Integral Sliding-Mode Control for Linear Time-Invariant Implicit Systems ⋆

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Abstract

We propose an integral sliding surface for linear time-invariant implicit systems (descriptor systems). We show that, under reasonable assumptions (regularity, stabilizability) it is possible to design a stabilizing controller that compensates the matched perturbations exactly. Higher-order sliding motions are required since, for the solutions of the implicit system to be well defined, special care must be taken on the degree of smoothness of the controller and the perturbations. The algorithm is tested on a system where the perturbation enters through an algebraic equation.

Key words: Sliding-mode control; Implicit systems; Descriptor systems; Robust stability.

1 Introduction

We consider perturbed implicit systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t) + Dw(t) \ , \qquad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $w(t) \in \mathbb{R}^p$ are the descriptor variable, control input and unknown perturbation at time t, respectively. The constant matrices E, A, B and D are of appropriate dimensions with B of full rank. We restrict our attention to the case where E is singular with $\operatorname{rank} E = l < n$ (otherwise (1) could be easily changed into an explicit system in state-space form).

When the perturbations $w(t)$ enter the implicit system through the same channel as $u(t)$ we say that the perturbations satisfy the *matching condition*. More precisely, the matching condition is satisfied when all the columns of D are linear combinations of the columns of B, a requirement that can be succinctly written as

$$\operatorname{rank} \begin{bmatrix} B & D \end{bmatrix} = \operatorname{rank} B \ . \qquad (2)$$

For state-space systems this condition is equivalent to the existence of a control law such that the closed-loop system is completely insensitive to $w(t)$ [1].

Our problem consists in designing a control law that robustly stabilizes (1), that is, that stabilizes it in spite of $w(t)$ and in spite of parametric uncertainty.

We note that, while the literature of both, implicit systems and sliding-mode control is vast (see e.g., [2] for implicit linear systems and [3] for sliding mode control), there is surprisingly little research connecting these two subjects. In order to bridge this gap, we propose an integral sliding-mode control law:

- For which we determine the minimal required sliding-order for the solutions of the implicit system to be well defined.
- That compensates the disturbances exactly if matched and smooth enough.
- That we combine with other robust techniques to address parametric uncertainty, both matched and unmatched (the particular case without parametric uncertainty was presented in [4]).

The following section introduces the main assumptions (regularity and stabilizability), while recalling some well-known facts about implicit systems. An integral sliding surface is proposed in section 3. It is shown that, by appropriately choosing the order of the sliding-mode controller, it is possible to compensate the matched

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perturbation exactly while respecting the required degree of smoothness. The example used to illustrate the method is simple but unique in the sense that the perturbation enters through an algebraic instead of a differential equation. Section 4 extends the results to the case where there is unmatched parametric uncertainty. The conclusions are given in Section 5.

2 Main assumptions

The qualitative behavior of (1) strongly depends on the structure of the matrix pencil $\lambda E - A$, $\lambda \in \mathbb{C}$. For ease of notation, let us write $(E, A) := \lambda E - A$.

Assumption 1 *The pencil (E, A) is regular, that is, $|\lambda E - A|$ is not identically zero.*

Recall that $|\lambda E - A|$ is a polynomial of the form $\phi \prod_{i=1}^{n_1} (\lambda - \lambda_i)$, where ϕ is a real constant and $n_1 \leq n$ with $n_1 = n$ if and only if E is nonsingular. We refer to $\Lambda(E, A) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n_1}\}$ as the *finite eigenvalues* of the pencil (E, A).

Regularity of (E, A) is important since it ensures that, for any admissible (smooth enough) input, the solutions of (1) exist and are unique. Furthermore, by an appropriate change of basis, a regular system can always be decomposed into the so-called Weierstrass form

$$\dot{x}_1(t) = Jx_1(t) + B_1 u(t) + D_1 w(t) \tag{3a}$$
$$N\dot{x}_2(t) = x_2(t) + B_2 u(t) + D_2 w(t)\ , \tag{3b}$$

where $x_1(t) \in \mathbb{R}^{n_1}$ is the state variable of the *slow subsystem* and $x_2(t) \in \mathbb{R}^{n_2}$, $n_2 = n - n_1$, is the descriptor variable of the *fast subsystem*. The matrix J is in Jordan's form and represents the finite structure of (E, A). More precisely, $\Lambda(I_{n_1}, J) = \Lambda(E, A)$. The matrix N is also in Jordan's form, it is nilpotent with index of nilpotence q and it represents the infinite structure of (E, A).

The solution of (3a) is well-known, while the solution of (3b) can be found by successively differentiating with respect to time and pre-multiplying by N:

$$x_2(t) = -\sum_{i=0}^{q-1} N^i \left(B_2 u^{(i)}(t) + D_2 w^{(i)}(t) \right) \tag{4}$$

In other words, the descriptor variable x_2 is a linear combination of the inputs and their first $q-1$ derivatives.

Remark 2 *It follows from (4) that, if we require continuity of the solutions, the system inputs (perturbations and controls) must be smooth enough. In a worst case scenario, the controls and the perturbations have to be $(q-1)$ times continuously differentiable ($\mathcal{C}^{q-1}$).*

Assumption 3 *Let q be the index of nilpotence of N in (9). The unknown perturbation $w(t)$ is $\mathcal{C}^{q-1}$ and satisfies the bound $\|w^{(q)}(t)\| \leq \bar{w}_q$ for all $t \geq 0$ and some known $\bar{w}_q \geq 0$.*

The implicit system (1) is called *impulse free* if $N = 0$. The name stems from the fact that, when solutions are taken in the distributional sense, the system's free response does not contain impulses. A system is impulse free if and only if $n_1 = l$ (see [2,5] for details). When the system is impulse free, Assumption 3 reduces to continuity of w and boundedness of its first derivative, a fairly reasonable assumption.

Let us recall some basic facts about stability and stabilizability of implicit systems. We refer the reader to [2,6] for more details.

Definition 4 *The autonomous regular implicit system*

$$E\dot{x}(t) = Ax(t) \tag{5}$$

is called stable *if there exist scalars $\alpha, \beta > 0$ such that $\|x(t)\|_2 \leq \alpha e^{-\beta t} \|x(0)\|_2$ for all $t > 0$.*

Theorem 5 *[2, p. 69] The autonomous regular implicit system (5) is stable if and only if $\Lambda(E, A) \in \mathbb{C}^-$, where $\mathbb{C}^-$ is the open left-half complex plane.*

Assumption 6 *The unperturbed regular implicit system*

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{6}$$

is stabilizable. That is, there exists a feedback control $u(t) = -Kx(t)$ such that $E\dot{x}(t) = (A - BK)x(t)$ is stable.

Theorem 7 *[2, p. 71–72] The following statements are equivalent:*

(i) *System (6) is stabilizable.*
(ii) *$\dot{x}_1(t) = Jx_1(t) + B_1 u(t)$, is stabilizable.*
(iii) $\mathrm{rank}[\lambda E - A \;\; B] = n$ *for all finite $\lambda \in \bar{\mathbb{C}}^+$.*

3 Compensating the matched perturbations

In this section we propose an integral sliding surface [7,8] that is suitable for the compensation of matched incertitude in implicit systems. The controllers that drive the system trajectories to the sliding surface are discussed afterwards.

Lemma 8 *Consider an implicit system (1) satisfying Assumptions 1, 6 and the matching condition (2). Split the control as $u(t) = u_{\mathrm{n}}(t) + u_{\mathrm{p}}(t)$, choose a nominal control $u_{\mathrm{n}}(t) = -Kx(t)$ such that $\Lambda(E, A - BK) \subset \mathbb{C}^-$ and define the integral sliding variable*

$$\sigma(t) = B^+ \left[Ex(t) - \int_0^t \big((A - BK)x(\tau) \big) \mathrm{d}\tau \right] \tag{7}$$

with B^+ the pseudo inverse of B. Then, the sliding variable: 1) Satisfies

$$\dot{\sigma}(t) = u_\mathrm{p}(t) + w_\mathrm{m}(t) \ , \tag{8}$$

where w_m is defined by $Dw = Bw_\mathrm{m}$. 2) Is a minimum-phase output.

PROOF. Statement 1) follows from direct differentiation of (7). The constraint $\sigma(t) \equiv 0$ implies $\dot{\sigma} = u_\mathrm{p}(t) + w_\mathrm{m}(t) \equiv 0$, that is, $u_\mathrm{p}(t) = -w_\mathrm{m}(t)$. This control results in the closed loop system $E\dot{x}(t) = (A - BK)x(t)$, which is stable since, by construction, all the finite eigenvalues of $(E, A - BK)$ have negative real parts. This proves Statement 2).

By cascading a chain of q integrators with the control, the desired $(q-1)$ degree of smoothness is achieved for $u_p(t)$. The required smoothness is at the expense of an increased relative degree. This is detailed in what follows: Consider the differential equation $\sigma^{(r)} = \gamma(t) + \delta(t)$, where $\gamma(t), \delta(t) \in \mathbb{R}$ are a control and an unknown perturbation, and $\sigma(t)$ is an output of relative degree r. We say that the system exhibits an *r-sliding mode* if the constraints $\sigma(t) = \dot{\sigma}(t) = \cdots = \sigma^{(r-1)}(t) = 0$ are satisfied identically after a finite period of time. If a bound on $\delta(t)$ is known, it is always possible to construct a discontinuous control that enforces an r-sliding mode, for an arbitrary r (see, e.g., the family of quasi-continuous controllers [9])

Now, let us define the auxiliary control $\gamma(t) = u^{(q)}(t)$. Computing the q-th time derivative of (8) gives the expression $\sigma^{(q+1)}(t) = \gamma(t) + \delta(t)$ with $\delta(t) = w_\mathrm{m}^{(q)}(t)$. Thus, if there exists a bound $\bar{w}_q$ such that $|w_\mathrm{m}^{(q)}(t)| \leq \bar{w}_q$, then it possible to construct a discontinuous $\gamma(t)$ enforcing a $(q+1)$-sliding motion in finite time. Since the actual input $u_\mathrm{p}(t)$ is obtained by q successive integrations of $\gamma(t)$, $u_\mathrm{p}(t)$ will be $\mathcal{C}^{q-1}$.

The following example illustrates the use of Lemma 8 in combination with higher-order sliding-mode control.

Example 9 *Consider the implicit system*

$$\dot{x}_2(t) = x_1(t) \tag{9a}$$
$$0 = x_2(t) + u(t) + w(t) \ . \tag{9b}$$

Notice that the solutions are $x_2(t) = -u(t) - w(t)$ and $x_1(t) = -\dot{u}(t) - \dot{w}(t)$: the system acts as a differentiator. Despite the system's simplicity, it is not obvious how to compensate $w(t)$, even if $x_2(t)$ is available.

The system has the form (1) *with*

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ , \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = D = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ .$$

We have $|\lambda E - A| \equiv 1$. Thus, the systems is regular with $n_1 = 0$ and $n_2 = 2$ (there is no slow subsystem). Moreover, the system is in the form (3b) *with $N = E$ and $q = 2$ ($E^2 = 0$). The index of nilpotence indicates that the inputs must be at least $\mathcal{C}^1$.*

The system is stabilizable since $\mathrm{rank}[\lambda E - A \ \ B] = 2$. *Indeed, the unforced system is already stable since there is no slow subsystem. However, we propose $u_\mathrm{n}(t) = -Kx(t)$ with $K = [-1 \ \ 0]$ in order to reduce q while maintaining the stability of the system. The new characteristic polynomial is $|\lambda E - A + BK| = 1 + \lambda$, which has a stable root at $\lambda = -1$ and for which $n_1 = l = 1$ (the system is impulse free) and $n_2 = 1$. Now, the value of q for the closed-loop system is one, so inputs are required to be continuous only (this reduces the required order of the sliding controller).*

Since $B^+ = [0 \ \ 1]$ and $B^+E = 0$, the sliding variable (7) *takes the form $\sigma(t) = -\int_0^t (x_1(\tau) + x_2(\tau_1))\,\mathrm{d}\tau_1$. Note that $\dot{\sigma}(t) = -x_1(t) - x_2(t) = u_\mathrm{p}(t) + w(t)$, as predicted by Lemma 8.*

To produce a continuous controller, we set

$$u_\mathrm{p}(t) = \int_0^t \gamma(\tau_1)\mathrm{d}\tau_1 \tag{10}$$

with $\gamma(\tau_1)$ a 2-sliding controller. We can use, e.g., the so-called quasi-continuous controller for $r = 2$:

$$\gamma(\tau_1) = -\alpha \frac{\dot{\sigma} + \sqrt{|\sigma|}\,\mathrm{sign}(\sigma)}{|\dot{\sigma}| + \sqrt{|\sigma|}}(\tau_1) \ ,$$

where α is a controller gain that has to be large enough (see [9] for details).

Fig. 1 shows the system response to an initial condition $x_1(0) = 0.5$ and a perturbation $w(t) = 1.5 + \sin(t)$ ($\bar{w}_1 = 1$). The initial condition $x_2(0)$ is determined by (9b). *The controller gain was set to $\alpha = 5$. It can be verified that $x_1(t)$ and $x_2(t)$ converge exponentially to the origin.*

4 Unmatched uncertainty

The decomposition of the control law into a pair of nominal and sliding-mode components allows for easy integration with other robust techniques. These techniques

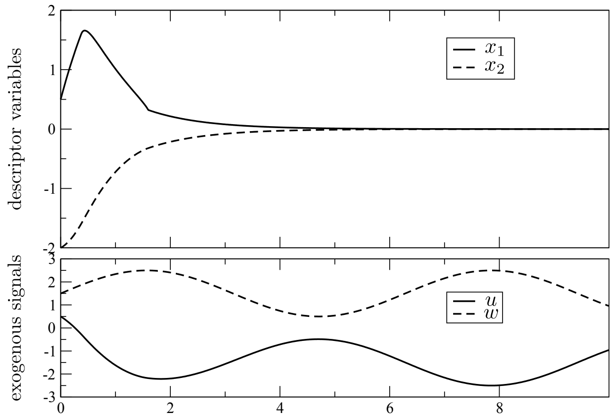

Fig. 1. Response of (9). The controller compensates the disturbance exactly. Exponential convergence to the origin can be observed, in spite of w.

can be used to address unmatched parametric uncertainty or unmatched external disturbances. As an example, let us recall a robust stability result.

Suppose that $A = A_0 + M\Delta N_1$ and $B = B_0 + M\Delta N_2$, where A_0, B_0, M, N_1 and N_2 are known and Δ is a parametric uncertainty matrix that satisfies $||\Delta|| \leq 1$.

Theorem 10 *[10, p. 59] The implicit uncertain system $E\dot{x}(t) = Ax(t) + Bu_{\mathrm{n}}(t)$ is generalized quadratically stabilizable if and only if there exists matrices $P > 0$, Q, Y and a scalar $\varepsilon > 0$ such that*

$$\begin{bmatrix} \Gamma(P,Q) + \varepsilon MM^\top & \Omega(P,Q)^\top N_1^\top + Y^\top N_2^\top \\ N_1\Omega(P,Q) + N_2 Y & -\varepsilon I \end{bmatrix} < 0 \,, \tag{11}$$

where

$$\Gamma(P,Q) = \Omega(P,Q)^\top A_0^\top + A_0\Omega(P,Q) + B_0 Y + Y^\top B_0^\top$$
$$\Omega(P,Q) = PE^\top + SQ$$

and $S \in \mathbb{R}^{n\times(n-l)}$ is any matrix with full column rank that satisfies $ES = 0$. Assume, without loss of generality, that $\Omega(P,Q)$ is nonsingular. Then, a desired robustly stabilizing state feedback controller can be chosen as

$$u_{\mathrm{n}}(t) = Y\Omega(P,Q)^{-1}x(t) \,. \tag{12}$$

By generalized quadratic stability, it is meant that there exists a common closed-loop quadratic Lyapunov function that is valid for all admissible Δ. The nonsingularity of $\Omega(P,Q)$ is without loss of generality in the sense that, if singular, it can always be replaced by a nonsingular $\hat{\Omega}(P,Q) = \Omega(P,Q) + \tilde{P}$ in which $\tilde{P}$ satisfies $E\tilde{P} = \tilde{P}^\top E^\top \geq 0$ and the generalized Lyapunov inequality $\hat{\Omega}^\top(P,Q)A_0^\top + A_0\hat{\Omega}(P,Q) + B_0Y + Y^\top B_0^\top < 0$ (see [10] for details).

It can be shown that generalized quadratic stability implies that the closed-loop system is regular and impulse free. Recall that the latter property implies that $N = 0$, which in turn implies that $q = 1$, so only continuity (as opposed to continuous differentiability) of the inputs is required for well posedness.

Let us now consider parametric uncertainties of the form

$$A = A_0 + M\Delta_A N_1 \quad \text{and} \quad B = B_0(I + \Delta_B) \,, \tag{13}$$

where $||\Delta_A|| \leq 1$, $||\Delta_B|| \leq e < 1$. The particular form of B implies that its uncertainty is matched. The uncertainty on A, on the other hand, is allowed to be unmatched. Indeed, let us write $M = M_B + M_{B^\perp}$, where

$$\operatorname{rank}[B_0 \;\; M_B] = \operatorname{rank} B_0 \tag{14}$$

and $\operatorname{rank}[B_0^\perp \;\; M_{B^\perp}] = \operatorname{rank} B_0^\perp$ with $B_0^\perp$ a full-rank matrix such that $B_0^\top B_0^\perp = 0$ (note that this decomposition is *always* possible for arbitrary M).

The uncertain terms $M_B\Delta_A N_1 x(t)$ and $B_0\Delta_B u_{\mathrm{n}}(t)$ are matched (but now state-dependent) and can be compensated by $u_{\mathrm{p}}(t)$. The remaining term, $M_{B^\perp}\Delta_A N_1 x(t)$, is unmatched and can be dealt with using Theorem 10. This is precisely stated in the following theorem.

Theorem 11 *Consider an implicit perturbed system* (1) *with parametric uncertainty of the form* (13)*. Suppose that:*

i) The matching condition (2) *holds.*
ii) The bound $||\dot{w}(t)|| \leq \bar{w}_1$ holds for some known $\bar{w}_1$.
iii) The LMI (11) *is solvable with*

$$M = M_{B^\perp} \quad \textit{and} \quad N_2 = 0 \,.$$

Then, there exists a continuous control that enforces $\|x(t)\|_2 \leq \alpha e^{-\beta t}\|x(0)\|_2$ for all $t > 0$.

PROOF. Define the sliding variable

$$\sigma(t) = B_0^+\left[Ex(t) - \int_0^t (A_0 - B_0K)x(\tau)\mathrm{d}\tau\right] \tag{15}$$

with $K = -Y\Omega(P,Q)^{-1}$. Split the control as $u(t) = -Kx(t) + u_{\mathrm{p}}(t)$. When the system is constrained to the sliding surface we have

$$\dot{\sigma}(t) = B_0^+\Big[\Big((M_{B_0} + M_{B_0^\perp})\Delta_A N_1 - B_0\Delta_B K\Big)x(t) + B_0(I + \Delta_B)u_{\mathrm{p}}(t) + Dw(t)\Big] = 0 \,. \tag{16}$$

Noting that $B_0^+ B_0 = I$, $B_0^+ M_{B_0^\perp} = 0$ and solving for $u_p(t)$ gives

$$u_p(t) = -(I+\Delta_B)^{-1}\big((B_0^+ M_{B_0}\Delta_A N_1 - \Delta_B K)x(t) + B_0^+ D w(t)\big) \ . \quad (17)$$

Note that $B_0 B_0^+[M_{B_0} \ D] = [M_{B_0} \ D]$ (cf. (2) and (14)), so substitution of (17) in (1) gives the sliding dynamics $E\dot{x}(t) = (A_0 - B_0 K + M_{B_0^\perp}\Delta_A N_1)x(t)$. Stability of this system is a direct consequence of Theorem 10. The Theorem also ensures that, for the closed-loop system $E\dot{x}(t) = (A - BK)x(t) + Bu_{\mathrm{p}}(t) + Dw(t)$, $q = 1$, so $x(t)$ will be continuous if $u_{\mathrm{p}}(t)$ and $w(t)$ are. A continuous controller that constrains the system to the sliding surface can be finally constructed by adding one integrator as described in Section 3 and applying a second-order quasi-continuous controller with state-dependent gain as described in [11].

Example 12 *Consider the uncertain system*

$$\dot{x}_2 = (1-2\delta_1)x_1(t) - 2\delta_1 x_2(t) \quad (18a)$$
$$-w(t) = \delta_1 x_1(t) + (1+\delta_1)x_2(t) + (1+\delta_2)u(t) \quad (18b)$$

with $|\delta_1| \leq 1$, $|\delta_2| \leq 0.1$ and $\bar{w}_1 = 1$. The nominal part is the same as in Example 9, while the parametric uncertainty is determined by $M = [-2 \ 1]^\top$, $N_1 = [1 \ 1]$, $\Delta_A = \delta_1$ and $\Delta_B = \delta_2$. It is not hard to verify that the open-loop system is unstable for all $-1 < \delta_1 < 0$ and that its finite eigenvalue tends to infinity as δ_1 tends to zero.

We have $M_B = [0 \ 1]^\top$ and $M_{B^\perp} = [-2 \ 0]^\top$. Using YALMIP over SeDuMi, the set of matrices

$$P = \begin{bmatrix} 0.32 & 0 \\ 0 & 0.68 \end{bmatrix} , \quad Q = \begin{bmatrix} 1.48 & 0 \end{bmatrix}$$
$$Y = \begin{bmatrix} -0.68 & -0.41 \end{bmatrix} , \quad \tilde{P} = 0.05E$$

was found to be a solution of the LMI (11) with M replaced by $M_{B^\perp}$ and $N_2 = 0$. Plugging the solution into (12) we obtain $u_{\mathrm{n}}(t) = -Kx(t)$, where $K = [8.3 \ 19]$. This control law robustly stabilizes the system vis-à-vis the unmatched parametric uncertainty $M_{B^\perp}\Delta_A N_1$. The matched uncertainty is compensated using (10) with the aid of the sliding variable (15), i.e.,

$$\sigma(t) = \int_0^t (K_1 x_1(\tau) + (K_2 - 1)x_2(\tau))\,\mathrm{d}\tau \ .$$

The second derivative is

$$\ddot{\sigma}(t) = K_1 \dot{x}_1(t) + (K_2 - 1)\dot{x}_2(t) \quad (19)$$

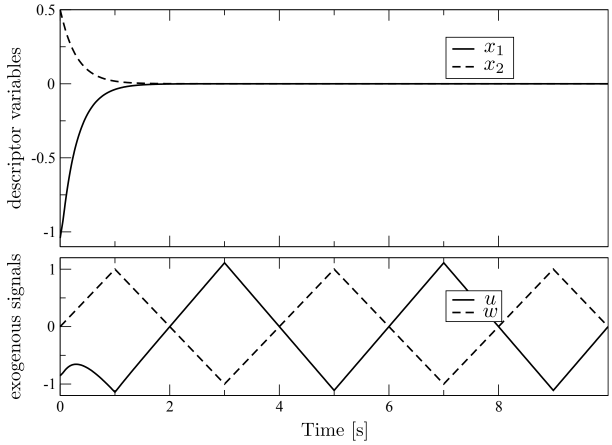

Fig. 2. Response of (18). The controller compensates w exactly and stabilizes the system, in spite of parametric uncertainty.

(note that $\dot{\sigma}(t)$ is readily available). By setting $\gamma(t) = u_{\mathrm{p}}^{(q)}(t) = \dot{u}_{\mathrm{p}}(t)$ and differentiating (18b) we obtain

$$\hat{K}_1 \dot{x}_1(t) = (1 - \hat{K}_2)\dot{x}_2(t) + (1+\delta_2)\gamma(t) + \dot{w}(t) \ ,$$

where $\hat{K}_1 = K_1(1+\delta_2) - \delta_1$ and $\hat{K}_2 = K_2(1+\delta_2) - \delta_1$.

Solving for $\dot{x}_1(t)$, substituting in (19) and using (18a) gives

$$\ddot{\sigma}(t) = h(x(t), \dot{w}(t), \delta) + g(\delta)\gamma(t) \ ,$$

where $h(x, \dot{w}, \delta) = a(\delta)\cdot x + k(\delta)\dot{w}$ and $g(\delta) = k(\delta)(1+\delta_2)$ with $k(\delta) = K_1/\hat{K}_1$ and

$$a(\delta) = \Big(k(\delta)(1-\hat{K}_2) + (K_2 - 1)\Big)\begin{bmatrix} 1 - 2\delta_1 & -2\delta_1 \end{bmatrix} \ .$$

Following [11], we propose the auxiliary control

$$\gamma(\tau_1) = -\alpha\Phi(x(\tau_1))\frac{\dot{\sigma} + \sqrt{|\sigma|}\,\mathrm{sign}(\sigma)}{|\dot{\sigma}| + \sqrt{|\sigma|}}(\tau_1) \quad (20)$$

in which $\Phi(x)$ is such that, for any $d > 0$, the inequality $\alpha g(\delta)\Phi(x) > |h(x, \dot{w}, \delta)| + d$ holds for sufficiently large α. After performing some basic bounding of g and h, it is not hard to see that $\Phi(x) = (5.3||x|| + 3)$ satisfies such condition with $\alpha = \max\{1, d\}$.

Fig. 2 shows the system response to an initial condition $x_2(0) = 0.5$ and a perturbation with a triangular waveform ($w(t)$ is not $\mathcal{C}^1$ but $|\dot{w}(t)|$ is bounded). The uncertain parameters were set as $\delta_1 = -0.5$ and $\delta_1 = -0.1$. The controller gain was set to $\alpha = 1$. It can be verified that $x_1(t)$ and $x_2(t)$ converge exponentially to the origin.

5 Conclusions

Sliding-mode control achieves robust stability by enforcing algebraic constraints in dynamical systems. It then makes sense to study the sliding-mode stabilization problem for systems that already include algebraic constraints, whether the constraints appear naturally in the physical process, whether they result from a model simplification (e.g., as in singular perturbation theory) or whether they result from a lower-level control loop (e.g., as in a hierarchical sliding-mode control setup).

The integral sliding surface proposed in Lemma 8 allows for the exact compensation of matched perturbations. The sliding surface is not much different from the one proposed in [8], but the required analysis is not straightforward because implicit systems involve subtle, yet important issues such as regularity, impulse freeness and admissibility of the inputs. In particular, it is shown that the control has to be $\mathcal{C}^{q-1}$. The examples presented are new in the sense that the perturbations enter the system through an algebraic constraint.

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