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On matrix Hurwitz type polynomials and their interrelations to Stieltjes positive definite sequences and orthogonal matrix polynomials

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A B S T R A C T

This paper is a direct continuation of the author's recent investigations [4] on the non-degenerate truncated matricial Stieltjes problem. Inspired by a characterization of scalar Hurwitz polynomials via continued fractions (see e.g. Gantmacher [15]) a class of matrix Hurwitz type polynomials is introduced. It is shown that this class is intimately related with Stieltjes positive definite sequences of matrices and can be expressed in terms of the associated Stieltjes quadruple of sequences of left orthogonal matrix polynomials. It is shown that a monic matrix polynomial is a matrix Hurwitz type polynomial if and only if the sequence of its Markov parameters is a Stieltjes positive definite sequence.

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1. Introduction

In the center of this paper stands a certain matrix generalization of the classical notion of Hurwitz polynomials. A real polynomial with all its roots in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C}: \operatorname{Re} z \in (-\infty, 0)\}$ of the complex plane is called *Hurwitz polynomial*. Apparently, such polynomials were first studied by J.C. Maxwell [25] in 1868 and later considered by E.J. Routh [28] and A. Hurwitz [19] in 1875 and 1895, respectively. Routh and Hurwitz gave a necessary and sufficient condition for a real polynomial to be a Hurwitz polynomial. Other tools for dealing with such polynomials are provided by Sturm's theorem [30] and Bezoutian approach [14]. The so-called Markov parameters (MP) approach was studied in [15]; see also [6,18] and references therein. Recently, by using Wall's continued fractions, a connection between the Schwarz matrices and rational functions associated with Hurwitz polynomials was studied in [31]. In [8] the description of functions generating infinite totally non-negative Hurwitz matrices via Stieltjes meromorphic functions is given. What concerns a detailed treatment of the theory of Hurwitz polynomials we refer the reader to the monographs Gantmacher [15, Chapter XV], Postnikov [26] and Rahman/Schmeisser [27, Section 11.4].

Starting with the classical work of Grommer [17] the theory of Hurwitz polynomials was extended to entire functions (see Chebotarev/Meiman [2], Krein [21], Levin [23, Chapter VII], Katsnelson [20]). The direct matricial generalization of the notion should be defined as follows: A $q \times q$ matrix polynomial P is called *Hurwitz polynomial* if the inclusion $\{z \in \mathbb{C}: \det P(z) = 0\} \subseteq \mathbb{C}^-$ is satisfied. In this paper, we follow another line of generalizing the classical notion. We are inspired by the investigations in Gantmacher [15, Chapter XV] on Hurwitz polynomials and related mechanical application in Gantmacher/Krein [16, Appendix II]. One of the central ideas there is to study interrelations between the even and odd parts of a polynomial. The membership of a polynomial to the class of Hurwitz polynomials is characterized in terms of particular continued fraction representations of the rational functions which are formed by the even and odd parts of the given polynomial (see Gantmacher [15, Chapter XV, Section 14, Theorem 16]). The shape of these continued fractions immediately establishes a bridge to the Stieltjes moment problem. These connections are discussed in [15, Chapter XV, Section 16].

This paper is based on the author's recent investigations [4] on the non-degenerate truncated matricial Stieltjes moment problems. In [4] we obtained distinguished matrix continued fraction expansions for the two extremal solutions to the above mentioned moment problem. Against to this background we introduce the notion "matrix Hurwitz type polynomial" which in the scalar case indeed coincides with the classical notion. In the matrix case the relation between "matrix Hurwitz type polynomials" and "matrix Hurwitz polynomials" is still uncovered.

A closer look at [4, Theorems 3.4 and 4.8] shows that a Stieltjes positive definite sequence produces matrix continued fractions which lead in a natural way to matrix Hurwitz type polynomials (see Theorem 5.4). Starting from this observation we establish a principle of constructing matrix Hurwitz type polynomials by using the Stieltjes quadruple of sequences of left orthogonal matrix polynomials associated with a Stieltjes positive definite sequence (see Theorem 6.1). Conversely, we show that each matrix Hurwitz type polynomial can be generated in this way (see Theorem 7.9). In this way a bijective correspondence between matrix Hurwitz type polynomials and finite Stieltjes positive definite sequences is established via Markov parameter sequences.

In Section 8, we obtain in the scalar case a criterion of asymptotic stability of a system of linear ordinary differential equations with constant coefficients in terms of the main results of this paper (see Theorem 8.2).

2. Matrix Hurwitz type polynomials

We introduce in this section a class of $q \times q$ matrix polynomials which turns out to coincide in the classical case $q = 1$ with the class of classical Hurwitz polynomials. In our further considerations we will often work with the odd and even parts of a polynomial, which will be introduced now.

Definition 2.1. Let $n \in \mathbb{N}$ and let $\mathbf{f}_n$ be a $q \times q$ matrix polynomial of degree n. For $z \in \mathbb{C}$ let $\mathbf{f}_n$ be given by

$$\mathbf{f}_n(z) = \sum_{k=0}^{n} A_k z^{n-k}. \tag{2.1}$$

Let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m-1$ is satisfied. Then the $q \times q$ matrix polynomial $\mathbf{h}_n : \mathbb{C} \to \mathbb{C}^{q\times q}$ defined by

$$\mathbf{h}_n(z) := \begin{cases} \sum_{\ell=0}^{m} A_{2\ell} z^{m-\ell}, & \text{if } n = 2m \\ \sum_{\ell=1}^{m} A_{2\ell-1} z^{m-\ell}, & \text{if } n = 2m-1 \end{cases} \tag{2.2}$$

is called the *even part of* $\mathbf{f}_n$, whereas the $q \times q$ matrix polynomial $\mathbf{g}_n : \mathbb{C} \to \mathbb{C}^{q\times q}$ defined by

$$\mathbf{g}_n(z) := \begin{cases} \sum_{\ell=1}^{m} A_{2\ell-1} z^{m-\ell}, & \text{if } n = 2m \\ \sum_{\ell=1}^{m} A_{2\ell-2} z^{m-\ell}, & \text{if } n = 2m-1 \end{cases} \tag{2.3}$$

is called the *odd part of* $\mathbf{f}_n$.

Remark 2.2. Let $n \in \mathbb{N}$ and let $\mathbf{f}_n$ be a $q \times q$ matrix polynomial of degree n. Denote by $\mathbf{h}_n$ and $\mathbf{g}_n$ the even and odd part of $\mathbf{f}_n$, respectively. Then it is easily checked by straightforward computation that the identity

$$\mathbf{f}_n(z) = \mathbf{h}_n(z^2) + z\mathbf{g}_n(z^2)$$

holds for $z \in \mathbb{C}$.

Remark 2.3. Let $n \in \mathbb{N}$ and let $\mathbf{f}_n$ be a $q \times q$ matrix polynomial of degree n. Denote by $\mathbf{h}_n$ and $\mathbf{g}_n$ the even and odd part of $\mathbf{f}_n$, respectively.

(a) Suppose that $n = 2m$ for some $m \in \mathbb{N}$. Then the leading coefficients of $\mathbf{f}_n$ and $\mathbf{h}_n$ coincide. In particular, $\mathbf{f}_n$ is monic if and only if $\mathbf{h}_n$ is monic, and furthermore $\mathbf{f}_n$ has non-singular leading coefficient if and only if $\mathbf{h}_n$ has non-singular leading coefficient.
(b) Suppose that $n = 2m-1$ for some $m \in \mathbb{N}$. Then the leading coefficients of $\mathbf{f}_n$ and $\mathbf{g}_n$ coincide. In particular, $\mathbf{f}_n$ is monic if and only if $\mathbf{g}_n$ is monic, and furthermore $\mathbf{f}_n$ has non-singular leading coefficient if and only if $\mathbf{g}_n$ has non-singular leading coefficient.

Remark 2.4. Let $m \in \mathbb{N}$.

(a) Let $\mathbf{h}$ be a $q \times q$ matrix polynomial of degree m and let $\mathbf{g}$ be a $q \times q$ matrix polynomial of degree at most $m-1$. Let $\mathbf{f}: \mathbb{C} \to \mathbb{C}$ be defined by $\mathbf{f}(z) := \mathbf{h}(z^2) + z\mathbf{g}(z^2)$. Then f is a $q \times q$ matrix polynomial of degree $2m$ with even part h and odd part $\mathbf{g}$.
(b) Let $\mathbf{h}$ be a $q \times q$ matrix polynomial of degree at most $m-1$ and let $\mathbf{g}$ be a $q \times q$ matrix polynomial of degree $m-1$. Let $\mathbf{f}: \mathbb{C} \to \mathbb{C}$ be defined by $\mathbf{f}(z) := \mathbf{h}(z^2) + z\mathbf{g}(z^2)$. Then f is a $q \times q$ matrix polynomial of degree $2m-1$ with even part h and odd part $\mathbf{g}$.

Lemma 2.5. *Let $n \in \mathbb{N}$ and let $\mathbf{f}_n$ be a $q \times q$ matrix polynomial of degree n with non-singular leading coefficient. Let $\mathcal{N}_{\mathbf{f}_n} := \{z \in \mathbb{C}: \det \mathbf{f}_n(z) = 0\}$. Then $\mathcal{N}_{\mathbf{f}_n}$ is a finite subset of $\mathbb{C}$.*

Proof. For $z \in \mathbb{C}$ let $\mathbf{f}_n$ be given by (2.1) and let $\mathbf{f}_n^\vee: \mathbb{C} \to \mathbb{C}^{q \times q}$ be defined by

$$\mathbf{f}_n^\vee(z) := \sum_{k=0}^{n} A_k^* z^k.$$

Then $\mathbf{f}_n^\vee(0) = A_0^*$. Hence, $\det[\mathbf{f}_n^\vee(0)] = \overline{\det A_0} \neq 0$. Thus, since $\det \mathbf{f}_n^\vee$ is a polynomial, the set $\mathcal{N}_{\mathbf{f}_n^\vee} := \{z \in \mathbb{C}: \det \mathbf{f}_n^\vee(z) = 0\}$ is a finite subset of $\mathbb{C}$. For $z \in \mathbb{C} \setminus \{0\}$ the identity

$$\mathbf{f}_n(z) = z^n \left[\mathbf{f}_n^\vee\left(\frac{1}{\bar{z}}\right)\right]^*$$

holds (see e.g. [7, Lemma 1.2.2]). Hence,

$$\det[\mathbf{f}_n(z)] = z^{nq}\overline{\det\left[\mathbf{f}_n^\vee\left(\frac{1}{\bar{z}}\right)\right]}.$$

This implies that $\det \mathbf{f}_n$ does not identically vanish in $\mathbb{C}$. Since $\det \mathbf{f}_n$ is a polynomial the set $\mathcal{N}_{\mathbf{f}_n}$ is a finite subset of $\mathbb{C}$. □

Lemma 2.6. *Let $n \in \mathbb{N}$ and let $\mathbf{f}_n$ be a $q \times q$ matrix polynomial of degree n with non-singular leading coefficient. Denote by $\mathbf{h}_n$ and $\mathbf{g}_n$ the even and odd parts of $\mathbf{f}_n$, respectively. Let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m - 1$ is satisfied. If $n = 2m$ then the set $\mathcal{N}_{\mathbf{h}_n} := \{z \in \mathbb{C}\colon \det \mathbf{h}_n(z) = 0\}$ is finite whereas in the case $n = 2m - 1$ the set $\mathcal{N}_{\mathbf{g}_n} := \{z \in \mathbb{C}\colon \det \mathbf{g}_n(z) = 0\}$ is finite.*

Proof. In the case $n = 2m$ (resp. $n = 2m - 1$) the application of Lemma 2.5 in combination with (2.2) (resp. (2.3)) yields the assertion. □

Now we introduce the central objects of this paper. What concerns the case of an even n we are inspired by a classical result due to Stieltjes [29] and its connections with classical Hurwitz polynomials (see Gantmacher [15, Chapter XV, Section 16, Propositions 15 and 16]). In the case of an odd number n our construction differs from that one which is suggested by the scalar result. The reason for this will be seen later.

In view of Lemma 2.6 the following definition is correct. For $A, B \in \mathbb{C}^{q\times q}$ with B invertible, set $\frac{A}{B} := AB^{-1}$.

Definition 2.7. Let $n \in \{2, 3, \dots\}$ and let $\mathbf{f}_n$ be a monic $q \times q$ matrix polynomial of degree n. Denote by $\mathbf{h}_n$ and $\mathbf{g}_n$ the even and odd parts of $\mathbf{f}_n$, respectively.

(a) Suppose $n = 2m$ for some $m \in \mathbb{N}$. Then $\mathbf{f}_n$ is called a *matrix Hurwitz type polynomial* (short *MHTP*) *of degree n* if there exists an ordered pair of sequences $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}]$ from $\mathbb{C}_>^{q\times q}$ such that the identity

$$\frac{\mathbf{g}_n(z)}{\mathbf{h}_n(z)} = \cfrac{I_q}{z\mathbf{c}_0 + \cfrac{I_q}{\mathbf{d}_0 + \cfrac{I_q}{\ddots \; \mathbf{d}_{m-2} + \cfrac{I_q}{z\mathbf{c}_{m-1} + \mathbf{d}_{m-1}^{-1}}}}} \tag{2.4}$$

holds for all $z \in \mathbb{C} \setminus \mathcal{N}_{\mathbf{h}_n}$. In this case the pair $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}]$ is called a *Hurwitz parametrization of* $\mathbf{f}_n$.

(b) Suppose $n = 2m + 1$ for some $m \in \mathbb{N}$. Then $\mathbf{f}_n$ is called a *matrix Hurwitz type polynomial* (short *MHTP*) *of degree n* if there exists an ordered pair of sequences $[(\mathbf{c}_k)_{k=0}^{m}, (\mathbf{d}_k)_{k=0}^{m-1}]$ from $\mathbb{C}_>^{q\times q}$ such that the identity

$$\frac{\mathbf{h}_n(z)}{z\mathbf{g}_n(z)} = \cfrac{I_q}{z\mathbf{c}_0 + \cfrac{I_q}{\mathbf{d}_0 + \cfrac{I_q}{\ddots \; z\mathbf{c}_{m-1} + \cfrac{I_q}{\mathbf{d}_{m-1} + z^{-1}\mathbf{c}_m^{-1}}}}} \tag{2.5}$$

holds for all $z \in \mathbb{C} \setminus [\{0\} \cup \mathcal{N}_{\mathbf{g}_n}]$. In this case the pair $[(\mathbf{c}_k)_{k=0}^{m}, (\mathbf{d}_k)_{k=0}^{m-1}]$ is called a *Hurwitz parametrization of* $\mathbf{f}_n$.

Remark 2.8. Let $m \in \mathbb{N}$ and let $n := 2m + 1$. Let $\mathbf{f}_n$ be a MHTP of degree n and let $[(\mathbf{c}_k)_{k=0}^{m}, (\mathbf{d}_k)_{k=0}^{m-1}]$ be a Hurwitz parametrization of $\mathbf{f}_n$. By multiplying (2.5) by z and taking the inverse of $\frac{\mathbf{h}_n(z)}{\mathbf{g}_n(z)}$, we see that

$$\frac{\mathbf{g}_n(z)}{\mathbf{h}_n(z)} = \mathbf{c}_0 + \cfrac{I_q}{z\mathbf{d}_0 + \cfrac{I_q}{\mathbf{c}_1 + \cfrac{I_q}{\ddots \, \mathbf{c}_{m-1} + \cfrac{I_q}{z\mathbf{d}_{m-1} + \mathbf{c}_m^{-1}}}}} \tag{2.6}$$

holds for all $z \in \mathbb{C} \setminus \mathcal{N}_{\mathbf{h}_n}$.

In the scalar case, i.e. for $q = 1$, the polynomials of Definition 2.7 coincide, in view of (2.4), (2.6) and [15, Chapter XV, Section 14, Theorem 16], with the classical Hurwitz polynomials.

In our subsequent considerations we will use a matricial generalization of the construction of Markov parameters of a polynomial (see Gantmacher [15, Chapter XV, Section 15]). Our approach is based on the following observation.

Lemma 2.9. *Let $n \in \mathbb{N}$ and let $\mathbf{f}_n$ be a $q \times q$ matrix polynomial of degree n with invertible leading coefficient. Denote by $\mathbf{h}_n$ and $\mathbf{g}_n$ the even and odd parts of $\mathbf{f}_n$, respectively.*

(a) *Suppose $n = 2m$ for some $m \in \mathbb{N}$. Let $\eta_n := \max\{|z| \colon z \in \mathcal{N}_{\mathbf{h}_n}\}$. Then there exists a unique sequence $(\tilde{s}_j)_{j=0}^{\infty}$ from $\mathbb{C}^{q\times q}$ such that*

$$\frac{\mathbf{g}_n(z)}{\mathbf{h}_n(z)} = \sum_{j=0}^{\infty} (-1)^j z^{-(j+1)} \tilde{s}_j$$

for all $z \in \mathbb{C}$ with $|z| > \eta_n$.

(b) *Suppose $n = 2m + 1$ for some $m \in \mathbb{N}$. Let $\rho_n := \max\{|z| \colon z \in \mathcal{N}_{\mathbf{g}_n}\}$. Then there exists a unique sequence $(\tilde{s}_j)_{j=0}^{\infty}$ from $\mathbb{C}^{q\times q}$ such that*

$$\frac{\mathbf{h}_n(z)}{z\mathbf{g}_n(z)} = \sum_{j=0}^{\infty} (-1)^j z^{-(j+1)} \tilde{s}_j$$

for all $z \in \mathbb{C}$ with $|z| > \rho_n$.

Proof. This follows by observing that in view of (2.2) and (2.3) we have $\deg \mathbf{h}_n = m$ and $\deg \mathbf{g}_n = m - 1$ in the case $n = 2m$, and furthermore $\deg \mathbf{h}_n = m$ and $\deg \mathbf{g}_n = m$ in the case $n = 2m + 1$. $\square$

Lemma 2.9 leads us to the following notion which will be important for our subsequent considerations. What concerns the following definition we meet the same situation as in Definition 2.7. Namely, in the case of even n we have a direct generalization of the classical notion of Markov parameters whereas in the odd case we prefer a different construction.

Definition 2.10. Let $n \in \mathbb{N}$ and let $\mathbf{f}_n$ be a $q \times q$ matrix polynomial of degree n with invertible leading coefficient. Denote by $(\tilde{s}_j)_{j=0}^{\infty}$ the sequence from Lemma 2.9. Then $(\tilde{s}_j)_{j=0}^{n}$ is called the *sequence of Markov parameters of* $\mathbf{f}_n$. Furthermore, $(\tilde{s}_j)_{j=0}^{\infty}$ is called the *extended sequence of Markov parameters of* $\mathbf{f}_n$

3. On matricial power moment problems and several classes of sequences of matrices

Our subsequent considerations will be closely related to matricial power moment problems. In order to give precise formulations of the moment problems under consideration we introduce some terminology. Let Ω be a Borelian subset of the real axis $\mathbb{R}$. Further, let $q \in \mathbb{N}$, let $\mathfrak{B}_\Omega$ be the σ-algebra of all Borelian subsets of Ω, and let $\mathcal{M}^q_{\geq}(\Omega)$ be the set of all non-negative Hermitian $q \times q$ measures on $(\Omega, \mathfrak{B}_\Omega)$. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $\mathcal{M}^q_{\geq,\kappa}(\Omega)$ be the set of all $\sigma \in \mathcal{M}^q_{\geq}(\Omega)$ such that the integral

$$s_j^{(\sigma)} := \int_\Omega t^j \sigma(\mathrm{d}t)$$

exists for all non-negative integers $j \leq \kappa$. Initiated in the scalar case $q = 1$ by M.G. Krein [22] the following truncated matrix moment problem is studied:

$\mathsf{M}[\Omega; (s_j)_{j=0}^m, \leq]$ Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}^q_{\geq}[\Omega; (s_j)_{j=0}^m, \leq]$ of all $\sigma \in \mathcal{M}^q_{\geq,m}(\Omega)$ for which $s_m - s_m^{(\sigma)}$ is non-negative Hermitian and, in the case $m > 0$, moreover $s_j^{(\sigma)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0,m-1}$.

There are many interesting interrelations between moment problems on $[0, \infty)$ on the one side and on $\mathbb{R}$ on the other side. Obviously each $\sigma \in \mathcal{M}^q_{\geq}[[0, \infty); (s_j)_{j=0}^m, \leq]$ determines by a natural zero continuation a measure $\bar{\sigma} \in \mathcal{M}^q_{\geq}[\mathbb{R}; (s_j)_{j=0}^m, \leq]$. So the case $\Omega = \mathbb{R}$ stands always in the background.

To discuss the solvability of the above mentioned moment problems, we introduce some classes of sequences of matrices. Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q \times q}$.

Then $(s_j)_{j=0}^{2n}$ is called *Hankel non-negative definite* (resp. *Hankel positive definite*) if the block Hankel matrix

$$H_n := (s_{j+k})_{j,k=0}^{n}$$

is non-negative Hermitian (resp. positive Hermitian). Denote by $\mathcal{H}_{q,2n}^{\geq}$ (resp. $\mathcal{H}_{q,2n}^{>}$) the set of all Hankel non-negative definite (resp. Hankel positive definite) sequences $(s_j)_{j=0}^{2n}$ from $\mathbb{C}^{q\times q}$.

Remark 3.1. Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$ (resp. $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$). Then for $k \in \mathbb{Z}_{0,n-1}$ it is obvious that $(s_j)_{j=0}^{2k} \in \mathcal{H}_{q,2k}^{\geq}$ (resp. $(s_j)_{j=0}^{2k} \in \mathcal{H}_{q,2k}^{>}$).

Remark 3.1 leads us to the following notion. A sequence $(s_j)_{j=0}^{\infty}$ from $\mathbb{C}^{q\times q}$ is called *Hankel non-negative definite* (resp. *Hankel positive definite*) if for all $n \in \mathbb{N}_0$ the sequence $(s_j)_{j=0}^{2n}$ is Hankel non-negative definite (resp. Hankel positive definite). We denote be $\mathcal{H}_{q,\infty}^{\geq}$ (resp. $\mathcal{H}_{q,\infty}^{>}$) the set of all Hankel non-negative definite (resp. Hankel positive definite) sequences from $\mathbb{C}^{q\times q}$. The following result shows the importance of the set $\mathcal{H}_{q,2n}^{\geq}$.

Theorem 3.2. *(See [3, Theorem 3.2].) Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_{\geq}^{q}[\mathbb{R}; (s_j)_{j=0}^{2n}, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$.*

The present work is closely related to the non-degenerate situation of Problem $\mathsf{M}[[0,\infty); (s_j)_{j=0}^{m}, \leq]$. To describe the solvability of this problem we introduce further classes of sequences of complex matrices. Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n}$ is called *Stieltjes non-negative definite* (resp. *Stieltjes positive definite*) if $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$ and $(s_{j+1})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq}$ (resp. $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{>}$ and $(s_{j+1})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{>}$). Let $(s_j)_{j=0}^{2n+1}$ be a sequence from $\mathbb{C}^{q\times q}$. Then $(s_j)_{j=0}^{2n+1}$ is called *Stieltjes non-negative definite* (resp. *Stieltjes positive definite*) if $\{(s_j)_{j=0}^{2n}, (s_{j+1})_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^{\geq}$ (resp. $\{(s_j)_{j=0}^{2n}, (s_{j+1})_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^{>}$). For $m \in \mathbb{N}$ the symbol $\mathcal{K}_{q,m}^{\geq}$ (resp. $\mathcal{K}_{q,m}^{>}$) stands for the set of all Stieltjes non-negative definite (resp. Stieltjes positive definite) sequences $(s_j)_{j=0}^{m}$ from $\mathbb{C}^{q\times q}$.

Remark 3.3. Let $m \in \mathbb{N}$ and let $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m}^{\geq}$ (resp. $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m}^{>}$). Then for $\ell \in \mathbb{Z}_{0,m}$ it is obvious that $(s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell}^{\geq}$ (resp. $(s_j)_{j=0}^{\ell} \in \mathcal{K}_{q,\ell}^{>}$).

Remark 3.3 leads us to the following notion. A sequence $(s_j)_{j=0}^{\infty}$ from $\mathbb{C}^{q\times q}$ is called *Stieltjes non-negative definite* (resp. *Stieltjes positive definite*) if for all $m \in \mathbb{N}_0$ the sequence $(s_j)_{j=0}^{m}$ is Stieltjes non-negative definite (resp. Stieltjes positive definite). We denote be $\mathcal{K}_{q,\infty}^{\geq}$ (resp. $\mathcal{K}_{q,\infty}^{>}$) the set of all Stieltjes non-negative definite (resp. Stieltjes positive definite) sequences from $\mathbb{C}^{q\times q}$. Concerning a detailed treatment of the theory of Stieltjes non-negative definite sequences we refer the reader to Dyukarev/Fritzsche/Kirstein/Mädler [10], Fritzsche/Kirstein/Mädler [11,12].

The following result provides the reasoning why we will work at many places in this paper with infinite Stieltjes positive definite sequences instead of finite ones.

Proposition 3.4. *(See [11, Propositions 4.13 and 4.14].) Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^>$. Then there exists a sequence $(\tilde{s}_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty}^>$ such that $(\tilde{s}_j)_{j=0}^m = (s_j)_{j=0}^m$.*

The following result shows the importance of the set $\mathcal{K}_{q,m}^{\geq}$.

Theorem 3.5. *(Cf. [10, Theorem 1.4].) Let $m \in \mathbb{N}$ and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_{\geq}^q[[0,\infty); (s_j)_{j=0}^m, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^{\geq}$.*

Remark 3.6. Let $m \in \mathbb{N}$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^>$. Then Theorem 3.5 shows that $\mathcal{M}_{\geq}^q[[0,\infty); (s_j)_{j=0}^m, \leq] \neq \emptyset$.

For a detailed description of the work on matrix versions of the Stieltjes moment problem we refer the reader to [12] and the references therein. This paper is intimately related to several aspects of the author's recent investigations [4] on Problem $\mathsf{M}[[0,\infty); (s_j)_{j=0}^m, \leq]$ in the case $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^>$. First we explain that, following the classical line, we did not study the original moment problem but an equivalent problem for holomorphic $q \times q$ matrix-valued functions in $\mathbb{C} \setminus [0,\infty)$.

Definition 3.7. Let $\sigma \in \mathcal{M}_{\geq}^q([0,\infty))$. Then the function $G_\sigma: \mathbb{C} \setminus [0,\infty) \to \mathbb{C}^{q\times q}$ defined by

$$G_\sigma(z) := \int_{[0,\infty)} \frac{1}{t-z} \sigma(\mathrm{d}t)$$

is called *Stieltjes transform of* σ.

In view of a matrix version of the Stieltjes–Perron inversion formula (see [5, Theorem 8.2]) a measure $\sigma \in \mathcal{M}_{\geq}^q([0,\infty))$ is uniquely determined by its Stieltjes transform. For this reason the solution set of the Stieltjes moment problem can be described in terms of Stieltjes transform s. In this way, Stieltjes [29] already handled the classical case.

In the case $(s_j)_{j=0}^m \in \mathcal{K}_{q,m}^>$ the set of all Stieltjes transforms of the measures belonging to the solution set of Problem $\mathsf{M}[[0,\infty); (s_j)_{j=0}^m, \leq]$ was parametrized by Yu.M. Dyukarev [9] with the aid of a linear fractional transformation (see also [4, Theorem 3.2]). Of particular importance for our subsequent considerations is the fact that the generating matrix-valued polynomial of this linear fractional transformation was expressed in [4, Theorem 4.6] in terms of a quadruple of $q \times q$ matrix polynomials with particular orthogonality properties.

In our subsequent considerations we will sometimes meet compactly supported matrix measures on $\mathbb{R}$. In this case the Stieltjes transform has a particularly simple shape.

Lemma 3.8. *Let $a, b \in \mathbb{R}$ with $a < b$ and let $\sigma \in \mathcal{M}^q_{\geq}([a,b])$. Then $\sigma \in \mathcal{M}^q_{\geq,\infty}([a,b])$ and the function $S_\sigma : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q\times q}$ given by*

$$S_\sigma(z) := \int\limits_{[a,b]} \frac{1}{t-z}\sigma(\mathrm{d}t)$$

admits for all $z \in \mathbb{C}$ with $|z| > \max\{|a|,|b|\}$ the representation

$$S_\sigma(z) = -\sum_{j=0}^{\infty} z^{-(j+1)} s_j^{(\sigma)}.$$

Proof. If $z \in \mathbb{C}$ satisfies $|z| > \max\{|a|,|b|\}$ then

$$\frac{1}{t-z} = -\frac{1}{z(1-t/z)} = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{t}{z}\right)^k = -\sum_{k=0}^{\infty} t^k z^{-(k+1)}$$

for all $t \in [a,b]$. Now changing integration and summation yields the assertion. □

4. The Dyukarev–Stieltjes parametrization of Stieltjes positive definite sequences

In this section we recall the Dyukarev–Stieltjes parametrization of sequences $(s_j)_{j=0}^{\infty} \in \mathcal{K}^{>}_{q,\infty}$ which was introduced in Fritzsche/Kirstein/Mädler [12, Section 8] inspired by Yu.M. Dyukarev's paper [9]. For a complex $p \times q$ matrix A, let A^* be the conjugate transpose of A and let $A^\dagger$ be the Moore–Penrose inverse of A, i.e., the unique matrix $X \in \mathbb{C}^{q\times p}$ which satisfies the four equations $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$. If $A \in \mathbb{C}^{q\times q}$, then let $\det A$ be the determinant of A. For a complex $q \times q$ matrix A with $\det A \neq 0$ let A^{-1} be the inverse of A. In this case, $A^\dagger = A^{-1}$.

Let $(s_j)_{j=0}^{\infty}$ be a sequence from $\mathbb{C}^{p\times q}$. First we introduce some sequences of matrices associated with $(s_j)_{j=0}^{\infty}$. Let

$$H_{1,j} := \begin{pmatrix} s_0 & s_1 & \dots & s_j \\ s_1 & s_2 & \dots & s_{j+1} \\ \vdots & \vdots & & \vdots \\ s_j & s_{j+1} & \dots & s_{2j} \end{pmatrix}, \qquad j \in \mathbb{N}_0, \tag{4.1}$$

$$H_{2,j} := \begin{pmatrix} s_1 & s_2 & \dots & s_{j+1} \\ s_2 & s_3 & \dots & s_{j+2} \\ \vdots & \vdots & & \vdots \\ s_{j+1} & s_{j+2} & \dots & s_{2j+1} \end{pmatrix}, \qquad j \in \mathbb{N}_0, \tag{4.2}$$

$$Y_{1,j} := \begin{pmatrix} s_j \\ s_{j+1} \\ \vdots \\ s_{2j-1} \end{pmatrix}, \qquad Z_{1,j} := (s_j, s_{j+1}, \dots, s_{2j-1}), \quad j \in \mathbb{N}, \tag{4.3}$$

$$Y_{2,j} := \begin{pmatrix} s_{j+1} \\ s_{j+2} \\ \vdots \\ s_{2j} \end{pmatrix}, \qquad Z_{2,j} := (s_{j+1}, s_{j+2}, \dots, s_{2j}), \quad j \in \mathbb{N}. \tag{4.4}$$

Furthermore, for $j \in \mathbb{N}_0$ we set

$$\widehat{H}_{1,j} := \begin{cases} s_0, & \text{if } j = 0 \\ s_{2j} - Z_{1,j} H^{\dagger}_{1,j-1} Y_{1,j}, & \text{if } j \geq 1 \end{cases}, \tag{4.5}$$

$$\widehat{H}_{2,j} := \begin{cases} s_1, & \text{if } j = 0 \\ s_{2j+1} - Z_{2,j} H^{\dagger}_{2,j-1} Y_{2,j}, & \text{if } j \geq 1 \end{cases}. \tag{4.6}$$

In other words, if $j \in \mathbb{N}$ then $\widehat{H}_{1,j}$ (resp. $\widehat{H}_{2,j}$) is the Schur complement of s_{2j} (resp. s_{2j+1}) in $H_{1,j}$ (resp. $H_{2,j}$). Now we summarize some basic properties of Hankel positive definite sequences which are mostly taken from [13, Section 3].

Remark 4.1. Let $(s_j)_{j=0}^{\infty} \in \mathcal{H}^{>}_{q,\infty}$. Then:

(a) If $j \in \mathbb{N}_0$, then $s_{2j} \in \mathbb{C}^{q\times q}_{>}$ and $s_{2j+1} \in \mathbb{C}^{q\times q}_{\mathrm{H}}$.
(b) If $j \in \mathbb{N}_0$, then $H_{1,j} \in \mathbb{C}^{(j+1)q\times(j+1)q}_{>}$ (and in particular $\det H_{1,j} \in (0, +\infty)$).
(c) If $j \in \mathbb{N}_0$, then $\widehat{H}_{1,j} \in \mathbb{C}^{q\times q}_{>}$ (and in particular $\det \widehat{H}_{1,j} \in (0, +\infty)$).

From Remark 4.1 and the definition of the set $\mathcal{K}^{>}_{q,\infty}$ we immediately obtain the following observations.

Remark 4.2. Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}^{>}_{q,\infty}$. Then:

(a) If $j \in \mathbb{N}_0$, then $s_j \in \mathbb{C}^{q\times q}_{>}$.
(b) If $k \in \{1, 2\}$ and $j \in \mathbb{N}_0$, then $H_{k,j} \in \mathbb{C}^{(j+1)q\times(j+1)q}_{>}$ (and in particular $\det H_{k,j} \in (0, +\infty)$).
(c) If $k \in \{1, 2\}$ and $j \in \mathbb{N}_0$, then $\widehat{H}_{k,j} \in \mathbb{C}^{q\times q}_{>}$ (and in particular $\det \widehat{H}_{k,j} \in (0, +\infty)$).

Let

$$v_0 := I_q, \qquad v_k := \begin{pmatrix} I_q \\ 0_{kq\times q} \end{pmatrix}, \qquad k \in \mathbb{N}. \tag{4.7}$$

Let $(s_j)_{j=0}^{\infty}$ be a sequence from $\mathbb{C}^{p\times q}$. Let $j, k \in \mathbb{N}_0$. Then, we set

$$y_{[j,k]} := \begin{cases} 0_{q\times q}, & \text{if } j > k \\ \begin{pmatrix} s_j \\ s_{j+1} \\ \vdots \\ s_k \end{pmatrix}, & \text{if } j \leq k \end{cases} \tag{4.8}$$

and

$$z_{[j,k]} := \begin{cases} 0_{q\times q}, & \text{if } j > k \\ (s_j, s_{j+1}, \dots, s_k), & \text{if } j \le k \end{cases}. \tag{4.9}$$

The following construction goes back to Yu.M. Dyukarev [9, p. 77].

Definition 4.3. (See [12, Definition 8.2].) Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ and let

$$\mathbf{M}_k := \begin{cases} s_0^{-1}, & \text{if } k = 0 \\ v_k^* H_{1,k}^{-1} v_k - v_{k-1}^* H_{1,k-1}^{-1} v_{k-1}, & \text{if } k \ge 1 \end{cases}$$

and

$$\mathbf{L}_k := \begin{cases} s_0 s_1^{-1} s_0, & \text{if } k = 0 \\ y_{[0,k]}^* H_{2,k}^{-1} y_{[0,k]} - y_{[0,k-1]}^* H_{2,k-1}^{-1} y_{[0,k-1]}, & \text{if } k \ge 1 \end{cases}$$

for all $k \in \mathbb{N}_0$. Then the ordered pair $[(\mathbf{L}_k)_{k=0}^{\infty}, (\mathbf{M}_k)_{k=0}^{\infty}]$ is called the *Dyukarev–Stieltjes parametrization* (shortly *DS-parametrization*) *of* $(s_j)_{j=0}^{\infty}$.

Remark 4.4. (See [12, Remark 8.23].) Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ with DS-parametrization $[(\mathbf{L}_k)_{k=0}^{\infty}, (\mathbf{M}_k)_{k=0}^{\infty}]$. Then, in view of [9, Theorem 7], the matrices $\mathbf{L}_k$ and $\mathbf{M}_k$ are positive Hermitian and, in particular, invertible for all $k \in \mathbb{N}_0$.

Remark 4.5. (See [12, Remark 8.3].) Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ with DS-parametrization $[(\mathbf{L}_k)_{k=0}^{\infty}, (\mathbf{M}_k)_{k=0}^{\infty}]$. In view of Definition 4.3, (4.7), (4.1), (4.8), and (4.2), we can easily see then, that, for all $k \in \mathbb{N}_0$, the matrix $\mathbf{M}_k$ only depends on the matrices $s_0, \dots, s_{2k}$ and that the matrix $\mathbf{L}_k$ only depends on the matrices $s_0, s_1, \dots, s_{2k+1}$.

Against to the background of Remark 4.5 we see that the notion “DS-parametrization” could also be analogously introduced for finite Stieltjes positive definite sequences. Remark 4.5 leads us to the following notion.

Definition 4.6. Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{n} \in \mathcal{K}_{q,n}^{>}$. Let $(\tilde{s}_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ be chosen such that $(\tilde{s}_j)_{j=0}^{n} = (s_j)_{j=0}^{n}$ is satisfied and let $[(\mathbf{L}_k)_{k=0}^{\infty}, (\mathbf{M}_k)_{k=0}^{\infty}]$ be the DS-parametrization of $(\tilde{s}_j)_{j=0}^{\infty}$. Let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m-1$ is satisfied. Then the ordered pairs $[(\mathbf{L}_k)_{k=0}^{m-1}, (\mathbf{M}_k)_{k=0}^{m}]$ and $[(\mathbf{L}_k)_{k=0}^{m-1}, (\mathbf{M}_k)_{k=0}^{m-1}]$ are called the *DS-parametrization of* $(s_j)_{j=0}^{2m}$ and $(s_j)_{j=0}^{2m-1}$, respectively.

Remark 4.7. Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{n} \in \mathcal{K}_{q,n}^{>}$. Let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m-1$ is satisfied. Denote by $[(\mathbf{L}_k)_{k=0}^{m-1}, (\mathbf{M}_k)_{k=0}^{m}]$ (resp. $[(\mathbf{L}_k)_{k=0}^{m-1}, (\mathbf{M}_k)_{k=0}^{m-1}]$) the DS-parametrization of $(s_j)_{j=0}^{n}$. Then, in view of Definition 4.6 and Remark 4.4, the matrices $\mathbf{L}_k$ and $\mathbf{M}_\ell$ are positive Hermitian and, in particular, invertible for all $k \in \mathbb{Z}_{0,m-1}$ and all $\ell \in \mathbb{Z}_{0,m}$ (resp. $\ell \in \mathbb{Z}_{0,m-1}$).

The following proposition plays a key role in the proof of Theorem 7.9 which is the second main result of present work. This proposition is a slightly modified version of the Fritzsche/Kirstein/Mädler's Proposition 8.27 in [12], which considers an infinite sequence of pairs $[(\mathbf{L}_k)_{k=0}^{\infty}, (\mathbf{M}_k)_{k=0}^{\infty}]$ instead of an analogue finite sequence. Our proof differs from the one in [12] since it is based on part d) of Theorem 4.12 in [11] which in turn uses the rank of the matrices $H_{1,j}$ and $H_{2,j}$. Let us first recall the following well-known result (see e.g. [7, Lemma 1.1.9] or [1, Proposition 8.2.4]).

Lemma 4.8. *Let $A := \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$ be a Hermitian $(n+m)\times(n+m)$ matrix with $n\times n$ block A_{11}. Then the following statements are equivalent:*

(i) $A \in \mathbb{C}_{>}^{(n+m)\times(n+m)}$.
(ii) $A_{11} \in \mathbb{C}_{>}^{n\times n}$ *and* $A_{22} - A_{12}^* A_{11}^{-1} A_{12} \in \mathbb{C}_{>}^{m\times m}$.
(iii) $A_{22} \in \mathbb{C}_{>}^{m\times m}$ *and* $A_{11} - A_{12} A_{22}^{-1} A_{12}^* \in \mathbb{C}_{>}^{n\times n}$.

Proposition 4.9. *Let $m \in \mathbb{N}$ and let $(\mathbf{L}_j)_{j=0}^{m-1}$ and $(\mathbf{M}_j)_{j=0}^{m}$ (resp. $(\mathbf{L}_j)_{j=0}^{m}$ and $(\mathbf{M}_j)_{j=0}^{m}$) be two sequences of positive Hermitian complex $q\times q$ matrices. Let the sequence $(s_j)_{j=0}^{2m}$ (resp. $(s_j)_{j=0}^{2m+1}$) be recursively defined by*

$$s_{2j} := \begin{cases} \mathbf{M}_0^{-1}, & \text{if } j=0 \\ Y_{1,j}^* H_{1,j-1}^{-1} Y_{1,j} + (\overrightarrow{\prod}_{k=0}^{j-1} \mathbf{M}_k\mathbf{L}_k)^{-*}\mathbf{M}_j^{-1}(\overrightarrow{\prod}_{k=0}^{j-1} \mathbf{M}_k\mathbf{L}_k)^{-1}, & \text{if } j\geq 1 \end{cases} \tag{4.10}$$

and

$$s_{2j+1} := \begin{cases} (\mathbf{M}_0\mathbf{L}_0)^{-*}\mathbf{L}_0(\mathbf{M}_0\mathbf{L}_0)^{-1}, & \text{if } j=0 \\ Y_{2,j}^* H_{2,j-1}^{-1} Y_{2,j} + (\overrightarrow{\prod}_{k=0}^{j} \mathbf{M}_k\mathbf{L}_k)^{-*}\mathbf{L}_j(\overrightarrow{\prod}_{k=0}^{j} \mathbf{M}_k\mathbf{L}_k)^{-1}, & \text{if } j\geq 1 \end{cases} \tag{4.11}$$

for $j=0,\dots,2m$ (resp. $j=0,\dots,2m+1$). Then $(s_j)_{j=0}^{2m}$ (resp. $(s_j)_{j=0}^{2m+1}$) is a Stieltjes positive definite sequence.

Proof. From (4.10) and (4.5), we obtain

$$\widehat{H}_{1,0} = \mathbf{M}_0^{-1}, \qquad \widehat{H}_{1,j} = \left(\overrightarrow{\prod_{k=0}^{j-1}} \mathbf{M}_k\mathbf{L}_k\right)^{-*} \mathbf{M}_j^{-1} \left(\overrightarrow{\prod_{k=0}^{j-1}} \mathbf{M}_k\mathbf{L}_k\right)^{-1}. \tag{4.12}$$

Respectively from (4.11) and (4.6) we get

$$\widehat{H}_{2,0} = \mathbf{M}_0^{-1}\mathbf{L}_0^{-1}\mathbf{M}_0^{-1}, \qquad \widehat{H}_{2,j} = \left(\overrightarrow{\prod_{k=0}^{j}} \mathbf{M}_k\mathbf{L}_k\right)^{-*} \mathbf{L}_j \left(\overrightarrow{\prod_{k=0}^{j}} \mathbf{M}_k\mathbf{L}_k\right)^{-1}. \tag{4.13}$$

Since $\mathbf{L}_j \in \mathbb{C}_>^{q\times q}$ and $\mathbf{M}_j \in \mathbb{C}_>^{q\times q}$ for all j, then

$$\mathbf{M}_0^{-1} \in \mathbb{C}_>^{q\times q}, \qquad \left(\overrightarrow{\prod_{k=0}^{j-1}} \mathbf{M}_k \mathbf{L}_k\right)^{-*} \mathbf{M}_j^{-1} \left(\overrightarrow{\prod_{k=0}^{j-1}} \mathbf{M}_k \mathbf{L}_k\right)^{-1} \in \mathbb{C}_>^{q\times q}, \tag{4.14}$$

$$\mathbf{M}_0^{-1}\mathbf{L}_0^{-1}\mathbf{M}_0^{-1} \in \mathbb{C}_>^{q\times q}, \qquad \left(\overrightarrow{\prod_{k=0}^{j}} \mathbf{M}_k \mathbf{L}_k\right)^{-*} \mathbf{L}_j \left(\overrightarrow{\prod_{k=0}^{j}} \mathbf{M}_k \mathbf{L}_k\right)^{-1} \in \mathbb{C}_>^{q\times q}. \tag{4.15}$$

By (4.12), (4.14) (resp. (4.13), (4.15)), we have

$$\widehat{H}_{1,j} \in \mathbb{C}_>^{q\times q}, \qquad \text{for all } j \in \mathbb{N}_0, \tag{4.16}$$

$$\widehat{H}_{2,j} \in \mathbb{C}_>^{q\times q}, \qquad \text{for all } j \in \mathbb{N}_0. \tag{4.17}$$

Let

$$H_{1,j} = \begin{pmatrix} H_{1,j-1} & Y_{1,j} \\ Y_{1,j}^* & s_{2j} \end{pmatrix}, \qquad H_{2,j-1} = \begin{pmatrix} H_{2,j-2} & Y_{2,j-1} \\ Y_{2,j-1}^* & s_{2j-1} \end{pmatrix}. \tag{4.18}$$

By using (4.16), (4.18) (resp. (4.17), (4.18)) and Lemma 4.8, we obtain $H_{1,m}$ which is positive Hermitian; (resp. $H_{2,m}$ is positive Hermitian). Consequently, the sequence $(s_j)_{j=0}^{2m}$ is a Stieltjes positive definite sequence. □

Note that equalities (4.12) and (4.13) were first proved in a different way in [4, Corollary 4.10] under the condition that $(s_j)_{j=0}^{m}$ for $m = 2n$ and $m = 2n + 1$ is Stieltjes positive definite. This result, proved using a different approach, later appears in part (a) of Proposition 8.28 in [12].

5. On the Stieltjes quadruple of sequences of left orthogonal matrix polynomials associated with a sequence $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$

Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$. Then we will recall the Stieltjes quadruple of sequences of left orthogonal matrix polynomials (or short SQSLOMP) associated with $(s_j)_{j=0}^{\infty}$. This notion originates in [4, Definition 4.1]. For this reason, first we introduce some notations. Let

$$T_0 := 0_{q\times q}, \qquad T_j := \begin{pmatrix} 0_{q\times jq} & 0_{q\times q} \\ I_{jq} & 0_{jq\times q} \end{pmatrix}, \qquad j \in \mathbb{N}. \tag{5.1}$$

From (5.1) it follows immediately that for $z \in \mathbb{C}$ we have $\det(I_{(j+1)q} - zT_j) = 1$. Thus, the function $R_j \colon \mathbb{C} \to \mathbb{C}^{(j+1)q\times(j+1)q}$ given by

$$R_j(z) := (I_{(j+1)q} - zT_j)^{-1} \tag{5.2}$$

is well-defined. Observe, that the matrix-valued function R_j is given for $z \in \mathbb{C}$ via

$$R_j(z) = \begin{pmatrix} I_q & 0_{q\times q} & 0_{q\times q} & \dots & 0_{q\times q} & 0_{q\times q} \\ zI_q & I_q & 0_{q\times q} & \dots & 0_{q\times q} & 0_{q\times q} \\ z^2 I_q & zI_q & I_q & \dots & 0_{q\times q} & 0_{q\times q} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z^j I_q & z^{j-1} I_q & z^{j-2} I_q & \dots & zI_q & I_q \end{pmatrix}. \tag{5.3}$$

Let $(s_j)_{j=0}^{\infty}$ be a sequence from $\mathbb{C}^{q\times q}$. Using (4.8) we set

$$u_{1,0} := 0_{q\times q}, \qquad u_{1,j} := \begin{pmatrix} 0_{q\times q} \\ -y_{[0,j-1]} \end{pmatrix}, \qquad j \in \mathbb{N}, \tag{5.4}$$

and

$$u_{2,j} := -y_{[0,j]}, \qquad j \in \mathbb{N}. \tag{5.5}$$

Definition 5.1. Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$. Let $z \in \mathbb{C}$. Let

$$P_{1,0}(z) := I_q, \qquad Q_{1,0}(z) := 0_{q\times q}, \qquad P_{2,0}(z) := I_q, \qquad Q_{2,0}(z) := s_0. \tag{5.6}$$

For $j \in \mathbb{N}$, let

$$P_{1,j}(z) := (-Y_{1,j}^* H_{1,j-1}^{-1}, I_q) R_j(z) v_j, \tag{5.7}$$

$$Q_{1,j}(z) := -(-Y_{1,j}^* H_{1,j-1}^{-1}, I_q) R_j(z) u_{1,j}, \tag{5.8}$$

$$P_{2,j}(z) := (-Y_{2,j}^* H_{2,j-1}^{-1}, I_q) R_j(z) v_j, \tag{5.9}$$

$$Q_{2,j}(z) := -(-Y_{2,j}^* H_{2,j-1}^{-1}, I_q) R_j(z) u_{2,j}. \tag{5.10}$$

Then $[(P_{1,j})_{j=0}^{\infty}, (Q_{1,j})_{j=0}^{\infty}, (P_{2,j})_{j=0}^{\infty}, (Q_{2,j})_{j=0}^{\infty}]$ is called the *Stieltjes quadruple of sequences of left orthogonal matrix polynomials* (or short SQSLOMP) *associated with* $(s_j)_{j=0}^{\infty}$.

The reasoning for the terminology chosen in Definition 5.1 was given in [4, Proposition 4.2]. From Definition 5.1 we immediately see the following observation.

Remark 5.2. Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ and let $[(P_{1,j})_{j=0}^{\infty}, (Q_{1,j})_{j=0}^{\infty}, (P_{2,j})_{j=0}^{\infty}, (Q_{2,j})_{j=0}^{\infty}]$ be the associated SQSLOMP.

(a) For all $j \in \mathbb{N}_0$, the matrix polynomials $P_{1,j}$ and $P_{2,j}$ are monic of degree j.
(b) For all $j \in \mathbb{N}_0$, the matrix polynomials $Q_{1,j+1}$ and $Q_{2,j}$ are of degree j.

Remark 5.3. Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{1,\infty}^{>}$ and let $[(P_{1,j})_{j=0}^{\infty}, (Q_{1,j})_{j=0}^{\infty}, (P_{2,j})_{j=0}^{\infty}, (Q_{2,j})_{j=0}^{\infty}]$ be the associated SQSLOMP. Then using part (a) of Remark 4.2 it is immediately seen that for $k \in \{1,2\}$ and $j \in \mathbb{N}_0$ the polynomials $P_{k,j}$ and $Q_{k,j}$ have real coefficients.

The role of the SQSLOMP associated with a sequence $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ was detailed studied in [4]. These polynomials have particular orthogonality properties (see [4, Proposition 4.2]) and can be used to describe the Stieltjes transforms of the measures of the solution sets $\mathcal{M}_{\geq}^{q}[[0,\infty); (s_j)_{j=0}^{m}, \leq]$ for all $m \in \mathbb{N}$ (see [4, Theorem 4.7]).

In the case $(s_j)_{j=0}^{m} \in \mathcal{K}_{q,m}^{>}$ the set $\mathcal{M}_{\geq}^{q}[[0,\infty); (s_j)_{j=0}^{m}, \leq]$ contains two distinguished elements $\sigma_{m,\min}$ and $\sigma_{m,\max}$ (see [4, formulas (3.12) and (3.13)] for the Stieltjes transforms of these measures). These Stieltjes transforms admit particular matrix continued fraction expansions, which will be recalled now. The following result establishes the bridge between [4] and the present paper.

Theorem 5.4. *(Cf. [4, Theorems 3.4 and 4.8].) Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ with DS-parametrization $[(\mathbf{L}_k)_{k=0}^{\infty}, (\mathbf{M}_k)_{k=0}^{\infty}]$. Denote by $[(P_{1,j})_{j=0}^{\infty}, (Q_{1,j})_{j=0}^{\infty}, (P_{2,j})_{j=0}^{\infty}, (Q_{2,j})_{j=0}^{\infty}]$ the SQSLOMP associated with $(s_j)_{j=0}^{\infty}$. Further let $n \in \mathbb{N}_0$. For $z \in \mathbb{C} \setminus [0,\infty)$, then*

$$G_{\sigma_{2n,\min}}(z) = -\frac{[Q_{2,n}(\bar{z})]^*}{z[P_{2,n}(\bar{z})]^*} = \cfrac{I_q}{-z\mathbf{M}_0 + \cfrac{I_q}{\mathbf{L}_0 + \cfrac{I_q}{\ddots + \cfrac{}{-z\mathbf{M}_{n-1} + \cfrac{I_q}{\mathbf{L}_{n-1} - z^{-1}\mathbf{M}_n^{-1}}}}}}$$

and

$$G_{\sigma_{2n,\max}}(z) = -\frac{[Q_{1,n}(\bar{z})]^*}{[P_{1,n}(\bar{z})]^*} = \cfrac{I_q}{-z\mathbf{M}_0 + \cfrac{I_q}{\mathbf{L}_0 + \cfrac{I_q}{\ddots + \cfrac{}{\mathbf{L}_{n-2} + \cfrac{I_q}{-z\mathbf{M}_{n-1} + \mathbf{L}_{n-1}^{-1}}}}}}.$$

A closer look at Theorem 5.4 shows now that a Stieltjes positive definite sequence produces matrix continued fractions of the type occurring in the construction of matrix Hurwitz type polynomials (see Definition 2.7). Furthermore, Theorem 5.4 indicates why we have chosen in the definitions of matrix Hurwitz type polynomials (see Definition 2.7) and of Markov parameters (see Definition 2.10) a version which slightly differs from the classical version. This is caused by the structure of the Stieltjes transform of the extremal solution $\sigma_{n,\min}$ of the non-degenerate truncated matricial Stieltjes moment problem.

6. Construction of matrix Hurwitz type polynomials on the basis of Stieltjes positive definite sequences

The direction of the investigations of this section is determined by Theorem 5.4 which provides us a principle of constructing matrix Hurwitz type polynomials with the aid of a Stieltjes positive definite sequence.

Theorem 6.1. *Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ with DS-parametrization $[(\mathbf{L}_k)_{k=0}^{\infty}, (\mathbf{M}_k)_{k=0}^{\infty}]$. Denote by $[(P_{1,j})_{j=0}^{\infty}, (Q_{1,j})_{j=0}^{\infty}, (P_{2,j})_{j=0}^{\infty}, (Q_{2,j})_{j=0}^{\infty}]$ the SQSLOMP associated with $(s_j)_{j=0}^{\infty}$. Let $m \in \mathbb{N}$.*

(a) *Let $\mathbf{f}_{2m}: \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by*

$$\mathbf{f}_{2m}(z) := (-1)^m \left[P_{1,m}(-\bar{z}^2)\right]^* + z(-1)^{m+1} \left[Q_{1,m}(-\bar{z}^2)\right]^* .$$

- (a1) *$\mathbf{f}_{2m}$ is a MHTP of degree $2m$.*
- (a2) *$[(\mathbf{M}_k)_{k=0}^{m-1}, (\mathbf{L}_k)_{k=0}^{m-1}]$ is a Hurwitz parametrization of $\mathbf{f}_{2m}$.*
- (a3) *Denote by $(s_j)_{j=0}^{\infty}$ the extended sequence of Markov parameters of $\mathbf{f}_{2m}$. Then $s_j = \tilde{s}_j$ for all $j \in \mathbb{Z}_{0,2m-1}$.*

(b) *Let $\mathbf{f}_{2m+1}: \mathbb{C} \to \mathbb{C}^{q\times q}$ be defined by*

$$\mathbf{f}_{2m+1}(z) := (-1)^m \left[Q_{2,m}(-\bar{z}^2)\right]^* + z(-1)^m \left[P_{2,m}(-\bar{z}^2)\right]^* .$$

- (b1) *$\mathbf{f}_{2m+1}$ is a MHTP of degree $2m+1$.*
- (b2) *$[(\mathbf{M}_k)_{k=0}^{m}, (\mathbf{L}_k)_{k=0}^{m-1}]$ is a Hurwitz parametrization of $\mathbf{f}_{2m+1}$.*
- (b3) *Denote by $(\tilde{s}_j)_{j=0}^{\infty}$ the extended sequence of Markov parameters of $\mathbf{f}_{2m+1}$. Then $s_j = \tilde{s}_j$ for all $j \in \mathbb{Z}_{0,2m}$.*

Proof. (a) From Remark 5.2 we can conclude that $\mathbf{f}_{2m}$ is a monic $q \times q$ matrix polynomial of degree $2m$. Furthermore, in view of Remark 5.2 and part (a) of Remark 2.4, the even and odd parts $\mathbf{h}_{2m}$ and $\mathbf{g}_{2m}$ of $\mathbf{f}_{2m}$ fulfill $\mathbf{h}_{2m}(z) = (-1)^m [P_{1,m}(-\bar{z})]^*$ and $\mathbf{g}_{2m}(z) = (-1)^{m+1}[Q_{1,m}(-\bar{z})]^*$ for all $z \in \mathbb{C}$. Taking additionally into account Theorem 5.4, we see that $\mathbf{f}_{2m}$ is a MHTP of degree $2m$ and that $[(\mathbf{M}_k)_{k=0}^{m-1}, (\mathbf{L}_k)_{k=0}^{m-1}]$ is a Hurwitz parametrization of $\mathbf{f}_{2m}$. In view of Theorem 5.4 and Definitions 2.7 and 2.10, we can conclude

$$G_{\sigma_{2m,\max}}(z) = -\frac{[Q_{1,m}(\bar{z})]^*}{[P_{1,m}(\bar{z})]^*} = \frac{\mathbf{g}_{2m}(-z)}{\mathbf{h}_{2m}(-z)} = -\sum_{j=0}^{\infty} z^{-(j+1)} \tilde{s}_j$$

for all $z \in \mathbb{C} \setminus [0, \infty)$ with $|z| > \eta_{2m}$, where $\eta_n := \max\{|z|: z \in \mathcal{N}_{\mathbf{h}_n}\}$. Since $\sigma_{2m,\max}$ belongs to $\mathcal{M}_{\geq}^{q}[[0,\infty); (s_j)_{j=0}^{2m}, \leq]$ and is concentrated on a finite set of points and hence compactly supported, we can via Lemma 3.8 conclude $s_j = \tilde{s}_j$ for all $j \in \mathbb{Z}_{0,2m-1}$.

(b) From Remark 5.2 we can conclude that $\mathbf{f}_{2m+1}$ is a monic $q \times q$ matrix polynomial of degree $2m+1$. Furthermore, in view of Remark 5.2 and part (b) of Remark 2.4, the even and odd part s $\mathbf{h}_{2m+1}$ and $\mathbf{g}_{2m+1}$ of $\mathbf{f}_{2m+1}$ fulfill $\mathbf{h}_{2m+1}(z) = (-1)^m [Q_{2,m}(-\bar{z})]^*$ and $\mathbf{g}_{2m+1}(z) = (-1)^m [P_{2,m}(-\bar{z})]^*$ for all $z \in \mathbb{C}$. Taking additionally into account Theorem 5.4, we see that $\mathbf{f}_{2m+1}$ is a MHTP of degree $2m+1$ and that $[(\mathbf{M}_k)_{k=0}^m, (\mathbf{L}_k)_{k=0}^{m-1}]$ is a Hurwitz parametrization of $\mathbf{f}_{2m+1}$. In view of Theorem 5.4 and Definitions 2.7 and 2.10, we can conclude

$$G_{\sigma_{2m,\min}}(z) = -\frac{[Q_{2,m}(\bar{z})]^*}{z[P_{2,m}(\bar{z})]^*} = \frac{\mathbf{h}_{2m+1}(-z)}{(-z)\mathbf{g}_{2m+1}(-z)} = -\sum_{j=0}^{\infty} z^{-(j+1)} \tilde{s}_j$$

for all $z \in \mathbb{C} \setminus [0, \infty)$ with $|z| > \rho_{2m+1}$, where $\rho_{2m+1} := \max\{|z| : z \in \mathcal{N}_{\mathbf{g}_{2m+1}}\}$. Since $\sigma_{2m,\min}$ belongs to $\mathcal{M}^q_{\geq}[[0,\infty); (s_j)_{j=0}^{2m+1}, \leq]$ and is concentrated on a finite set of points and hence compactly supported, we can via Lemma 3.8 conclude $s_j = \tilde{s}_j$ for all $j \in \mathbb{Z}_{0,2m}$. $\square$

Theorem 6.1 leads us to the following notion.

Definition 6.2. Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}^{>}_{q,\infty}$, let $n \in \{2, 3, \ldots\}$, and let $\mathbf{f}_n$ be defined as in Theorem 6.1. Then $\mathbf{f}_n$ is called the *MHTP of degree n associated with* $(s_j)_{j=0}^{\infty}$.

Remark 6.3. Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}^{>}_{q,\infty}$, let $n \in \{2, 3, \ldots\}$, and let $\mathbf{f}_n$ be the MHTP of degree n associated with $(s_j)_{j=0}^{\infty}$. In view of (5.7)–(5.10), then $\mathbf{f}_n$ only depends on $(s_j)_{j=0}^{n-1}$.

Remark 6.3 leads us to the following notion.

Definition 6.4. Let $n \in \{2, 3, \ldots\}$ and let $(s_j)_{j=0}^{n-1} \in \mathcal{K}^{>}_{q,n-1}$. Let $(\tilde{s}_j)_{j=0}^{\infty} \in \mathcal{K}^{>}_{q,\infty}$ be chosen such that $(\tilde{s}_j)_{j=0}^{n-1} = (s_j)_{j=0}^{n-1}$. Then by the *MHTP of degree n associated with* $(s_j)_{j=0}^{n-1}$ we mean the MHTP of degree n associated with $(\tilde{s}_j)_{j=0}^{\infty}$.

7. On associating a Stieltjes positive definite sequence $(s_j)_{j=0}^{n-1}$ with a matrix Hurwitz type polynomial of degree n

Definition 6.4 leads us to the following inverse problem. Let $n \in \{2, 3, \ldots\}$ and let $\mathbf{f}_n$ be a matrix Hurwitz type polynomial of degree n. Then we want to show that there exists a unique sequence $(s_j)_{j=0}^{n-1} \in \mathcal{K}^{>}_{q,n-1}$ such that $\mathbf{f}_n$ coincides with the matrix Hurwitz type polynomial of degree n associated with $(s_j)_{j=0}^{n-1}$. We will show that the sequence $(s_j)_{j=0}^{n-1}$ is given by the sequence of Markov parameters of $\mathbf{f}_n$.

First we introduce some notations. Let $\kappa \in \mathbb{N}_0 \cup \{\infty\}$ and let $(s_j)_{j=0}^{\kappa}$ and $(A_k)_{k=0}^{\kappa}$ be sequences of complex $q \times q$ matrices. For all $j \in \mathbb{Z}_{0,\kappa}$, let

$$S_{[0,j]} := \begin{pmatrix} s_0 & s_1 & \dots & s_{j-1} & s_j \\ 0_{q\times q} & s_0 & \dots & s_{j-2} & s_{j-1} \\ \vdots & & \ddots & \vdots & \vdots \\ 0_{q\times q} & 0_{q\times q} & & s_0 & s_1 \\ 0_{q\times q} & 0_{q\times q} & \dots & 0_{q\times q} & s_0 \end{pmatrix}. \tag{7.1}$$

Furthermore, let

$$A_{[j,k]} := \begin{pmatrix} A_{2j} \\ A_{2j-2} \\ \vdots \\ A_{2(k+1)} \\ A_{2k} \end{pmatrix} \tag{7.2}$$

for all $j,k \in \mathbb{N}_0$ with $k \le j$ and $2j \le \kappa$, and let

$$A^{[j,k]} := \begin{pmatrix} A_{2j+1} \\ A_{2j-1} \\ \vdots \\ A_{2k+3} \\ A_{2k+1} \end{pmatrix} \tag{7.3}$$

for all $j,k \in \mathbb{N}_0$ with $k \le j$ and $2j+1 \le \kappa$. For all $m \in \mathbb{N}$ let

$$J_m := \operatorname{diag}\left((-1)^{m-1}I_q, (-1)^{m-2}I_q, \dots, (-1)^2 I_q, (-1)^1 I_q, (-1)^0 I_q\right). \tag{7.4}$$

Observe that

$$J_m = \begin{pmatrix} -J_{m-1} & 0_{(m-1)q\times q} \\ 0_{q\times(m-1)q} & I_q \end{pmatrix} = \begin{pmatrix} (-1)^{m-1} I_q & 0_{q\times(m-1)q} \\ 0_{(m-1)q\times q} & J_{m-1} \end{pmatrix} \tag{7.5}$$

for all $m \in \{2,3,\dots\}$.

Lemma 7.1. *Let $n \in \{2,3,\dots\}$ and let $\mathbf{f}_n$ be a monic $q \times q$ matrix polynomial of degree n given for all $z \in \mathbb{C}$ by (2.1) with extended sequence of Markov parameters $(s_j)_{j=0}^{\infty}$. Let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m+1$ is satisfied. Then*

$$A^{[m-1,0]} = J_m S_{[0,m-1]} J_m A_{[m-1,0]}, \tag{7.6}$$

$$Y_{1,m} = H_{1,m-1} J_m A_{[m,1]} \tag{7.7}$$

if $n = 2m$, *and*

$$A^{[m,0]} = J_{m+1} S_{[0,m]} J_{m+1} A_{[m,0]}, \tag{7.8}$$

$$Y_{2,m} = H_{2,m-1} J_m A_{[m,1]} \tag{7.9}$$

if $n = 2m + 1$.

Proof. Denote by $\mathbf{h}_n$ and $\mathbf{g}_n$ the even and odd parts of $\mathbf{f}_n$, respectively. Suppose $n = 2m$ and let $\eta_n := \max\{|z|: z \in \mathcal{N}_{\mathbf{h}_n}\}$. In view of Definition 2.10 and part (a) of Lemma 2.9, then $\frac{\mathbf{g}_n(z)}{\mathbf{h}_n(z)} = \sum_{j=0}^{\infty} (-1)^j z^{-(j+1)} \tilde{s}_j$ for all $z \in \mathbb{C}$ with $|z| > \eta_n$. Multiplying both sides by $\mathbf{h}_n(z)$ results in

$$\mathbf{g}_n(z) = \left(\frac{s_0}{z} - \frac{s_1}{z^2} + \cdots + \frac{s_{2m}}{z^{2m+1}} - \cdots \right) \mathbf{h}_n(z)$$

for all $z \in \mathbb{C}$ with $|z| > \eta_n$. By equating coefficients of equal powers of z on both sides of this identity, we get

$$\begin{cases} A_1 = s_0 A_0, \\ A_3 = s_0 A_2 - s_1 A_0, \\ \quad \vdots \\ A_{2m-1} = s_0 A_{2m-2} - s_1 A_{2m-4} + \cdots + (-1)^{m-1} s_{m-1} A_0, \end{cases} \tag{7.10}$$

and

$$0_{q\times q} = s_k A_{2m} - s_{k+1} A_{2m-2} + \cdots + (-1)^m s_{k+m} A_0, \qquad \text{for all } k \in \mathbb{N}_0. \tag{7.11}$$

By rewriting (7.10) and (7.11) in matrix form, where k runs from 0 to $m-1$, yields

$$\begin{pmatrix} A_{2m-1} \\ A_{2m-3} \\ \vdots \\ A_1 \end{pmatrix} = \begin{pmatrix} s_0 & -s_1 & \dots & (-1)^{m-1} s_{m-1} \\ \vdots & \ddots & & \vdots \\ 0_{q\times q} & & \ddots & -s_1 \\ 0_{q\times q} & 0_{q\times q} & \dots & s_0 \end{pmatrix} \begin{pmatrix} A_{2m-2} \\ A_{2m-4} \\ \vdots \\ A_0 \end{pmatrix} \tag{7.12}$$

and

$$\begin{pmatrix} s_m A_0 \\ s_{m+1} A_0 \\ \dots \\ s_{2m-1} A_0 \end{pmatrix} = \begin{pmatrix} s_0 & s_1 & \dots & s_{m-1} \\ s_1 & s_0 & \dots & s_{m-2} \\ \vdots & \vdots & & \vdots \\ s_{m-1} & s_m & \dots & s_{2m-2} \end{pmatrix} \begin{pmatrix} (-1)^{m-1} A_{2m} \\ (-1)^{m-2} A_{2m-2} \\ \vdots \\ A_2 \end{pmatrix}. \tag{7.13}$$

From (7.3), (7.12), (7.4), (7.1), and (7.2), we obtain (7.6). Taking into account $A_0 = I_q$ we can conclude (7.7) from (4.3), (7.13), (4.1), (7.4), and (7.2). In the case $n = 2m + 1$ one proves (7.8) and (7.9) in a similar way. $\square$

Remark 7.2. Let $n \in \{2, 3, \ldots\}$ and let $(A_k)_{k=0}^{n}$ be a sequence of complex $q \times q$ matrices. In view of (7.5), (7.2), and (7.3), then

$$J_{m+1}A_{[m,0]} = \begin{pmatrix} -J_m A_{[m,1]} \\ A_0 \end{pmatrix} = \begin{pmatrix} (-1)^m A_{2m} \\ J_m A_{[m-1,0]} \end{pmatrix} \tag{7.14}$$

for all $m \in \mathbb{N}$ with $2m \leq n$, and

$$J_{m+1}A^{[m,0]} = \begin{pmatrix} -J_m A^{[m,1]} \\ A_1 \end{pmatrix} = \begin{pmatrix} (-1)^m A_{2m+1} \\ J_m A^{[m-1,0]} \end{pmatrix} \tag{7.15}$$

for all $m \in \mathbb{N}$ with $2m + 1 \leq n$.

Remark 7.3. Let $m \in \mathbb{N}$. According to (7.5) and (5.1), then

$$T_m^* J_{m+1} = -J_{m+1} T_m^*, \tag{7.16}$$

which, in view of (5.2), implies

$$R_m^*(\bar{z}) J_{m+1} = J_{m+1} R_m^*(-\bar{z}) \tag{7.17}$$

for all $z \in \mathbb{C}$.

Remark 7.4. Let $m \in \mathbb{N}$ and let $(s_j)_{j=0}^{m}$ be a sequence of complex $q \times q$ matrices. Using (5.4), (5.5), (5.3), (4.7), (5.1), and (7.1), one can prove by direct calculations that

$$u_{1,m}^* R_m^*(\bar{z}) = -v_m^* R_m^*(\bar{z}) T_m^* S_{[0,m]} \tag{7.18}$$

and

$$u_{2,m}^* R_m^*(\bar{z}) = -v_m^* R_m^*(\bar{z}) S_{[0,m]} \tag{7.19}$$

for all $z \in \mathbb{C}$.

Lemma 7.5. *Let $m \in \mathbb{N}$, and let $(s_j)_{j=0}^{m}$ and $(A_k)_{k=0}^{2m}$ be sequences of complex $q \times q$ matrices. For all $z \in \mathbb{C}$, then*

$$v_m^* T_m^* R_m^*(\bar{z}) J_{m+1} S_{[0,m]} J_{m+1} A_{[m,0]} = v_{m-1}^* R_{m-1}^*(\bar{z}) J_m S_{[0,m-1]} J_m A_{[m-1,0]}. \tag{7.20}$$

Proof. This is proved by using (4.8), the second equality of (7.5), (7.14) and $S_{[0,m]} = \left(\begin{smallmatrix} s_0 & y_{[1,m]}^* \\ 0_{qm \times q} & S_{[0,m-1]} \end{smallmatrix}\right)$. $\square$

Lemma 7.6. *Let $n \in \{2, 3, \ldots\}$ and let $\mathbf{f}_n$ be a monic $q \times q$ matrix polynomial of degree n given for all $z \in \mathbb{C}$ by (2.1) with extended sequence of Markov parameters $(s_j)_{j=0}^{\infty}$. Let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m + 1$ is satisfied. Then*

$$J_{m+1}A_{[m,0]} = \begin{pmatrix} -H_{1,m-1}^{-1}Y_{1,m} \\ I_q \end{pmatrix} \tag{7.21}$$

if $n = 2m$, and

$$J_{m+1}A_{[m,0]} = \begin{pmatrix} -H_{2,m-1}^{-1}Y_{2,m} \\ I_q \end{pmatrix} \tag{7.22}$$

if $n = 2m + 1$.

Proof. In view of $A_0 = I_q$, Eqs. (7.21) and (7.22) follow from (7.5), (7.2), (7.7) and (7.5), (7.2), (7.9), respectively. □

Lemma 7.7. *Let $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ and let $[(P_{1,j})_{j=0}^{\infty}, (Q_{1,j})_{j=0}^{\infty}, (P_{2,j})_{j=0}^{\infty}, (Q_{2,j})_{j=0}^{\infty}]$ be the associated SQSLOMP. For all $j \in \mathbb{N}_0$ and all $z \in \mathbb{C}$, then*

$$Q_{1,j}^{*}(\bar{z}) = v_j^{*}R_j^{*}(\bar{z})T_j^{*}S_{[0,j]} \begin{pmatrix} -H_{1,j-1}^{-1}Y_{1,j} \\ I_q \end{pmatrix} \tag{7.23}$$

and

$$Q_{2,j}^{*}(\bar{z}) = v_j^{*}R_j^{*}(\bar{z})S_{[0,j]} \begin{pmatrix} -H_{2,j-1}^{-1}Y_{2,j} \\ I_q \end{pmatrix}. \tag{7.24}$$

Proof. The identities (7.23) and (7.24) are proved by employing (5.8), (7.18) and (5.10), (7.19), respectively. □

Proposition 7.8. *Let $n \in \{2, 3, \ldots\}$ and let $\mathbf{f}_n$ be a monic $q \times q$ matrix polynomial of degree n with sequence of Markov parameters $(\tilde{s}_j)_{j=0}^{n}$ such that $(\tilde{s}_j)_{j=0}^{n-1} \in \mathcal{K}_{q,n-1}^{>}$. Then $\mathbf{f}_n$ is the MHTP of degree n associated with $(\tilde{s}_j)_{j=0}^{n-1}$.*

Proof. For $z \in \mathbb{C}$ let $\mathbf{f}_n$ be given by (2.1). Denote by $\mathbf{h}_n$ and $\mathbf{g}_n$ the even and odd part of $\mathbf{f}_n$, respectively. In view of Proposition 3.4, there exists a sequence $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ with

$$s_j = \tilde{s}_j \qquad \text{for all } j \in \mathbb{Z}_{0,n-1}. \tag{7.25}$$

Denote by $[(P_{1,j})_{j=0}^{\infty}, (Q_{1,j})_{j=0}^{\infty}, (P_{2,j})_{j=0}^{\infty}, (Q_{2,j})_{j=0}^{\infty}]$ the SQSLOMP associated with $(s_j)_{j=0}^{\infty}$. Let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m - 1$ is satisfied. Then the matrices $H_{1,m-1}$ and $H_{2,m-1}$ are both invertible, according to part (b) of Remark 4.2. Let $z \in \mathbb{C}$.

Suppose $n = 2m$. We show that

$$\mathbf{h}_n(z) = (-1)^m P^*_{1,m}(-\bar{z}) \qquad \text{and} \qquad \mathbf{g}_n(z) = (-1)^{m+1} Q^*_{1,m}(-\bar{z}).$$

To this end,

$$\begin{aligned}
\mathbf{h}_n(z) &= (I_q, zI_q, \ldots, z^m I_q) \begin{pmatrix} A_{2m} \\ A_{2m-2} \\ \vdots \\ A_0 \end{pmatrix} = v^*_m R^*_m(\bar{z}) A_{[m,0]} = v^*_m R^*_m(\bar{z}) J_{m+1} J_{m+1} A_{[m,0]} \\
&= v^*_m J_{m+1} R^*_m(-\bar{z}) \begin{pmatrix} -J_m A_{[m,1]} \\ A_0 \end{pmatrix} = (-1)^m v^*_m R^*_m(-\bar{z}) \begin{pmatrix} -H^{-1}_{1,m-1} Y_{1,m} \\ I_q \end{pmatrix} \\
&= (-1)^m P^*_{1,m}(-\bar{z}),
\end{aligned}$$

where the 1st equality is due to (2.2), the 2nd equality is due to (4.7), (5.3) and (7.2), the 3rd equality is due to (7.4), the 4th equality is due to (7.17) and (7.14), the 5th equality is due to (4.7), (7.5), (7.25), (7.7) and $A_0 = I_q$, and the last equality is due to (5.7). Analogously,

$$\begin{aligned}
\mathbf{g}_n(z) &= (I_q, zI_q, \ldots, z^{m-1} I_q) \begin{pmatrix} A_{2m-1} \\ A_{2m-3} \\ \vdots \\ A_1 \end{pmatrix} = v^*_{m-1} R^*_{m-1}(\bar{z}) A^{[m-1,0]} \\
&= v^*_{m-1} R^*_{m-1}(\bar{z}) J_m S_{[0,m-1]} J_m A_{[m-1,0]} = v^*_m T^*_m R^*_m(\bar{z}) J_{m+1} S_{[0,m]} J_{m+1} A_{[m,0]} \\
&= v^*_m T^*_m J_{m+1} R^*_m(-\bar{z}) S_{[0,m]} J_{m+1} A_{[m,0]} = -v^*_m J_{m+1} T^*_m R^*_m(-\bar{z}) S_{[0,m]} J_{m+1} A_{[m,0]} \\
&= -(-1)^m v^*_m T^*_m R^*_m(-\bar{z}) S_{[0,m]} \begin{pmatrix} -H^{-1}_{1,m-1} Y_{1,m} \\ I_q \end{pmatrix} \\
&= (-1)^{m+1} v^*_m R^*_m(-\bar{z}) T^*_m S_{[0,m]} \begin{pmatrix} -H^{-1}_{1,m-1} Y_{1,m} \\ I_q \end{pmatrix} = (-1)^{m+1} Q^*_{1,m}(-\bar{z}),
\end{aligned}$$

where the 1st equality is due to (2.3), the 2nd equality is due to (4.7), (5.3) and (7.3), the 3rd equality is due to (7.25) and (7.6), the 4th equality is due to (7.20), the 5th equality is due to (7.17), the 6th equality is due to (7.16), the 7th equality is due to (4.7), (7.5), (7.25) and (7.21), the 8th equality is due to (5.2), and the last equality is due to (7.23). Hence,

$$\mathbf{f}_n(z) = (-1)^m P^*_{1,m}(-\bar{z}^2) + z(-1)^{m+1} Q^*_{1,m}(-\bar{z}^2).$$

Since, in view of (5.7) and (5.8), the matrix polynomials $P_{1,m}$ and $Q_{1,m}$ only depend on the matrices $s_0, s_1, \ldots, s_{2m-1}$, which by (7.25) coincide with the matrices $\tilde{s}_0, \tilde{s}_1, \ldots, \tilde{s}_{2m-1}$, this shows that $\mathbf{f}_n$ is the MHTP of degree n associated with $(\tilde{s}_j)_{j=0}^{n-1}$.

Suppose $n = 2m + 1$. We show that

$$\mathbf{h}_n(z) = (-1)^m Q^*_{2,m}(-\bar{z}) \qquad \text{and} \qquad \mathbf{g}_n(z) = (-1)^m P^*_{2,m}(-\bar{z}).$$

To this end,

$$\begin{aligned}
\mathbf{h}_n(z) &= (I_q, zI_q, \dots, z^m I_q) \begin{pmatrix} A_{2m+1} \\ A_{2m-1} \\ \vdots \\ A_1 \end{pmatrix} = v^*_m R^*_m(\bar{z}) A^{[m,0]} \\
&= v^*_m R^*_m(\bar{z}) J_{m+1} S_{[0,m]} J_{m+1} A_{[m,0]} = v^*_m J_{m+1} R^*_m(-\bar{z}) S_{[0,m]} J_{m+1} A_{[m,0]} \\
&= (-1)^m v^*_m R^*_m(-\bar{z}) S_{[0,m]} \begin{pmatrix} -H^{-1}_{2,m-1} Y_{2,m} \\ I_q \end{pmatrix} = (-1)^m Q^*_{2,m}(-\bar{z}),
\end{aligned}$$

where the 1st equality is due to (2.2), the 2nd equality is due to (4.7), (5.3) and (7.3), the 3rd equality is due to (7.25) and (7.8), the 4th equality is due to (7.17), the 5th equality is due to (4.7), (7.5), (7.25) and (7.22), and the last equality is due to (7.24). Analogously,

$$\begin{aligned}
\mathbf{g}_n(z) &= (I_q, zI_q, \dots, z^m I_q) \begin{pmatrix} A_{2m} \\ A_{2m-2} \\ \vdots \\ A_0 \end{pmatrix} = v^*_m R^*_m(\bar{z}) A_{[m,0]} = v^*_m R^*_m(\bar{z}) J_{m+1} J_{m+1} A_{[m,0]} \\
&= v^*_m J_{m+1} R^*_m(-\bar{z}) \begin{pmatrix} -J_m A_{[m,1]} \\ A_0 \end{pmatrix} = (-1)^m v^*_m R^*_m(-\bar{z}) \begin{pmatrix} -H^{-1}_{2,m-1} Y_{2,m} \\ I_q \end{pmatrix} \\
&= (-1)^m P^*_{2,m}(-\bar{z}),
\end{aligned}$$

where the 1st equality is due to (2.3), the 2nd equality is due to (4.7), (5.3) and (7.2), the 3rd equality is due to (7.4), the 4th equality is due to (7.17) and (7.14), the 5th equality is due to (4.7), (7.5), (7.25), (7.9) and $A_0 = I_q$, and the last equality is due to (5.9). Hence,

$$\mathbf{f}_n(z) = (-1)^m Q^*_{2,m}(-\bar{z}^2) + z(-1)^m P^*_{2,m}(-\bar{z}^2).$$

Since, in view of (5.10) and (5.9), the matrix polynomials $Q_{2,m}$ and $P_{2,m}$ only depend on the matrices $s_0, s_1, \dots, s_{2m}$, which by (7.25) coincide with the matrices $\tilde{s}_0, \tilde{s}_1, \dots, \tilde{s}_{2m}$, this shows that $\mathbf{f}_n$ is the MHTP of degree n associated with $(\tilde{s}_j)_{j=0}^{n-1}$. $\square$

Now we state the main result of this section.

Theorem 7.9. *Let $n \in \{2, 3, \dots\}$ and let $\mathbf{f}_n$ be a MHTP of degree n with sequence of Markov parameters $(\tilde{s}_j)_{j=0}^{n}$. Let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m + 1$ is satisfied.*

(a) *Suppose* $n = 2m$ *and let* $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}]$ *be a Hurwitz parametrization of* $\mathbf{f}_n$*. Then* $(\tilde{s}_j)_{j=0}^{n-1} \in \mathcal{K}_{q,n-1}^{>}$ *with DS-parametrization* $[(\mathbf{d}_k)_{k=0}^{m-1}, (\mathbf{c}_k)_{k=0}^{m-1}]$ *and* $\mathbf{f}_n$ *is the MHTP of degree* n *associated with* $(\tilde{s}_j)_{j=0}^{n-1}$*.*

(b) *Suppose* $n = 2m + 1$ *and let* $[(\mathbf{c}_k)_{k=0}^{m}, (\mathbf{d}_k)_{k=0}^{m-1}]$ *be a Hurwitz parametrization of* $\mathbf{f}_n$*. Then* $(\tilde{s}_j)_{j=0}^{n-1} \in \mathcal{K}_{q,n-1}^{>}$ *with DS-parametrization* $[(\mathbf{d}_k)_{k=0}^{m-1}, (\mathbf{c}_k)_{k=0}^{m}]$ *and* $\mathbf{f}_n$ *is the MHTP of degree* n *associated with* $(\tilde{s}_j)_{j=0}^{n-1}$*.*

Proof. Denote by $\mathbf{h}_n$ and $\mathbf{g}_n$ the even and odd parts of $\mathbf{f}_n$, respectively.

(a) Let $\mathbf{c}_k := I_q$ and $\mathbf{d}_k := I_q$ for all $k \in \{m, m+1, \ldots\}$. Then $(\mathbf{c}_k)_{k=0}^{\infty}$ and $(\mathbf{d}_k)_{k=0}^{\infty}$ are sequences from $\mathbb{C}_{>}^{q\times q}$. According to [12, Proposition 8.27] there exists a sequence $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ with DS-parametrization $[(\mathbf{d}_k)_{k=0}^{\infty}, (\mathbf{c}_k)_{k=0}^{\infty}]$. Using Theorem 5.4 and Definitions 2.7 and 2.10, we obtain

$$G_{\sigma_{2m,\max}}(z) = \frac{\mathbf{g}_{2m}(-z)}{\mathbf{h}_{2m}(-z)} = -\sum_{j=0}^{\infty} z^{-(j+1)}\tilde{s}_j$$

for all $z \in \mathbb{C} \setminus [0,\infty)$ with $|z| > \eta_{2m}$, where $\eta_n := \max\{|z| : z \in \mathcal{N}_{\mathbf{h}_n}\}$. Since $\sigma_{2m,\max}$ belongs to $\mathcal{M}_{\geq}^{q}[[0,\infty); (s_j)_{j=0}^{2m}, \leq]$ and is concentrated on a finite set of points and hence compactly supported, we can via Lemma 3.8 conclude $s_j = \tilde{s}_j$ for all $j \in \mathbb{Z}_{0,2m-1}$. In particular, $(\tilde{s}_j)_{j=0}^{n-1} \in \mathcal{K}_{q,n-1}^{>}$ with DS-parametrization $[(\mathbf{d}_k)_{k=0}^{m-1}, (\mathbf{c}_k)_{k=0}^{m-1}]$. Proposition 7.8 yields then that $\mathbf{f}_n$ is the MHTP of degree n associated with $(\tilde{s}_j)_{j=0}^{n-1}$.

(b) Let $\mathbf{c}_k := I_q$ for all $k \in \{m+1, m+2, \ldots\}$ and let $\mathbf{d}_k := I_q$ for all $k \in \{m, m+1, \ldots\}$. Then $(\mathbf{c}_k)_{k=0}^{\infty}$ and $(\mathbf{d}_k)_{k=0}^{\infty}$ are sequences from $\mathbb{C}_{>}^{q\times q}$. According to [12, Proposition 8.27] there exists a sequence $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty}^{>}$ with DS-parametrization $[(\mathbf{d}_k)_{k=0}^{\infty}, (\mathbf{c}_k)_{k=0}^{\infty}]$. Using Theorem 5.4 and Definitions 2.7 and 2.10, we obtain

$$G_{\sigma_{2m,\min}}(z) = \frac{\mathbf{h}_{2m+1}(-z)}{(-z)\mathbf{g}_{2m+1}(-z)} = -\sum_{j=0}^{\infty} z^{-(j+1)}\tilde{s}_j$$

for all $z \in \mathbb{C} \setminus [0,\infty)$ with $|z| > \rho_{2m+1}$, where $\rho_{2m+1} := \max\{|z| : z \in \mathcal{N}_{\mathbf{g}_{2m+1}}\}$. Since $\sigma_{2m,\min}$ belongs to $\mathcal{M}_{\geq}^{q}[[0,\infty); (s_j)_{j=0}^{2m+1}, \leq]$ and is concentrated on a finite set of points and hence compactly supported, we can via Lemma 3.8 conclude $s_j = \tilde{s}_j$ for all $j \in \mathbb{Z}_{0,2m}$. In particular, $(\tilde{s}_j)_{j=0}^{n-1} \in \mathcal{K}_{q,n-1}^{>}$ with DS-parametrization $[(\mathbf{d}_k)_{k=0}^{m-1}, (\mathbf{c}_k)_{k=0}^{m}]$. Proposition 7.8 yields then that $\mathbf{f}_n$ is the MHTP of degree n associated with $(\tilde{s}_j)_{j=0}^{n-1}$. $\square$

Now we are able to state a useful characterization of th Hurwitz type property of monic $q \times q$ matrix polynomials in terms of the sequence of Markov parameters.

Theorem 7.10. *Let* $n \in \{2, 3, \ldots\}$ *and let* $\mathbf{f}_n$ *be a monic* $q \times q$ *matrix polynomial of degree* n *with sequence of Markov parameters* $(\tilde{s}_j)_{j=0}^{n}$*. Then the following statements are equivalent:*

(i) $\mathbf{f}_n$ *is a MHTP of degree* n*.*

(ii) $(\tilde{s}_j)_{j=0}^{n-1} \in \mathcal{K}_{q,n-1}^{>}$.

Proof. "(i) ⇒ (ii)": This follows from Theorem 7.9.
"(ii) ⇒ (i)": This follows from Proposition 7.8. □

Now we show that the correspondence between matrix Hurwitz type polynomials and finite Stieltjes positive definite sequences is bijective.

Theorem 7.11. *Let $n \in \{2, 3, \ldots\}$ and let $\mathbf{f}_n$ be a MHTP of degree n with sequence of Markov parameters $(\tilde{s}_j)_{j=0}^{n}$. Then there is a unique sequence $(s_j)_{j=0}^{n-1} \in \mathcal{K}_{q,n-1}^{>}$ such that $\mathbf{f}_n$ is the MHTP of degree n associated with $(s_j)_{j=0}^{n-1}$ namely $(s_j)_{j=0}^{n-1} = (\tilde{s}_j)_{j=0}^{n-1}$.*

Proof. From Theorem 7.9 we obtain $(\tilde{s}_j)_{j=0}^{n-1} \in \mathcal{K}_{q,n-1}^{>}$ and that $\mathbf{f}_n$ is the MHTP of degree n associated with $(\tilde{s}_j)_{j=0}^{n-1}$. If $(s_j)_{j=0}^{n-1}$ is an arbitrary sequence from $\mathcal{K}_{q,n-1}^{>}$ such that $\mathbf{f}_n$ is the MHTP of degree n associated with $(s_j)_{j=0}^{n-1}$, then Definition 6.4 and Theorem 6.1 yield $(s_j)_{j=0}^{n-1} = (\tilde{s}_j)_{j=0}^{n-1}$. □

Now we demonstrate that a matrix Hurwitz type polynomial admits a unique Hurwitz parametrization.

Theorem 7.12. *Let $n \in \{2, 3, \ldots\}$ and let $\mathbf{f}_n$ be a MHTP of degree n with sequence of Markov parameters $(\tilde{s}_j)_{j=0}^{n}$. Let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m + 1$ is satisfied.*

(a) *Suppose $n = 2m$ and let $[(\mathbf{L}_k)_{k=0}^{m-1}, (\mathbf{M}_k)_{k=0}^{m-1}]$ be the DS-parametrization of $(\tilde{s}_j)_{j=0}^{n-1}$. Then there is a unique Hurwitz parametrization $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}]$ of $\mathbf{f}_n$, namely $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}] = [(\mathbf{M}_k)_{k=0}^{m-1}, (\mathbf{L}_k)_{k=0}^{m-1}]$.*
(b) *Suppose $n = 2m + 1$ and let $[(\mathbf{L}_k)_{k=0}^{m-1}, (\mathbf{M}_k)_{k=0}^{m}]$ be the DS-parametrization of $(\tilde{s}_j)_{j=0}^{n-1}$. Then there is a unique Hurwitz parametrization $[(\mathbf{c}_k)_{k=0}^{m}, (\mathbf{d}_k)_{k=0}^{m-1}]$ of $\mathbf{f}_n$, namely $[(\mathbf{c}_k)_{k=0}^{m}, (\mathbf{d}_k)_{k=0}^{m-1}] = [(\mathbf{M}_k)_{k=0}^{m}, (\mathbf{L}_k)_{k=0}^{m-1}]$.*

Proof. From Theorem 7.9 we obtain $(\tilde{s}_j)_{j=0}^{n-1} \in \mathcal{K}_{q,n-1}^{>}$ and that $\mathbf{f}_n$ is the MHTP of degree n associated with $(\tilde{s}_j)_{j=0}^{n-1}$.

(a) In view of Definitions 6.4 and 4.6, we can conclude from part (a) of Theorem 6.1 that $[(\mathbf{M}_k)_{k=0}^{m-1}, (\mathbf{L}_k)_{k=0}^{m-1}]$ is a Hurwitz parametrization of $\mathbf{f}_n$. If $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}]$ is an arbitrary Hurwitz parametrization of $\mathbf{f}_n$, then part (a) of Theorem 7.9 shows that $[(\mathbf{d}_k)_{k=0}^{m-1}, (\mathbf{c}_k)_{k=0}^{m-1}]$ is the DS-parametrization of $(\tilde{s}_j)_{j=0}^{n-1}$ and hence $[(\mathbf{c}_k)_{k=0}^{m-1}, (\mathbf{d}_k)_{k=0}^{m-1}] = [(\mathbf{M}_k)_{k=0}^{m-1}, (\mathbf{L}_k)_{k=0}^{m-1}]$.

(b) In view of Definitions 6.4 and 4.6, we can conclude from part (b) of Theorem 6.1 that $[(\mathbf{M}_k)_{k=0}^{m}, (\mathbf{L}_k)_{k=0}^{m-1}]$ is a Hurwitz parametrization of $\mathbf{f}_n$. If $[(\mathbf{c}_k)_{k=0}^{m}, (\mathbf{d}_k)_{k=0}^{m-1}]$ is an arbitrary Hurwitz parametrization of $\mathbf{f}_n$, then part (b) of Theorem 7.9 shows that $[(\mathbf{d}_k)_{k=0}^{m-1}, (\mathbf{c}_k)_{k=0}^{m}]$ is the DS-parametrization of $(\tilde{s}_j)_{j=0}^{n-1}$ and hence $[(\mathbf{c}_k)_{k=0}^{m}, (\mathbf{d}_k)_{k=0}^{m-1}] = [(\mathbf{M}_k)_{k=0}^{m}, (\mathbf{L}_k)_{k=0}^{m-1}]$. □

8. On asymptotic stability of ODEs

As an application of the obtained results, we consider in this section the well-known Lyapunov's stability criterion [24] for ordinary differential equations with constant coefficients in terms of orthogonal polynomials on $[0, \infty)$ and their second kind polynomials.

Let $n \in \{2, 3, \dots\}$ and consider the differential equation

$$\dot{x} = Ax, \tag{8.1}$$

where x is an $n \times 1$ vector, and A is an $n \times n$ matrix whose entries are real constants.

Remark 8.1. Let $R_A: \mathbb{C} \to \mathbb{C}$ defined by $R_A(z) := \det(zI_n - A)$ be the characteristic polynomial of the matrix A, which is a monic polynomial of degree n. Recall that the system (8.1) is asymptotically stable in the sense of Lyapunov if and only if all the zeros of the characteristic polynomial R_A have a negative real part, i.e., if and only if R_A is a Hurwitz polynomial.

Theorem 8.2. *Let $n \in \{2, 3, \dots\}$ and let A be a real $n \times n$ matrix with characteristic polynomial R_A. Denote by $(\tilde{s}_j)_{j=0}^{n}$ the sequence of Markov parameters of R_A. Then, the system (8.1) is asymptotically stable if and only if $(\tilde{s}_j)_{j=0}^{n-1}$ is Stieltjes positive definite. In this case, the set $\mathcal{M}^1_{\geq}[[0, \infty); (\tilde{s}_j)_{j=0}^{n-1}, \leq]$ is non-empty and R_A is the matrix Hurwitz type polynomial of degree n associated with $(\tilde{s}_j)_{j=0}^{n-1}$.*

Proof. This follows, in view of Remark 8.1, from Theorem 7.10, Remark 3.6, and Proposition 7.8. $\square$

Suppose the system (8.1) is asymptotically stable. Then, $(\tilde{s}_j)_{j=0}^{n-1}$ is Stieltjes positive definite by Theorem 8.2. In view of Proposition 3.4, there exists a sequence $(s_j)_{j=0}^{\infty} \in \mathcal{K}^{>}_{1,\infty}$ such that $(\tilde{s}_j)_{j=0}^{n-1} = (s_j)_{j=0}^{n-1}$. Denote by $[(p_{1,j})_{j=0}^{\infty}, (q_{1,j})_{j=0}^{\infty}, (p_{2,j})_{j=0}^{\infty}, (q_{2,j})_{j=0}^{\infty}]$ the SQSLOMP associated with $(s_j)_{j=0}^{\infty}$ (see Definition 5.1) and let $m \in \mathbb{N}$ be chosen such that $n = 2m$ or $n = 2m + 1$ is satisfied. From Theorem 8.2 and Definition 6.4 we then can conclude that

$$R_A(z) = \begin{cases} (-1)^m p_{1,m}(-\bar{z}^2) - z(-1)^m q_{1,m}(-\bar{z}^2), & \text{if } n = 2m \\ (-1)^m q_{2,m}(-\bar{z}^2) + z(-1)^m p_{2,m}(-\bar{z}^2), & \text{if } n = 2m + 1 \end{cases} \tag{8.2}$$

holds for all $z \in \mathbb{C}$. Observe that the SQSLOMP $[(p_{1,j})_{j=0}^{\infty}, (q_{1,j})_{j=0}^{\infty}, (p_{2,j})_{j=0}^{\infty}, (q_{2,j})_{j=0}^{\infty}]$ associated with $(s_j)_{j=0}^{\infty}$ can be interpreted in terms of orthogonal polynomials with respect to $(s_j)_{j=0}^{\infty}$ and other related polynomials (see [4, Proposition 4.2]). Note that these polynomials then also have the corresponding orthogonality properties with respect to each $\sigma \in \mathcal{M}^q_{\geq}[[0, \infty); (s_j)_{j=0}^{\infty}, =]$. We refer to [4, Section 4, Appendices D and E] for a detailed treatment of this topic.

9. An example

In this section we consider the scalar case $q = 1$. For $j \in \mathbb{N}_0$ let $s_j := j!$. Then $(s_j)_{j=0}^{\infty}$ is the sequence of power moments of the exponential distribution with parameter 1. This is the measure on $\mathfrak{B}_{[0,\infty)}$ which is given via the density $h\colon [0,\infty) \to [0,\infty)$ defined by $h(u) := \exp\{-u\}$. Thus, we have $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{1,\infty}^{>}$. Denote by $[(P_{1,j})_{j=0}^{\infty}, (Q_{1,j})_{j=0}^{\infty}, (P_{2,j})_{j=0}^{\infty}, (Q_{2,j})_{j=0}^{\infty}]$ the SQSLOMP associated with $(s_j)_{j=0}^{\infty}$. (Observe that $(P_{1,j})_{j=0}^{\infty}$ is the sequence of classical Laguerre polynomials.) Then the first polynomials of the sequence are given by

$$\begin{aligned}
P_{1,0}(z) &= 1, & P_{2,0}(z) &= 1,\\
P_{1,1}(z) &= z-1, & P_{2,1}(z) &= z-2,\\
P_{1,2}(z) &= z^2-4z+2, & P_{2,2}(z) &= z^2-6z+6,\\
P_{1,3}(z) &= z^3-9z^2+18z-6, & P_{2,3}(z) &= z^3-12z^2+36z-24,\\
Q_{1,0}(z) &= 0, & Q_{2,0}(z) &= 1,\\
Q_{1,1}(z) &= 1, & Q_{2,1}(z) &= z-1,\\
Q_{1,2}(z) &= z-3, & Q_{2,2}(z) &= z^2-5z+2,\\
Q_{1,3}(z) &= z^2-8z^2+11, & Q_{2,3}(z) &= z^3-11z^2+26z-6.
\end{aligned}$$

For $n \in \{2, 3, \ldots\}$ we denote by $\mathbf{f}_n$ the Hurwitz type polynomial of degree n associated with $(s_j)_{j=0}^{\infty}$. Then the first seven polynomials are given by

$$\begin{aligned}
\mathbf{f}_1(z) &= z+1, & &\\
\mathbf{f}_2(z) &= z^2+z+1, & \mathbf{f}_5(z) &= z^5+z^4+6z^3+5z^2+6z+2,\\
\mathbf{f}_3(z) &= z^3+z^2+2z+1, & \mathbf{f}_6(z) &= z^6+z^5+9z^4+8z^3+18z^2+11z+6,\\
\mathbf{f}_4(z) &= z^4+z^3+4z^2+3z+2, & \mathbf{f}_7(z) &= z^7+z^6+12z^5+11z^4+36z^3+26z^2\\
& & &\quad +24z+6.
\end{aligned}$$

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