{Im}\,C^\top$.

Equivalently, η_1 satisfies the following inequality

$$\langle \eta_1 + \rho, \sigma - \xi\rangle \leq \hat{\Phi}(\sigma) - \hat{\Phi}(\xi),$$

for all $\sigma \in \text{Dom}\,\partial\hat{\Phi}$, and for all $\rho \in \text{Im}\,C^\top \cap \varepsilon\mathcal{B}_n$ so that

$$\langle \eta_1, \sigma - \xi\rangle - \hat{\Phi}(\sigma) + \hat{\Phi}(\xi) \leq -\theta(\sigma - \xi),$$

where $\theta(\sigma - \xi) := \sup_{\rho \in \text{Im}\,C^\top \cap \varepsilon\mathcal{B}_n}\langle \rho, \sigma - \xi\rangle$. Using Lemma 1 in [8] we have that $\theta(\sigma - \xi) = 0$ if and only if $\sigma - \xi \in \left(\text{Im}\,C^\top\right)^\perp = \text{Ker}\,C$ (see, e.g., [23]), i.e., $\theta(\sigma - \xi) = 0$ if and only if $\sigma \in \mathcal{C}$, and $\theta(\sigma - \xi) = \beta > 0$ for all $\sigma \notin \mathcal{C}$.

Assuming that $x_1 \notin \mathcal{C}$, and taking $\sigma = x_1 \in \text{Dom}\,\hat{\Phi}$ in the previous inequality, leads us to

$$\begin{aligned}\langle \eta_1, x_1\rangle - \hat{\Phi}(x_1) &\leq \langle \eta_1, \xi\rangle - \hat{\Phi}(\xi) - \theta(x_1 - \xi)\\ &< \langle \eta_1, \xi\rangle - \hat{\Phi}(\xi),\end{aligned}$$

which is a contradiction in view of (9). Hence, $x_1 \in \mathcal{C}$.

Now assume that there exists an $x_2 \in \mathcal{E}(v,S)$ such that $x_2 \notin \mathcal{C}$, thus by definition of the set $\mathcal{E}(v,S)$ we have

$$\langle \eta_2, \sigma - x_2\rangle \leq \hat{\Phi}(\sigma) - \hat{\Phi}(x_2)$$

subtracting and adding η_1 on the left-hand side of the inner product and setting $\sigma = x_1$ results in

$$\langle -PA(x_1 - x_2), x_1 - x_2\rangle + \langle \eta_1, x_1 - x_2\rangle \leq \hat{\Phi}(x_1) - \hat{\Phi}(x_2)$$

Recalling the maximal monotonicity of the $-PA$ map we have

$$\hat{\Phi}(x_2) - \hat{\Phi}(x_1) \leq \langle \eta_1, x_2 - x_1\rangle \leq \hat{\Phi}(x_2) - \hat{\Phi}(x_1)$$

therefore

$$\begin{aligned}\langle\eta_1, x_2\rangle - \hat{\Phi}(x_2) &= \langle\eta_1, x_1\rangle - \hat{\Phi}(x_1)\\ &= \sup_{\sigma\in\operatorname{Dom}\partial\hat{\Phi}}\left\{\langle\eta_1,\sigma\rangle - \hat{\Phi}(\sigma)\right\}.\end{aligned}$$

Applying the same argument as before, we conclude that $x_2 \in \mathcal{C}$ and therefore $\mathcal{E}(v, S) \subset \mathcal{C}$. ■

Now, we turn our study to robustness under exogenous bounded disturbances. We consider the matched perturbation case ($B_1 = B_2$).

Theorem 1 (Main Theorem): Assume that (1) and (2) are fulfilled, consider the closed-loop system described by (6) with $B_1 = B_2$. Moreover, assume that the conditions of Lemma 2 are satisfied for some $v \in \mathbb{R}^m$. Then, for some $0 \leq R < \infty$ and all disturbances ν such that $\|\nu\| \leq R$, there exists $v^* = v + \vartheta$ with $\vartheta \in N_S(y_\mathrm{d})$, such that the set of equilibrium points $\mathcal{E}(v, S)$ is stable and y converges to y_d.

Proof: Consider the closed-loop system (6) and the following candidate Lyapunov function for the set $\mathcal{E}(v, S)$,

$$V(x) = \frac{1}{2}\operatorname{dist}_P^2\left(x, \mathcal{E}(v, S)\right),$$

where $\operatorname{dist}_P(\zeta, \mathcal{A})$ accounts for the distance from the point ζ to the set $\mathcal{A}$ in norm P, with P satisfying (1) and (2). In other words,

$$\operatorname{dist}_P^2(\zeta, \mathcal{A}) = \min_{\xi\in\mathcal{A}}\|\zeta - \xi\|_P^2 = \min_{\xi\in\mathcal{A}}(\zeta-\xi)^\top P(\zeta-\xi).$$

Taking the derivative of V along system trajectories, leads us to

$$\begin{aligned}\dot{V} =& -\frac{1}{2}(x-x_*)^\top Q(x-x_*) - (x-x_*)^\top\phi_1\\ &+ (x-x_*)^\top P(Ax_* + B_1 v) + (x-x_*)^\top PB_1\nu,\end{aligned}$$

where $0 \leq Q = -(A^\top P + PA)$, $\phi_1 \in \partial\hat{\Phi}(x)$ and we use the fact that $\nabla V(x) = P(x - x_*)$ (see, e.g., [5, Prop. 2.33]) with x_* the projection of the point x into the closed, convex set $\mathcal{E}(v, S)$ using norm P, i.e.,

$$x_* = \operatorname*{arg\,min}_{\xi\in\mathcal{E}(v,S)}\|x - \xi\|_P \in \mathcal{E}(v, S).$$

Thus, $PAx_* + PB_1 v = \phi_2$, for some $\phi_2 \in \partial\hat{\Phi}(x_*)$, then the subdifferential's definition leads to

$$\begin{aligned}\langle\phi_1, \sigma_1 - x\rangle &\leq \hat{\Phi}(\sigma_1) - \hat{\Phi}(x)\\ \langle\phi_2, \sigma_2 - x_*\rangle &\leq \hat{\Phi}(\sigma_2) - \hat{\Phi}(x_*).\end{aligned}$$

Taking $\sigma_1 = x_* \in \operatorname{Dom}\partial\hat{\Phi}$, $\sigma_2 = x \in \operatorname{Dom}\partial\hat{\Phi}$ and substituting in $\dot{V}$ yields

$$\dot{V} \leq -\frac{1}{2}\lambda_{\min}(Q)\|x - x_*\|^2 + \langle\vartheta + \nu, Cx - Cx_*\rangle,$$

where $Cx \in S$ and $Cx_* = y_\mathrm{d}$. From Lemma 1, we obtain $\vartheta + \nu \in \operatorname{int} N_S(y_\mathrm{d})$ for some $\vartheta \in N_S(y_\mathrm{d})$. Recall that we are in the matched case ($B_1 = B_2$), so the term ϑ can be viewed as a component of the control input u. Hence, there exist $\varepsilon > 0$ such that $\dot{V}(x) \leq -\varepsilon\|Cx - Cx_*\|$ for all $x \in \mathbb{R}^n$. Moreover, $\dot{V}(x) < 0$ for all $x \notin \mathcal{C}$, consequently $x \to \mathcal{C}$ (see [24, Ch. 1.12]). ■

Remark 2: It is worth to mention that in the case of strict passivity, we have a unique equilibrium point x_* satisfying (8), which furthermore is globally asymptotically stable (because the matrix Q is positive definite).

In addition, under the strictly passive assumption it is possible to regulate the output even in the presence of unmatched disturbances of small magnitude. Indeed, the time derivative of V satisfies

$$\begin{aligned}\dot{V} &\leq -\frac{1}{2}\lambda_{\min}(Q)\|x - x_*\|^2 + \langle PB_2 v, x - x_*\rangle\\ &\leq -\frac{1}{2}\left(\lambda_{\min}(Q) - 2\|PB_2\|\|\nu\|\right)\|x - x_*\|^2,\end{aligned}$$

Hence, we obtain asymptotic stability of x_* whenever $\|\nu\| < \lambda_{\min}Q/2\|PB_2\|$.

Remark 3: From Lemma 2 we have that, for some $x_1 \in \mathcal{E}(v, S)$, the term $\eta_1 \in \operatorname{rint} C^\top\partial\Phi(y_\mathrm{d})$. Likewise, for any $x_2 \in \mathcal{E}(v, S)$, the term $\eta_2 \in C^\top\partial\Phi(y_\mathrm{d})$ with $\eta_i = P(Ax_i + B_2 v)$, $i = 1, 2$. Hence, for all $\lambda \in (0, 1]$ we have $\eta_\lambda \in \operatorname{rint} C^\top\partial\Phi(y_\mathrm{d})$, where $\eta_\lambda = \lambda\eta_1 + (1 - \lambda)\eta_2$.

Motivated by the previous observation, we investigate whether there exist special cases where it is not be necessary to add the term ϑ to the controller in order to obtain output convergence if the external disturbance is small enough. Namely, if $\eta_i \in \operatorname{rint} C^\top\partial\Phi(y_\mathrm{d})$ for any $x_i \in \mathcal{E}(v, S)$, then the time derivative of V turns into

$$\dot{V} \leq -\frac{1}{2}\lambda_{\min}(Q)\|x - x_*\|^2 + \langle\nu, y - y_\mathrm{d}\rangle - \theta(x - x_*).$$

Now Lemma 1 in [8] and Corollary 3.22 in [25] lead us to $\theta(\zeta) = \varepsilon\|\operatorname{Proj}_{\operatorname{Im}C^\top}(\zeta)\|$. Moreover, because $\operatorname{Im}C^\top$ is a subspace we have $\zeta - \operatorname{Proj}_{\operatorname{Im}C^\top(\zeta)} \in (\operatorname{Im}C^\top)^\perp = \operatorname{Ker}C$ and then $C\zeta = C\operatorname{Proj}_{\operatorname{Im}C^\top}(\zeta)$, which implies $\|C\zeta\|^2 = \|C\operatorname{Proj}_{\operatorname{Im}C^\top}(\zeta)\|^2 \leq \lambda_{\max}(C^\top C)\|\operatorname{Proj}_{\operatorname{Im}C^\top}(\zeta)\|^2$. Finally we obtain the inequality $\theta(\zeta)\sqrt{\lambda_{\max}(C^\top C)} \geq \varepsilon\|C\zeta\|$ and we get

$$\begin{aligned}\dot{V} \leq& -\frac{1}{2}\lambda_{\min}(Q)\|x - x_*\|^2\\ &- \left(\frac{\varepsilon}{\sqrt{\lambda_{\max}(C^\top C)}} - \|\nu\|\right)\|y - y_\mathrm{d}\|,\end{aligned}$$

from where it is clear that robust output regulation is obtained whenever $\|\nu\| < \frac{\varepsilon}{\sqrt{\lambda_{\max}(C^\top C)}}$.

Remark 4: Notice that Theorem 1 does not guarantee asymptotic stability of the equilibrium set. Direct application of LaSalle's invariance principle assures that the state will converge to the greatest invariant set contained in $\{x \in \mathbb{R}^n : \dot{V}(x) = 0\} \subset \mathcal{C}$, which is not necessarily equal to $\mathcal{E}(v, S)$. However, the output regulation is attained.

B. Implementation issues

As it was mentioned before, the subdifferential of a proper, convex, lsc function $\Phi : \mathbb{R}^m \to \mathbb{R}$ defines a maximally monotone operator. This kind of operators enjoy the property of approximation by singled-valued, Lipschitz continuous, maximally monotones maps (see Remark 1).

In our case we are dealing with maximally monotone controls of the form $-u \in \partial\Phi(y)$ where $\Phi = \Psi_S + \varphi$

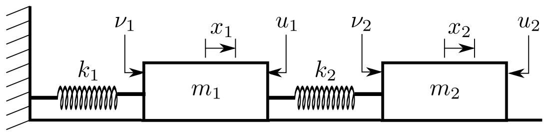

Fig. 1. Mass spring system.

for some function φ which is proper, convex and lsc. If in addition, we consider the case where φ is a continuously differentiable function then the control u takes the form $-u(t) - \nabla\varphi(y(t)) \in N_S(y(t))$. Recalling Remark 1, it follows that the Moreau-Yosida approximation f_ε of the indicator function Ψ_S satisfies $f_\varepsilon(y) = \frac{1}{2\varepsilon}\operatorname{dist}^2(y, S)$.

Using once again Proposition 2.33 of [5], we obtain the Yosida approximation to the normal cone as the gradient $\nabla f_\varepsilon = \frac{y - \mathrm{Proj}_S(y)}{\varepsilon}$. Thus, we can approximate the control input u by:

$$-u_\varepsilon(t) = \frac{y(t) - \mathrm{Proj}_S(y(t))}{\varepsilon} + \nabla\varphi(y(t)), \qquad (10)$$

from where we obtain a control which does not depend directly neither on the state nor the system parameters. Moreover, there exists a unique solution to

$$\dot{x}(t) = Ax(t) + B_1(u_\varepsilon(t) + v + \nu(t)) \qquad (11)$$

with $x(0) = x_0$. In order to prove convergence of the solution of differential equation (11) to the solution of the differential inclusion (6), we use the change of variables proposed in [7], i.e., $z = P^{1/2}x$, which transforms (6) into the inclusion $-\dot{z} + \bar{B}_2 v(t) \in -\bar{A}z(t) + \partial\hat{\Phi}(z(t))$ with $\bar{B}_2 = P^{1/2}B_2$, $\bar{A} = P^{1/2}AP^{-1/2}$ and $\hat{\Phi} = \Phi \circ CP^{-1/2}$. Recalling that the system satisfies Assumption 1, it is clear that $-\bar{A}$ is a maximally monotone operator and therefore the right-hand side of the inclusion is too. Finally, making use of Proposition 3.11 in [2], we obtain the desired result; i.e., the sequence of solutions to (11) approaches the solution of (6) as $\varepsilon \downarrow 0$.

The use of the Yosida approximation has the advantage that the solution of (11) is defined even in the case when $Cx(0) \notin S$ as is shown in the following example.

Example 1: Consider a mass-spring system consisting of two objects with masses m_1 and m_2 that are coupled through springs with constants k_1 and k_2 as shown in Fig. 1, and ν_2 account for external bounded disturbances. Taking x_1, x_2, x_3 and x_4 as the elongation of spring k_1, the elongation of spring k_2, the velocity of mass m_1 and the velocity of mass m_2 respectively, we obtain a state space representation in the following form

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} &= \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & -\frac{\mu_1}{m_1} & 0\\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & 0 & -\frac{\mu_2}{m_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}\\ &+ \begin{bmatrix}0 & 0\\ 0 & 0\\ \frac{1}{m_1} & 0\\ 0 & \frac{1}{m_2}\end{bmatrix}\left(\begin{bmatrix}u_1\\ u_2\end{bmatrix} + \begin{bmatrix}\nu_1\\ \nu_2\end{bmatrix}\right)\end{aligned} \qquad (12)$$

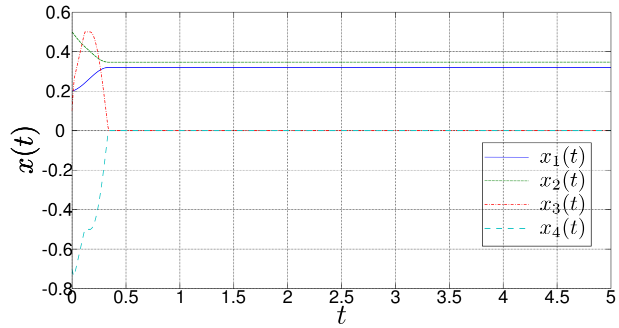

Fig. 2. State trajectories showing convergence to the equilibrium $x_* = [0.31\ 0.34\ 0\ 0]^\top \in \mathcal{C}$.

TABLE I

m_1	m_2	k_1	k_2	μ_1	μ_2
1 kg.	1.5 kg.	8 N/m	40 N/m	0.8	0.5

with output

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}.$$

Here, μ_i denote the friction coefficients related to masses m_i, L_i are the lengths of springs k_i in the absence of external forces, $i = 1, 2$.

The control objective is to maintain both masses without movement ($y_\mathrm{d} = [0, 0]^\top$) even in the presence of the forces ν_i, $i = 1, 2$. Notice that system (12) meets Assumption 1 with

$$P = \begin{bmatrix}k_1 + k_2 & -k_2 & 0 & 0\\ -k_2 & k_2 & 0 & 0\\ 0 & 0 & m_1 & 0\\ 0 & 0 & 0 & m_2\end{bmatrix},$$

which clearly satisfies $P = P^\top > 0$. Consider the set-valued control $-u(t) \in \partial\Phi(y(t))$ with $\Phi(y) = \Psi_S(y)$ which is proper, convex and lsc; and set $v = 0$. In view of $y_\mathrm{d} = [0, 0]^\top$ we propose the set S as

$$S = \operatorname{conv}\left\{\begin{bmatrix}0\\ 0\end{bmatrix}, \begin{bmatrix}-0.5\\ -0.5\end{bmatrix}, \begin{bmatrix}0\\ -1\end{bmatrix}, \begin{bmatrix}0.5\\ -0.5\end{bmatrix}\right\}.$$

Recall that the proposed set S restricts the output y into the polyhedral specified above. The set of equilibrium points is

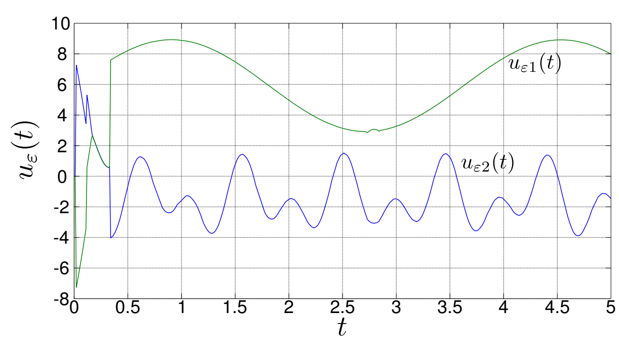

Fig. 3. Control signal compensating the external disturbance ν.

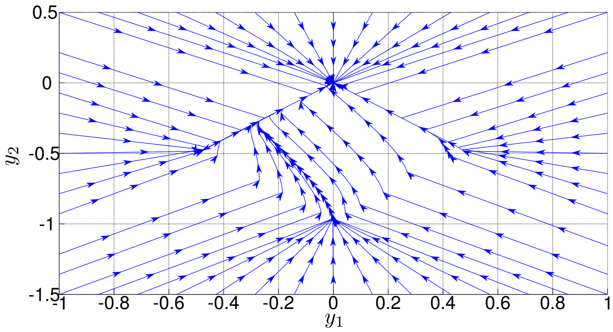

Fig. 4. Output phase plane for system (12), showing the convergence of all trajectories to the origin when regularized control (10) is used.

given by the inclusion $PAx_* \in C^\top \partial\Phi(Cx_*)$. Substituting parameters, we have

$$\begin{bmatrix} k_1 x_{3*} + k_2(x_{3*} - x_{4*}) \\ -k_2(x_{3*} - x_{4*}) \\ -k_1 x_{1*} - k_2(x_{1*} - x_{2*}) - \mu_1 x_{3*} \\ k_2(x_{1*} - x_{2*}) - \mu_2 x_{4*} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \rho_1 \\ \rho_2 \end{bmatrix}$$

where $\rho = \begin{bmatrix} \rho_1 & \rho_2 \end{bmatrix}^\top \in \partial\Phi(Cx_*) \subset \mathbb{R}^2$. From the first two equations we obtain $x_{3*} = x_{4*} = 0$ which implies $Cx_* = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ and clearly $\mathcal{E}(v, S) \subset \mathcal{C}$. Therefore, all conditions of Theorem 1 are satisfied and consequently we can achieve robust output regulation for all bounded external disturbances ν adding an appropriate vector $\vartheta_2 \in N_S(y_{\rm d})$ which strongly depends on the bound of the external perturbation ν. Figs. 2 and 3 show the state trajectories and the control signals, respectively, when system (12) is simulated with the parameters shown in Table 1 and the external disturbance $\nu = 3\begin{bmatrix} \cos(10t)\sin(\pi t) & \sin(\sqrt{3}t) \end{bmatrix}^\top$. Finally, Fig. 4 shows the output phase plane when the set S (specified above) is used.

IV. Conclusions and Further Research

The output regulation problem with external disturbances is treated. We start with a set-valued maximally monotone control and prove the existence and stability of the equilibrium set. Moreover, the conditions under which the equilibrium set provides the desired output are established. Some examples support the theoretical results and opens an opportunity to study the unmatched case.

A future research line appears in the case when the $Cx(0) \in S$ assumption is not satisfied, in this case a possible solution is to take the control approach presented in this note as a second stage of a switched control law. *Id est*, we first design a control law that assures the convergence of the output to the set S and then we switch to the multivalued controller $u(t) + v$ with $u(t)$ and v satisfying (5) and (7) respectively.

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