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CINVESTAV

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Positive Defined **Quadratic Form** (A.M. Lyapunov, 1892) for *smooth right-hand sides*(RHS)
The Smoothness of the Lyapunov functions

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- Positive Defined **Continuos** Function Decreasing Along The Trajectories of The System (V. Zubov, 1957) for *continous RHS* (necessary and sufficient condition)
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- Positive Defined **Absolute Continous** Function (V.I. Utkin, 1971) for *discontinuous RHS* (Sliding Mode Systems)
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- Positive Defined **Absolute Continous** Function (V.I. Utkin, 1971) for *discontinous RHS* (Sliding Mode Systems)
- Positive Defined **Semi-Continous** Function (,) for *Lipshitz continous RHS*
For the first-order sliding mode systems, the simplest representative of which is
\[
\dot{x}(t) = -r \text{ sign}[x(t)], \quad r > 0, \quad t \geq 0 \quad (1)
\]
with
\[
\text{sign}[x] := \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0 \\
\in [-1, 1] & \text{if } x = 0
\end{cases} \quad (2)
\]
evidently, the nonsmooth Lyapunov function is
\[
V(x) = |x|
\]
and hence, its time-derivative (for $x \neq 0$) taken on the trajectories of (1) satisfies
\[
\dot{V}(x(t)) = -r
\]
that implies $0 \leq V(x(t)) = V(x(0)) - rt$ and defines
\[
t_{\text{reach}} = V(x(0)) / r
Consider the system

\[
\begin{align*}
\dot{x}_1 &= -2 \text{sign}[x_1] - \text{sign}[x_2] \\
\dot{x}_2 &= -2 \text{sign}[x_1] + \text{sign}[x_2]
\end{align*}
\]

For the Lyapunov function

\[ V(x_1, x_2) = 4|x_1| + |x_2| \]

\[
\frac{dV}{dt} = \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 = -7 - 6 \text{sign}[x_1 x_2] \leq -1
\]
Lyapunov Functions for Second Order Sliding Systems

Consider the second order sliding system (A. Levant, 1990)

\[ \ddot{x}(t) = -r_1 \text{sign}[x(t)] - r_2 \text{sign}[\dot{x}(t)], \quad t \geq 0 \]  (3)

where \( x \in \mathbb{R}, r_1 > r_2 > 0. \)

The Lyapunov Function this system has the form (Y. Orlov, 2006)

\[ V(x, \dot{x}) = r_1 |x| + \dot{x}^2 / 2 \]  (4)

The time derivative of \( V(x, \dot{x}) \) in terms of the system (3)

\[ \frac{dV}{dt} = \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial \dot{x}} \ddot{x} = -r_2 |\dot{x}| \leq 0 \]

Presented Lypunov Function does not provide the finite-time convergence and does not allow to obtain the reaching time estimation.
the unified method of the Lyapunov function design has never been presented for 2-sliding mode systems;
Motivation

- the unified method of the Lyapunov function design has never been presented for 2-sliding mode systems;
- the geometrical approach for the finite-time convergence analysis of the trajectories the second order sliding mode systems can not be definitely extended to multidimensional case;
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the geometrical approach for the finite-time convergence analysis of the trajectories the second order sliding mode systems can not be definitely extended to multidimensional case;

the existed Lyapunov functions for 2-sliding mode systems do not guarantee a finite time convergence providing it only asymptotically;
the unified method of the Lyapunov function design has never been presented for 2-sliding mode systems;

the geometrical approach for the finite-time convergence analysis of the trajectories the second order sliding mode systems can not be definitely extended to multidimensional case;

the existed Lyapunov functions for 2-sliding mode systems do not guarantee a finite time convergence providing it only asymptotically;

the use of non-smooth Lyapunov function design permits to estimate the reaching time even in the presence of a bounded noise.
Problem Formulation

Let us consider an affine control system

\[
\begin{align*}
\dot{x} &= g(x, y) \\
\dot{y} &= a(x, y) + b(x, y)u + f(t, x, y)
\end{align*}
\]  

(5)

where \( x \in \mathbb{R}^k \), \( y \in \mathbb{R}^n \) are components of the state-space vector, \( g : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^k \), \( a : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n \) are smooth system vector functions, \( u \in \mathbb{R}^m \) is vector of control inputs, \( b : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^{n \times m} \) are control-gain matrix, and \( f : \mathbb{R}_+ \times \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n \) is an uncertain measurable but bounded function, i.e.,

\[
|f_j(t, x, y)| \leq C_j
\]  

(6)

for \( \forall x \in \mathbb{R}^k \), \( \forall y \in \mathbb{R}^n \), \( \forall t \geq 0, j = 1, n \). Let \( u \) is already designed as

\[
u = \bar{u}(x, y)
\]  

(7)

Problem (1)

To propose the method of the Lyapunov function design for finite-time convergence analysis of the system (5)-(7).
The Total Derivative Estimation

\[
\frac{dV(x, y)}{dt} = \sum_{i=1}^{k} \frac{\partial V}{\partial x_i} \dot{x}_i + \sum_{j=1}^{n} \frac{\partial V}{\partial y_j} \dot{y}_j = k \sum_{i=1}^{k} \frac{\partial V}{\partial x_i} g_i(x, y) + \\
\sum_{j=1}^{n} \frac{\partial V}{\partial y_j} (a_j(x, y) + b_j(x, y) \bar{u}(x, y)) + \sum_{j=1}^{n} \frac{\partial V}{\partial y_j} f_j(t, x, y) \leq \\
k \sum_{i=1}^{k} \frac{\partial V}{\partial x_i} g_i(x, y) + \sum_{j=1}^{n} \frac{\partial V}{\partial y_j} (a_j(x, y) + b_j(x, y) \bar{u}(x, y)) + \sum_{j=1}^{n} \frac{\partial V}{\partial y_j} C_j
\]

Denoting

\[
h_j(x, y, \gamma_j) := a_j(x, y) + b_j(x, y) \bar{u}(x, y) + \gamma_j \]  \tag{8}

\[
\gamma_j := C_j \text{sign}[\frac{\partial V}{\partial y_j}], j = 1, n \]  \tag{9}

we will have

\[
\frac{dV}{dt} \leq \sum_{i=1}^{k} \frac{\partial V}{\partial x_i} g_i(x, y) + \sum_{j=1}^{k} \frac{\partial V}{\partial y_j} h_j(x, y, \gamma_j) \]  \tag{10}
Let us try to find the Lyapunov function $V(x, y)$ as solution of
\[
\sum_{i=1}^{k} g_i(x, y) \frac{\partial V}{\partial x_i} + \sum_{j=1}^{k} h_j(x, y, \gamma_j) \frac{\partial V}{\partial y_j} = -qV^\rho, \tag{11}
\]
\[
\frac{dV}{dt} \leq -qV^\rho \tag{12}
\]

a) In the case $0 < \rho < 1$ we obtain the Lyapunov function with finite
time convergence $t_{reach} = \frac{1}{q(1-\rho)}[V(x(0), y(0))]^{1-\rho}$ since
\[
V(x(t), y(t)) = ([V(x(0), y(0))]^{1-\rho} - (1 - \rho)qt)^{\frac{1}{1-\rho}}
\]

b) For $\rho \geq 1$ the positive definite solution of (11) implies only an
asymptotic convergence since
\[
V(x(t), y(t)) = \begin{cases} 
V(x(0), y(0))e^{-qt} & \text{if } \rho=1 \\
\frac{1}{(\frac{1}{V(x(0), y(0))^{\rho-1}} + (\rho-1)qt)^{\frac{1}{\rho-1}}} & \text{if } \rho>1 
\end{cases}
\]
Lemma (1)

If an absolute continuous positive definite function \( V(x, y) \) satisfies the following system of ODE (Characteristic system)

\[
\frac{dx_1}{g_1(x, y)} = \cdots = \frac{dx_k}{g_k(x, y)} = \frac{dy_1}{h_1(x, y, \gamma_1)} = \cdots = \frac{dy_n}{h_n(x, y, \gamma_n)} = \frac{dV}{-qV^\rho}
\]

for \( \|x\|^2 + \|y\|^2 > 0 \), then the same function \( V(x, y) \) is a solution of (35).

Using the first integrals of the Characteristic system

\[
\varphi_i(V, x, y, \gamma, q, \rho) = \text{const} := c_i, \quad i = 1, n + k
\]

(13)

the solution \( V(x, y) \) can be found from

\[
\Phi(\varphi_1(V, x, y, \gamma, q, \rho), \ldots, \varphi_{n+k}(V, x, y, \gamma, q, \rho)) = 0
\]

(14)

where \( \Phi(\varphi_1, \ldots, \varphi_{n+k}) \) is an arbitrary function.
The summary of the method

- to find an appropriate estimation of the total derivative for an arbitrary function $V(x, y)$

$$\frac{dV}{dt} \leq RHS$$
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The summary of the method

- to find an appropriate estimation of the total derivative for an arbitrary function $V(x, y)$

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- to obtain the PDE

  $$RHS = -qV^\rho$$

- to apply the characteristics method and find an absolute continous positive defined solution $V(x, y)$ of PDE

- to prove that the discontinuity set of partial derivatives of $V(x, y)$ does not contain the integral manifolds of the initial system
Lyapunov Function design for "twistig" controller

Consider the controlled system given by

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= f(t, x, y) + u(t)
\end{align*}
\]

(15)

where \(x, y \in \mathbb{R}\) are the scalar state variables, \(f(t, x, y)\) is a measurable but unknown function, \(u \in \mathbb{R}\) is the, so-called, "twisting" control (A. Levant)

\[
u(t) = -r_1 \text{sign}[x(t)] - r_2 \text{sign}[y(t)]
\]

(16)

where \(r_1, r_2 > 0\) are control parameters. The solution of (15) is understood in the Filippov’s sense.

It is supposed that

\[|f(t, x, y)| \leq C \text{ for } \forall x, y \in \mathbb{R} \text{ and } \forall t \geq 0\]

(17)

with a known constant \(C\).

Problem (2)

To find a Lyapunov function with a finite-time convergence for the system (15)-(17).
PDE for "Twisting" System

For an arbitrary absolute continuous function $V(x,y)$ we have

$$
\frac{dV}{dt} = \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial y} \dot{y} = y \frac{\partial V}{\partial x} + \left( -r_1 \text{sign}[x] - r_2 \text{sign}[y] + f(t,x,y) \right) \frac{\partial V}{\partial y} \leq y \frac{\partial V}{\partial x} + \left( -r_1 \text{sign}[x] - r_2 \text{sign}[y] + C \text{sign}[\frac{\partial V}{\partial y}] \right) \frac{\partial V}{\partial y} = y \frac{\partial V}{\partial x} - \text{sign}[x] \gamma \frac{\partial V}{\partial y}
$$

where

$$\gamma = r_1 + r_2 \text{sign}[xy] - C \text{sign}[x \frac{\partial V}{\partial y}] \tag{18}$$

According the proposed method the equation (11) becomes to be as

$$y \frac{\partial V}{\partial x} - \text{sign}[x] \gamma \frac{\partial V}{\partial y} = -k V^{1/2} \tag{19}$$

where $k > 0$ is a positive parameter.
The corresponding systems of the characteristics ODE is

\[
\frac{dx}{y} = \frac{dy}{-\text{sign}[x] \gamma} = \frac{dV}{-k \sqrt{V}}
\]

(20)

Its first integrals are as follows

\[
\varphi_1(x, y) = |x| + \frac{y^2}{2 \gamma}
\]

\[
\varphi_2(V, y) = \frac{y \text{sign}[x]}{\gamma} - \frac{2 \sqrt{V}}{k}
\]

(21)

Select \( \Phi \) as

\[
\Phi(\varphi_1, \varphi_2) = k_0 \sqrt{\varphi_1} + \varphi_2 = 0
\]

(22)

where \( k_0 \) is a real parameter.
Substitution (21) into (22) leads to the equation

$$\frac{2\sqrt{V}}{k} = \frac{y \text{ sign}[x]}{\gamma} + k_0 \sqrt{|x|} + \frac{y^2}{2\gamma}$$  \hspace{1cm} (23)

Obviously, the right-hand side of (23) makes sense for all $x^2 + y^2 > 0$ if and only if $\gamma > 0$, or equivalently, when $r_1 > r_2 + C$. Since the left-hand side of (23) should be positive when $\|x\|^2 + \|y\|^2 > 0$, it follows that

**Condition (1)**

$$k_0 > -\frac{\text{sign}[xy] \sqrt{2} \gamma}{\sqrt{1 + 2\gamma|x|/y^2}}$$  \hspace{1cm} (24)

So, the function $V(x, y)$ can be found from (23) as

$$V(x, y) = \frac{k^2}{4} \left( \frac{y \text{ sign}[x]}{\gamma} + k_0 \sqrt{|x| + \frac{y^2}{2\gamma}} \right)^2$$  \hspace{1cm} (25)
Removing of Discontinuities

Considering the partial limits of the function $V(x, y)$ when $x$ tends to zero for any fixed $y$ and when $y$ tends to zero for any fixed $x$, one derive

$$
\begin{align*}
\text{if } x \to 0 & \text{ then } V(x, y) \to \frac{k^2y^2}{4\gamma} \left( \frac{\text{sign}[xy]}{\sqrt{\gamma}} + \frac{k_0}{\sqrt{2}} \right)^2 \\
\text{if } y \to 0 & \text{ then } V(x, y) \to \frac{k^2k_0^2}{4} |x|
\end{align*}
$$

Hence, to eliminate the discontinuities on the lines $x = 0$ and $y = 0$ it is sufficient to resolve the following system

$$
\begin{cases}
k^2 \left( \frac{\text{sign}[xy]}{\gamma} + \frac{k_0}{\sqrt{2}\gamma} \right)^2 = \bar{k}^2 \\
k^2k_0^2 = 1
\end{cases}
$$

for some positive value $\bar{k}$. This gives

$$
k = \sqrt{\frac{\gamma}{2}} |\sqrt{2\gamma \bar{k}} - 1| > 0
$$

$$
k_0 = \sqrt{\frac{2}{\gamma} \frac{\text{sign}[xy]}{\sqrt{2}\gamma \bar{k} - 1}}
$$

(26)
Lemma (2)

If \( r_1 > r_2 + C \) and \( r_2 > C \) and the parameter \( \bar{k} \) in (26) is selected in such a way that the following inequalities hold

\[
\frac{1}{\sqrt{2(r_1 + r_2 - C)}} < \bar{k} < \frac{1}{\sqrt{2(r_1 - r_2 + C)}} \tag{27}
\]

then the condition (24) for \( xy \neq 0 \) holds too.

So, the Lyapunov function (25) becomes

\[
V = \begin{cases} 
\frac{k^2}{4} \left( \frac{y \text{sign}[x]}{\gamma} + k_0 \sqrt{|x| + \frac{y^2}{2\gamma}} \right)^2 & xy \neq 0 \\
\bar{k}^2 y^2 / 4 & x = 0 \\
|x| / 4 & y = 0 
\end{cases} \tag{28}
\]

with \( k, k_0 \) are as in (26) and \( \bar{k} \) satisfying (27).
Theorem (1)

If \( r_1 > r_2 + C \), \( r_2 > C \) then the function \( V(x, y) \) (28) with
\[
\gamma := r_1 + (r_2 - C) \text{sign}[xy]
\]
has the following properties:
1) \( V(x, y) \) is positive defined in all space \( \mathbb{R}^2 \);
2) \( V(x, y) \) is absolute continuous in all space \( \mathbb{R}^2 \) and continuously differentiable when \( xy \neq 0 \);
3) the time derivative of \( V(x, y) \) on the trajectories of (15) satisfies
\[
\frac{dV}{dt} \leq -k_{\text{min}} \sqrt{V}
\]
almost everywhere with
\[
k_{\text{min}} := \frac{r_1 - r_2 - C}{r_1 - r_2 + C} \min_{\delta \in \{-1, 1\}} \frac{r_1 + (r_2 - C)\delta}{\sqrt{2}} \left| k \sqrt{2} - \frac{1}{\sqrt{r_1 + (r_2 - C)\delta}} \right|
\]
3) the guaranteed reaching time is \( t_{\text{reach}} \leq \frac{2}{k_{\text{min}}} \sqrt{V(x(0), y(0))} \)
Example 1

Let $r_1 = 1.8, r_2 = 1, C = 0.2$

Fig. 1 The Lyapunov Function For "Twisting" System

Fig. 2 The Level Lines of The Lyapunov Function
Let us consider the system

\[
\dot{x}(t) = u(t) + \varphi(t)
\]  \hspace{1cm} (29)

with the super-twisting control[A. Levant]

\[
u(t) = u_1(t) + u_2(t)
\]  \hspace{1cm} (30)

where

\[
\begin{align*}
  u_1(t) &= -\alpha \sqrt{|x(t)|} \text{sign}[x(t)] \\
  \dot{u}_2(t) &= -\beta \text{sign}[x(t)]
\end{align*}
\]  \hspace{1cm} (31)

\[\alpha > 0 \text{ and } \beta > 0\]

and \(\varphi(t)\) is unknown function with bounded derivative

\[|\dot{\varphi}(t)| \leq L, \ L > 0.\]  \hspace{1cm} (32)
Reduction to The 2-Dimensional Case

Let us denote

\[ y(t) = \varphi(t) - \beta \int_{0}^{t} \text{sign}[x(\tau)] d\tau \]  \hspace{1cm} (33)

then

\[
\begin{cases}
\dot{x}(t) = -\alpha \sqrt{|x(t)|} \text{sign}[x(t)] + y(t) \\
\dot{y}(t) = \dot{\varphi}(t) - \beta \text{sign}[x(t)]
\end{cases}
\]  \hspace{1cm} (34)

Problem (3)

To find a Lyapunov function with a finite-time convergence for the system (34).

If \( x = 0 \) and \( y = 0 \) then \( \dot{x} = 0 \) and we have the second order sliding mode on variable \( x \).
For an arbitrary function $V(x, y)$

$$
\frac{dV(x, y)}{dt} = \frac{\partial V(x, y)}{\partial x} \dot{x} + \frac{\partial V(x, y)}{\partial y} \dot{y} = \\
= \frac{\partial V(x, y)}{\partial x} \dot{x} + \frac{\partial V(x, y)}{\partial y} (\dot{\phi} - \beta \text{sign}[x]) \leq \\
\leq \frac{\partial V(x, y)}{\partial x} \dot{x} + \frac{\partial V(x, y)}{\partial y} \left( L \text{sign} \left[ \frac{\partial V(x, y)}{\partial y} \right] - \beta \text{sign}[x] \right)
$$

Hence, to find the Lyapunov function with fine-time convergence it is needed to solve the partial differential equation

$$
(y - \alpha \sqrt{|x|} \text{sign}[x]) \frac{\partial V(x, y)}{\partial x} - \gamma \text{sign}[x] \frac{\partial V(x, y)}{\partial y} = -k \sqrt{V(x, y)} \tag{35}
$$

where $k > 0$ and $\gamma = \beta - L \text{sign}[x \frac{\partial V(x, y)}{\partial y}], \gamma \in \{\beta + L, \beta - L\}$. 
In our case the characteristics equations have the form

\[
\frac{dx}{y - \alpha \sqrt{|x|} \text{sign}[x]} = \frac{dy}{-\gamma \text{sign}[x]} = \frac{dV}{-k\sqrt{V}} \tag{36}
\]

1. The first equation

\[
\frac{dx}{y - \alpha \sqrt{|x|} \text{sign}[x]} = \frac{dy}{-\gamma \text{sign}[x]} \tag{37}
\]

Let us denote \( z = \frac{\sqrt{|x|} \text{sign}[x]}{y} \)

\[-\frac{2\gamma z \, dz}{2\gamma z^2 - \alpha z + 1} = \frac{dy}{y} \tag{38}\]
The first integrals (2)

The general solution of the first equation has the form

\[ \psi_1(x, y) = \begin{cases} \frac{\ln s(x, y)}{2} + \frac{1}{\sqrt{g-1}} \arctan \left( \frac{\alpha g \sqrt{|x| \text{sign}[x] - 2y}}{2 \sqrt{g-1}y} \right) & g > 1 \\ \frac{\ln |s(x, y)|}{2} + \frac{0.5}{\sqrt{1-g}} \ln \left| \frac{\sqrt{|x| \text{sign}[x] - 1 + \sqrt{1-g}}}{\frac{\alpha g/2}{\sqrt{|x| \text{sign}[x] - 1 - \sqrt{1-g}} y} \right| & g < 1 \end{cases} \]  

(39)

where \( g = 8\gamma/\alpha^2 \), \( s(x, y) = 2\gamma|x| - \alpha \sqrt{|x| \text{sign}[x]} y + y^2 \).

2. The second equation

\[ \frac{dy}{-\gamma \text{sign}[x]} = \frac{dV}{-k \sqrt{V}} \]  

(40)

The general solution of the last system has a form

\[ \psi_2(y, V) = \frac{y \text{sign}[x]}{\gamma} - \frac{2\sqrt{V}}{k} = \text{const} \]  

(41)
Let us consider the case \( g > 1 \) (i.e. \( \alpha^2 < 8(\beta - L) \)) and define

\[
\Phi(\psi_1, \psi_2) = k_0 e^{\psi_1} + \psi_2 = 0
\]  

(42)

Hence, using the expressions for the first integrals we will have

\[
k_0 e^{m(x,y)} \sqrt{s(x,y)} + \frac{y \text{sign}[x]}{\gamma} = \frac{2\sqrt{V}}{k}
\]  

(43)

where \( g = 8\gamma/\alpha^2 \)

\[
m(x, y) = \frac{1}{\sqrt{g - 1}} \arctan \left( \frac{\alpha g \sqrt{|x|} \text{sign}[x] - 2y}{2\sqrt{g - 1}y} \right)
\]  

(44)

\[
s(x, y) = 2\gamma|x| - \alpha \sqrt{|x|} \text{sign}[x]y + y^2
\]  

(45)
The Lyapunov Function Candidate

\( V(x, y) = \frac{k^2}{4} \left[ \frac{y \cdot \text{sign}[x]}{\gamma} + k_0 e^{m(x,y)} \sqrt{s(x, y)} \right]^2 \)  \( (46) \)

and the formal calculation shows

\[
\frac{dV}{dt} = \frac{k^2}{2} V_0 \left( \frac{\partial V_0}{\partial x} \dot{x}(t) + \frac{\partial V_0}{\partial y} \dot{y} \right) \\
\leq k \sqrt{V} \left( \frac{\partial V_0}{\partial x} \dot{x}(t) - \frac{\partial V_0}{\partial y} \gamma \cdot \text{sign}[x] \right) = -k \sqrt{V}
\]

where

\( V_0(x, y) = \frac{y \cdot \text{sign}[x]}{\gamma} + k_0 e^{m(x,y)} \sqrt{s(x, y)} \)  \( (47) \)

\[
\frac{\partial V_0}{\partial x} = k_0 \gamma \cdot \text{sign}[x] \frac{e^{m(x,y)}}{\sqrt{s(x, y)}}
\]  \( (48) \)

\[
\frac{\partial V_0}{\partial y} = \frac{\text{sign}[x]}{\gamma} + k_0 (y - \alpha \sqrt{|x| \cdot \text{sign}[x]}) \frac{e^{m(x,y)}}{\sqrt{s(x, y)}}
\]  \( (49) \)
Removing of discontinuities

if \( x \to 0 \) \[ V(x, y) \to \frac{k^2}{4} \left( \frac{\text{sign}[xy]}{\gamma} + k_0 e^{\frac{1}{\sqrt{g-1}}} \arctan\left(-\frac{1}{\sqrt{g-1}}\right) \right)^2 y^2 \]

if \( y \to 0 \) \[ V(x, y) \to \frac{1}{2} k^2 k_0^2 e^{\frac{\pi \text{sign}[xy]}{\sqrt{g-1}}} \gamma |x| \]

\[ k_0 = \frac{\text{sign}[xy]}{\gamma \left( \bar{k} \sqrt{g e^{\frac{\pi \text{sign}[xy]}{2\sqrt{g-1}}} - e^{\frac{1}{\sqrt{g-1}}} \arctan\left(-\frac{1}{\sqrt{g-1}}\right)} \right)} \] \hspace{1cm} (50)

\[ k = \sqrt{\gamma} \left| \bar{k} \sqrt{g} - e^{\frac{1}{\sqrt{g-1}}} \arctan\left(-\frac{1}{\sqrt{g-1}}\right) - \frac{\pi \text{sign}[xy]}{2\sqrt{g-1}} \right| > 0 \] \hspace{1cm} (51)

where \( \bar{k} \equiv \text{const} > 0 \).

\[ V(0, y) = \frac{2\bar{k}^2 y^2}{\alpha^2} \text{ and } V(x, 0) = \frac{|x|}{2} \] \hspace{1cm} (52)
Lemma (3)

If

\[ \bar{k} \in \left( \frac{1}{\sqrt{g} - 1} \left( -\frac{\pi}{2} + \arctan \left( -\frac{1}{\sqrt{g} - 1} \right) \right), \frac{1}{\sqrt{g} - 1} \left( \frac{\pi}{2} + \arctan \left( -\frac{1}{\sqrt{g} - 1} \right) \right) \right) \]  

(53)

and \( k_0 \) has the form (50) then Condition 2 holds.

Lemma (4)

If

\[ \bar{k} > \frac{2}{g} + e^{\frac{1}{\sqrt{g} - 1} \left( -\frac{\pi}{2} + \arctan \left( -\frac{1}{\sqrt{g} - 1} \right) \right)} \]  

(54)

then \( \text{sign} \left[ \frac{\partial V}{\partial y} \right] = \text{sign}[y] \) and

\[ \gamma = \beta - L \text{sign}[x \frac{\partial V}{\partial y}] = \beta - L \text{sign}[xy] \]  

(55)
From Lemmas 1 and 2 it follows that $\bar{k} \in I(g)$, where

$$I(g) = \left( \frac{2}{g} + \frac{e^{\frac{1}{\sqrt{g-1}}} \left(-\frac{\pi}{2} + \arctan\left(\frac{-1}{\sqrt{g-1}}\right)\right)}{\sqrt{g}}, \frac{e^{\frac{1}{\sqrt{g-1}}} \left(\frac{\pi}{2} + \arctan\left(\frac{-1}{\sqrt{g-1}}\right)\right)}{\sqrt{g}} \right)$$

(56)

**Lemma**

*If $g > 1$ then the interval $I(g)$ is non empty.*

But $g = 8\gamma/\alpha^2$, $\gamma \in \{\beta - L, \beta + L\}$, so

$$\bar{k} \in I(g^-) \cap I(g^+)$$

(57)

where $g^- = 8(\beta - L)\alpha^{-2}$, $g^+ = 8(\beta + L)\alpha^{-2}$. 
Theorem 2

If $\alpha^2 < 8(\beta - L)$ and $I(g^-) \cap I(g^+) \neq \emptyset$ then the designed Lyapunov function $V(x, y)$ has the following properties:

- $V(x, y)$ is positive defined function in $\mathbb{R}^2$;
- $V(x, y)$ is continuous in $\mathbb{R}^2$ and continuously differentiable for $xy \neq 0$;
- the total derivative of $V(x, y)$ in terms of the system (34) is

$$\frac{dV}{dt} \leq -k\sqrt{V} \leq k_{\text{min}}\sqrt{V}$$

$$k_{\text{min}} := \frac{\alpha}{\sqrt{8}} \min_{g \in \{g^-, g^+\}} \left\{ \sqrt{g} \left| k\sqrt{g} - e^{-\frac{1}{\sqrt{g} - 1}} \arctan \frac{1}{\sqrt{g} - 1} - \frac{\pi(\beta - \alpha^2 g)}{16L\sqrt{g - 1}} \right| \right\}$$

- the guaranteed reaching time is $t_{\text{reach}} \leq \frac{2}{k_{\text{min}}} \sqrt{V(x(0), y(0))}$
Corollary (1)

If $\beta > 5L$ and $4L < \alpha^2 / 8 < \beta - L$ then the designed function $V(x, y)$ is the Lyapunov function.

Corollary (2)

If $\dot{\varphi}(t) \equiv L$ and $\alpha^2 < 8(\beta - L)$, then the designed function $V(x, y)$ is the Lyapunov function.

Remark

The estimation (58) of the time derivative of the Lyapunov function is unimprovable. The equality in (58) will take place for a piece-wise linear function $\varphi(t) : \dot{\varphi}(t) \equiv L \text{sign}[y(t)]$. 
Example 2

Let $\alpha = 1.8$, $\beta = 1$, $L = 0.2$ and $\varphi(t) = \begin{cases} 
0.1t & \text{if } 0 \leq t \leq 23 \\
-0.1t + 4.6 & \text{if } t > 23
\end{cases}$

Fig. 3 The Lyapunov Function For "Super-Twisting" System

Fig. 4 The Level Lines of The Lyapunov Function
Example 2 (Conveging time)

$t_{	ext{reach}}^{ex} = 26.2 \quad \text{and} \quad t_{	ext{reach}} \leq 2\sqrt{V(10,-20)/k_{\text{min}}} = 26.7$

Fig. 5 Converging trajectory (x-coordinate)

Fig. 6 Converging trajectory (y-coordinate)
The new method of the Lyapunov function design for finite-time convergence analysis of affine control system is proposed.
The new method of the Lyapunov function design for finite-time convergence analysis of affine control system is proposed. Using proposed method the Lyapunov Functions with finite-time convergence for "twisting" and "super-twisting" systems are designed in explicit forms.

The reaching time estimations of considered controllers are presented for the first time. For the "super-twisting" system is obtained probably unimprovable reaching time estimation.
Conclusion

- The new method of the Lyapunov function design for finite-time convergence analysis of affine control system is proposed.
- Using proposed method the Lyapunov Functions with finite-time convergence for "twisting" and "super-twisting" systems are designed in explicit forms.
- The reaching time estimations of considered controllers are presented for the first time.
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