Integral Sliding Mode Control for Linear Time-Invariant Implicit Descriptions

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***Abstract*— We propose an integral sliding surface for linear time-invariant implicit descriptions (descriptor systems). We show that, under reasonable assumptions (regularity, stabilizability and a corresponding matching condition), it is possible to design a controller that drives the descriptor variables to zero, even in the presence of disturbances. Higher-order sliding motions are required since, for the solutions of the implicit description to be well defined, special care must be taken on the degree of smoothness of the controller and the perturbations.**

I. INTRODUCTION

Implicit systems are ubiquitous in nature. They appear naturally in the context of mechanics and circuit theory [1]. When dealing with complex dynamic systems, it is common in scientific and engineering practice to decompose a model into several, simpler submodels. These submodels, which typically consist of ordinary differential equations, are then interconnected to construct a model for the original aggregate system. Interconnecting the submodels amounts to imposing a set of algebraic constraints, so the overall resulting model is typically an implicit description. Implicit descriptions also reveal themselves as time-domain realizations of improper transfer functions.

In many cases, it is possible to convert an implicit description into an explicit state-space equation. Anyhow, being able to do analysis and engineering design directly on the original implicit equations would lead to faster design and allow for more complex models [2].

Implicit systems even originate by designer's choice. Consider in particular the case of sliding-mode control [3]. In sliding mode control the design cycle consists of two stages. First, a sliding surface is designed such that, when the system trajectories are restricted to the sliding surface, the system meets the control objectives (e.g., stability). During the second stage, a (possibly discontinuous) control is designed to drive and constrain the system trajectories to the sliding surface, irrespectively of the disturbances acting on the system. The robustness property against disturbances and the ease of implementation are probably the most attractive features of sliding mode control.

This work was supported by CONACyT (Consejo Nacional de Ciencia y Tecnología), Mexico, grant 132125 and CVU 331039. Programa de Apoyo a Proyectos de Investigación e Inovación Tecnológica (PAPIIT) UNAM, grant 117211.

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The sliding surface is typically given in the form of an algebraic equation on the system states. Thus, while the system is in the sliding motion, it behaves as a dynamical system subject to an algebraic constraint, a phenomenon that is most naturally modelled by an implicit description.

A. Motivation

While the literature of both, implicit descriptions and sliding-mode control is vast (see e.g., [4], [5] for linear descriptions, [6], [7], [8], [9] for nonlinear descriptions and [3] for sliding mode control), there is surprisingly little research connecting these two subjects.

We believe that both subjects can benefit from each other: sliding-mode control applied to implicit descriptions can bring in robustness and ease of implementation, and implicit description theory can bring insight into the design and analysis of sliding-mode controllers.

B. Contributions

In this paper we address the basic issues of sliding-mode control of implicit descriptions:

- We study the minimal required sliding-order for the solutions of the implicit description to be well defined.
- We derive conditions under which the disturbances can be compensated exactly (matching conditions for implicit descriptions).
- We propose a higher-order integral sliding-mode controller (HOISMC) that compensates perturbations under relaxed matching assumptions[1]

II. PROBLEM STATEMENT

We consider perturbed implicit descriptions of the form

$$E\dot{x}(t) = Ax(t) + Bv(t) \ , \quad v(t) = u(t) + w(t) \ , \qquad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $w(t) \in \mathbb{R}^m$ are the descriptor variable, control input and unknown perturbation at time t, respectively. The constant matrices E, A and B are of appropriate dimensions with B of full rank ($\operatorname{rank} B = m$). We restrict our attention to the case where E is singular (otherwise, equation (1) could be easily changed into an equation describing an explicit system in state-space form).

When $w(t)$ and $u(t)$ enter the implicit description through the same channel (as in (1)) we say that the perturbations satisfy the *matching condition*. For state-space systems, this

[1]In the context of this paper, a HOISMC is obtained by first defining the sliding surface as a function of the integral of the trajectories of the unperturbed system (as in [10]), and then enforcing a higher-order sliding motion on that surface. For other notions of HOISM see [11], [12].

condition is equivalent to the existence of a control law such that the closed-loop system is completely insensitive to $w(t)$ [13].

Our control objective is the following:

> *To design a stabilizing control law that drives the descriptor variable $x(t)$ to zero irrespectively of the perturbation $w(t)$.*

This problem will be stated more precisely as we go along the following section and introduce the appropriate assumptions (cf. Assumptions 1, 2 and 3).

III. BACKGROUND

A. Implicit descriptions

In this section we recall some basic facts about stability and stabilizability of implicit descriptions. We refer the reader, e.g., to [5], [14] for more details.

The qualitative behavior of (1) strongly depends on the structure of the matrix pencil $\lambda E - A$, $\lambda \in \mathbb{C}$. For ease of notation, let us write $(E, A) := \lambda E - A$.

Definition 1: An implicit description (1) is called a *regular system* if the pencil (E, A) is regular, i.e., if

$$|\lambda E - A| \not\equiv 0 \ . \tag{2}$$

In other words, a pencil (E, A) is regular if there exists a λ such that $|\lambda E - A| \neq 0$. The regularity of (E, A) is important since it ensures that, for any admissible input, the solutions of (1) exist and are unique.

Assumption 1: The pencil (E, A) is regular.

The determinant in (2) can be written as

$$|\lambda E - A| = k \Pi_{i=1}^{n_1} (\lambda - \lambda_i) \ ,$$

where $n_1 \leq n$ ($n_1 = n$ if and only if E is nonsingular) and k is a real constant. We refer to $\Lambda(E, A) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n_1}\}$ as the *finite eigenvalues* of the pencil (E, A)[2].

By an appropriate change of basis, a regular system can always be decomposed into the so-called Weierstrass form

$$\begin{aligned} \dot{x}_1(t) &= J x_1(t) + B_1 v(t) & \text{(3a)} \\ N \dot{x}_2(t) &= x_2(t) + B_2 v(t) \ , & \text{(3b)} \end{aligned}$$

where $x_1(t) \in \mathbb{R}^{n_1}$ is the state variable of the *dynamic subsystem* and $x_2(t) \in \mathbb{R}^{n_2}$, $n_2 = n - n_1$, is the descriptor variable of the *differential subsystem*. The matrix J is in Jordan's form and represents the finite structure of (E, A). More precisely,

$$\Lambda(I_{n_1}, J) = \Lambda(E, A) \ .$$

The matrix N is also in Jordan's form, it is nilpotent with index of nilpotence q and it represents the infinite structure of (E, A).

The solution of (3a) can be written explicitly as

$$x_1(t) = e^{Jt} x_1(0) + \int_0^t e^{J(t-\tau)} B_1 u(\tau) \mathrm{d}\tau \ ,$$

[2]The finite eigenvalues of the pencil (I_n, A) coincide with the usual eigenvalues of A.

while the solution of (3b) can be found by successively differentiating (3b) with respect to time and pre-multiplying by N:

$$\begin{aligned} N \dot{x}_2(t) &= x_2(t) + B_2 v(t) \\ N^2 \ddot{x}_2(t) &= N \dot{x}_2(t) + N B_2 \dot{v}(t) \\ &\vdots \\ N^q x_2^q(t) &= N^{q-1} x_2^{q-1}(t) + N^{q-1} B_2 v^{(q-1)}(t) \ . \end{aligned}$$

Adding all this equations and noting that $N^q = 0$ gives

$$x_2(t) = -\sum_{i=0}^{q-1} N^i B_2 v^{(i)}(t) \ . \tag{4}$$

In other words, the descriptor variable x_2 can be written explicitly as a linear combination of the input and its first $q-1$ derivatives. Clearly, the set of admissible inputs must be contained in the set of $(q-1)$-times continuously differentiable functions.

Definition 2: The implicit description (1) is called *stable* if there exist scalars $\alpha, \beta > 0$ such that, when $v(t) \equiv 0$ for $t > 0$, its descriptor variable $x(t)$ satisfies

$$\|x(t)\|_2 \leq \alpha e^{-\beta t} \|x(0)\|_2 \ , \quad t > 0 \ .$$

Since $v(t) \equiv 0 \Rightarrow \dot{v}(t) \equiv 0 \Rightarrow \ldots$, it is clear from (4) that $x_2(t) \equiv 0$ for $t > 0$. Thus, the stability of (1) depends on the dynamics of x_1 only, i.e., on the finite structure of (E, A).

Theorem 1 ([5]): The regular system (1) is stable if and only if

$$\Lambda(E, A) \in \mathbb{C}^- \ ,$$

where $\mathbb{C}^-$ is the open left-half complex plane.

Definition 3: The regular system (1) is *stabilizable* if there exists a state feedback $u(t) = -Kx(t)$ such that the closed-loop system

$$E \dot{x} = (A - BK) x(t) + B w(t)$$

with input $w(t)$ is stable.

There exists an algebraic characterization of stabilizable systems which is reminiscent of Popov-Belevitch-Hautus criterion for stabilizability of state-space systems.

Theorem 2 ([5]): The following statements are equivalent:

(i) The regular system (1) is stabilizable.
(ii) $\operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = n$ for all finite $\lambda \in \bar{\mathbb{C}}^+$, where $\bar{\mathbb{C}}^+$ is the closed right-half complex plane.
(iii) The state-space system (3a) is stabilizable.
(iv) $\operatorname{rank} \begin{bmatrix} \lambda I_{n_1} - J & B_1 \end{bmatrix} = n_1$ for all $\lambda \in \bar{\mathbb{C}}^+$.

Assumption 2: The regular system (1) is stabilizable

B. Higher-Order Sliding-Mode Control

In this section we recall that, with a higher-order sliding-mode controller, it is possible to achieve robustness against perturbations with any desired degree of smoothness for the controller.

Consider first a simple controlled and perturbed system

$$\dot{\sigma}(t) = u(t) + w(t) \ , \tag{5}$$

where $u(t), w(t) \in \mathbb{R}$ are, again, the control and the unknown perturbation, and $\sigma(t)$ is an output of relative degree one that we wish to drive to zero. Suppose that the perturbation is bounded by a known constant $\bar{w}_0$, i.e., $|w(t)| \leq \bar{w}_0$ for all $t \geq 0$. A discontinuous control law

$$u(t) = -L \operatorname{sign}(\sigma(t)) , \quad L \geq \bar{w}_0 + \delta_0 , \quad \delta_0 > 0 \qquad (6)$$

ensures that the constraint $\sigma(t) = 0$ is attained in finite time and maintained thereafter (the solutions of discontinuous differential equations are taken in Filippov's sense [15]). This can be verified by means of the Lyapunov function $V(\sigma) = \sigma^2/2$, which has the following time derivative along the solutions of (5):

$$\begin{aligned}\dot{V}(\sigma(t)) &= \sigma(t)\left(-L \operatorname{sign}(\sigma(t)) - w(t)\right) \\ &\leq -|\sigma(t)|(L - \bar{w}_0) \leq -\delta_0 \sqrt{V(\sigma(t))} \ .\end{aligned}$$

It can be readily shown that the solution of

$$\dot{W}(t) = -\delta_0 \sqrt{W(t)}$$

satisfies $W(T) = 0$ for some finite T dependent on $W(0)$. According to the standard theory of differential equations, if $W(0) = V(\sigma(0))$, then $W(t) \geq V(\sigma(t))$ for all $t \geq 0$ [16]. Since V is non negative, $W(T) = 0$ implies $V(\sigma(T)) = 0$ and $\sigma(T) = 0$. Because $\dot{V}$ is non positive, $V(\sigma(t)) = 0$ and $\sigma(t) = 0$ for all $t \geq T$. We say that (6) enforces a *first order sliding mode* (1-sliding mode for short) in finite time.

Consider now a system of relative degree two:

$$\begin{aligned}\begin{bmatrix}\dot{\xi}_1(t) \\ \dot{\xi}_2(t)\end{bmatrix} &= \begin{bmatrix}\xi_2(t) \\ 0\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix}(u(t) + w(t)) \ , \\ \sigma(t) &= \xi_1(t) \ ,\end{aligned}$$

i.e.,

$$\ddot{\sigma}(t) = u(t) + w(t) \ . \qquad (7)$$

We say that the system exhibits a finite-time 2-sliding mode if the constraints $\sigma(t) = \dot{\sigma}(t) = 0$ are satisfied for all t greater than some finite and positive T. Many finite-time 2-sliding mode controllers can be found in the literature [17], [18]. An example is the so-called *twisting* algorithm, a discontinuous control of the form

$$\begin{aligned}u(t) &= -L_1 \operatorname{sign}(\xi_1(t)) - L_2 \operatorname{sign}(\xi_2(t)) \\ &= -L_1 \operatorname{sign}(\sigma(t)) - L_2 \operatorname{sign}(\dot{\sigma}(t)) \ ,\end{aligned}$$

with $L_2 > \bar{w}_0$ and $L_1 > L_2 + \bar{w}_0$ (see [17] for details).

More generally, consider a system of relative degree r:

$$\begin{aligned}\begin{bmatrix}\dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \vdots \\ \dot{\xi}_{r-1}(t) \\ \dot{\xi}_r(t)\end{bmatrix} &= \begin{bmatrix}\xi_2(t) \\ \xi_3(t) \\ \vdots \\ \xi_{r-2}(t) \\ 0\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ \vdots \\ 0 \\ 1\end{bmatrix}(u(t) + w(t)) \ , \\ \sigma(t) &= \xi_1(t) \ ,\end{aligned}$$

i.e.,

$$\sigma^{(r)} = u(t) + w(t) \ , \qquad (8)$$

We say that the system exhibits an *r-sliding mode* if the constraints $\sigma(t) = \dot{\sigma}(t) = \cdots = \sigma^{(r-1)}(t) = 0$ are satisfied identically after a finite period of time. For an arbitrary r, it is always possible to construct a discontinuous control that enforces an r-sliding motion on (8) [17], [19].

By cascading a chain of k integrators with the control, the relative degree of the system is artificially increased while, at the same time, a $k-1$ degree of smoothness is achieved for $u(t)$ (i.e., $u(t)$ is made $(k-1)$-times continuously differentiable). Let us introduce a new set of state variables $\xi_{r+1}, \ldots, \xi_{r+k+1}$ and define its dynamics by

$$\begin{aligned}\begin{bmatrix}\dot{\xi}_{r+1}(t) \\ \dot{\xi}_{r+2}(t) \\ \vdots \\ \dot{\xi}_{r+k}(t) \\ \dot{\xi}_{r+k+1}(t)\end{bmatrix} &= \begin{bmatrix}\xi_{r+2}(t) \\ \xi_{r+3}(t) \\ \vdots \\ \xi_{r+k+1}(t) \\ 0\end{bmatrix} + \begin{bmatrix}0 \\ 0 \\ \vdots \\ 0 \\ 1\end{bmatrix}\gamma(t) \ , \\ u(t) &= \xi_{r+1}(t)\end{aligned}$$

with $\gamma(t)$ a new virtual control, i.e.,

$$u^{(k)}(t) = \gamma(t) \ . \qquad (9)$$

It follows from (8) and (9) that

$$\sigma^{(r+k)} = \gamma(t) + w^{(k)}(t) \ . \qquad (10)$$

Thus, if there exists a bound $\bar{w}_k$ such that $|w^{(k)}(t)| \leq \bar{w}_k$ for all $t \geq 0$, then it possible to construct a discontinuous virtual control $\gamma(t)$ enforcing an $(r+k)$-sliding motion in finite time. Since the actual input $u(t)$ is obtained by k successive integrations of $\gamma(t)$, $u(t)$ will be $(k-1)$-times continuously differentiable.

Remark 1: All the terms in equation (10) are real valued. The vector case $\sigma(t), \gamma(t), w(t) \in \mathbb{R}^m$ can be dealt with simply by considering m copies of (10).

In order to construct an admissible (sufficiently smooth) controller for (1), we will need the following assumption.

Assumption 3: Let q be the index of nilpotency of N in (3b). The unknown perturbation $w(t)$ satisfies the bound

$$\|w^{(q)}(t)\| \leq \bar{w}_q \ , \quad t \geq 0$$

for some known constant $\bar{w}_q$.

IV. COMPENSATING THE PERTURBATIONS EXACTLY

Let us divide the control effort in two parts:

$$u(t) = u_0(t) + u_1(t) \ , \qquad (11)$$

where $u_0(t)$ is a linear feedback responsible of stabilizing the unperturbed description

$$E\dot{x}(t) = Ax(t) + Bu_0(t)$$

and $u_1(t)$ is a highly nonlinear control responsible of compensating $w(t)$.

Notice that the control and the perturbation enter simultaneously in both, the dynamic subsystem (3a) and the differential one (3b). Suppose, only for the sake of argument, that $\operatorname{rank} B_1 = m$ (this implies $n_1 \geq m$). In this case, we can forget about the differential subsystem and apply standard techniques to stabilize the dynamic subsystem. A fairly obvious procedure to robustly stabilize (1) would be:

1) Choose a matrix $C \in \mathbb{R}^{m \times n_1}$ such that CB_1 is non singular and such that the motion of x_1 along the constraint $\sigma(t) = (CB_1)^{-1} C x_1(t) = 0$ is stable (i.e., such that $\sigma(t)$ is an minimum-phase output). A matrix C can always be found when (3a) is stabilizable and when $\operatorname{rank} B_1 = m$ [3].
2) Set $u_0(t) = -(CB_1)^{-1} C J x_1(t)$. The dynamics of the sliding variable becomes

$$\dot{\sigma}(t) = (CB_1)^{-1} C \Big(J x_1(t) - B_1 (CB_1)^{-1} C J x_1(t) + \\ + CB_1 \big(u_1(t) + w(t)\big) \Big) ,$$

which simplifies to

$$\dot{\sigma}(t) = u_1(t) + w(t) \;, \tag{12}$$

i.e., the sliding variable has relative degree equal to one.
3) Set $u_1^{(q)}(t) = \gamma(t)$ and design $\gamma(t)$ as a $(1+q)$-sliding mode controller for $\sigma^{(1+q)} = \gamma(t) + w^q(t)$.

For $q \geq 1$, equation (12) implies that, when the $(1+q)$-sliding motion occurs, we have $u_1(t) \equiv -w(t)$ [4]. Hence,

$$v(t) = u_0(t) + u_1(t) + w(t) = -(CB_1)^{-1} C x_1(t) \;.$$

By construction, $x_1(t)$ and its first $q-1$ derivatives will go to zero. This implies that $x_2(t)$ goes to zero too (cf. (4)).

Example 1: Consider the implicit description

$$\begin{aligned} \dot{x}_1(t) &= x_1(t) + u(t) + w(t) & \text{(13a)} \\ 0 &= x_2(t) + u(t) + w(t) \;, & \text{(13b)} \end{aligned}$$

which can be written as in (1) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

We have

$$|\lambda E - A| = \left| \begin{bmatrix} \lambda - 1 & 0 \\ 0 & -1 \end{bmatrix} \right| = -(\lambda - 1) \;,$$

from which we conclude that the system is regular, that $n_1 = n_2 = 1$ and that the system is unstable, since it has the finite eigenvalue $\lambda = 1 \notin \mathbb{C}^-$. Clearly,

$$\operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \lambda - 1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} = 2 = n$$

for all λ, which confirms that the obvious fact that system is stabilizable. The system is already in Weierstrass form with $N = 0$, $J = 1$, $B_1 = 1$ and $B_2 = 1$. Also, we have $\operatorname{rank} B_1 = 1 = m$, so steps 1) to 3) can be applied.

The scalar $C = 1$ trivially satisfies the conditions of step 1), so we set $\sigma(t) = x_1(t)$ and $u_0(t) = -x_1(t)$.

[3]Such C can be easily computed if (3a) is first put into the so-called *regular form* [3].

[4]By definition, we set $q = 0$ when E is nonsingular ($n_2 = 0$). When $q = 0$, $u_1(t)$ is discontinuous and has to be replaced by the so-called *equivalent control* [3].

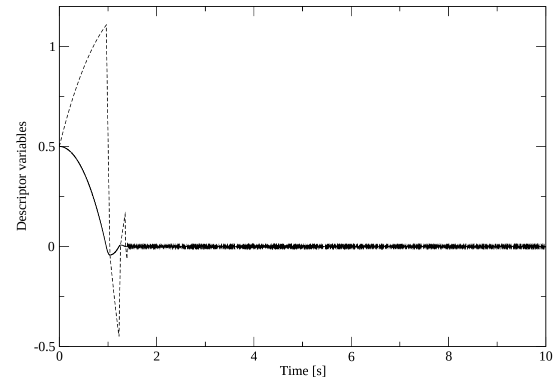

Fig. 1. Response of (13), x_1–solid, x_2–dashed.

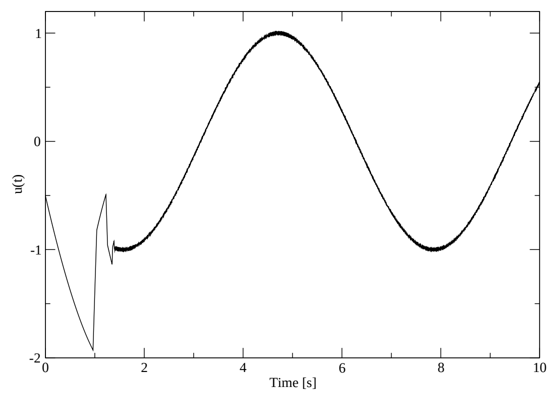

Fig. 2. Response of (13), control action.

The index of nilpotency of $N = 0$ is one ($q = 1$), so the system will be robust against continuously differentiable perturbations that satisfy the bound $|\dot{w}(t)| \leq \bar{w}_1$. For the sake of concreteness, let us assume that $\bar{w}_1 = 1$. To achieve a 2-sliding motion we propose to use the twisting controller

$$\gamma(t) = -8 \operatorname{sign}(x_1(t)) - 6 \operatorname{sign}(\dot{x}_1(t)) \;, \tag{14}$$

so the actual control is given by the continuous function

$$u(t) = -x_1(t) - \int_0^t \left(8 \operatorname{sign}(x_1(\tau)) + 6 \operatorname{sign}(\dot{x}_1(\tau)) \right) \mathrm{d}\tau \;.$$

Figures 1 and 2 show the system's response to an initial condition $x_1(0) = 0.5$ and a perturbation $w(t) = \sin(t)$. It can be verified that x_1 and x_2 reach the origin in finite time.

A. Main result, higher-order integral sliding-mode control

The previous procedure requires to put (1) in Weierstrass form (this is not overly complicated if the appropriate software is available) and more importantly, it requires $\operatorname{rank} B_1 = m$, which can be seen as an additional requirement (a stringent one) to the usual matching condition. This drawback can be circumvented by defining an *integral* sliding variable. Roughly speaking, the idea is to split the control action as in (11) and to define the sliding variable as a linear function of the difference between the actual value of the descriptor variables and the 'value that this variables would have in the absence of disturbances' (see [10], [20]).

Theorem 3: Consider an implicit description of the form (1) satisfying Assumptions 1 and 2. Split the control as in (11), choose a nominal control $u_0(t) = -Kx(t)$ such that

$$\Lambda(E, A - BK) \subset \mathbb{C}^-$$

and define the integral sliding variable

$$\sigma(t) = B^+ \left[Ex(t) - \int_0^t \left((A - BK)x(\tau) \right) \mathrm{d}\tau \right] , \qquad (15)$$

where $B^+ = \left(B^\top B \right)^{-1} B^\top$ is B's Moore-Penrose pseudo-inverse. Then, the sliding variable:

1) Satisfies the differential equation (12).
2) Is a minimum-phase output (the solutions of (1) converge to zero when the constraint $\sigma(t) \equiv 0$ is enforced).

Remark 2: Once a minimum-phase output with relative degree one is obtained, it is possible to increase the relative degree by q. Then, Assumption 3 ensures the possibility of rejecting the perturbations by enforcing a $(q+1)$-sliding motion with a $(q-1)$-times differentiable feedback $u_1(t)$.

Proof: Statement 1) follows from direct differentiation of (15):

$$\dot{\sigma}(t) = B^+ \left(E\dot{x}(t) - (A - BK)x(t) \right) . \qquad (16)$$

Substitution of (1) in (16) gives

$$\dot{\sigma}(t) = B^+ (Bu_1(t) + w(t)) = u_1(t) + w(t) .$$

The constraint $\sigma(t) \equiv 0$ implies $\dot{\sigma} = u_1(t) + w(t) \equiv 0$, that is, $u_1(t) = -w(t)$. This control results in the closed loop system

$$\begin{aligned} E\dot{x} &= Ax(t) + B\left(-Kx(t) + w(t) - w(t) \right) \\ &= (A - BK)x(t) , \end{aligned}$$

which is stable since the finite eigenvalues of $(E, A - BK)$ have negative real part. ∎

Example 2: Consider the implicit description

$$\dot{x}_2(t) = x_1(t) \qquad (17a)$$
$$0 = x_2(t) + u(t) + w(t) . \qquad (17b)$$

This description has the form (1) with

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

(Notice that $u(t)$ and $w(t)$ act on the algebraic constraint only, so steps 1) to 3) in p. 4 cannot be applied.) We have

$$|\lambda E - A| = \left| \begin{bmatrix} -1 & \lambda \\ 0 & -1 \end{bmatrix} \right| \equiv 1 .$$

Thus, the system is regular, $n_1 = 0$ and $n_2 = 2$ (there is no dynamical part). The system is in the form (3b) with $N = E$ and $q = 2$ ($E^2 = 0$). The system is stabilizable since

$$\operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} -1 & \lambda & 0 \\ 0 & -1 & 1 \end{bmatrix} = 2 = n .$$

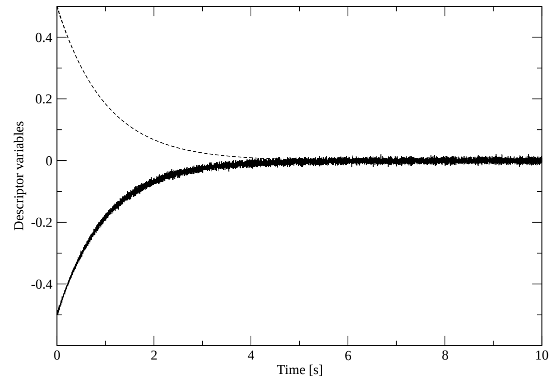

Fig. 3. Response of (17), x_1–solid, x_2–dashed.

Indeed, the system is already stable since there is no dynamical subsystem. However, we can use, e.g.,

$$u_0(t) = -\begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1(t)$$

to reduce q while maintaining the stability of the system. The new characteristic polynomial is

$$|\lambda E - (A - BK)| = \left| \begin{bmatrix} -1 & \lambda \\ -1 & -1 \end{bmatrix} \right| = 1 + \lambda .$$

which has the stable root $\lambda_1 = -1$ and for which $n_1 = n_2 = 1$.

We have $B^+ = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $B^+ E = 0$, so the sliding variable is

$$\begin{aligned} \sigma(t) &= -\int_0^t B^+ (A - BK)x(\tau)\mathrm{d}\tau \\ &= -\int_0^t (x_1(\tau) + x_2(\tau))\mathrm{d}\tau \end{aligned}$$

or, equivalently,

$$\dot{\sigma}(t) = -x_1(t) - x_2(t) = u_1(t) + w(t) .$$

Suppose, for simplicity, that $\bar{w}_1 = 1$. To enforce a 2-sliding mode we can use the twisting controller (14) from the previous example. The resulting controller is

$$u(t) = u_0(t) + u_1(t) = x_1(t) + \int_0^t \gamma(\tau)\mathrm{d}\tau$$

which, upon substitution gives

$$\begin{aligned} u(t) = x_1(t) + \int_0^t \Big(& 8 \operatorname{sign} \Big(\int_0^{\tau_2} (x_1(\tau_1) + x_2(\tau_1))\mathrm{d}\tau_1 \Big) \\ & + 6 \operatorname{sign}(x_1(\tau_2) + x_2(\tau_2)) \Big) \mathrm{d}\tau_2 . \end{aligned}$$

Figures 3 and 4 show the system's response to an initial condition $x_2(0) = 0.5$ and a perturbation $w(t) = \sin(t)$. It can be verified that x_1 and x_2 asymptotically converge to the origin.

V. ACKNOWLEDGMENTS

Our sincerest gratitude to Prof. Moisés Bonilla, whose invaluable help made the preparation of this paper possible.

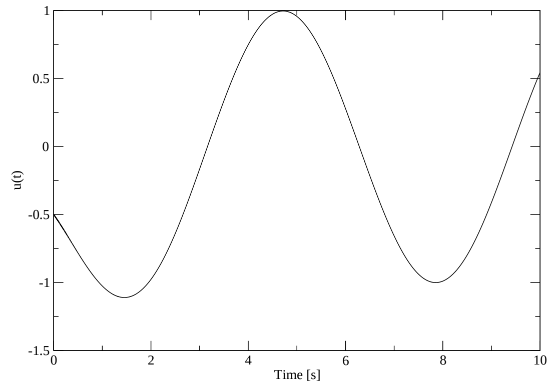

Fig. 4. Response of (17), control action.

VI. CONCLUSIONS AND FUTURE WORKS

We have proposed two higher-order sliding-mode controllers for linear implicit descriptions. The controllers drive the descriptor variables to the origin, irrespectively of sufficiently smooth perturbations satisfying a matching condition. In one of the schemes, the sliding variable is a simple linear combination of the descriptor variables, but an additional matching condition is required ($\operatorname{rank} B_1 = m$). The second scheme requires the integration of the descriptor variables to generate the sliding surface, but no additional matching condition is necessary.

Further investigation is required to determine if it is possible to define a non integral sliding surface with out the extra rank condition. Higher-order sliding-mode control of nonlinear implicit descriptions is another line for future work.

REFERENCES

[1] R. W. Newcomb and B. Dziurla, "Some circuits and systems applications of semistate theory," *IEEE Trans. Autom. Control*, vol. 8, pp. 235 – 260, 1989.

[2] S. L. Campbell, "Linearization of DAEs along trajectories," *Z ange Math Phys*, vol. 46, pp. 71 – 84, 1995.

[3] V. Utkin, J. Guldner, and J. Shi, *Sliding Modes in Electromechanical Systems*. London, U.K.: Taylor & Francis, 1999.

[4] F. L. Lewis, "A survey of linear singular systems," *Circuits Syst. Signal Process.*, vol. 5, pp. 3 – 36, 1986.

[5] L. Dai, *Singular Control Systems*. Berlin: Springer-Verlag, 1989.

[6] K. E. Brenan, S. L. Campbell, and L. R. Petzold, *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. New York: North-Holland, 1989.

[7] W. C. Rheinboldt, "On the existence and uniqueness of solutions of nonlinear semi-implicit differential-algebraic equations," *Nonlinear Analysis. Theory, Methods and Applications*, vol. 16, pp. 647 – 661, 1991.

[8] S. Reich, "On a geometrical interpretation of differential-algebraic equations," *Circuits Syst. Signal Process.*, vol. 9, pp. 367 – 382, 1990.

[9] ——, "On an existence and uniqueness theory for nonlinear differential-algebraic equations," *Circuits Syst. Signal Process.*, vol. 10, pp. 343 – 359, 1991.

[10] V. Utkin and J. Shi, "Integral sliding mode in systems operating under uncertainty conditions," in *Proc. Conference on Decision and Control*, Kobe, Japan, Dec. 1996, pp. 4591–4596.

[11] A. Levant and L. Alelishvili, "Integral high-order sliding modes," *IEEE Trans. Autom. Control*, vol. 52, pp. 1278 – 1282, 2007.

[12] S. Laghrouche, F. Plestan, and A. Glumineaub, "Higher order sliding mode control based on integral sliding mode," *Automatica*, vol. 43, pp. 531 – 537, 2007.

[13] B. Draženović, "The invariance conditions in variable structure systems," *Automatica*, vol. 5, pp. 287 – 295, 1969.

[14] F. L. Lewis, "A tutorial on the geometric analysis of linear time-invariant implicit systems," *Automatica*, vol. 28, pp. 119 – 137, 1992.

[15] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*. Dordrecht, The Netherlands: Kluwer Academic Publishers, 1988.

[16] P. Hartman, *Ordinary Differential Equations*. Philadelphia: Society for Industrial and Applied Mathematics, 2002.

[17] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control," *Int. J. Control*, vol. 76, pp. 924–941, 2003.

[18] ——, "Principles of 2-sliding mode design," *Automatica*, vol. 43, pp. 576 – 586, 2007.

[19] ——, "Quasi-continuous high-order sliding-mode controllers," *IEEE Trans. Autom. Control*, vol. 50, pp. 1812 – 1816, Nov. 2005.

[20] F. Castaños and L. Fridman, "Analysis and design of integral sliding manifolds for systems with unmatched perturbations," *IEEE Trans. Autom. Control*, vol. 51, pp. 853 – 858, May 2006.