Optimal Control of Hybrid Systems: Theory and Algorithms

Peter E. Caines
peterc@cim.mcgill.ca

Department of Electrical and Computer Engineering
and
Center for Intelligent Machines
McGill University
Montréal, Canada and Technische Universitaet Berlin
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Outline

• Theory
  • Description of Hybrid Systems.
  • Hybrid System Examples.
  • Optimal Control of Hybrid Systems: Necessary Conditions.

• Algorithms I
  Development of efficient hybrid optimization algorithms along with their proofs of convergence.

• Algorithms II
  Introduction of Optimality Zones and development of associated algorithms.
  (Exponential complexity to linear complexity.)
Hybrid Systems

Shall consider hybrid systems of the form:

\[ u(t) \in U, \text{ control value set } U \subset \mathbb{R}^n, \text{ where either} \]

(i) \( U \) open bounded, or (ii) compact with \( U = \bar{U} \).

\[ \mathcal{U} = \{u(\cdot) \in L_\infty(U)\} \text{ in cases (i) and (ii) respectively.} \]

\[ \dot{x}_{qi}(t) = f_{qi}(x_{qi}(t), u(t)), \quad \text{a.e. } t \in [t_i, t_{i+1}), \quad i = 0, 1, \ldots, L, \]

\[ x_{qi+1}(t_{i+1}) = \lim_\limits{t \uparrow t_{i+1}} x_{qi}(t), \quad i = 0, 1, \ldots, L. \]

Discrete Dynamics:

autonomous \( q_i \rightarrow q_{i+1} \) if \( m_{q_i,q_{i+1}}(x(t_{i+1})) = 0, \)

or

controlled \( \Gamma_c(q_i, \sigma_{i+1}(t_{i+1})) \equiv \Gamma_c(q_i, \sigma_{i+1}) = q_{i+1}, \quad \sigma_{i+1} \in \Sigma \cup \phi. \)
\( \mathcal{H} = \{ H \triangle Q \times \mathbb{R}^n, A, I \triangle \Sigma \times U, F, \Gamma, \mathcal{M} \}, \)
$\mathcal{H} = \{H \triangle Q \times \mathbb{R}^n, A, I \triangle \Sigma \times U, F, \Gamma, \mathcal{M}\},$

\[\dot{x} = f_q(x, u)\]

\[(t_0, x_0)\]

Free (controlled) switch

\[Q = \{1, 2, \ldots, |Q|\}\]

\[\mathbb{R}^n\]

$\mathcal{M} = \{m_{q_1q_2}(x) = 0\}$

Forced (autonomous) switch

\[\dot{x} = f_{q_2}(x, u)\]

\[(t_f, x_f)\]
Hybrid State

Hybrid State = \( h(t) = \left[ \begin{array}{c} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{array} \right] , q(t-) \] \in \mathbb{R}^n \times Q.
A hybrid system with continuous state $x \in \mathbb{R}^n$ and discrete state set $Q = \{1, 2, 3, 4\}$. 
A linear hybrid automaton with continuous state $[x \ y \ z]^T \in \mathbb{R}^3$ and with three discrete states (locations).
Linear Hybrid Automaton Trajectory

Trajectory of linear hybrid automaton.
A Hybrid Automaton Example
Reactor Temperature Control

\[ \begin{align*}
& r_{od_1} \\
& x \geq 510 \\
& \dot{x} \in [-5, -1] \\
& \dot{y}_1 = \dot{y}_2 = 1 \\
& l_1 \\
& x = 510 \\
& y_1 := 0 \\
& x = 550 \\
& y_1 \geq 20 \\
& l_2 \\
& no_{-rod} \\
& x \leq 550 \\
& \dot{x} \in [1, 5] \\
& \dot{y}_1 = \dot{y}_2 = 1 \\
& l_3 \\
& x = 510 \\
& y_2 := 0 \\
& x = 550 \\
& y_2 \geq 20 \\
& \dot{y}_1 = \dot{y}_2 = 1 \\
& l_3 \\
& r_{od_2} \\
& x \geq 510 \\
& \dot{x} \in [-9, -5] \\
& \dot{y}_1 = \dot{y}_2 = 1
\end{align*} \]
A Typical Hybrid System
Walking Robot
(a) Walking robot with feet in single support with foot flat on the ground, rotation of the foot about the toe, and double support.

(b) Two-mode model corresponding to the gait in (a). If rotation of the foot is allowed, a hybrid model has more than two modes depending upon contact conditions of the foot with the ground.
Examples of Hybrid Systems Control and Computation Problems:

- Computation of reachable sets,
- Safety verification,
- Stability analysis,
- Optimal control.
Single autonomous switching case.

\[ q_1 : \quad \dot{x} = x + xu, \]
\[ q_2 : \quad \dot{x} = -x + xu, \]

\[ J(u) = \frac{1}{2} \int_0^2 u^2(s) \, ds, \]

\[ m(x, t) = x - et = 0, \]

Initial values:
\[ t_s = 0.5, \quad x(t_s) = 2. \]
Hybrid Optimal Control Problem (HOCP)

Consider the hybrid system:

\[ u(t) \in U, \text{ control value set } U \subset \mathbb{R}^n, \text{ where either} \]

(i) \( U \) open bounded, or (ii) compact with \( U = \overline{U} \).

\[ \mathcal{U} = \{ u(\cdot) \in L_{\infty}(U) \} \text{ in cases (i) and (ii) respectively.} \]

\[ \dot{x}_{q_i}(t) = f_{q_i}(x_{q_i}(t), u(t)), \quad \text{a.e. } t \in [t_i, t_{i+1}), \quad i = 0, 1, \ldots, L, \]

\[ x_{q_{i+1}}(t_{i+1}) = \lim_{t \uparrow t_{i+1}} x_{q_i}(t), \quad i = 0, 1, \ldots, L. \]

Discrete Dynamics:

autonomous \( q_i \rightarrow q_{i+1} \) if \( m_{q_i,q_{i+1}}(x(t_{i+1})) = 0, \)

or

controlled \( \Gamma_c(q_i, \sigma_{i+1}(t_{i+1})) \equiv \Gamma_c(q_i, \sigma_{i+1}) = q_{i+1} \quad \sigma_{i+1} \in \Sigma \cup \phi. \)
Hybrid Optimal Control Problem (HOCP) (continued)

Find a switching sequence

$$S_L \triangleq \{ (\sigma_0, t_0), (\sigma_1, t_{s_1}), \cdots, (\sigma_L, t_{s_L}) \},$$

and control $u \in U$, where $U = U^0$ or $U = U^{cpt}$, such that $I_{L(\bar{L})} \triangleq (S_L, u)$ minimizes:

$$J(t_0, t_f, h_0; I_L, \bar{L}, U) \triangleq \sum_{i=0}^{L(\bar{L})} \int_{t_i}^{t_{i+1}} l_{q_i}(x_{q_i}(s), u(s)) \, ds + g(x_{q_L}(t_f)),$$

$$t_{L+1} = t_f < \infty, \quad L \leq \bar{L} \leq \infty,$$
Hybrid Minimum Principle

• Let a hybrid input trajectory $I_{L_0}^0$ be optimal for \( \text{HOCP}(t_0, x_0; t_f, x_f, \bar{L}, \mathcal{U}) \) and $L^0 + 2 = L_a + L_c + 2 \leq \bar{L}$.

• Let every point $x$ on the optimal trajectory $x^0(\cdot)$ be a small time tubular fountain (STTF) w.r.t. $x^0(\cdot)$.

(i) Then there exists a piecewise absolutely continuous adjoint process $\lambda^0$ such that

$$
\dot{\lambda}^0 = -\frac{\partial H}{\partial x}(h_0, u_0, \sigma_0, \lambda^0), \quad t \in [t_{s_i}, t_{s_{i+1}}), \quad i \in \{0, 1, 2, \ldots, L^0\},
$$

$$
\lambda^0(t_f) = \nabla_x g(h_0(t_f)),
$$

where

$$
H(h, u, \sigma, \lambda) = \lambda^T f_\sigma(q_j)(x(h), u) + l_\sigma(q_j)(x(h), u), \quad \lambda \in \mathbb{R}^n, \sigma \neq \phi
$$

$$
H(h, u, \sigma, \lambda) = \lambda^T f_{q_j}(x(h), u) + l_{q_j}(x(h), u), \quad \lambda \in \mathbb{R}^n, \sigma = \phi.
$$
(ii) The Hamiltonian minimization conditions are satisfied, i.e.

(a) (Classical a.e. condition)

\[ H(h^0(t), u^0(t), \sigma^0(t), \lambda^0(t)) \leq H(h^0(t), u, \sigma^0(t), \lambda^0(t)), \quad u \in U, \quad \sigma^0(t) = \phi \]

(b) (Hybrid condition: at switching times)

If \( L_a + L_c + 2 \leq \bar{L} \), then

\[ H(h^0(t), u^0(t), \sigma^0(t), \lambda^0(t)) \leq H(h^0(t), u^0(t), \sigma, \lambda^0(t)), \quad \sigma \neq \phi \]
Hybrid Minimum Principle (continued)

(iii) Let $t_{s_i}$ be a controlled switching time. Then the following Hamiltonian continuity and adjoint continuity conditions hold at $t = t_{s_i}$

$$
\lambda^0(t_{s_i}^-) = \lambda^0(t_{s_i}) = \lambda^0(t_{s_i}^+),
$$

$$
H(t_{s_i}^-) = H(t_{s_i}) = H(t_{s_i}^+), \quad i \in \{0, 1, \ldots, L^0\}.
$$

(iv) Let $t_{s_i}$ be an autonomous switching time satisfying $m_{i,j}(x(t_{s_i})) = 0$. Then there exists $p \in \mathbb{R}$ such that the following transversality conditions hold at $t = t_{s_i}$

$$
\lambda^0(t_{s_i}^-) = \lambda^0(t_{s_i}) + \nabla_x m_{i,j}\big|_{t=t_{s_i}} p,
$$

$$
H(t_{s_i}^-) = H(t_{s_i}) = H(t_{s_i}^+), \quad i \in \{0, 1, \ldots, L^0\}. \quad \square
$$
Hybrid Systems

\[ H = \{ H \triangle Q \times \mathbb{R}^n, I \triangle \Sigma \times U, F, \Gamma, M \} \]
Ray of light passes from one medium to another. Velocity $v_1$ in medium 1, $v_2$ in medium 2. Minimize time of travel (Fermat’s Principle):

$$J(\theta_1, \theta_2) = \int_{0}^{t_s} dt + \int_{t_s}^{t_f} dt.$$
Snell’s Law (continued)

At each time the ray forms an angle $\theta_1(t)$, and $\theta_2(t)$, $-\frac{\pi}{2} < \theta_i(t) < \frac{\pi}{2}$, $t_1 \leq t \leq t_f$ with the normal to $M$. Then

\[
\begin{align*}
\dot{x}_1 &= v_1 \sin \theta_1(t), \\
\dot{y}_1 &= v_1 \cos \theta_1(t), \\
\dot{x}_2 &= v_2 \sin \theta_2(t), \\
\dot{y}_2 &= v_2 \cos \theta_2(t).
\end{align*}
\]

The Hamiltonian in each medium is given by:

\[H_i = 1 + \lambda_i v_i \sin \theta_i(t) + \mu_i v_i \cos \theta_i(t), \quad i = 1, 2,\]

$\lambda_i$: adjoint to $x_i$, $\mu_i$: adjoint to $y_i$, $i = 1, 2$.

$\theta_i^0$ optimal $\Rightarrow$ $\frac{\partial H_i}{\partial \theta_i} \bigg|_{\theta_i=\theta_i^0} = \lambda_i v_i \cos \theta_i^0 - \mu_i v_i \sin \theta_i^0 = 0$

$\Rightarrow$ $\tan \theta_i^0 = \frac{\lambda_i}{\mu_i}$. 
Snell’s Law (continued)

Along the optimal trajectory,

\[
\dot{\lambda}_i = -\frac{\partial H_i}{\partial x_i} = 0 \quad \text{and} \quad \dot{\mu}_i = -\frac{\partial H_i}{\partial y_i} = 0, \quad i = 1, 2
\]

\[\Rightarrow \lambda_i \text{ and } \mu_i \text{ constant} \Rightarrow \theta_i \text{ constant} \quad (1)\]

Hamiltonian continuity:

\[
H_1(t_s-) = H_2(t_s)
\]

\[\Rightarrow \lambda_1 v_1 \sin \theta_1 + \mu_1 v_1 \cos \theta_1 = \lambda_2 v_2 \sin \theta_2 + \mu_2 v_2 \cos \theta_2. \quad (2)\]
Snell’s Law (continued)

Adjoint transversality:

\[ \lambda_1 = \lambda_2 + \nabla_x m \quad \Rightarrow \quad \lambda_1 = \lambda_2, \tag{3} \]

\[ \mu_1 = \mu_2 + \nabla_y m \quad \Rightarrow \quad \mu_1 = \mu_2 + p. \tag{4} \]

(??) \quad \Rightarrow \quad \text{Path of ray in each medium is a straight line.}

(??) (??) (??) and (??) \quad \Rightarrow \quad \frac{v_1}{v_2} = \frac{\sin \theta_1}{\sin \theta_2}. \]
HMP Based Hybrid Optimization Algorithms: Convergence, Combinatoric Search and Optimality Zones
HMP Based HOCP Algorithms (Multiple Autonomous Switchings (MAS))

1. Compute $Q_k \triangleq \begin{pmatrix} \nabla_t m(t_{s}^{k-1}, x_{s}^{k-1}) \\ \nabla_x m(t_{s}^{k-1}, x_{s}^{k-1}) \end{pmatrix}$.

Set

$$\eta_k \triangleq \left\| \begin{pmatrix} H_1^{k}(t_{s}^{k-1}) - H_2^{k}(t_{s}^{k-1}) \\ \lambda_2^{k}(t_{s}^{k-1}) - \lambda_1^{k}(t_{s}^{k-1}) \end{pmatrix} - Q_k p_k \right\|^2 + \left\| m(t_{s}^{k-1}, x_{s}^{k-1}) \right\|^2 .$$

Compute $p^k = (Q_k^T Q_k)^{-1} Q_k^T \begin{pmatrix} H_1^{k}(t_{s}^{k-1}) - H_2^{k}(t_{s}^{k-1}) \\ \lambda_2^{k}(t_{s}^{k-1}) - \lambda_1^{k}(t_{s}^{k-1}) \end{pmatrix}$.

2. Set $t_{s}^{k} = t_{s}^{k-1} - \epsilon_1 \left( H_1^{k}(t_{s}^{k-1}) - H_2^{k}(t_{s}^{k-1}) - \nabla_t m(t_{s}^{k-1}, x_{s}^{k-1}) p_k \right) - \epsilon_1 \nabla_t m(t_{s}^{k-1}, x_{s}^{k-1}) m(t_{s}^{k-1}, x_{s}^{k-1}) .$

3. Set $x_{s}^{k} = x_{s}^{k-1} - \epsilon_2 \left( \lambda_2^{k}(t_{s}^{k-1}) - \lambda_1^{k}(t_{s}^{k-1}) - \nabla_x m(t_{s}^{k-1}, x_{s}^{k-1}) p_k \right) - \epsilon_2 \nabla_x m(t_{s}^{k-1}, x_{s}^{k-1}) m(t_{s}^{k-1}, x_{s}^{k-1}) . \square
Comments on HMPMAS

\[
\left( H^k_1(t_{s}^{k-1}) - H^k_2(t_{s}^{k-1}) - \nabla_t m(t_{s}^{k-1}, x_{s}^{k-1})p^k \right)
\]

and

\[
\left( \lambda^k_2(t_{s}^{k-1}) - \lambda^k_1(t_{s}^{k-1}) - \nabla_x m(t_{s}^{k-1}, x_{s}^{k-1})p^k \right)
\]

approximate \( \frac{\partial J}{\partial t}(t_{s}^{k-1}, x_{s}^{k-1}) \), \( \frac{\partial J}{\partial x}(t_{s}^{k-1}, x_{s}^{k-1}) \), respectively, in a neighbourhood of \((t_{s}^{0}, x_{s}^{0})\).

- Algorithm MAS generalizes to the multiple autonomous switchings case and specializes to the controlled switchings case.

- Convergence analysis of MAS: Performed using the penalty function methods and Ekeland’s variational principle.
Single Autonomous Switching Case Example

Figure 1: Single autonomous switching case.

\[ \begin{align*}
q_1 : \quad & \dot{x} = x + xu, \\
q_2 : \quad & \dot{x} = -x + xu, \\
J(u) & = \frac{1}{2} \int_0^2 u^2(s) \, ds, \\
m(x, t) & = x - et = 0, \\
\text{Initial values:} & \quad t_s = 0.5, \ x(t_s) = 2.
\end{align*} \]
Single Autonomous Switching CaseExample (continued)

System successively occupies:

\[ q_1 : \dot{x} = x + xu, \quad q_2 : \dot{x} = -x + xu, \]

\[ t_0 = 0, \quad t_f = 2, \quad x(0) = 1, \quad x(2) = 1, \]

\[ J(u) = \frac{1}{2} \int_0^2 u^2(s) \, ds, \]

\[ m(x, t) = x - et = 0, \quad (e = 2.7183...). \]

Initial values: \( t_s = 0.5, \quad x(t_s) = 2, \]

Unique optimal control: \( u^0 \equiv 0, \quad t^0_s = 1, \quad x^0_s = e \) result in: \( J^0 = 0. \)

Figure 1 shows convergence to \( t^0_s = 1, \quad x^0_s = e. \)

Hardware: Pentium III 550 MHz with 128MB of SDRAM.
Software: Matlab 6.0 under Redhat Linux 6.2.
CPU time: 222.97 seconds.
Single Autonomous Switching Case

We apply Algorithm HMPMAS to the system of Example 3 in: Optimal control of switched systems based on parameterization of the switching instants by X. Xu and P.J. Antsaklis


Two discrete modes:

\[ q_1 : \dot{x} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad q_2 : \dot{x} = \begin{bmatrix} 0.5 & 0.866 \\ 0.866 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \]

\[ t_0 = 0, \ t_f = 2, \ x_0 = [1 \ 1]^T. \]

Switching manifold \((q_1 \rightarrow q_2)\): \(m(x) = x_1 + x_2 - 7 = 0.\)

Cost function:

\[ J(u) = \frac{1}{2} (x_1(2) - 10)^2 + \frac{1}{2} (x_2(2) - 6)^2 + \frac{1}{2} \int_0^2 u^2(s) \, ds. \]
Single Autonomous Switching Case

Initial guess \( t_s = 1.5, \ x_s = [4.5 \ 2.5]^T \),
S.C. numbers: \( t_s^0 = 1.1621, \ x_s^0 = [4.5556 \ 2.4444]^T, \ J^0 = 0.1132 \).
X.A. numbers: \( t_s^0 = 1.1624, \ J^0 = 0.1130 \).
S.C. computation time: 29.58 seconds.
X.A. computation time: 34 minutes.
Multiple Controlled Switchings (MCS) Example

Figure 2: Multiple controlled switchings case.

\( q_1 : \dot{x} = x + xu, \)
\( q_2 : \dot{x} = -x + xu, \)
\( q_3 : \dot{x} = x + u, \)
\( t_0 = 0, \quad t_f = 3, \)
\( x(0) = 1, \quad x(3) = e, \)

\[ J(u) = \frac{1}{2} \int_0^3 u^2(s) \, ds, \]

Initial values:
\( t_{s1} = 0.8, \quad x(t_{s1}) = 2.5, \)
\( t_{s2} = 2.2, \quad x(t_{s2}) = 0.8. \)
Single Controlled Switchings Case

We apply Algorithm HMPMCS to the system of Example 2 in:
Optimal control of switched systems based on parameterization of the switching instants
by X. Xu and P.J. Antsaklis

Two discrete modes:

\[ q_1 : \dot{x} = \begin{bmatrix} 0.6 & 1.2 \\ -0.8 & 3.4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \quad q_2 : \dot{x} = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u, \]

\[ t_0 = 0, \ t_f = 2, \ x_0 = [0 \ 2]^T. \]

Cost function:

\[ J(u) = \frac{1}{2}(x_1(2) - 4)^2 + \frac{1}{2}(x_2(2) - 2)^2 + \frac{1}{2} \int_0^2 ((x_2(s) - 2)^2 + u^2(s)) \, ds. \]
Single Controlled Switchings Case

Initial guess \( t_s = 1 \),
S.C. numbers: \( t_s^0 = 0.18876, \quad x_s^0 = [-1.5626 \quad 1.3231]^T, \)
\( J^0 = 9.7666 \). X.A. numbers: \( t_s^0 = 0.1897, \quad J^0 = 9.7667 \).
S.C. computation time: 69.52 seconds.
X.A. computation time: 30.75 seconds.

State Trajectory.

Combinatoric Search Example
Search in $N_{\leq 2}(1,1,1,1,1,1)$

$q_1 : \dot{x} = x + xu,$
$q_2 : \dot{x} = -x + xu,$
$q_3 : \dot{x} = x + u,$

$J(u) = \frac{1}{2} (x(5) - e)^2 + \frac{1}{2} \int_0^5 u^2(s) \, ds$

Figure 3: Trajectories for locally 2-opt sequence (2,1,1,2,1).
Combinatoric Search Example

Optimum = (2, 1, 1, 2, 1) ∈ \( N_{\leq 2}(1, 1, 1, 1, 1) \)

Each string in \( N_{\leq 2}(s_0) \) is converted to a ternary number as:
\[
\sum_{j=0}^{N-1} (q_j - 1) 3^{(N-1-j)}.
\]
Then normalized and plotted against the corresponding cost.

Figure: Cost for sequences in \( N_{\leq 2}(1, 1, 1, 1, 1) \).
Optimality Zones and Location Sequences
Optimality Zones (OZ) and HMPZ Algorithms

The Algorithm HMPZ = Algorithm HMP + choice of discrete state on each time interval at each iteration. Selection made using OZ precomputation (wrt automata constraint on the sequence of discrete states).

One Switching Time Example

Value function: \( v(t, x, q) \) of HOCP is bdd. and cts. in \((t, x, q)\).

Consider the case of:

(i) two locations, \( Q = \{q_1, q_2\} \),
(ii) controlled switch at \((t_s, x_s)\),
(iii) fixed initial and final times, 0 and \( t_f \), and
(iv) fixed initial and final states, \( x_0 \) and \( x_f \).

Optimality zone boundaries:

\[
\partial Z \triangleq \{(t, x) : J_1^0((t, x), (t_f, x_f)) = J_2^0((t, x), (t_f, x_f))\}.
\]
Figure: Optimality Zones and execution of Algorithm HMPZ:
LQ case with fixed terminal state;
Yellow (*): \((q_1, q_1)\), Blue (x): \((q_1, q_2)\), Green (o): \((q_2, q_1)\).
Convergence of Algorithm HMPZ

The convergence theory (strong assumptions) establishes that the HMP optimizes globally with respect to \( \{(t_{s_i}, x_{s_i}); 1 \leq i \leq L\} \) for any given schedule sequence.

\[ \Rightarrow \]

If there are no automata constraints on discrete state trajectories, the HMPZ will converge to a global trajectory \( x^0 \) and optimum value \( J^0 \) in a single \( \{(t^k_s, x^k_s); k \geq 1\} \) run without location sequence repetitions.
Hybrid Systems Optimal Control
Computational Complexity

HMPZ vs. Combinatorial Search HMP

\[ \alpha |Q| M^2|x|^1 + \beta k |\text{Pont}(|x|)|(N+1) < \gamma k |\text{Pont}(|x|)|(N+1)|Q|^N \]

Legend:
\(\alpha, \beta, \gamma = \text{Constants}\)
\(k = \text{Number of Problems}\)
\(M = \text{Space-time sample point density}\)
\(N = \text{Number of switchings}\)
\(|Q| = \text{Cardinality of } Q\)
\(|x| = \text{Dimension of } x\)
Optimality Zones Example 1

Consider system with two locations $q_1$ and $q_2$, linear dynamics and quadratic cost criteria in each location. The system and weighting matrices are:

$q_1: \dot{x} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \end{bmatrix} u, \quad q_2: \dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \end{bmatrix} u,$

Cost: $J_i = \frac{1}{2} \int_0^2 (x^T Q_i x + u^T R_i u) \, dt,$

where $B_1 = B_2 = Q_1 = Q_2 = R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R_2 = 1.6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$

Fixed switching time: $t_s = 1$
Initial and final times, states: $t_0 = 0, t_f = 2, x_0 = [2 \ 4]^T, x_f = \text{free}$
Figure: Optimality Zones and Convergence of Algorithm HMPZ: LQ case with free terminal state;
Red (+): \((q_2, q_2)\), Blue (o): \((q_2, q_1)\).
Optimality Zones Example 2

Consider system with two locations $q_1$ and $q_2$, linear dynamics and quadratic cost criteria in each location. The system and weighting matrices are:

$q_1: \dot{x} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \end{bmatrix} u, \quad q_2: \dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \end{bmatrix} u,$

Cost: $J_i = \frac{1}{2} \int_0^2 (x^T Q_i x + u^T R_i u) \, dt,$

where $B_1 = B_2 = Q_1 = Q_2 = R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_2 = 1.6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Fixed switching time: $t_s = 1$

Fixed initial and final times and states: $t_0 = 0, t_f = 2,$

$x_0 = [2 \ 4]^T, x_f = [2 \ 4]^T$
Figure: Optimality Zones and convergence of Algorithm HMPZ: LQ case with fixed terminal state;
Yellow (*):$(q_1, q_1)$, Blue (x):$(q_1, q_2)$, Green (o):$(q_2, q_1)$. 
Figure: Slices of Optimality Zones at $t_s = 0.9, 1.0, 1.1$. LQ case with fixed terminal state; Blue (+): $(q_1, q_1)$, Green (o): $(q_1, q_2)$, Red (*): $(q_2, q_1)$. 
OZ Example: Bilinear System

Dynamics: \( q_1 : \dot{x} = x + xu, \quad q_2 : \dot{x} = -x + xu. \)

Cost function: \( J(u) = \frac{1}{2} \int_0^2 u^2(s) \, ds. \)

Initial location sequence: \((1, 1, 1)\).

Initial values: \( t_{s_1} = 0.5, \quad t_{s_2} = 1.5, \quad x(t_{s_1}) = 1, \quad x(t_{s_2}) = 1.5. \)

Final location sequence: \((1, 2, 2)\).

Optimal cost: 0.16674.
OZ boundary for $x_1, x_2, t_2$ varying with $t_1 = 0.7$. Asterisks indicate joint time-state switching pairs at HMPZ iterations 3, 4, 5 and 6.
OZ Example: Bilinear System

HMPZ algorithm can solve HOCPs with large number switchings.

Initial data: \( t_0 = 0 \), \( t_f = 2 \), \( x_0 = 2.4 \), \( x_f = 2.6 \)

Number of switchings = 10.

The algorithm initially computes

(i) ten uniformly distributed switching times between \( t_0 \) and \( t_f \),
(ii) ten randomly distributed switching states between \( x_0 \) and \( x_f \), and

(iii) the initial switching sequence: \((1, 1, 1, 1, 2, 1, 1, 2, 2, 1, 1)\)

Running time for HMPZ = 45 seconds.

No set-up computation cost: linear OZ boundaries!
### OZ Example: Bilinear System

Execution of Algorithm HMPZ: Ten switchings case

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Location sequence</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(1,1,1,1,2,1,1,2,2,1,1)$</td>
<td>0.75653</td>
</tr>
<tr>
<td>3</td>
<td>$(1,1,1,1,2,1,1,2,2,1,1)$</td>
<td>0.68563</td>
</tr>
<tr>
<td>5</td>
<td>$(1,1,1,1,2,1,1,2,2,1,1)$</td>
<td>0.61678</td>
</tr>
<tr>
<td>7</td>
<td>$(1,1,1,1,2,1,1,2,2,1,1)$</td>
<td>0.58548</td>
</tr>
<tr>
<td>9</td>
<td>$(1,1,1,1,2,1,1,2,2,1,1)$</td>
<td>0.49985</td>
</tr>
<tr>
<td>11</td>
<td>$(1,1,1,1,2,1,1,2,2,1,1)$</td>
<td>0.43679</td>
</tr>
<tr>
<td>13</td>
<td>$(1,1,1,1,2,1,1,2,2,1,1)$</td>
<td>0.35672</td>
</tr>
<tr>
<td>15</td>
<td>$(1,1,1,1,2,1,1,2,2,1,1)$</td>
<td>0.31957</td>
</tr>
<tr>
<td>17</td>
<td>$(1,1,2,2,1,1,2,1,2,1,1)$</td>
<td>0.21897</td>
</tr>
<tr>
<td>18</td>
<td>$(1,1,2,2,1,1,2,1,2,1,1)$</td>
<td>0.21897</td>
</tr>
</tbody>
</table>
Conclusion

- Formulation of the hybrid optimal control problem (HOCP).
- Development of necessary conditions, or Minimum Principle for hybrid systems optimality.
- Development of efficient hybrid optimization algorithms (HMP) for fixed location schedules along with their proofs of convergence.
- Introduction of Optimality Zones and development of associated efficient algorithms.