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Dimension for Alexandrov Spaces

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ABSTRACT

This paper continues the study of the topological model of the support of a digital image published by Kronheimer in 1992. There, he interpreted the generation of the support D of the image from a topological space S by means of some "discretization" as the construction of a quotient space Δ of S, which represents the set D and has a reasonable (non-discrete) topology. Under some conditions the space Δ is an Alexandrov space. Having in mind the practical example $S = \mathbb{R}^n$ and $D = \mathbb{Z}^n$ we speak of "n-dimensional images", although there is no dimension on the space Δ. We define in this paper a so-called Alexandrov dimension for arbitrary Alexandrov spaces. Under this definition an image which was sampled from a function defined on $\mathbb{R}^n$ has dimension n. If the Alexandrov space Δ is T_0, then it corresponds to a canonical partially ordered set $(\Delta, \leq)$. We prove, that in this case the Alexandrov dimension coincides with the height of $(\Delta, \leq)$.

1. Introduction

This work is motivated by recent studies of topological models of digital images. A digital image is a function defined on a "discrete" set D . Usually this digital image is produced by sampling a continuous function defined on a topological space S [11]. We have in mind the example $D = \mathbb{Z}^2$ and $S = \mathbb{R}^2$. The relative topology on the subspace $(\mathbb{Z}^2, \tau_{\mathbb{Z}^2})$ of $(\mathbb{R}^2, \tau_{\mathbb{R}^2})$ is discrete and so in order to deal with connectivity and other topological properties of the image, a reasonable topology on D has to be constructed.

The first attempts to introduce topological concepts in D were those of Rosenfeld, Kak and others [11] where they developed the theory of neighborhood graphs on $D = \mathbb{Z}^2$ and $D = \mathbb{Z}^3$. Based on graph theory, combinatorics, and using some ideas of topology, this model was generalized to a theory of incidence structures by Voss [14], which can be used to describe n-dimensional images, and involves the theory of neighborhood structures developed by Klette and Voss [4]. On the other hand, using combinatorial topology and homotopy theory, topological structures were constructed on neighborhood graphs by Kong et al for

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representing 2- and 3-dimensional images [6]. An other idea is the interpretation of D as cellular complex and the use of algebraic topology (see papers [8] and [10]). During the past few years topological models have been used to describe the structure on D. The Khalimsky topology on $\mathbb{Z}$ has been used succesfully in this regard (see papers [5] and [7]), and in 1992 Kronheimer [9] described a model which treats D as an open quotient of S. The basic idea of Kronheimer for modelling the support of the digital image consists of two steps:

The first is the identification of the points $i \in D$ with open disjoint sets $W_i \subseteq S$ in such a way, that the union of all W_i is dense in S. He calls the family of open sets $\{W_i\}$ a fenestration of S. In our example of the first paragraph we can identify each $z \in \mathbb{Z}^2$ with the unit open square in $\mathbb{R}^2$, whose center is z.

The second step consists of constructing a decomposition space X of S, which contains the fenestration used in the first step, and whose natural projection map is open. Such a decomposition X is called a grid of S. It is clear that even for a fixed fenestration many grids can be obtained. For this reason Kronheimer imposes a minimality condition on that grid, which will model the support D of the digital image. For our example, the minimal grid X on $\mathbb{R}^2$ consists of all unit open squares centered at points with integer coordinates, and all unit open lines and vertices bounding these squares. In fact, this is one of the cellular complexes proposed by Kovalevsky [8] to represent the support of a two-dimensional digital image. Finally it seems to be reasonable to require that the space X be locally finite. In this case Kronheimer proved that the minimal grid is an Alexandrov space. Recall that an Alexandrov space is a topological space with the property that the intersection of an arbitrary number of open sets is open. Consequently there is a minimal neighborhood for any point of the space. Alexandrov spaces were first considered by Alexandrov [1] under the name of "discrete spaces".

The motivation of this paper is the following question:
If we digitize an image defined on $\mathbb{R}^n$, carring out the steps 1 and 2 in order to generate a minimal locally finite grid X of $\mathbb{R}^n$, will the support of the image modelled by X have dimension n, given some reasonable definition of dimension for locally finite grids ?

We are immediately confronted with the problem of defining the dimension of the Alexandrov space X. Alexandrov spaces are only interesting if they are not T_1, since a T_1 Alexandrov space is discrete. Hence we cannot apply any of the classical dimension functions ind, Ind, and dim (covering dimension) [3].

An Alexandrov space (X, τ) with the T_0 property is representable by a canonical partially ordered set $(X, \leq)$, where $x \leq y \Leftrightarrow x \in Cl(\{y\})$ [1]. Hence we can relate the dimension problem to the height of $(X, \leq)$ simply defined as the supremum of all lengths of chains in $(X, \leq)$.

Our aim is to define a topological dimension function for Alexandrov spaces, which can be applied to the minimal locally finite grids due to Kronheimer. This function should serve to give a sense to the concept of "n-dimensional digital image". We will call this dimension Alexandrov dimension.
This paper is organized as follows:

In section 2 we will define the Alexandrov dimension and resume its basic properties derived and proved in [15].

Section 4 will deal with the relation between the Alexandrov dimension and the height of the canonical partially ordered set, which we call partial order dimension. For a T_0 Alexandrov space both dimensions will be equal. Some preliminaries about those spaces are reviewed in section 3.

In section 5 we will answer the dimension problem for discrete images, based on the model of Kronheimer. We will show that the minimal locally finite grid of the standard fenestration of $\mathbb{R}^n$ has dimension n. It is necessary to explain the construction due to Kronheimer in some detail at the beginning.

In the following (X, τ) is an Alexandrov space with the T_0 property. $U(x)$ denotes the minimal neighborhood of $x \in X$. Int_X, Cl_X, and Fr_X stand for the interior, the closure, and the frontier in the space (X, τ), respectively. But we omit subscripts when confusion is not possible.

2. A topological dimension function for Alexandrov spaces

Our dimension function DIM for the Alexandrov space (X,τ) is defined inductively in terms of a local dimension determined by the minimal neighborhoods.

Definition 1.:

$$
\begin{aligned}
& Let\ n \in I\!N. \\
(i)\ & DIM\,X = -1 \iff X = \phi. \\
(ii)\ & If\ X \neq \phi\ then\ define \\
& DIM\,X = sup\{DILx, x \in X\},\ where\ for\ x \in X\ \ we\ define \\
& \quad DILx \leq n \iff DIM(FrU(x)) \leq n-1, \\
& \quad DILx = n \iff DILx \leq n \wedge DILx \nleq n-1, \\
& \quad DILx = \infty \iff DILx \leq n\ is\ false\ \forall n. \\
& (We\ call\ DIM\,X\ the\ Alexandrov\ dimension\ of\ X.)
\end{aligned}
$$

Note that $DIM(FrU(x))$ is the supremum of the local dimensions of the elements of $FrU(x)$ measured in the subspace $(FrU(x), \tau_{FrU(x)})$ with the relative topology. We can write more exactly

$$DIL_X x \leq n \iff sup\{DIL_{FrU(x)} y,\ y \in FrU(x)\} \leq n-1,$$

but we omit the indices when dealing only with the space (X,τ).

It is easy to see, that two homeomorphic Alexandrov spaces have the same dimension. Obviously a discrete space has dimension 0, and for a T_0 Alexandrov space the converse is also true [15]. The authors proved that the function DIM has properties like those of the classical topological dimension functions [15]. We will review here these properties without proofs :

Proposition 1.:

a) $If\ \ A \subseteq X\ and\ (A,\tau_A)\ is\ the\ relative\ subspace\ of\ (X,\tau),\ then\ \ DIM\,A \leq DIM\,X.$

b) $If\ (X,\tau_X),\ (Y,\tau_Y)\ are\ Alexandrov\ spaces,\ with\ X, Y\ non-empty\ and\ disjoint,$
$then\ \ DIM((X,\tau_X) \oplus (Y,\tau_Y)) = max\{DIM\,X, DIM\,Y\}.$

c) $If\ X,\ Y\ are\ non-empty\ T_0\ Alexandrov\ spaces,\ then$
  i) $DIM(X \times Y) \leq DIM\,X + DIM\,Y,$
  ii) $DIM(X \times Y) = DIM\,X + DIM\,Y,\ if\ DIM\,X = 0\ or\ DIM\,Y = 0.$

3. Alexandrov spaces as partially ordered sets

In this section we review some properties of T_0 Alexandrov spaces. All these have been stated, and many proved by Alexandrov in his paper "Diskrete Räume" [1].

Recall that (X,τ) is T_0 and for $x \in X\ \ U(x) = \bigcap\{A \in \tau :\ x \in A\} \ \ \in \tau$.

Lemma 1.:

$$
\begin{aligned}
& If\ x, y \in X,\ then \\
& a)\ x \leq y \iff U(x) \supseteq U(y) \quad defines\ a\ partial\ order\ relation\ on\ X, \\
& b)\ and\ \ x \leq y \iff y \in U(x) \iff x \in Cl\{y\}.
\end{aligned}
$$

Now we will consider the correspondence between Alexandrov spaces and partially ordered sets.

Proposition 2.: (Alexandrov [1])

(i) $If\ (M, \leq)\ is\ a\ partially\ ordered\ set,\ then$
$\{\ \{\ y \in M:\ x \leq y\},\ x \in M\}\ is\ a\ base\ of\ a\ topology\ \tau\ on\ M,$
$and\ (M, \tau)\ is\ a\ T_0\ Alexandrov\ space.$

(ii) $If\ (X, \tau)\ is\ a\ T_0\ Alexandrov\ space,\ then$
$x \leq y \iff U(x) \supseteq U(y)\ defines\ a\ partial\ order\ on\ X,$
$and\ the\ topological\ space\ constructed\ from\ (X, \leq)\ as\ in\ (i)\ is\ homeomorphic\ to\ (X, \tau).$

(for a proof see also [15])

Proposition 2 implies that the relation $\leq$ on the T_0 Alexandrov space (X, τ) is a canonical partial order. Obviously the minimal neighborhoods have the form $U(x) = \{y \in X:\ x \leq y\}$. Consequently the interior of a set $M \subseteq X$ can be expressed as

$$Int(M) = \{m \in M:\ m \leq n\ implies\ n \in M\}.$$

Also the closure and the frontier of a set can be characterized using the canonical partial order.

Lemma 2.:

$Let\ \leq\ the\ canonical\ partial\ order\ of\ (X, \tau).$
$a)\ If\ M \subseteq X\ \ then\ \ Cl(M) = \{y \in X:\ \exists m \in M\ such\ that\ y \leq m\}.$
$b)\ For\ x \in X\quad \{x\}\ is\ closed \iff y \leq x\ implies\ y = x.$
$c)\ If\ x, y \in X\ with\ x < y\ \ then\ x \in FrU(y).$

4. The relation between DIM and the partial order dimension

Let $(X, \leq)$ be the canonical partially ordered set generated by (X, τ) and let $x \in X$. The dimension or height of x is defined to be the supremum of all lengths l of chains of the form $a_l < a_{l-1} < \cdots < a_0 = x$ of elements of X, where $a < b \iff a \leq b$ and $a \neq b$. We will denote this dimension of x by $ODIL\, x$. Then the dimension or height of X can be defined as

Definition 2.:

$$\begin{aligned} ODIM(X) &= sup\ \{ODIL\, x,\ x \in X\} \\ &= \infty\quad if\ \{ODIL\, x,\ x \in X\}\ is\ unbounded. \end{aligned}$$
$$(We\ call\ ODIM\, X\ the\ partial\ order\ dimension\ of\ X.)$$

Before analyzing the relation between the partial order dimension $ODIM$ and the Alexandrov dimension DIM , we derive a useful property of the local 0 - dimensionality of (X, τ).

Lemma 3.:

$$\begin{aligned} & Let\ x \in X. \\ DIL\,x = 0\ \ implies: \quad & a)\ \{x\}\ is\ closed, \\ & b)\ \ y \leq x, y \in X\ implies\ y = x. \end{aligned}$$

Proof:
Obviously $a) \Leftrightarrow b)$. We will prove $Cl\,\{x\} = \{x\}$: Let $y \in Cl\,\{x\}$ and suppose $y \neq x$. By lemma 2 $y \leq x$ and $y \in Fr\,U(x)$, and so $Fr\,U(x) \neq \phi$, which contradicts $DIL\,x = 0$.

□

The converse of this lemma is false, as the following example shows.

Example 1.:

Let $X = \{a, b, c\}$ and $\tau = \{\{a, b\}, \{b, c\}, \{b\}, \phi, X\}$. Then $\{a\}$ is closed, but $DILa = 1 \neq 0$.

Corollary 1.:

$$If\ \exists\ y \neq x\ such\ that\ y \leq x\ \ then\ \ DILx \geq 1.$$

Now let us investigate the correspondence between $ODIM$ and DIM. First we will consider both dimensions locally.

Proposition 3.:

$$DIL\,x \leq n\ \ implies\ \ ODIL\,x \leq n \quad for\ any\ x \in X,\ n \geq 0\,.$$

Proof:
Let $x \in X$ with $DIL\,x \leq n$, $n \geq 0$, and $a_0, a_1, \cdots, a_m \in X$ with $a_0 = x$ and $m \geq 0$ such that $a_m < a_{m-1} < \cdots < a_0$.
It is sufficient to show that $m \leq n$ (by induction on n).
1) Let $n = 0$: Obviously $m \leq 0$, using lemma 3.
2) Suppose now that if $n \leq k$ then $DIL\,x \leq n$ implies $ODIL\,x \leq n$ and let $n = k + 1$.
From $DIL\,x \leq k + 1$ it follows $DIL_{FrU(x)}y \leq k\ \forall\, y \in FrU(x)$.
Because of the transitivity of $\leq$ we have $a_i < x\ \forall\ i = 1, \cdots, m$. Hence by lemma 2 $a_i \in FrU(x)$ holds $\forall\, i = 1, \cdots, m$. Furthermore $Cl_X(\{a_i\}) = Cl_{FrU(x)}(\{a_i\})$ for any i, because $FrU(x)$ is a closed subspace of (X, τ).
Since $DIL_{FrU(x)}a_1 \leq k$ it follows from the hypothesis that $ODIL\,a_1 \leq k$. Consequently $a_l < a_{l-1} < \cdots < a_2 < a_1$ implies $l \leq k$, where $a_r < a_{r-1} \ \Leftrightarrow\ a_r \in Cl_{FrU(x)}(\{a_r - 1\}) \ \Leftrightarrow\ a_r \in Cl_X(\{a_r - 1\})$. Hence $m \leq l + 1 \leq k + 1$.

□

Again, the converse of this proposition is false, as the following example shows.

Example 2.:

Let (X, τ) be the minimal grid of the standard fenestration of $\mathbb{R}^2$ (see section 5). Consider x a point element, y an unit line element, and z an unit plane element such that $x < y < z$. Then $ODIL\,x = 0$, $ODIL\,y = 1$, $ODIL\,z = 2$, but $DIL\,x = DIL\,y = DIL\,z = 2$.

Proposition 4.:

$$For\ any\ x \in X,\ if\ \ ODIL\,y \leq n\ \forall\, y \in U(x)\ \ then\ \ DIL\,x \leq n.$$

Proof:
Suppose $x \in X$ and $ODIL\,y \leq n\ \forall\, y \in U(x)$.
We prove $DIM\,FrU(x) \leq n-1$ by induction on n.
1) Let $n = 0$: Then $ODIL\,x = 0$ and by lemma 2 $\{x\}$ is closed, which implies $DIM\,FrU(x) = -1$, since $FrU(x) = \phi$. (For, suppose $s \in FrU(x)$; then since $s \in Cl(U(x))$ by lemma 2 $\exists\, t \in U(x)$ such that $s \leq t$. But $\{y\}$ is closed, hence $s = t$ and $t \in U(x)^c$, which is a contradiction.)
2) Suppose that conjecture holds for $n \leq k$, and let $n = k+1$.
Let $f \in FrU(x)$, and $a_m < a_{m-1} < \cdots < a_1 < a_0 = f$ be a chain in $FrU(x)$. If $m \geq k+1$ then the above chain exists in X, because $FrU(x)$ forms a closed subspace of X (see step 2 of the proof of proposition 3). But $f \in Cl\,U(x)$, and so there is $s \in U(x)$ with $f \leq s$. But $f = s$ is not possible, since $f \in U(x)^c$; hence $f < s$, which implies that $a_m < \cdots < a_1 < f < s$. Hence $ODIL\,s \geq k+2$ which is a contradiction. Consequently $m \leq k$ and $ODIL\,f \leq k$ in the space $FrU(x)$. From the hypothesis it follows that $DIL_{FrU(x)}f \leq k$ and hence $DIM\,FrU(x) \leq k$.

□

The following example shows, that the converse of proposition 4 is false.

Example 3.:

Let $(X, \tau) = (\{x, y\}, \{\{x\}, \phi, X\})$. Then $DIL\,y = 0$, but $x \in U(y)$ and $ODIL\,x = 1$, because $y < x$.

We can resume propositions 3 and 4 and their corollaries to state the following conditions for the local dimensions.

Proposition 5.:

Let $x \in X$.
a) $If\ \ ODIL\,x \geq n\ \ and\ \ ODIL\,y \leq n\ \forall\, y \in U(x),\ then\ \ DIL\,x = n.$
b) $If\ \ DIL\,x = n,\ then\ \ ODIL\,x \leq n\ \ and\ \ \exists y \in U(x)\ with\ ODIL\,y \geq n.$

Neither a) nor b) are invertible. Nevertheless we have the following:

Proposition 6.:

$$DIM\,X = n \quad \Longleftrightarrow \quad ODIM\,X = n.$$

Proof:
" $\Rightarrow$ ": $DIM\,X = n \Rightarrow DIL\,x \leq n\ \forall x \in X \wedge \exists x^* \in X$ with $DIL\,x^* \geq n$. This implies that $ODIL\,x \leq n\ \forall x \in X$ by proposition 3, and $\exists y \in U(x^*)$ such that $ODIL\,y \leq n$ by proposition 4. Clearly $y \in X$, hence $sup\{ODIL\,x;\ x \in X\} = ODIM\,X = n$.
" $\Leftarrow$ ": $ODIM\,X = n \Rightarrow ODIL\,x \leq n\ \forall x \in X \wedge \exists x^* \in X$ with $ODIL\,x^* \geq n$. This implies that $DIL\,x^* \geq n$ by proposition 3. Let $z \in X$; clearly $ODIL\,y \leq n$ holds $\forall y \in U(z)$, hence by proposition 4 $DIL\,z \leq n$. Thus $DIM\,X = n$.

□

5. N - dimensional discrete images

Let us define first some concepts used in the introduction and develop the construction of a topological model of the support of a digital image due to Kronheimer [9]. Let (S, τ_S) be a topological space. The image is to be defined on a set D, which is generated by some "discretization" [11] of S. Usually $S = \mathbb{R}^n$ and $D = \mathbb{Z}^n$, but in the case $S = \mathbb{R}^2$ we have as D the grid point set of one of the three homogeneous grids in the plane : of the orthogonal grid ($D = \mathbb{Z}^2$), of the hexagonal grid, or of the triangular grid [4]. The main idea of Kronheimer is the construction of a decomposition space of (S, τ_S), which gives a reasonable (non-discrete) topology on D.

Definition 3.:

$$\begin{gathered} A\ family\ \ E = \{W_i\}_{i \in I}\ \ of\ open\ disjoint\ sets\ is\ called \\ a\ fenestration\ of\ S,\ iff\ \ \cup\{W_i,\, i \in I\}\ \ is\ dense\ in\ S. \end{gathered}$$

Each element of D is indentified with a uniquely determined element of a fenestration E of S, then E is extended to a decomposition of S.

Definition 4.:

$$\begin{gathered} \textit{A decomposition } \Delta \textit{ of } S \textit{ is called an } E{-}grid\ \ iff\ \ E \subset \Delta \\ \textit{and the natural projection map }\ \pi : \Delta \longrightarrow S\ \ \textit{is open.} \end{gathered}$$

The set Δ forms a topological space with the quotient topology

$$\tau_\Delta = \{\, M \subseteq \Delta :\ \pi^{-1}(M) = \cup\{N \in \Delta :\ N \in M\}\ \in \tau_s\}.$$

The relative topology τ_E on E with respect to τ_Δ is discrete, and the subspace (E, τ_E) is dense in (Δ, τ_Δ). In general, for a fixed fenestration E there are various E-grids. Let us consider an example.

Example 4.: (standard fenestration of $\mathbb{R}^n$)

Let $S = \mathbb{R}^n$ and $D = \mathbb{Z}^n$. The set E of all open unit cubes in $\mathbb{R}^n$ centered in points with integer coordinates is a fenestration of $\mathbb{R}^n$. We identify each point p of $\mathbb{Z}^n$ with the cube whose center is p.

$$\begin{aligned} \Delta =& E \cup \{\textit{all unit cubes which are open in } \mathbb{R}^{n-k} \textit{ and bound} \\ & \textit{some element of } E;\ k = 1, 2, \cdots, n-1\} \cup \{\textit{all vertices of the elements of } E\} \end{aligned}$$

is an E-grid. Let $d \in \Delta$, then $U_\Delta(d)$ is the set of all those cubes which are bounded*1 by d. Note that for $x, y \in \Delta$, x bounds y iff $x \leq y$, where $\leq$ is the canonical partial order generated by the Alexandrov space (Δ, τ_Δ). An other E-grid is

$$\Delta_t = E \cup \{\{x\}\ :\ x \in \mathbb{R}^n \wedge\ \nexists e \in E \textit{ such that } x \in e\}\ \ (\textit{trivial grid}).$$

Observe that we can define an open continuous map f from Δ_t onto Δ by $f(e) = e$, if $e \in E$, and for $x \in \Delta_t \setminus E$ $f(\{x\}) = d$, where $x \in d$. This map is one-to-one on E, but it identifies all points of $\mathbb{R}^n$ which belong to the same open cube in $\mathbb{R}^{n-k}$ $(k = 1, 2, \cdots, n)$. It is easy to see that if we construct such a map from Δ onto another E-grid, we obtain a homeormorphism.

*1 An element e of E is a n-dimensional cube in $\mathbb{R}^n$. A (n-k)-dimensional cube bounds e, if it is a side of e, in the sense of polyhedras. The bounding relation on the set of unit cubes in $\mathbb{R}^n$ is reflexive, antisymmetric, and transitive. A point of $\mathbb{R}^n$ is considered a 0-dimensional unit cube.

Definition 5.:

$$\begin{aligned}&An\ E{-}grid\ \Delta\ is\ said\ to\ be\ minimal\ \ iff\\&any\ continuous\ open\ map\ from\ \Delta\ onto\ another\ E{-}grid\\&being\ one{-}to{-}one\ on\ E\ is\ a\ homeormorphism.\end{aligned}$$

Kronheimer has proved that for any fenestration E there exists a minimal E-grid, which is by definition 5 unique up to homeormorphism. In example 4 Δ is the minimal grid of the standard fenestration of $\mathbb{R}^n$. The concept of an E-grid is generalized to that of a trace space.

Definition 6.:

$$\begin{aligned}&For\ a\ topological\ space\ (\Delta, \tau_\Delta)\ the\ set\\&Tr\Delta = \{x \in \Delta\ :\ \{x\} \in \tau_\Delta\}\ is\ called\ the\ trace\ of\ \Delta.\\&If\ Tr\Delta\ is\ dense\ in\ \Delta,\ then\ \Delta\ is\ called\ a\ trace\ space.\end{aligned}$$

Clearly, an E-grid is a trace space with trace E. A minimal trace space is defined by analogy with definition 5. Kronheimer proved that for a trace space the T_0 property is necessary but not sufficient for minimality, and that any semiregular*2 T_0 trace space is minimal. He established as a topological model of the support of a digital image a locally finite "digital space" defined in the following manner.

Definition 7.:

$$\begin{aligned}&\Delta\ is\ said\ to\ be\ a\ digital\ space\ \ iff\ \ \ \Delta\ is\ a\ semiregular\ trace\ space\\&(or,\ equivalently:\ iff\ \ \Delta\ is\ a\ minimal\ trace\ space).\end{aligned}$$

A locally finite digital space is a T_0 Alexandrov space. An E-grid Δ is a locally finite digital space if the following set of conditions is satisfied (from [9]):
1) E is locally finite, that is, for each element of S there is a neighborhood which intersects only a finite number of elements of E.
2) The elements of E are connected and regularly open in S.
3) Δ is a minimal E-grid, and the space (Δ, τ_Δ) is semiregular.

Now we apply the Alexandrov dimension DIM to a locally finite digital space, in particular to the minimal E-grid Δ of a fenestration which fulfills the above conditions. Let us return to the problem of "n-dimensional images". We have $S = \mathbb{R}^n$ and $D = \mathbb{Z}^n$ identified with E and represented by Δ. It is easy to see that for the minimal grid Δ of the standard fenestration of $\mathbb{R}^n$ introduced in example 4, (Δ, τ_Δ) is a locally finite digital space. Then the following holds.

Proposition 7.:

$$The\ minimal\ grid\ \Delta\ of\ the\ standard\ fenestration\ E\ of\ \mathbb{R}^n\ has\ Alexandrov\ dimension\ \ n.$$

Proof:
We suppose $n \geq 1$. Let $\leq$ be the canonical partial order generated by (Δ, τ_Δ).
If $x \in E$, then for any $y \in \Delta$, $x \leq y$ implies $x = y$, which means that x is a maximal element. But for the n-dimensional cube x there are k-dimensional cubes y_k, $k = 0, 1, \cdots, n-1$, such that $y_0 < y_1 < \cdots < y_{n-1} < y_n$, hence $ODIL\, x = n$.

*2 A topological space is said to be semiregular, if there is a base of regularly open sets. In the Alexandrov space X this is equivalent to $Int(Cl\, U(x)) = U(x)\ \forall\, x \in X$.

If $x \in \Delta \setminus E$, then $U(x) \cap E \neq \phi$, because E is dense in Δ. Hence $ODIL\, x \leq n$, since exists a maximal element y such that $x \leq y$. Consequently $ODIM\, \Delta = n$, and by proposition 6 it follows $DIM\, \Delta = n$.

□

In [15] the authors have discussed generalizations of proposition 7 to other fenestrations and special types of minimal grids of subsets of $I\!R^n$, which have dimension n. Note that for such a subset dimension means *ind* or *Ind* or *dim* (covering dimension) all of which coincide in $I\!R^n$ [3], and in order to guarantee that it has dimension n, it is necessary and sufficient that it has interior points [2].

6. Final comments

The Alexandrov dimension has properties similar to those of the classical dimensions used in topology. But it is related also to other dimensions used in lattice and ring theory. For T_0 spaces the Alexandrov dimension coincides with the height of the canonical partially ordered set generated by the space, and also with the Krull dimension defined for arbitrary topological spaces using suitable lattices of closed subsets [12]. (This latter is called lattice dimension and graduated dimension by other authors.) For full details we refer the readers to [15].

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