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# THE TIME OPTIMAL CONTROL AS AN INTERPOLATION PROBLEM

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**Abstract.** The time optimal control problem occupies a central place in the control theory. In this paper we show with the help of the Potapov Fundamental Matrix Inequality that the solution of the time optimal control problem in the canonical linear system case can be given in terms of the solution of an interpolation problem in a certain class of holomorphic functions.

## 1 Introduction

Let $A$ be a real $n\times n$ matrix and $b$ a given constant vector in $\mathbb{R}^n$. Consider the following completely controllable system

$$\dot{x} = Ax + b\tilde{u} \tag{1.1}$$

for vector functions $x\in\mathbb{R}^n$: For a given initial vector $x_0 \neq 0$ find the minimal possible time $\theta_{\min}$ of the transfer (or the optimal time) and the optimal control $\tilde{u} = \tilde{u}_{x_0}(t)$ with $|\tilde{u}| \leq 1$ such that the trajectory of the closed system $\dot{x} = Ax + b\tilde{u}_{x_0}$ starting at $x_0$ terminates at the origin at time $\theta_{\min}$, i.e., $x(\theta_{\min}) = 0$. Such control problem is called the time optimal control (TOC), see [10].

Apparently N.N. Krasovskii [7] was first who proposed the use of moment problem methods for solving optimal control problems (OCP). He reduced linear OCP to moment problems by interpreting the cost function as a norm, which in fact was an application of the Krein $L$–moment problem [8]. An analytical solution of the TOC problem was given by V.I. Korobov and G.M. Sklyar [5], [6] on the basis of a treatment of an equivalent Markov power moment problem, the so-called Markov moment min-problem.

In the present paper we obtain the solution of the TOC problem in the canonical system case, that is, for

$$A := \{\delta_{j,k+1}\}_{j,k=1}^{n}, \quad b := (1,0,\dots,0)^T, \tag{1.2}$$

where $\delta_{j,k}$ is the Kronecker symbol. Our solution method is based on some deep results for classical moment and interpolation problems.

First we reduce the TOC problem to a Hausdorff moment problem. In its turn, the Hausdorff moment problem is translated into a Nevanlinna–Pick interpolation problem in the class of Nevanlinna functions, i.e., the class of holomorphic functions in the upper half–plane that have a nonnegative imaginary part. The interpolation problem is handled by

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using the Fundamental Matrix Inequality (FMI) of Potapov. This method is based on transforming the original problem into equivalent matrix inequalities.

In contrast to [5] and [6], the solution we give to the TOC problem is explicit and not recursive. Here we do not use in advance the facts that the optimal control takes the extremal values 1 or $-1$ and that the number of switches does not exceed $n-1$, where $n$ is the dimension of the system.

To apply our approach and give a complete solution of the mentioned problem, technically one needs to find a) a root of a polynomial to determine the minimal time, b) a vector $\nu \neq 0$ such that $H_1\nu = 0$ and/or $H_2\nu = 0$ where the matrices $H_1$ and $H_2$ are given in Definitions 2.3 and 2.4.

We find the optimal time $\theta_{\min}(x_0)$ by Theorem 3.6 below as the maximal solution of the equations $\det H_1(\theta, x_0) = 0$ and $\det H_1(\theta, x_0) = 0$. Then we find the vectors $\nu \neq 0$ such that $H_1(\theta_{\min}(x_0), x_0)\nu = 0$ or $H_2(\theta_{\min}(x_0), x_0)\nu = 0$. Knowing $\theta_{\min}(x_0)$ and vectors $\nu$, we can construct rational functions $s(z)$ for all $z \in \mathbb{C} \setminus [0, \theta_{\min}(x_0)]$ by Theorem 2.10. On the other hand, knowing $s(z)$ we can find the optimal control $\tilde{u}(t)$ for $t \in [0, \theta_{\min}(x_0)]$ making use of Theorem 3.7. It remains to substitute the optimal time $\theta_{\min}(x_0)$ and the optimal control $\tilde{u}(t)$ into (3.1).

Moreover, Lemma 3.1 allows us to claim that every TOC problem for the canonical system is equivalent to a degenerate Hausdorff moment problem and, conversely, every degenerate Hausdorff moment problem corresponds to a TOC problem.

The FMI approach allowed in [4] to solve the Admissible Control problem, i.e., the problem of finding the set of all bounded controls $u_{x_0,\theta}(t)$ that drive a starting point $x_0$ to the origin at time $\theta > \theta_{\min}$, where $\theta_{min}$ denotes the minimal possible time of this transfer.

# 2 Preliminaries

We start with some necessary notions related to moment problems on $[0, \theta]$.

## 2.1 $L$–Markov and Hausdorff moment problems

Let $\mathcal{C}_{0,L}$ denote the set of all measurable functions $f : [0, \theta] \to \mathbb{R}$ such that $0 \leq f(\tau) \leq L$ for all $\tau \in [0, \theta]$, and let $\mathcal{M}[0, \theta]$ stand for the set of all nonnegative measures on $[0, \theta]$ identifying with nondecreasing functions $\sigma : [0, \theta] \to \mathbb{R}$.

The $L$–Markov moment problem (MMP) for an interval $[0, \theta]$ is stated as follows: Let a finite sequence of real numbers $\{c_j\}_{j=0}^k$ be given. Find the set of functions $f \in \mathcal{C}_{0,L}$ such that $c_j = \int\limits_0^\theta \tau^j f(\tau)\, d\tau$ for all $j \in \{0, \cdots, k\}$.

The Hausdorff moment problem (HMP) for an interval $[0, \theta]$ is stated as follows: Let a finite sequence of real numbers $\{s_j\}_{j=0}^k$ be given. Find the set of measures $\sigma \in \mathcal{M}[0, \theta]$ such that $s_j = \int\limits_0^\theta \tau^j\, d\sigma(\tau)$ for all $j \in \{0, \cdots, k\}$. We denote by $\mathcal{M}\left([0, \theta], \{s_j\}_{j=0}^k\right)$ the set of solutions of this HMP.

## 2.2 The class $\mathcal{R}[0, \theta]$ of holomorphic functions

The TOC problem can be restated in the language of functions in the class $\mathcal{R}[0, \theta]$. This class consists of functions $s : \mathbb{C} \setminus [0, \theta] \to \mathbb{C}$ that satisfy the following conditions: (i) $s$ is holomorphic in $\mathbb{C} \setminus [0, \theta]$, (ii) $\operatorname{Im} s(z) \geq 0$ for $\operatorname{Im} z > 0$, (iii) $s(t) \geq 0$ for each $t \in (-\infty, 0)$, (iv) $-s(t) \geq 0$ for each $t \in (\theta, +\infty)$.

Functions in $\mathcal{R}[0, \theta]$ admit the following integral representation (see [9, Theorem A6]).

**Theorem 2.1.** *A function $s$ belongs to $\mathcal{R}[0, \theta]$ if and only if there exists a measure $\sigma \in \mathcal{M}[0, \theta]$ such that*

$$s(z) = \int_0^\theta (\tau - z)^{-1} d\sigma(\tau) \quad \text{for all } z \in \mathbb{C} \setminus [0, \theta]. \tag{2.1}$$

The holomorphic function $z \mapsto s(z)$ defined by (2.1) is called the Stieltjes transform of the measure $\sigma \in \mathcal{M}[0,\theta]$. Thus, by Theorem 2.1, the Stieltjes transform of a measure $\sigma \in \mathcal{M}[0,\theta]$ belongs to the class $\mathcal{R}[0,\theta]$.

### 2.3 The Stieltjes inverse formula

Given $s \in \mathcal{R}[0,\theta]$, the measure $\sigma$ satisfying the equation

$$s(z) = \int_0^\theta (\tau - z)^{-1} d\sigma(\tau)$$

and normalized by the conditions

$$\sigma(t) = \big(\sigma(t+0) - \sigma(t-0)\big)/2, \ \ \sigma(0) = 0,$$

is uniquely determined by the following Stieltjes inverse formula:

$$\sigma(t) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_0^t \operatorname{Im} s(x + i\varepsilon)\, dx, \ \ t \in [0,\theta]. \tag{2.2}$$

With this notation the HMP can be reformulated in the form: Describe the set $\mathcal{R}\big([0,\theta], \{s_j\}_{j=0}^k\big)$ of the Stieltjes transforms of all nonnegative measures in $\mathcal{M}\big([0,\theta], \{s_j\}_{j=0}^k\big)$.

### 2.4 Relation between the $L$–Markov moment problem and the finite Hausdorff moment problem

There is a bijection (see [9]) between the set $\mathcal{C}_{0,L}$ and the measures $\sigma \in \mathcal{M}[0,\theta]$ satisfying $\displaystyle\int_0^\theta d\sigma(\tau) = 1$, which is given by

$$\int_0^\theta \frac{d\sigma(\tau)}{\tau - z} = -\frac{1}{z} \exp\left( \frac{1}{L} \int_0^\theta \frac{f(\tau) d\tau}{z - \tau} \right). \tag{2.3}$$

The formal asymptotic expansions of the left- and right-hand sides of (2.3) determine an explicit relation between $\{c_j\}_{j=0}^{n-1}$ and $\{s_j\}_{j=0}^n$ (see [1]):

$$s_j = \frac{1}{j! L^j} \begin{vmatrix} c_0 & -L & 0 & \cdots & 0 & 0 \\ 2c_1 & c_0 & -2L & \cdots & 0 & 0 \\ 3c_2 & 2c_1 & c_0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ (j-1)c_{j-2} & (j-2)c_{j-3} & (j-3)c_{j-4} & \ddots & c_0 & -(j-1)L \\ jc_{j-1} & (j-1)c_{j-2} & (j-2)c_{j-3} & \cdots & 2c_1 & c_0 \end{vmatrix} \tag{2.4}$$

$$= \frac{b_{j,1}}{L} + \frac{b_{j,2}}{L^2} + \ldots + \frac{b_{j,j}}{L^j} \quad \text{for} \ \ j = 1, 2, \ldots,$$

and

$$s_0 = 1 \quad \text{and} \ \ c_j = 0, \ s_j = 0 \ \ \text{for} \ \ j < 0, \tag{2.5}$$

where

$$b_{j,1} = c_{j-1}, \quad b_{j,2} = \begin{cases} \sum_{m=0}^{k-2} c_m c_{j-2-m} + c_{k-1}^2/2 & \text{if} \ \ j = 2k, \\ \sum_{m=0}^{k-1} c_m c_{j-2-m} & \text{if} \ \ j = 2k+1, \end{cases} \quad \ldots, \quad b_{j,j} = \frac{c_0^j}{j!}.$$

If the sequence $\{s_j\}_{j \geq 0}$ satisfies (2.4)–(2.5), then, applying Laplace's expansion with respect to the last row of determinant (2.4) and mathematical induction, we obtain the following identity:

$$Ljs_j = s_{j-1}c_0 + 2s_{j-2}c_1 + 3s_{j-3}c_2 + \ldots + js_0 c_{j-1}, \ \ j \geq 1. \tag{2.6}$$

**Theorem 2.2.** [9, Theorem 2.1] *The L–Markov moment problem for the interval* $[0,\theta]$ *with data* $\{c_{j-1}(\theta,x_0)\}_{j=1}^{n}$ *is solvable if and only if the Hausdorff moment problem for the interval* $[0,\theta]$ *with data* $\{s_j(\theta,x_0)\}_{j=0}^{n}$ *is solvable.*

Due to bijective relation (2.4), the $L$–Markov moment problem for $[0,\theta]$ can be solved in terms of the Hausdorff moment problem for $[0,\theta]$. We carry out the treatment of the latter problem making use of Potapov's FMI approach (cf. [2], [3]). Note that in [2] and [3] an explicit solution of the nondegenerate matrix version of the Hausdorff moment problem was given.

## 2.5 Potapov's Fundamental Matrix Inequality

V.P. Potapov developed a powerful approach to matrix interpolation problems, which we now use in its scalar version. This approach is based on a generalization of a classical lemma by H.A. Schwarz and a modification of this result which goes back to G. Pick. Potapov converted the original problem into an equivalent matrix inequality, FMI. In the case, where the so-called information block of this inequality is nondegenerate (see the matrices $H_1$ and $H_2$ in Definitions 2.3 and 2.4), he created an ingenious factorization method which allows one to determine the solution set for the matrix inequality and, consequently, for the original HMP.

Note that, in the construction of the solution, there is a remarkable difference between the cases of even and odd number of data (see Theorem 2.10 below). Taking this into account, we first introduce the matrices which appear in the FMI in the even case (scalar version).

**Definition 2.3.** Let $n = 2p+1$. Using the moments $s_0, s_1, \ldots, s_{2p+1}$, we construct the following matrices:

$$\begin{aligned}
&H_1 := [s_{j+k+1}]_{j,k=0}^{p}, \;\; H_2 := [\theta s_{j+k} - s_{j+k+1}]_{j,k=0}^{p},\\
&T := [\delta_{j,k+1}]_{j,k=1}^{p+1}, \;\; R_T(z) := (I - zT)^{-1},\\
&u := (-s_0, -s_1, \ldots, -s_p)^T, \;\; \upsilon := (1, 0, \ldots, 0)^T,\\
&u_1 := u, \;\; u_2 = -u + \theta T u,
\end{aligned}$$

where $u, \upsilon \in \mathbb{R}^{p+1}$ and $I$ is the identity matrix of corresponding dimension. Further, we introduce two auxiliary holomorphic functions

$$\tilde{s}_1(z) := z s(z), \;\; \tilde{s}_2(z) := (\theta - z) s(z), \quad z \in \mathbb{C} \setminus [0,\theta], \tag{2.7}$$

where $s(z)$ is the Stieltjes transform of a measure $\sigma \in \mathcal{M}[0,\theta]$.

In a similar way, we introduce the matrices that appear in Potapov's FMI in the odd case.

**Definition 2.4.** Let $n = 2p$. Using the moments $s_0, s_1, \ldots, s_{2p}$, we construct the following matrices:

$$\begin{aligned}
&H_1 := [s_{j+k}]_{j,k=0}^{p}, \;\; H_2 := [\theta s_{j+k+1} - s_{j+k+2}]_{j,k=0}^{p-1},\\
&T_1 := [\delta_{j,k+1}]_{j,k=1}^{p+1}, \;\; T_2 := [\delta_{j,k+1}]_{j,k=1}^{p},\\
&R_{T_k}(z) := (I - zT_k)^{-1}, \;\; k = 1, 2,\\
&u_1 := (0, -s_0, \ldots, -s_{p-1})^T, \;\; \upsilon_1 = (1, 0, \ldots, 0)^T,\\
&\tilde{u}_1 := (-s_0, -s_1, \ldots, -s_{p-1})^T, \;\; \tilde{u}_3 := (-s_1, -s_2, \ldots, -s_p)^T,\\
&u_2 := \theta \tilde{u}_1 - \tilde{u}_3, \;\; \upsilon_2 := (1, 0, \ldots, 0)^T,
\end{aligned}$$

where $u_1, \upsilon_1 \in \mathbb{R}^{p+1}$, $u_2, \upsilon_2 \in \mathbb{R}^{p}$. We also introduce two auxiliary holomorphic functions

$$\tilde{s}_1(z) := s(z), \;\; \tilde{s}_2(z) := (\theta - z) z s(z) - s_0 z, \quad z \in \mathbb{C} \setminus [0,\theta], \tag{2.8}$$

where $s(z)$ is the Stieltjes transform of a measure $\sigma \in \mathcal{M}[0,\theta]$.

The following result gives a solvability criterion for the Hausdorff moment problem (see [9]).

**Proposition 2.5.** *The Hausdorff moment problem is solvable if and only if $H_1 \geq 0$ and $H_2 \geq 0$, i.e., the matrices $H_r$ for $r = 1,2$ are positive semidefinite.*

We now define the systems of Potapov's FMI for the even and odd cases (see [2],[3]). In what follows, for $n = 2p+1$, we set $T_1 = T_2 = T$ and $\upsilon_1 = \upsilon_2 = \upsilon$. The complex conjugate of a number $z \in \mathbb{C}$ and the Hermitian conjugate of a matrix function $w \in \mathbb{C}^{m\times n}$ is denoted by $\overline{z}$ and $w^*$, respectively.

**Definition 2.6.** A function $s$ is called a solution of the associated system of Potapov's Fundamental Matrix Inequality, if $s$ satisfies the following conditions: i) $s$ is holomorphic in $\mathbb{C}\setminus[0,\theta]$, ii) for every $r = 1,2$ the inequality

$$\left[\begin{array}{c|c} H_r & R_{T_r}(z)\left[\upsilon_r\tilde{s}_r(z)-u_r\right] \\ \hline (R_{T_r}(z)\left[\upsilon_r\tilde{s}_r(z)-u_r\right])^* & (s_r(z)-\tilde{s}_r^*(z))/(z-\overline{z}) \end{array}\right] \geq 0 \tag{2.9}$$

holds.

It turns out that the treatment of the matrix moment problem is equivalent to finding all solutions of the corresponding system of FMI (see [2],[3]):

**Theorem 2.7.** *The function $s(z)$ is a Stieltjes transform of a measure $\sigma \in \mathcal{M}\left([0,\theta],\{s_j\}_{j=0}^k\right)$ if and only if $s(z)$ is a solution of the system of Potapov's Fundamental Matrix Inequalities* (2.9)*.*

This theorem holds for both the even and odd cases of data. In this way, the problem of finding the Stieltjes transform of $\sigma$ is reduced to the problem of finding the holomorphic function $s(z)$ in Definition 2.6.

**Definition 2.8.** The Hausdorff moment problem is called degenerate if one of the determinants $\det H_r$, $r = 1,2$, is equal to zero.

Below we use the following important result (see [9, Theorem 4.1]).

**Theorem 2.9.** *The degenerate Hausdorff moment problem has a unique solution.*

Following the Potapov scheme, the solution of the degenerate HMP is given by the following result.

**Theorem 2.10.** *The holomorphic function $s(z)$, $z \in \mathbb{C}\setminus[0,\theta]$, associated to the positive measure $\sigma$ of the degenerate HMP has the following rational representation:*

*1) if $n = 2p+1$, then $s(z)$ is equal to*

$$\frac{P_1(z)}{Q_1(z)} := \frac{\nu^* R_T(z)u_1}{z\nu^* R_T(z)\upsilon}, \quad \text{or} \quad \frac{P_2(z)}{Q_2(z)} := \frac{\nu^* R_T(z)u_2}{(\theta - z)\nu^* R_T(z)\upsilon}; \tag{2.10}$$

*2) if $n = 2p$, then $s(z)$ is equal to*

$$\frac{P_3(z)}{Q_3(z)} := \frac{\nu^* R_{T_1}(z)u_1}{\nu^* R_{T_1}(z)\upsilon_1}, \quad \text{or} \quad \frac{P_4(z)}{Q_4(z)} := \frac{\nu^* R_{T_2}(z)(u_2 + s_0 z\upsilon_2)}{(\theta - z)z\nu^* R_{T_2}(z)\upsilon_2}. \tag{2.11}$$

*Proof.* Denote by $\hat{H}$ the matrix $H_r$ (even or odd case) such that $\det H_r = 0$ for $r = 1$ or $r = 2$. Put $M := \begin{bmatrix} \nu & 0 \\ 0 & 1 \end{bmatrix}$, where $\nu \in \mathbb{R}^{p+1}$ or $\nu \in \mathbb{R}^p$ is such that $\hat{H}\nu = 0$. Hence $M \in \mathbb{R}^{(p+2)\times 2}$ or $M \in \mathbb{R}^{(p+1)\times 2}$. Taking into account Proposition 2.2 and Theorem 2.7, we consider the FMI (2.9). Write the inequality (2.9) in the equivalent form

$$M^*\left[\begin{array}{c|c} \hat{H} & R_{T_r}(z)\left[\upsilon_r\tilde{s}_r(z)-u_r\right] \\ \hline (R_{T_r}(z)\left[\upsilon_r\tilde{s}_r(z)-u_r\right])^* & (\tilde{s}_r(z)-\tilde{s}_r^*(z))/(z-\overline{z}) \end{array}\right] M$$

$$= \left[\begin{array}{c|c} \nu^*\hat{H}\nu & \nu^* R_{T_r}(z)\left[\upsilon_r\tilde{s}_r(z)-u_r\right] \\ \hline (R_{T_r}(z)\left[\upsilon_r\tilde{s}_r(z)-u_r\right])^*\nu & (\tilde{s}_r(z)-\tilde{s}_r^*(z))/(z-\overline{z}) \end{array}\right] \geq 0. \tag{2.12}$$

Since $\nu^*\hat{H}\nu = 0$ and since all the eigenvalues of the (Hermite) positive semidefinite matrix are nonnegative, we infer from (2.12) that $-|\nu^* R_{T_r}(z)\,[\upsilon_r \tilde{s}_r(z) - u_r\,]|^2 \geq 0$. Consequently, $\nu^* R_{T_r}(z)\,[\upsilon_r \tilde{s}_r(z) - u_r\,] = 0$, and therefore

$$\tilde{s}_r(z) = \frac{\nu^* R_{T_r}(z) u_r}{\nu^* R_{T_r}(z)\upsilon_r}. \tag{2.13}$$

Hence, for the even case ($n = 2p+1$), taking into account (2.7) and (2.13), we obtain (2.10). Similarly, for the odd case ($n = 2p$), from (2.8) and (2.13) it follows (2.11). □

Note that all functions $P_k$ and $Q_k$ for $k = 1,2,3,4$ are polynomials. Let $\dfrac{P(t)}{Q(t)}$ denote one of the rational fractions $\dfrac{P_k(t)}{Q_k(t)}$ ($k = 1,2,3,4$) corresponding to the condition $\det \hat{H} = 0$.

# 3 Solution of the TOC problem

In this section we give a solution of the TOC problem. In what follows let $L = 1$ and let $x_{k,j}$ denote the $j$-entry of any vector $x_k \in \mathbb{R}^m$. Thus, $x_k = \{x_{k,j}\}_{j=1}^m$.

**Lemma 3.1.** *The TOC problem for the canonical system is equivalent to a degenerate Hausdorff moment problem.*

*Proof.* Because of the complete controllability of (1.1), there exists a $\theta$ such that $x(\theta) = 0$. The system (1.1) with initial condition $x(0) = x_0$ has the unique solution

$$x(t) = e^{At}\left(x_0 + \int_0^t e^{-A\tau} b\tilde{u}(\tau)d\tau\right) \quad \text{for all } \; t \in [0,\theta]. \tag{3.1}$$

Then condition $x(\theta) = 0$ is equivalent to

$$-x_0 = \int_0^\theta e^{-A\tau} b\tilde{u}(\tau)d\tau. \tag{3.2}$$

Using the fact that $A$ and $b$ are canonical, and hence $A^n = 0$ due to (1.2), we conclude that the equality (3.2) can be rewritten in the equivalent form

$$-x_{0,j} = \frac{(-1)^{j-1}}{(j-1)!}\int_0^\theta \tau^{j-1}\tilde{u}(\tau)d\tau, \quad j = 1,2,\ldots,n, \tag{3.3}$$

where $x_{0,j}$ is the $j$-entry of the vector $x_0 \in \mathbb{R}^n$. Setting $f := (\tilde{u}+1)/2$, we get

$$\frac{\theta^j + (-1)^j j!\, x_{0,j}}{2j} = \int_0^\theta \tau^{j-1} f(\tau)d\tau, \quad j = 1,2,\ldots,n.$$

Denoting

$$c_{j-1}(\theta, x_0) := \frac{\theta^j + (-1)^j j!\, x_{0,j}}{2j}, \quad j = 1,2,\ldots,n, \tag{3.4}$$

the TOC problem is reduced to a Markov moment problem, i.e., to the problem of finding a set of functions $f$ such that $0 \leq f(\tau) \leq 1$ for $\tau \in [0,\theta]$ and

$$c_{j-1}(\theta, x_0) = \int_0^\theta \tau^{j-1} f(\tau)d\tau \quad \text{for all } \; j = 1,2,\ldots,n. \tag{3.5}$$

Applying now the relation (2.4) with $L=1$, we obtain the data moments $s_j(\theta,x_0)$ ($j=1,2,\dots,n$) of the classical Hausdorff moment problem for the interval $[0,\theta]$,

$$s_j(\theta,x_0)=\int_0^\theta t^j d\sigma(t),\quad j=0,\dots,n, \tag{3.6}$$

with additional condition that $\theta$ should be the minimal possible value such that (3.6) has a solution. This condition takes place when $\det H_r(\theta,x_0)=0$ (the existence of such $\theta$ in the canonical case (1.2) follows from Theorems 3.6 and 3.7), i.e., in the case of degenerate Hausdorff moment problem.

As to the sufficiency part of the proof, we use the Theorems 2.3 and 2.4 in [9, page 62], which say that the Hausdorff moment problem with even or odd number of given moments $\{s_j(\theta,x_0)\}_{j=0}^k$ is solvable if and only if their corresponding matrix functions $H_1(\theta,x_0)$ and $H_2(\theta,x_0)$ are nonnegative. If the Hausdorff moment problem is degenerate, we can find the optimal time $\theta_{\min}(x_0)$. Taking $\theta=\theta_{\min}(x_0)$, we can get the optimal control $\tilde{u}(t)$ for $t\in[0,\theta_{\min}(x_0)]$ from the equations (3.3) or (3.2). To obtain the solution of the TOC problem, it remains to substitute $\tilde{u}(t)$ in (3.1). □

**Corollary 3.2.** *If $\tilde{x}_j=(0,\dots,0,x_{0,j},\dots,x_{0,n})\in\mathbb{R}^n$ where $x_{0,j}\neq 0$, then*

$$s_j(0,\tilde{x}_j)=c_{j-1}(0,x_0)=\frac{(-1)^j(j-1)!\,x_{0,j}}{2},\quad j\geq 1. \tag{3.7}$$

*Proof.* It is sufficient to apply (2.6) with $L=1$, and then (3.4). □

**Theorem 3.3.** *Let $x_0\in\mathbb{R}^n$ be a fixed state and let $\theta_1>0$ be sufficiently large. Then for all $\theta>\theta_1$ and for both even and odd cases, $H_1(\theta,x_0)>0$ and $H_2(\theta,x_0)>0$, i.e., the matrices $H_r(\theta,x_0)$ for $r=1,2$ are positive definite.*

*Proof.* Applying (2.4) in the case $L=1$ with $c_{j-1}(\theta,x_0)$ given by (3.4), we obtain

$$s_j(\theta,x_0)=\frac{1}{j!}\begin{vmatrix} \frac{\theta}{2}-\frac{x_{0,1}}{2} & -1 & \cdots & 0\\ \frac{\theta^2}{2}+\frac{2!x_{0,2}}{2} & \frac{\theta}{2}-\frac{x_{0,1}}{2} & \ddots & 0\\ \vdots & \ddots & \ddots & \vdots\\ \frac{\theta^{j-1}}{2}+\frac{(-1)^{j-1}(j-1)!x_{0,j-1}}{2} & \frac{\theta^{j-2}}{2}+\frac{(-1)^{j-2}(j-2)!x_{0,j-2}}{2} & \cdots & -(j-1)\\ \frac{\theta^j}{2}+\frac{(-1)^j j!x_{0,j}}{2} & \frac{\theta^{j-1}}{2}+\frac{(-1)^{j-1}(j-1)!x_{0,j-1}}{2} & \cdots & \frac{\theta}{2}-\frac{x_{0,1}}{2} \end{vmatrix}$$

$$=\frac{1}{2^j j!}\begin{vmatrix} \theta & -2 & \cdots & 0 & 0\\ \theta^2 & \theta & \ddots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \theta^{j-1} & \theta^{j-2} & \cdots & \theta & -2(j-1)\\ \theta^j & \theta^{j-1} & \cdots & \theta^2 & \theta \end{vmatrix}+P_{j-1}(\theta,x_0)$$

$$:=\frac{1}{2^j j!}\Delta_j(\theta)+P_{j-1}(\theta,x_0), \tag{3.8}$$

where $P_{j-1}(\theta,x_0)$ is a polynomial of degree $j-1$ in $\theta$. To calculate the determinant $\Delta_j(\theta)$ we use the recursive formula $\Delta_j(\theta)=(2j-1)\theta\Delta_{j-1}(\theta)$, which is obtained by applying Laplace's expansion with respect to the last column. As a result, we get $\Delta_j(\theta)=(2j-1)!!\,\theta^j$, whence (3.8) implies that

$$s_j(\theta,x_0)=\frac{(2j-1)!!}{2^j j!}\,\theta^j+P_{j-1}(\theta,x_0), \tag{3.9}$$

Now we consider the case $n=2p+1$ and $r=1$. Let $y\in\mathbb{R}^{p+1}$. Taking $s_j=s_j(\theta,x_0)$, we deduce from Definition 2.3 and (2.4) that

$$\big(H_1(\theta,x_0)y,y\big)=\big([s_{j+k+1}]_{j,k=0}^{p}y,y\big)=\sum_{k=1}^{p+1}\left(2\sum_{i=1}^{k-1}s_{k+i-1}y_iy_k+s_{2k-1}y_k^2\right), \tag{3.10}$$

By (3.9), each summand in (3.10) is of the form

$$2\sum_{i=1}^{k-1}s_{k+i-1}y_iy_k+s_{2k-1}y_k^2=\frac{(4k-3)!!}{2^{2k-1}(2k-1)!}\,\theta^{2k-1}y_k^2+R_{2k-2}(\theta,x_0,y)y_k,$$

where $R_{2k-2}(\theta,x_0,y)$ is a polynomial of degree $2k-2$ in $\theta$. Thus, we infer from (3.10) that

$$\big(H_1(\theta,x_0)y,y\big)=\sum_{k=1}^{p+1}\left(\frac{(4k-3)!!}{2^{2k-1}(2k-1)!}\,\theta^{2k-1}y_k^2+R_{2k-2}(\theta,x_0,y)y_k\right).$$

Hence there is a $\theta_1>0$ such that for all $\theta>\theta_1$ and all $y\neq 0$, we have $\big(H_1(\theta,x_0)y,y\big)>0$, which means that $H_1(\theta,x_0)$ is positive definite. In a similar way $\big(H_2(\theta,x_0)y,y\big)>0$ for all sufficiently large $\theta$ and all $y\neq 0$. The same arguments are valid for the odd case of given moments. □

**Lemma 3.4.** *If $n=2p+1$, then for $\theta=0$ and all $j=1,2,\dots,p$ we have the following:*

$$\big(H_1(0,x_0)y_1,y_1\big)=-\big(H_2(0,x_0)y_1,y_1\big)=-\frac{1}{2}\,x_{0,1}, \tag{3.11}$$

$$\big(H_1(0,\tilde{x}_{2j})y_{j+1},y_{j+1}\big)=-\big(H_2(0,\tilde{x}_{2j})y_{j+1},y_{j+1}\big)=-\frac{(2j)!}{2}\,x_{0,2j+1}, \tag{3.12}$$

$$\big(H_1(0,\tilde{x}_{2j})\tilde{y}_j,\tilde{y}_j\big)=-\big(H_2(0,\tilde{x}_{2j})\tilde{y}_j,\tilde{y}_j\big)=(2j-1)!\,x_{0,2j}-\frac{(2j)!}{2}\,x_{0,2j+1}, \tag{3.13}$$

*where $y_j=(0,\dots,0,1,0,\dots,0)^T\in\mathbb{R}^{p+1}$ with $y_{j,j}=1$ for $j=1,2,\dots,p+1$, and $\tilde{y}_j=(0,\dots,0,1,1,0,\dots,0)^T\in\mathbb{R}^{p+1}$ with $\tilde{y}_{j,j}=\tilde{y}_{j,j+1}=1$ for $j=1,2,\dots,p$.*

*Proof.* By Definition 2.3 and Corollary 3.2, we obtain

$$\big(H_1(0,x_0)y_1,y_1\big)=-\big(H_2(0,x_0)y_1,y_1\big)=s_1(0,x_0)=-\frac{x_{0,1}}{2},$$

which gives (3.11). Further, for all $j=1,2,\dots,p$, Definition 2.3 and (2.6) imply that

$$\begin{aligned}\big(H_1(0,\tilde{x}_{2j})y_{j+1},y_{j+1}\big)&=-\big(H_2(0,\tilde{x}_{2j})y_{j+1},y_{j+1}\big)=s_{2j+1}(0,\tilde{x}_{2j})\\&=\frac{2j}{2j+1}\,s_1(0,\tilde{x}_{2j})c_{2j}(0,\tilde{x}_{2j})+c_{2j}(0,\tilde{x}_{2j}).\end{aligned} \tag{3.14}$$

Since $s_1(0,\tilde{x}_{2j})=-\frac{1}{2}\tilde{x}_{2j,1}=0$ in view of the first equality in (3.8) and since $c_{2j}(0,\tilde{x}_{2j})=-\frac{(2j)!}{2}\,x_{0,2j+1}$, we infer (3.12) from (3.14).

On the other hand, from Definition 2.3 it follows that

$$\begin{aligned}\big(H_1(0,\tilde{x}_{2j})\tilde{y}_j,\tilde{y}_j\big)&=-\big(H_2(0,\tilde{x}_{2j})\tilde{y}_j,\tilde{y}_j\big)\\&=s_{2j-1}(0,\tilde{x}_{2j})+2s_{2j}(0,\tilde{x}_{2j})+s_{2j+1}(0,\tilde{x}_{2j}).\end{aligned} \tag{3.15}$$

By Corollary 3.2 and (2.6), we conclude that

$$s_{2j-1}(0,\tilde{x}_{2j})=0,\quad 2s_{2j}(0,\tilde{x}_{2j})=(2j-1)!\,x_{0,2j}, \tag{3.16}$$

and

$$s_{2j+1}(0,\tilde{x}_{2j}) = \frac{2j}{2j+1}\, s_1(0,\tilde{x}_{2j})c_{2j-1}(0,\tilde{x}_{2j}) + c_{2j}(0,\tilde{x}_{2j})$$
$$= c_{2j}(0,\tilde{x}_{2j}) = -\frac{(2j)!\, x_{0,2j+1}}{2},. \tag{3.17}$$

Combining (3.15)–(3.17), we get (3.13). □

**Lemma 3.5.** *If $n = 2p$, then for $\theta = 0$ the following relations hold:*

$$\big(H_1(0,x_0)y_2,y_2\big) = -\big(H_2(0,x_0)\hat{y}_1,\hat{y}_1\big) = \frac{1}{8}\,(x_{0,1}^2 + 4x_{0,2}), \tag{3.18}$$

$$\big(H_1(0,x_0)y_0,y_0\big) = 1 - x_{0,1}y_{0,2} + \frac{1}{8}\,(x_{0,1}^2 + 4x_{0,2})y_{0,2}^2, \tag{3.19}$$

$$\big(H_1(0,\tilde{x}_{2j-1})y_{j+1},y_{j+1}\big) = -\big(H_2(0,\tilde{x}_{2j-1})\hat{y}_j,\hat{y}_j\big) = \frac{(2j-1)!}{2}\, x_{0,2j}, \tag{3.20}$$

$$\big(H_1(0,\tilde{x}_{2j-1})\tilde{y}_j,\tilde{y}_j\big) = -\big(H_2(0,\tilde{x}_{2j-1})\check{y}_{j-1},\check{y}_{j-1}\big) = -(2j-2)!\, x_{0,2j-1} + \frac{(2j-1)!}{2}\, x_{0,2j}, \tag{3.21}$$

*where $y_0 = (1,y_{0,2},0,\ldots,0)^T \in \mathbb{R}^{p+1}$, $\hat{y}_j = (0,\ldots,0,1,0,\ldots,0)^T \in \mathbb{R}^p$ with $\hat{y}_{j,j} = 1$ for $j = 2,3,\ldots,p$, and $\check{y}_{j-1} = (0,\ldots,0,1,1,0,\ldots,0)^T \in \mathbb{R}^p$ with $\check{y}_{j-1,j-1} = \check{y}_{j-1,j} = 1$ for $j = 2,3,\ldots,p$.*

*Proof.* Applying Definition 2.4, (2.6) and the first equality in (3.8), we deduce that

$$\big(H_1(0,x_0)y_2,y_2\big) = -\big(H_2(0,x_0)\hat{y}_1,\hat{y}_1\big) = s_2(0,x_0) = \frac{1}{8}\,(x_{0,1}^2 + 4x_{0,2}),$$

which gives (3.18). Analogously,

$$\big(H_1(0,x_0)y_0,y_0\big) = 1 + 2s_1(0,x_0)y_{0,2} + s_2(0,x_0)y_{0,2}^2$$
$$= 1 - x_{0,1}y_{0,2} + \frac{1}{8}\,(x_{0,1}^2 + 4x_{0,2})y_{0,2}^2,$$

which proves (3.19).

Further, we infer from Definition 2.4, (2.6) and Corollary 3.2 that for all $j = 2,3,\ldots,p$,

$$\big(H_1(0,\tilde{x}_{2j-1})y_{j+1},y_{j+1}\big) = -\big(H_2(0,\tilde{x}_{2j-1})\hat{y}_j,\hat{y}_j\big) = s_{2j}(0,\tilde{x}_{2j-1})$$
$$= \frac{2j-1}{2j}\, s_1(0,\tilde{x}_{2j-1})c_{2j-2}(0,\tilde{x}_{2j-1}) + c_{2j-1}(0,\tilde{x}_{2j-1})$$
$$= \frac{(2j-1)!\, x_{0,2j}}{2},$$

which gives (3.20). On the other hand, by analogy with (3.13), from Definition 2.4, Corollary 3.2 and (2.6) it follows that for all $j = 2,3\ldots,p$,

$$\big(H_1(0,\tilde{x}_{2j-1})\tilde{y}_j,\tilde{y}_j\big) = -\big(H_2(0,\tilde{x}_{2j-1})\check{y}_{j-1},\check{y}_{j-1}\big)$$
$$= s_{2j-2}(0,\tilde{x}_{2j-1}) + 2s_{2j-1}(0,\tilde{x}_{2j-1}) + s_{2j}(0,\tilde{x}_{2j-1})$$
$$= -(2j-2)!\, x_{0,2j-1} + \frac{(2j-1)!\, x_{0,2j}}{2},$$

which proves (3.21). □

**Theorem 3.6.** *For every $n \in \mathbb{N}$ and every nonzero vector $x_0 \in \mathbb{R}^n$ there exists a maximal positive solution (root) $\theta_n(x_0)$ of* $\det H_1(\theta, x_0) = 0$ *or/and* $\det H_2(\theta, x_0) = 0$.

*Proof.* It follows from Theorem 3.3 that $(H_r(\theta, x_0)y, y) > 0$ for all sufficiently large $\theta > 0$ and all $y \neq 0$. On the other hand, by Lemma 3.4, for $n = 2p+1$ we have the following. Relations (3.11) imply that one of the quadratic forms $\big(H_r(0,x_0)y_1, y_1\big)$ for $r = 1,2$ has the negative value $-|x_{0,1}|/2$ at the point $x_0$, if $x_{0,1} \neq 0$. If $x_{0,1} = 0$, then applying the same arguments we infer from (3.12) for $j = 1$ that one of the quadratic forms $\big(H_r(0,x_0)y_2, y_2\big) = \big(H_r(0,\tilde{x}_2)y_2, y_2\big)$ for $r = 1,2$ has a negative value $-|x_{0,3}|$ in case $x_{0,3} \neq 0$. If $x_{0,1} = x_{0,3} = 0$, then from (3.13) it follows that again one of the quadratic forms $\big(H_r(0,x_0)\tilde{y}_1, \tilde{y}_1\big) = \big(H_r(0,\tilde{x}_2)y_2, y_2\big)$ for $r = 1,2$ has a negative value $-|x_{0,2}|$ whenever $x_{0,2} \neq 0$. Thus, we may substitute $x_0$ by $\tilde{x}_4$ and repeat our arguments, by applying first (3.12) for $j = 2$ and then (3.13) for $j = 2$, and so on. Hence, for $x_{0,1} = 0$ and every $j = 1, 2, \dots, p$ we successively obtain

$$\begin{aligned} \big|\big(H_r(0,x_0)y_{j+1}, y_{j+1}\big)\big| &= \big|\big(H_r(0,\tilde{x}_{2j})y_{j+1}, y_{j+1}\big)\big| = -\frac{(2j)!}{2}\,|x_{0,2j+1}|, \\ \big|\big(H_r(0,x_0)\tilde{y}_j, \tilde{y}_j\big)\big| &= \big|\big(H_r(0,\tilde{x}_{2j})\tilde{y}_j, \tilde{y}_j\big)\big| = -(2j-1)!\,|x_{0,2j}|, \end{aligned}$$

which implies that in the case $n = 2p+1$ for every nonzero vector $x_0 \in \mathbb{R}^n$ at least one of the quadratic forms $\big(H_r(0,x_0)y, y\big)$ for $r = 1,2$ is negative for some $y \in \mathbb{R}^{p+1}$.

In the case $n = 2p$ we use Lemma 3.5. If $x_{0,2}^2 + 4x_{0,1} \neq 0$, then by (3.18) one of the quadratic forms $\big(H_1(0,x_0)y_2, y_2\big)$ or $\big(H_2(0,x_0)\hat{y}_1, \hat{y}_1\big)$ has the negative value $-|x_{0,2}^2 + 4x_{0,1}|$. If $x_{0,2}^2 + 4x_{0,1} = 0$, then from (3.19) it follows that

$$\big(H_1(0,x_0)y_0, y_0\big) = 1 - x_{0,1}y_{0,2},$$

which is negative for $x_{0,1} \neq 0$ and $y_{0,2} = 2/x_{0,1}$. If $x_{0,2}^2 + 4x_{0,1} = 0$ and $x_{0,1} = 0$, then $x_{0,2} = 0$ as well. Let $j = 2$. Applying now (3.20) we infer that for $j = 2$ one of the quadratic forms $\big(H_1(0,\tilde{x}_{2j-1})y_{j+1}, y_{j+1}\big)$ or $\big(H_2(0,\tilde{x}_{2j-1})\hat{y}_j, \hat{y}_j\big)$ has the negative value $-\frac{(2j-1)!}{2}\,|x_{0,2j}|$ if $x_{0,2j} \neq 0$. If $x_{0,2j} = 0$, then (3.21) implies that one of the quadratic forms $\big(H_1(0,\tilde{x}_{2j-1})\tilde{y}_j, \tilde{y}_j\big)$ or $\big(H_2(0,\tilde{x}_{2j-1})\check{y}_{j-1}, \check{y}_{j-1}\big)$ has the negative value $-(2j-2)!\,|x_{0,2j-1}|$ if $x_{0,2j} \neq 0$. Thus, now it is sufficient to consider $x_0 = \tilde{x}_{2j-1}$ for all $j \geq 3$. Then we again use (3.20), and then (3.21), and so on. Finally, we conclude that in the case $n = 2p$ similarly to $n = 2p+1$ for every nonzero vector $x_0 \in \mathbb{R}^n$ at least one of the quadratic forms $\big(H_r(0,x_0)y, y\big)$ for $r = 1,2$ is negative for some $y \in \mathbb{R}^{p+1}$ if $r = 1$ and $y \in \mathbb{R}^p$ if $r = 2$.

Since both functions $\big(H_r(\theta,x_0)y, y\big)$ $(r = 1,2)$, for even or odd cases of $n$, are continuous in $\theta$ and $y$ for any fixed $x_0 \neq 0$, we infer that for every $n$ there exists the maximal value $\theta_n(x_0)$ such that $\big(H_r(\theta_n(x_0), x_0)y, y\big) = 0$ for some $r = 1,2$ and some $y \neq 0$. Hence either $\det H_1(\theta_n(x_0), x_0) = 0$, or $\det H_2(\theta_n(x_0), x_0) = 0$, which means in view of Theorem 3.3 that both matrices $H_r(\theta_n(x_0), x_0)$ are positive semidefinite and at least one of the determinants $\det H_r(\theta_n(x_0), x_0)$ equals 0 for $r = 1,2$. $\square$

Taking into account the existence of $\theta_n(x_0)$ for every $n \in \mathbb{N}$ and every nonzero vector $x_0 \in \mathbb{R}^n$ due to Theorem 3.6, we will call $\theta_n(x_0)$ the optimal time $\theta_{\min}(x_0)$ of the system (1.1).

The condition $\det H_1 = 0$ or $\det H_2 = 0$ says that the considered Hausdorff moment problem for an interval $[0, \theta]$ is *degenerate*. Consequently, due to Theorem 2.9, this HMP has a unique solution.

By virtue of the proved equivalence between the HMP and TOC problems (see Lemma 3.1) and Proposition 2.9, there is a unique solution of the TOC problem. In each case (even and odd) we have two rational functions (2.10) and (2.11), respectively, which give the same solution of the TOC problem.

Now we find the optimal control $\tilde{u}(t)$ related to the optimal time $\theta = \theta_{\min}(x_0)$.

**Theorem 3.7.** *The time optimal control of system* (1.1) *is given by*

$$\tilde{u}(t) = -\operatorname{sign}\frac{P(t)}{Q(t)}, \quad t \in [0, \theta_{\min}(x_0)], \tag{3.22}$$

*where $P$ and $Q$ are the polynomials of the rational function $s = P/Q$ associated to the degenerate HMP.*

*Proof.* Let for definiteness $n = 2p+1$, and assume that $\det H_1(\theta_{\min}(x_0), x_0) = 0$. By Theorem 2.10, the holomorphic function $s(z)$ associated to the solution of the corresponding HMP is a rational function. Let, for example, $s(z) = \frac{P_1(z)}{Q_1(z)}$. Due to (2.3) and the properties of this solution (see [9, page 244]), we have

$$-z\,\frac{P_1(z)}{Q_1(z)} = \frac{(z-\xi_1)\cdots(z-\xi_m)}{(z-\eta_1)\cdots(z-\eta_m)} = \exp\left(\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\frac{dt}{z-t}\right), \tag{3.23}$$

where $0 = \xi_1 < \eta_1 < \xi_2 < \eta_2 < \ldots < \xi_m < \eta_m (\le \theta_{\min}(x_0))$.

By (3.23), the solution $f$ has the following form:

$$f(t) = \begin{cases} 1 & \text{if } t \in (\xi_j, \eta_j), \\ 0 & \text{if } t \in (\eta_j, \xi_{j+1}), \end{cases} \quad \text{or, equivalently,} \quad f(t) = \frac{1}{2}\left(1 - \operatorname{sign}\frac{P_1(t)}{Q_1(t)}\right),$$

which completes the proof. □

*Remark* 3.8. The switching points of (3.22) are given by the roots of $P(t)Q(t)$. From (2.10)–(2.11) and Definitions 2.3–2.4 it follows that control (3.22) does not have more than $(n-1)$ points of switching. By virtue of [5, Lemma 9], this control is optimal.

One of the advantages of using the Potapov Method for solving the TOC problems is precisely the determination of switching points of the optimal control without recursive operations.

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