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A Fixed-Point State observer with Steffensen-Aitken accelerated convergence

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Abstract

In this work, we present an alternative algorithm for the state estimation of discrete-time nonlinear systems, which we have called Fixed-Point state observer with Steffensen-Aitken accelerated convergence. This algorithm decomposes the state estimation task into a set of consecutive fixed point iteration problems. In other words, it considers a system of nonlinear equations for each time instant and uses a fixed point iteration method to solve it, such that the solution given by the method is actually a state estimation of the discrete-time nonlinear system at the current time instant. To increase the convergence speed of the fixed point iteration method, we propose to incorporate the Δ^2-Aitken method. Nonetheless, later we show that is possible to increase even more this speed by means of the Steffensen's method. The main advantage of our algorithm is the lack of complex calculations, such as the Jacobian matrix and its inverse, which are necessary for similar algorithms such as the Newton observer. Therefore, our proposal has a low computational cost, is free of singularities, and is easy to implement. Furthermore, unlike conventional estimators like the Luenberger observer and the Sliding Mode observer, it does not require to calculate gains. To prove the effectiveness of the Fixed-Point state observer, we estimate the unknown states of a modified Chua chaotic attractor and compare the numerical results with those obtained by Luenberger, Sliding Mode, and Newton observers.

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1. Introduction

In the real world applications, the state variables of any industrial process or system are crucial for the design of a controller. However, these variables might be totally or partially unknown due to economic restrictions or technological limitations. Therefore, the design of mathematical algorithms that allow estimating these variables from the available measurements is extremely important. Such algorithms are known as state estimators or observers.

In the literature, we can find a great number of estimators applied to all kinds of discrete-time dynamic systems. The usual ones are the Luenberger-type observers [1,2] and the sliding mode observers [3,4], very useful for fault detection. Some less common but really interesting approaches are the interval observers, especially useful for systems with bounded disturbances [5–7], and graph theory-based observers, widely used in multi-agent problems [8,9].

All the observers mentioned above consist of a copy of the dynamic system with additional corrective terms. However, Grizzle and Moraal presented an alternative idea in their conference paper [10]. These authors suggested tackling the state estimation problem for discrete-time nonlinear systems by solving a set of consecutive systems of nonlinear equations. That is, they consider the estimation problem as a set of nonlinear inversion problems (one problem for each time instant) and propose to find the solution through an iteration method [11]. In particular, they use the Newton's iteration method and named this algorithm as Newton observer. The idea is to iterate long enough and then propagate the solution to obtain the state estimate at the current time instant. Subsequently, this solution is considered as initial guess for the iteration method of the following time instant [10,12].

The Newton observer is a structurally robust algorithm that has been successfully applied to several dynamic systems: water flow in industrial tanks [13], the magnetic flux in magnetic bearings [14], the glucose level in intensive care unit patients [15], robot's kinematic model [16,17], etc. However, this observer has two implementation problems: 1) If the initial guess is not close enough to the solution, the iteration method does not converge, and 2) it is necessary to evaluate a Jacobian matrix and compute its inverse at each iteration and for every time instant. More over, the Jacobian matrix could be ill-conditioned and, in general, is computationally expensive. For these reasons, the Newton observer can have numerical errors and its computational cost might be elevated [18–20].

To overcome this issues, one can ensure the convergence of the iteration method by finding an acceptable initial guess through mathematical analysis. On the other hand, to reduce the computational effort, several strategies exist, e.g.: in [19], the authors proposed to use the secant approximation method along with a continuous-time filter scheme. In [15], Pontryagin's maximum principle was applied to calculate the Jacobian matrix by considering a performance indicator. Meanwhile, the finite difference approximation and the Broyden-Fletcher-Goldfarb-Shanno algorithm were used to design a free Jacobian observer in [20]. Although these strategies can reduce the computational effort, a great amount of additional parameters needs to be identified.

In this context, we propose an alternative algorithm for the state estimation of discrete-time nonlinear systems that seeks to overcome the second implementation problem that we mentioned before, i.e., to avoid the computation of derivatives and inverse maps. We have called this algorithms as Fixed-Point state observer with Steffensen-Aitken accelerated convergence. Thus, our proposal presents a series of advantages and contributions that are listed below:

(1) To increase the convergence speed of the fixed point iteration method used by the estimator, we propose to implement a) the Δ^2-Aitken method and b) the Steffensen's method.
(2) The Fixed-Point observer does not require complex calculations, such as the Jacobian matrix and its inverse, used in similar algorithms such as the Newton observer [18–20], this traduces in a lower computational cost.
(3) The implementation of the Steffensen's method increases considerably the convergence speed of the fixed point iteration method used by the observer and at the same time, it still exhibits a lower computational cost than the Newton observer.
(4) Unlike conventional observers, our proposal does not require the computation of gains or a set of several other parameters.
(5) The corresponding mathematical formulation and analysis to ensure convergence are presented. As far as we know, these have not been previously reported in the literature.
(6) The Fixed-Point observer has a low computational cost, and since does not involve a Jacobian matrix, can be easily apply to systems whose derivatives computation is to complex.

The remainder of this work is organized as follows: Section 2 contains the necessary theoretical framework for the problem statement and the design of the observer. In Section 3, we introduce the Fixed-Point observer and the proposed convergence schemes. Subsequently, the modified Chua chaotic attractor is considered in Section 4 to exemplify the effectiveness of the proposed algorithm. Furthermore, the numerical results are compared with the performance of Newton, Luenberger, and sliding mode observers. Finally, the conclusions of this work are given in Section 5.

2. Preliminaries

Whenever one works with real-world applications, the available measurements are discrete-time signals. In these cases, the best option to design a state observer is using an exact discrete-time model of the system. However, most of the time, only a continuous-time model is available. Since discrete-time models are commonly unknown or hard to obtain, it is advisable to use a discretization method, such as in [18,21,22]. Depending on the discretization approach that is used, the simulation results will correspond better or not with the behaviour of the original continuous system. The most common approach is the well known Euler method, due to its simplicity, although recent advances have shown that there exist better alternatives. Nonetheless, this discussion is beyond the scope of this work, and the reader is advised to look for [23], where further details about this topic can be found.

Once a discrete-time model is available, we can find a finite sequence of the output signal as follows.

2.1. Output signal's finite sequence of a continuous-time system

Consider the following continuous-time nonlinear dynamic system

$$\begin{aligned} \dot{x} &= \mathcal{F}(x), \\ y &= h(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$,$y \in \mathbb{R}^p$, h is an analytic function and $\mathcal{F}$ is a Lipschitz function. For the sake of simplicity, we assume that $p = 1$, henceforth. Besides, suppose a sampling time $T > 0$ and let $f(t, x_0)$ be solution of (1) with the initial condition x_0. Then, the T-sampled system of (1) is given by

$$\begin{aligned} x_{k+1} &= f(T, x_k), \\ y_k &= h(x_k) \end{aligned} \tag{2}$$

where $x_k = (\mathrm{x}_{1,k}, \mathrm{x}_{2,k}, \ldots, \mathrm{x}_{n,k})^T$ is the discrete-time state vector at the time instant k and y_k is the corresponding system's output.

On the other hand, let δ be a delay operator such that $\delta\varphi_k = \varphi_{k-1}$, and δ^{-1} be the corresponding advance operator. Therefore $\delta^{-\mu}\varphi_k = \varphi_{k+\mu}$ and similarly $\delta^{\mu}\varphi_k = \varphi_{k-\mu}$ for any $\mu > 0$. Let $\hat{\delta}^{\mu}\varphi_k$ define the collection $\hat{\delta}^{\mu}\varphi_k = \{\delta\varphi_k, \ldots, \delta^{\mu}\varphi_k\}$. Meanwhile, $\hat{\delta}^{-\mu}\varphi_k$ stands for the collection $\hat{\delta}^{-\mu}\varphi_k = \{\varphi_k, \delta^{-1}\varphi_k, \ldots, \delta^{-\mu}\varphi_k\}$. Evidently, $\delta^0 = \hat{\delta}^0 = 1$ and $\hat{\delta}^1 = \delta$.

One can note that system (2) is equivalent to $x_k = \delta f(x_k) = f(\delta x_k) = f(x_{k-1})$ and $\delta x_{k+1} = x_k = f(x_{k-1})$. Additionally, $x_{k-1} = f(x_{k-2}) = f(\delta x_{k-1}) = f(\delta^2 x_k)$ and in the same way, $x_k = f(f(\delta^2 x_k))$. Therefore, the expression $f^{\mu}(\delta^{\mu} x_k)$ for $\mu > 0$, should be clear from the recursion

$$\begin{aligned} f^i(\delta^i x_k) &= f\big(f^{i-1}(\delta^i x_k)\big) \\ f^1(\delta x_k) &= f(\delta x_k) \end{aligned}$$

for some $i \in \mathbb{N}$. Notice that, the operators δ and $\hat{\delta}$ satisfy the following relation: $\delta^i \hat{\delta}^{-i}\varphi_k = \{\varphi_k, \hat{\delta}^i\varphi_k\}$ since $\delta^i\{\varphi_k, \delta^{-1}\varphi_k, \ldots, \delta^{-i}\varphi_k\} = \{\delta^i\varphi_k, \delta^{i-1}\varphi_k, \ldots, \delta\varphi_k, \varphi_k\}$. Then, the evolution of the state vector is equivalent to the next sequence

$$\begin{aligned} x_{k+1} &= \delta^{-1}x_k = f(x_k), \\ x_{k+2} &= \delta^{-2}x_k = f(f(x_k)), \\ &\vdots \\ x_{k+i} &= \delta^{-i}x_k = f(f^{i-1}(x_k)) \end{aligned} \tag{3}$$

furthermore, by considering (3) in an iterative form, we obtain

$$\begin{aligned} x_k &= \delta f(x_k) = f(\delta x_k), \\ x_k &= f(\delta(f(\delta x_k))) = f(f(\delta^2 x_k)), \\ &\vdots \\ x_k &= f(f^{i-1}(\delta^i x_k)) \end{aligned} \tag{4}$$

In a similar way, we can define the following finite sequence of the output signal

$$\begin{aligned} y_k &= h(x_k) = h(f^0(x_k)), \\ y_{k+1} &= \delta^{-1}(h(x_k)) = h(f(x_k)), \\ y_{k+2} &= \delta^{-1}h(f(x_k)) = h(f^2(x_k)), \\ &\vdots \\ y_{k+i} &= h(f^i(x_k)) \end{aligned} \tag{5}$$

which can be express as

$$Y_{[k,k+i]} = \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+i} \end{bmatrix} = \begin{bmatrix} h(x_k) \\ h(f(x_k)) \\ \vdots \\ h(f^i(x_k)) \end{bmatrix} = H(x_k) \tag{6}$$

In a further step of this work, the expression (6) will be useful for the state observer design.

2.2. Fixed point iteration problem

In numerical analysis, the roots or solution of a system of nonlinear equations are commonly found with a linear approximation. In addition, the problem can be directly addressed by some algorithms, such as Newton's method, Whittaker's method, Halley's method, etc. Nonetheless, these algorithms are computationally expensive due to a continuous evaluation of one or more derivatives and the computation of a map inverse.

As an alternative, we can use the fixed point iteration method or successive approximation method. Briefly speaking, this method consists in expressing a system of nonlinear equations $F(x) = 0$ as $x = G(x)$, e.g., with $G(x) = x + aF(x)$. Then, an iteration method can be defined as $x^{(l+1)} = G(x^{(l)})$, where $l \in \mathbb{N}$ and $x^{(i)}$ is the i-th approximation or iteration of x. This process is repeated for $l \geq 0$ until a value p is found, for which we have $G(p) = p$. Thus, the solution of the original system of nonlinear equations is precisely p, i.e., $F(p) = 0$.

Remark 1. Strictly speaking, every iteration method has the form $x^{(l+1)} = x^{(l)} + \Psi(x^{(l)})F(x^{(l)})$. If $\Psi = -J^{-1}$, where J is the Jacobian matrix of F, then we obtain the Newton's iteration method. Here, we work with a class of iteration methods, where Ψ is a constant, or in our case, equal to an identity. This methods are known as fixed point iteration methods [24,25].

Definition 1 ([24])**.** A function $G(x)$ of $D \subset \mathbb{R}^n$ on $\mathbb{R}^n$ has a fixed point in $p \in D$ if $G(p) = p$.

Theorem 1 ([24])**.** *Let* $D = \{x = (\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{x}_n)^T | a_i \leq \mathrm{x}_i \leq b_i, \forall i = 1, 2, \ldots, n\}$ *for some set of constants* $a_1, a_2, \ldots, a_n$ *and* $b_1, b_2, \ldots, b_n$. *Suppose that* G *is a continuous function of* $D \subset \mathbb{R}^n$ *on* $\mathbb{R}^n$ *such that* $G(x) \in D$ *for any* $x \in D$. *Then* G *has a fixed point at* D. *Suppose further that* G *has continuous partial derivatives and that there is a constant* $|K| < 1$ *such that*

$$\left| \frac{\partial g_i(x)}{\partial \mathrm{x}_j} \right| \leq K, \quad \forall x \in D,$$

for any $j = 1, 2, \ldots, n$ *and every function* g_i, *component of* G. *Then, the sequence* $\{x^{(l)}\}_{l=0}^{\infty}$, *with arbitrary initial value* $x^{(0)} \in D$, *and generated by*

$$x^{(l+1)} = G\big(x^{(l)}\big), \quad l \in \mathbb{N},$$

converges to a unique fixed point $p \in D$ *and*

$$\left\| x^{(l)} - p \right\| \leq \frac{K^l}{1 - K} \left\| x^{(1)} - x^{(0)} \right\|$$

Remark 2. The condition imposed to the partial derivatives implies that G is a contraction map on D. This is an immediate consequence of the mean value theorem and further details can be found in [25].

Also, let us introduce the following definition that will be useful in a later section.

Definition 2 ([26])**.** A sequence $\{x^{(l)}\}_{l=0}^{\infty}$ is said to converge linearly to p with rate $\rho \in (0, 1)$ if there is a constant $c > 0$ such that

$$\left\|x^{(l)} - p\right\| \leq c\rho^{l}, \quad \forall l \in \mathbb{N} \tag{7}$$

3. Fixed-Point state observer

In what follows, we show that the fixed point iteration method can be used to design a state estimator by considering the representation (6) of the output signal.

Let us consider once more the system (2), with x_k partially known for any time instant k. Besides, consider the following assumptions:

A1. The T-sampled system (2) is observable for the output sampling conditions and obeys the Theorem in [27], i.e., there exist $T_0 > 0$ such that the system is observable for any $T \leq T_0$.

A2. The $n-1$ previous measurements of any time instant k are retained, i.e., $y_{k-n+1}, y_{k-n+2}, \ldots, y_{k-1}$ are available.

To estimate the full state from the system's output y_k and a finite number of previous measurements, we propose the following methodology:

Let $Y_{[k-n+1,k]}$ denote the vector of n consecutive measurements from time instant $k-n+1$ to instant k, i.e., $Y_{[k-n+1,k]} = [y_{k-n+1}, y_{k-n+2}, \ldots, y_k]^T$. Note that, by considering the representation (6), we have that

$$Y_{[k-n+1,k]} = \begin{bmatrix} y_{k-n+1} \\ y_{k-n+2} \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} h(x_{k-n+1}) \\ h(f(x_{k-n+1})) \\ \vdots \\ h(f^{n-1}(x_{k-n+1})) \end{bmatrix} = H(x_{k-n+1}) \tag{8}$$

We can reorganize the last expression as the following root search problem,

$$F(x_{k-n+1}) = Y_{[k-n+1,k]} - H(x_{k-n+1}) = 0 \tag{9}$$

To solve the last problem, let us express (9) as

$$Y_{[k-n+1,k]} - H(x_{k-n+1}) + x_{k-n+1} - x_{k-n+1} = 0 \tag{10}$$

such that

$$x_{k-n+1} = x_{k-n+1} + Y_{[k-n+1,k]} - H(x_{k-n+1}) = G(x_{k-n+1}) \tag{11}$$

For now, let us assume that (11) has solution $x^*_{k-n+1} \in D$. Therefore, given an initial condition or guess $x^{(0)}_{k-n+1} \in D$, all iterated values $x^{(l)}_{k-n+1} \in D$ and converge to the fixed point x^*_{k-n+1} (Theorem 1), i.e.,

$$x^*_{k-n+1} = \lim_{l\to\infty} x^{(l+1)}_{k-n+1} = \lim_{l\to\infty} G\left(x^{(l)}_{k-n+1}\right) = G\left(\lim_{l\to\infty} x^{(l)}_{k-n+1}\right) = G\left(x^*_{k-n+1}\right) \tag{12}$$

where the iterated values $x^{(l)}_{k-n+1}$ are obtained from the following iteration method

$$x^{(l+1)}_{k-n+1} = x^{(l)}_{k-n+1} + Y_{[k-n+1,k]} - H\left(x^{(l)}_{k-n+1}\right), \quad l \in \mathbb{N} \tag{13}$$

Under this assumptions, we note that every value in the sequence $\left\{x^{(l)}_{k-n+1}\right\}_{l=0}^{\infty}$, including x^{*}_{k-n+1}, is an approximation of the solution of (9). In other words, these values are actually an estimation of the real state x_{k-n+1}.

Thus, in accordance with the usual notation from control theory for estimated values, let us express the iteration method (13) as

$$\hat{x}^{(l+1)}_{k-n+1} = \hat{x}^{(l)}_{k-n+1} + Y_{[k-n+1,k]} - H(\hat{x}^{(l)}_{k-n+1}), \quad l \in \mathbb{N} \tag{14}$$

and the fixed point as $\hat{x}^{*}_{k-n+1}$. Notice that the expression (14) has the general form of the iteration methods, with $F = Y_{[k-n+1,k]} - H$ and $\Psi = I$ (remark 1).

Furthermore, since $\hat{x}^{*}_{k-n+1}$ is the estimate at $k-n+1$, we obtain the estimate at the time instant k by propagating the fixed point $n-1$ steps forward, i.e.,

$$\hat{x}_k = f^{n-1}\left(\hat{x}^{*}_{k-n+1}\right) \tag{15}$$

Observe that the root search problem (9) needs to be solved for every time instant. Therefore, it is said that the Fixed-Point state observer algorithm consists in decomposing the state estimation into a set of consecutive fixed point iteration problems (one problem for each time instant). Based on the above, we establish the following theorem.

Theorem 2. *If the iteration method (14) obeys Theorem 1, then for any time instant k, the sequence $\{\hat{x}^{(l)}_{k-n+1}\}_{l=0}^{\infty}$ converges to $\hat{x}^{*}_{k-n+1}$, solution of the root search problem (9). That is to say, the expression (14) is a Fixed-Point state observer for the discrete-time nonlinear system (2).*

Proof. Let us consider a given time instant k. Then, since h is analytic, the function $G(\hat{x}_{k-n+1}) = \hat{x}_{k-n+1} + Y_{[k-n+1,k]} - H(\hat{x}_{k-n+1})$ is continuous. On the other hand, according to Theorem 1 we have that $G(\hat{x}_{k-n+1}) \in D$ for every $\hat{x}_{k-n+1} \in D$. Moreover, G is a contraction map, such that

$$\|G\left(\hat{x}^{(i)}_{k-n+1}\right) - G\left(\hat{x}^{(j)}_{k-n+1}\right)\| \leq K\|\hat{x}^{(i)}_{k-n+1} - \hat{x}^{(j)}_{k-n+1}\|, \qquad \forall \hat{x}^{i}_{k-n+1}, \hat{x}^{(j)}_{k-n+1} \in D, i \neq j \tag{16}$$

with $|K| < 1$. Therefore, we have

$$\begin{aligned}\|\hat{x}^{(l+1)}_{k-n+1} - \hat{x}^{(l)}_{k-n+1}\| &= \|G\left(\hat{x}^{(l)}_{k-n+1}\right) - G\left(\hat{x}^{(l-1)}_{k-n+1}\right)\| \\ &\leq K\|\hat{x}^{(l)}_{k-n+1} - \hat{x}^{(l-1)}_{k-n+1}\|\end{aligned} \tag{17}$$

Then, by induction

$$\|\hat{x}^{(l+c)}_{k-n+1} - \hat{x}^{(l+c-1)}_{k-n+1}\| \leq K\|\hat{x}^{(l+c-1)}_{k-n+1} - \hat{x}^{(l+c-2)}_{k-n+1}\| \leq \cdots \leq K^c\|\hat{x}^{(l)}_{k-n+1} - \hat{x}^{(l-1)}_{k-n+1}\| \tag{18}$$

for every $c \geq 1$ and $l \geq 0$. Later, we have that

$$\sum_{i=1}^{c} \|\hat{x}^{(l+i)}_{k-n+1} - \hat{x}^{(l+i-1)}_{k-n+1}\| \leq \left(K^c + K^{c-1} + \cdots + K\right)\|\hat{x}^{(l)}_{k-n+1} - \hat{x}^{(l-1)}_{k-n+1}\| \tag{19}$$

such that

$$\|\hat{x}^{(l+c)}_{k-n+1} - \hat{x}^{(l)}_{k-n+1}\| \leq \left(K^c + K^{c-1} + \cdots + K\right)K^{l-1}\|\hat{x}^{(1)}_{k-n+1} - \hat{x}^{(0)}_{k-n+1}\| \tag{20}$$

thus, we prove that $\{\hat{x}^{(l)}_{k-n+1}\}_{l=0}^{\infty}$ is a Cauchy sequence and therefore it has a limit, such that $\lim_{l\to\infty} G(\hat{x}^{(l)}_{k-n+1}) = G(\hat{x}^{*}_{k-n+1}) = \hat{x}^{*}_{k-n+1}$. Then, if $c \to \infty$, it happens that

$$\|\hat{x}^{(l)}_{k-n+1} - \hat{x}^{*}_{k-n+1}\| \leq \frac{K^l}{1-K}\|\hat{x}^{(1)}_{k-n+1} - \hat{x}^{(0)}_{k-n+1}\| \tag{21}$$

If $l \to \infty$, then $\frac{K^l}{1-K} \to 0$, therefore the sequence converges to the fixed point. Furthermore, we can prove that $\hat{x}^{*}_{k-n+1}$ is unique. Suppose that $\hat{x}^{(l)}_{k-n+1} \to \hat{x}^{*}_{k-n+1}$ and $\hat{x}^{(l)}_{k-n+1} \to \hat{z}^{*}_{k-n+1}$ as $l \to \infty$. Thus,

$$\begin{aligned}\|\hat{x}^{*}_{k-n+1} - \hat{z}^{*}_{k-n+1}\| &= \|\hat{x}^{*}_{k-n+1} - \hat{x}^{(l)}_{k-n+1} + \hat{x}^{(l)}_{k-n+1} - \hat{z}^{*}_{k-n+1}\| \\ &\leq \|\hat{x}^{*}_{k-n+1} - \hat{x}^{(l)}_{k-n+1}\| + \|\hat{x}^{(l)}_{k-n+1} - \hat{z}^{*}_{k-n+1}\| \to 0\end{aligned} \tag{22}$$

such that $\hat{x}^{*}_{k-n+1} = \hat{z}^{*}_{k-n+1}$, i.e., the fixed point is unique. □

Notice that the function G is a contraction map if $\hat{x}^{(0)}_{k-n+1}$ is close enough to the fixed point $\hat{x}^{*}_{k-n+1}$ and in general, if

$$\left|\frac{\partial g_i(\hat{x}_{k-n+1})}{\partial \hat{\mathrm{x}}_{k,j}}\right| \leq K, \quad j = 1, 2, \ldots, n \tag{23}$$

On the other hand, once the root search problem has been solved for the current time instant, we can set the initial guess for the following time instant as $\hat{x}^{(0)}_{k-n+2} = \hat{x}^{*}_{k-n+1}$. The logic behind this choice is that $\hat{x}^{*}_{k-n+1}$ and the solution of the following equation system $\hat{x}_{k-n+2}$ belong to the state trajectory of the discrete-time system (2). Since we assume that the continuous-time system that originates the sampled time version is given by a Lipschitz function, it is reasonable that $\hat{x}_{k-n+1}$ is close enough to the solution and therefore, G is a contraction map.

In a further step, for simplicity we will denote $\hat{x}^{(i)}_{k-n+1}$ as $\omega^{(i)}_k$ and $Y_{[k-n+1,k]}$ just as Y_k, such that (14) is:

$$\omega^{(l+1)}_k = \omega^{(l)}_k + Y_k - H\left(\omega^{(l)}_k\right), \quad l \in \mathbb{N} \tag{24}$$

where $\omega^{(i)}_k = (\mathrm{w}^{(i)}_{1,k}, \mathrm{w}^{(i)}_{2,k}, \ldots, \mathrm{w}^{(i)}_{n,k})^T$. The last representation will be particularly useful to avoid confusion during the numerical simulation.

3.1. Δ^2-Aitken accelerated convergence

The main disadvantage of the fixed point iteration method, and therefore of the Fixed-Point observer, is its low convergence speed. Unlike the Newton observer (with quadratic convergence), the proposed algorithm presents a linearly convergent sequence. Therefore, let us incorporate the Δ^2-Aitken method to increase the convergence speed.

Consider the following iteration method

$$x^{(l+1)} = G\left(x^{(l)}\right), \quad l \in \mathbb{N} \tag{25}$$

Suppose that (25) generates a convergent sequence $\{x^{(l)}\}_{l=0}^{\infty}$ and that the iteration function G satisfies conditions of Theorem 1. Then, let us define the convergence error as

$$e^{(l)} = x^{(l)} - p \tag{26}$$

where p is a fixed point. First, let us assume that the iterative method converges geometrically, i.e.,

$$e^{(l+1)} = \mathbb{K}e^{(l)}, \quad |\mathbb{K}| < 1 \tag{27}$$

such that

$$\frac{e^{(l+2)}}{e^{(l+1)}} = \frac{e^{(l+1)}}{e^{l}} \tag{28}$$

then, we have

$$\frac{x^{(l+2)} - p}{x^{(l+1)} - p} = \frac{x^{(l+1)} - p}{x^{(l)} - p} \tag{29}$$

and by expanding terms, we obtain

$$\begin{aligned}
&\left(x^{(l+2)} - p\right)\left(x^{(l)} - p\right) = \left(x^{(l+1)} - p\right)^2 \\
&x^{(l+2)}x^{(l)} - px^{(l+2)} - px^{(l)} + p^2 = \left(x^{(l+1)}\right)^2 - 2px^{(l+1)} + p^2 \\
&2x^{(l+1)}p - x^{(l+2)}p - x^{(l)}p = \left(x^{(l+1)}\right)^2 - x^{(l+2)}x^{(l)} \\
&p\left(2x^{(l+1)} - x^{(l+2)} - x^{(l)}\right) = \left(x^{(l+1)}\right)^2 - x^{(l+2)}x^{(l)}
\end{aligned}$$

this is

$$p = \frac{x^{(l+2)}x^{(l)} - \left(x^{(l+1)}\right)^2}{x^{(l+2)} - 2x^{(l+1)} + x^{(l)}} \tag{30}$$

or equivalently

$$p = x^{(l)} - \frac{\left(x^{(l+1)} - x^{(l)}\right)^2}{x^{(l+2)} - 2x^{(l+1)} + x^{(l)}} \tag{31}$$

We can notice that under this geometric convergence assumption, the value of p can be obtained with only three consecutive values of the sequence $\{x^{(l)}\}_{l=0}^{\infty}$. However, in practice, the solution of a system of nonlinear equations rarely exhibits this class of convergence. Instead, it is much more common a convergence of the following type,

$$e^{(l+1)} = \bar{K}e^{(l)}, \quad \bar{K} = \mathbb{K} + \bar{\delta}^{(l)}, \quad l \in \mathbb{N} \tag{32}$$

Clearly, the sequence $\{e^{(l)}\}_{l=0}^{\infty}$ converges linearly to zero if and only if $|\bar{K}| < 1$. Therefore, the sequence of $\bar{\delta}^{(l)}$ should tend to zero as $l \to \infty$. Since $0 < 1 - \mathbb{K} < 1$, there exists $\bar{\delta}^{(l)}$ such that

$$0 \leq \bar{\delta}^{(l)} < 1 - \mathbb{K} < 1, \quad \forall l \in \mathbb{N} \tag{33}$$

that is due to the Archimedean property. Thus, let $\bar{\delta}^{(l)}$ be

$$\bar{\delta}^{(l)} = \frac{1 - \mathbb{K}}{\bar{c}^{l}}, \quad l \in \mathbb{N} \tag{34}$$

with constant $\bar{c} > 1$. Evidently, $\bar{\delta}^{(l)} \to 0$ as $l \to \infty$. On the other hand, according to Definition 2 we know that $x^{(l)}$ converges to the fixed point p (or $e^{(l)}$ to zero) if

$$\left\| x^{(l)} - p \right\| = \left\| e^{(l)} \right\| \leq c\rho^l, \quad \forall l \in \mathbb{N} \tag{35}$$

with $c > 0$ and $0 < \rho < 1$. Let us propose $\rho = 1 - \mathbb{K}$, such that

$$\left\| e^{(l)} \right\| \leq c(1 - \mathbb{K})^l \tag{36}$$

or equivalently

$$\left\| \bar{K} e^{(l)} \right\| \leq c(1 - \mathbb{K})^{l+1} \tag{37}$$

i.e.,

$$\left\| \left(\mathbb{K} + \frac{1 - \mathbb{K}}{\bar{c}^l} \right) e^{(l)} \right\| \leq c(1 - \mathbb{K})^{l+1} \tag{38}$$

For simplicity, let us consider $\bar{c} = c$, such that

$$\left\| \left(\mathbb{K} + \frac{1 - \mathbb{K}}{c^l} \right) e^{(l)} \right\| \leq c(1 - \mathbb{K})^{l+1} = \varepsilon^{(l)} \tag{39}$$

Then, the sequence $\varepsilon^{(l)}$ is sufficiently large for all $l \in \mathbb{N}$, if $c >> 1$. Therefore, as a direct consequence, we have that the convergence will be linear if $|\mathbb{K}| >> \bar{\delta}^{(l)}$.

Remark 3. We can notice that, since $\bar{\delta}^{(l)}$ converges to zero, for some $l_1 > l \in \mathbb{N}$ sufficiently large we have that $\bar{K} = \mathbb{K}$. Therefore, the expression (28) is valid and the proof continues in the same way.

Now, in the following we expect the quantities

$$z^{(l)} = x^{(l)} - \frac{\left(x^{(l+1)} - x^{(l)} \right)^2}{x^{(l+2)} - 2x^{(l+1)} + x^{(l)}} \tag{40}$$

to be closer to p than $x^{(l)}$. The expression (40) can be simplified with the following forward difference operator

$$\Delta x^{(l)} = x^{(l+1)} - x^{(l)}, \quad l \in \mathbb{N} \tag{41}$$

such that

$$\begin{aligned} \Delta^2 x^{(l)} =& \Delta(\Delta x^{(l)}) \\ =& \Delta(x^{(l+1)} - x^{(l)}) \\ =& \Delta x^{(l+1)} - \Delta x^{(l)} \\ =& \left(x^{(l+2)} - x^{(l+1)} \right) - \left(x^{(l+1)} - x^{(l)} \right) \\ =& x^{(l+2)} - 2x^{(l+1)} + x^{(l)} \end{aligned}$$

Therefore, expression (40) is equivalent to

$$z^{(l)} = x^{(l)} - \frac{\left(\Delta x^{(l)} \right)^2}{\Delta^2 x^{(l)}}, \quad l \in \mathbb{N} \tag{42}$$

The last expression is the well-known Δ^2-Aitken method and is illustrated by Fig. 1. Now, let us consider the following assumptions:

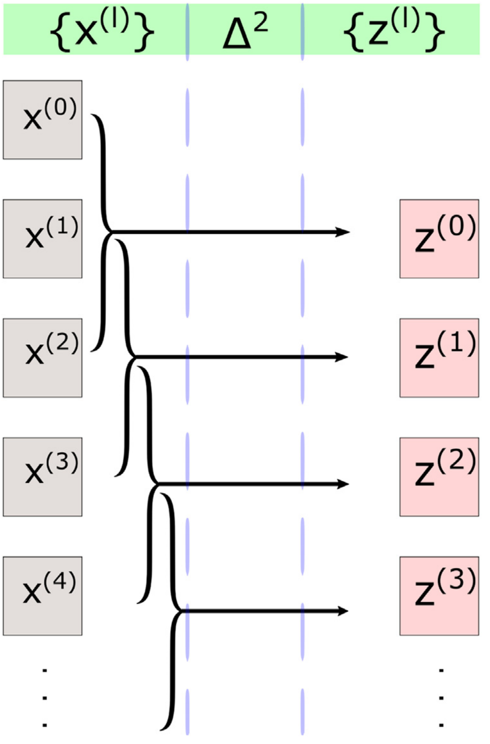


Fig. 1. Graphical scheme of the Δ^2-Aitken method: sequence $\{x^{(l)}\}_{l=0}^{\infty}$ is used to obtain a second sequence $\{z^{(l)}\}_{l=0}^{\infty}$, such that the convergence speed increases.

- $\{x^{(l)}\}_{l=0}^{\infty}$ is a sequence that converges to p
- $\{z^{(l)}\}_{l=0}^{\infty}$ is the sequence generated by the iterative method (42)

Let us consider the following variables

$$e^{(l)} = x^{(l)} - p \tag{43}$$

and

$$\tau^{(l)} = z^{(l)} - p \tag{44}$$

Moreover, suppose that

$$e^{(l+1)} = \bar{K}e^{(l)} = \left(\mathbb{K} + \bar{\delta}^{(l)}\right)e^{(l)} \neq 0, \quad \forall l \in \mathbb{N} \tag{45}$$

where $|\bar{K}| < 1$ and $\bar{\delta}^{(l)} \to 0$ as $l \to \infty$.

Then, the following result ensures that the Δ^2-Aitken method accelerates the linear type convergence of the iteration method.

Theorem 3. *The sequence* $\{z^{(l)}\}_{l=0}^{\infty}$ *converges faster than sequence* $\{x^{(l)}\}_{l=0}^{\infty}$ *in the sense*

$$\frac{\tau^{(l)}}{e^{(l)}} = \frac{z^{(l)} - p}{x^{(l)} - p} \to 0 \quad \text{as} \quad l \to \infty \tag{46}$$

Proof. From (43) and (44), we have that

$$x^{(l)} = e^{(l)} + p \tag{47}$$

and

$$z^{(l)} = \tau^{(l)} + p = x^{(l)} - \frac{\left(\bar{\delta}^{(l)} x^{(l)}\right)^2}{\Delta^2 x^{(l)}} \tag{48}$$

$$= x^{(l)} - \frac{\left(x^{(l+1)} - x^{(l)}\right)^2}{x^{(l+2)} - 2x^{(l+1)} + x^{(l)}} \tag{49}$$

Then, substituting (47) in (48) we get

$$z^{(l)} = e^{(l)} + p - \frac{\left(e^{(l+1)} - e^{(l)}\right)^2}{e^{(l+2)} - 2e^{(l+1)} + e^{(l)}} \tag{50}$$

such that

$$\tau^{(l)} = e^{(l)} - \frac{\left(e^{(l+1)} - e^{(l)}\right)^2}{e^{(l+2)} - 2e^{(l+1)} + e^{(l)}} \tag{51}$$

On the other hand

$$e^{(l+1)} - e^{(l)} = \left(\mathbb{K} + \bar{\delta}^{(l)}\right)e^{(l)} - e^{(l)} = \left[(\mathbb{K} - 1) + \bar{\delta}^{(l)}\right]e^{(l)} \tag{52}$$

therefore, we have

$$e^{(l+2)} - 2e^{(l+1)} + e^{(l)} = \left(\mathbb{K} + \bar{\delta}^{(l+1)}\right)e^{(l+1)} - 2\left(\mathbb{K} + \bar{\delta}^{(l)}\right)e^{(l)} + e^{(l)} \tag{53}$$

Then, since $e^{(l+1)} = (\mathbb{K} + \bar{\delta}^{(l)})e^{(l)}$, we have that

$$\begin{aligned} e^{(l+2)} - 2e^{(l+1)} + e^{(l)} &= \left(\mathbb{K} + \bar{\delta}^{(l+1)}\right)\left(\mathbb{K} + \bar{\delta}^{(l)}\right)e^{(l)} - 2\left(\mathbb{K} + \bar{\delta}^{(l)}\right)e^{(l)} + e^{(l)} \\ &= \left[\mathbb{K}^2 + \bar{\delta}^{(l)}\mathbb{K} + \bar{\delta}^{(l+1)}\mathbb{K} + \bar{\delta}^{(l+1)}\bar{\delta}^{(l)} - 2\mathbb{K} - 2\bar{\delta}^{(l)} + 1\right]e^{(l)} \\ &= \left[(\mathbb{K} - 1)^2 + \mathbb{K}(\bar{\delta}^{(l)} + \bar{\delta}^{(l+1)}) + \bar{\delta}^{(l+1)}\bar{\delta}^{(l)} - 2\bar{\delta}^{(l)}\right]e^{(l)} \\ &= \left[(\mathbb{K} - 1)^2 + \lambda^{(l)}\right]e^{(l)} \end{aligned} \tag{54}$$

where $\lambda^{(l)} = \mathbb{K}(\bar{\delta}^{(l)} + \bar{\delta}^{(l+1)}) + \bar{\delta}^{(l+1)}\bar{\delta}^{(l)} - 2\bar{\delta}^{(l)}$ and satisfies

$\lambda^{(l)} \to 0 \quad \text{as} \quad l \to \infty$

By substituting (52) and (54) in (51), we obtain

$$\begin{aligned} \tau^{(l)} &= e^{(l)} - \frac{\left[(\mathbb{K} - 1) + \bar{\delta}^{(l)}\right]^2 \left(e^{(l)}\right)^2}{\left[(\mathbb{K} - 1)^2 + \lambda^{(l)}\right]e^{(l)}} \\ &= e^{(l)} - \frac{\left[(\mathbb{K} - 1) + \bar{\delta}^{(l)}\right]^2}{(\mathbb{K} - 1)^2 + \lambda^{(l)}} e^{(l)} \\ &= \frac{\left[(\mathbb{K} - 1)^2 + \lambda^{(l)} - (\mathbb{K} - 1)^2 - 2\bar{\delta}^{(l)}(\mathbb{K} - 1) - \left(\bar{\delta}^{(l)}\right)^2\right]e^{(l)}}{(\mathbb{K} - 1)^2 + \lambda^{(l)}} \end{aligned} \tag{55}$$

this is

$$\tau^{(l)} = \frac{\lambda^{(l)} - 2\bar{\delta}^{(l)}(\mathbb{K} - 1) - \left(\bar{\delta}^{(l)}\right)^2}{(\mathbb{K} - 1)^2 + \lambda^{(l)}} e^{(l)} \tag{56}$$

such that

$$\frac{\tau^{(l)}}{e^{(l)}} = \frac{\lambda^{(l)} - 2\bar{\delta}^{(l)}(\mathbb{K}-1) - \left(\bar{\delta}^{(l)}\right)^2}{(\mathbb{K}-1)^2 + \lambda^{(l)}} \to 0 \quad \text{as} \quad l \to \infty \tag{57}$$

Hence, the expression (57) implies that $\{\tau^{(l)}\}_{l=0}^{\infty} \to 0$ faster than $\{e^{(l)}\}_{l=0}^{\infty}$. □

3.2. Steffensen accelerate convergence

In the above proof, we can note that if $\bar{\delta}^{(l)}$ is such that $\lim_{l\to 0} \frac{\bar{\delta}^{(l+1)}}{\bar{\delta}^{(l)}} = \beta$, $\beta \in \mathbb{R}$, then the convergence of sequence $\{z^{(l)}\}_{l=0}^{\infty}$ can be accelerated once more by means of the Δ^2-Aitken Method. Therefore, we propose to implement the Steffensen method, which is described below.

Remark 4. Observe that only in this subsection, the subindex of $x_i^{(j)}$ denotes that the three-element sequence $\{x_i^{(j)}\}_{j=0}^{2}$ was generated after applying $i-1$ times the Δ^2-Aitken method.

Given a initial guess $x_1^{(0)}$, we calculated two additional values by means of the iterative method (25), i.e., we obtain the three-element sequence $\{x_1^{(0)}, x_1^{(1)}, x_1^{(2)}\}$. Then, the Δ^2-Aitken method is applied, such that

$$x_2^{(0)} = x_1^{(0)} - \frac{\left(x_1^{(1)} - x_1^{(0)}\right)^2}{x_1^{(2)} - 2x_1^{(1)} + x_1^{(0)}} \tag{58}$$

Next, the value $x_2^{(0)}$ is used as new initial guess, such that two additional values are obtained, i.e.,

$$x_2^{(1)} = G\left(x_2^{(0)}\right),$$
$$x_2^{(2)} = G\left(x_2^{(1)}\right),$$

and the Aitken method is applied again

$$x_3^{(0)} = x_2^{(0)} - \frac{\left(x_2^{(1)} - x_2^{(0)}\right)^2}{x_2^{(2)} - 2x_2^{(1)} + x_2^{(0)}} \tag{59}$$

Thus, this process repeats until the fixed point is obtained. Fig. 2 graphically describes this process.

3.3. Stop criterion

The results shown above prove that the sequence generated by the iteration method of the state observer will converge to a unique fixed point. However, this analysis is valid just for the set of real numbers, and when the method is implemented in a digital computer, due to finite precision issues, the method will inevitably have a persistent error, such as is shown in [28]. In other words, depending on the precision of the floating point number's representation, the sequence will converge to a value close to the fixed point, and in the best scenario, to the fixed point itself.

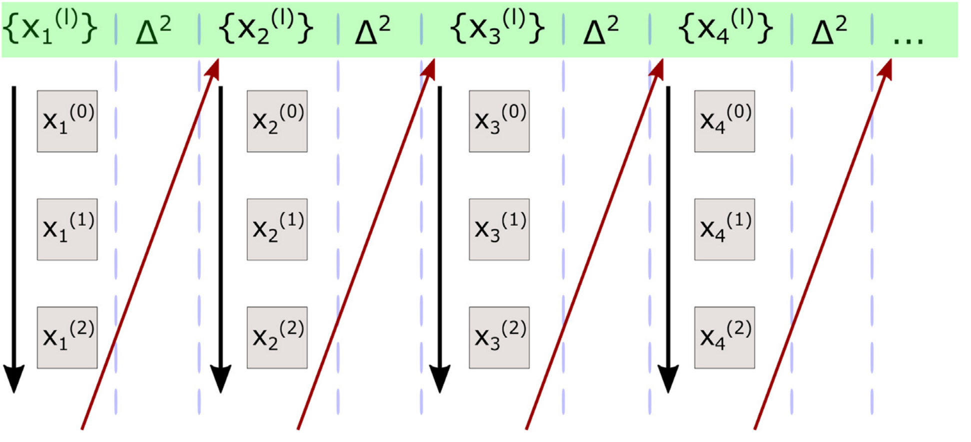


Fig. 2. Proposed Steffensen method. A three-element sequence is generated and then the Δ^2-Aitken method is applied. The result is then used to generate a new sequence of three elements and the Aitken method is applied one more time and so on.

Regardless of whether the method converges to the real fixed point or a near value, notice that the iteration method of the Fixed-Point observer does not specify a maximum number of iterations, and therefore, the method (24) could continue indefinitely. Thus, to avoid unnecessary iterations, we propose to implement a stop criterion.

Let $G : [a, b] \to [a, b]$ be a Lipschitz function with Lipschitz constant L, $0 \leq L < 1$ and consider the iterative method (25) with fixed point p, such that

$$|x^{(l+q)} - p| = |G(x^{(l+q-1)}) - G(p)| \leq L|x^{(l+q-1)} - p| \tag{60}$$

i.e.,

$$L|x^{(l+q-1)} - p| = L|G(x^{(l+q-2)}) - G(p)| \leq L^2|x^{(l+q-2)} - p| \tag{61}$$

hence recursively, we have

$$|x^{(l+q)} - p| \leq L^q|x^{(l)} - p| \tag{62}$$

Thus, if $G'(p) \neq 0$, the number of iterations necessary to reduce the convergence error by a factor of $10^{-\bar{m}}$ satisfies

$$L^q \leq 10^{-\bar{m}} \tag{63}$$

such that

$$q \leq \frac{\bar{m}}{\log_{10}\left(\frac{1}{L}\right)} \tag{64}$$

Remark 5. The expression (64) indicates the number of iterations q necessary to reach a desired accuracy $\bar{m}$. In [18–20], a similar criterion is established. However, the authors define this criterion considering a fractional base logarithm since they take into account specific compact sets.

Additionally, we can establish a second criterion based on the following observations

Remark 6. Let $\varepsilon > 0$, such that

$$|x^{(l+1)} - x^{(l)}| \leq (1 - L)\varepsilon \tag{65}$$

therefore

$$|x^{(l)} - p| \leq \varepsilon \tag{66}$$

The proof is immediate, since

$$\begin{aligned} |x^{(l)} - p| &= |x^{(l)} - G(x^{(l)}) + G(x^{(l)}) - p| \\ &\leq |x^{(l)} - x^{(l+1)}| + L|x^{(l)} - p| \end{aligned} \tag{67}$$

i.e.,

$$|x^{(l)} - p|(1 - L) \leq |x^{(l)} - x^{(l+1)}| \tag{68}$$

Then, since $0 \leq L < 1 \Rightarrow (1 - L) > 0$, we have

$$|x^{(l)} - p| \leq \frac{1}{1-L}|x^{(l+1)} - x^{(l)}| \leq \frac{1}{1-L}(1-L)\varepsilon = \varepsilon \quad \square \tag{69}$$

Remark 7. Since the convergence error monotonically tends to zero, we have that

$$|x^{(l+1)} - p| = |G(x^{(l)}) - G(p)| \leq L|x^{(l)} - p| < |x^{(l)} - p| \tag{70}$$

therefore, the previous iteration has a greater convergence error than the current iteration. □

Based on the above, we can establish that

$$|x^{(l+1)} - x^{(l)}| = |G(x^{(l)}) - G(x^{(l-1)})| \leq L|x^{(l)} - x^{(l-1)}| < |x^{(l)} - x^{(l-1)}| \tag{71}$$

In practice, due to finite precision errors, it may be the case that

$$|x^{(l+1)} - x^{(l)}| \geq |x^{(l)} - x^{(l-1)}| \tag{72}$$

for some l. In such scenario, the method is stopped. In general, the iteration method will stop when

$$|x^{(l)} - x^{(l-1)}| < 10^{-\bar{m}} \tag{73}$$

Remark 8. Note that if the Aitken's method is included in the Fixed-Point observer scheme, we could also have the following scenarios: 1) If the exact solution is reached, then two iterations later the Aitken method will have a singularity (division by zero). 2) The method can have an iteration loop due to floating-point issues (finite precision). Therefore, whenever the sequence presents three equal consecutive values, the algorithm also must be stopped.

3.4. Newton observer

For comparison purposes, we briefly introduce the Newton observer algorithm [10]. This is based on the well-known Newton-Raphson method and is given by

$$\begin{aligned} \hat{x}_{k-n+1}^{(l+1)} = \hat{x}_{k-n+1}^{(l)} + \left[\frac{\partial H\left(\hat{x}_{k-n+1}^{(l)}\right)}{\partial \hat{x}_{k-n+1}}\right]^{-1} \left(Y_{[k-n+1,k]} - H\left(\hat{x}_{k-n+1}^{(l)}\right)\right), \\ l = 0, 1, 2, \ldots, r-1 \end{aligned} \tag{74}$$

where $\hat{x}^{(i)}_{k-n+1} = (\hat{x}^{(i)}_{1,k-n+1}, \hat{x}^{(i)}_{2,k-n+1}, \dots, \hat{x}^{(i)}_{n,k-n+1})^T$ and the design parameter r is the maximum number of iterations. As in the Fixed-Point observer, the state estimation at the time instant k is found by propagating $n-1$ steps forward the value $\hat{x}^{(r)}_{k-n+1}$, i.e.,

$$\hat{x}_k = f^{n-1}\left(\hat{x}^{(r)}_{k-n+1}\right) \tag{75}$$

and then, for the following time instant, we select $\hat{x}^{(r)}_{k-n+1}$ as initial guess of the iteration method.

To avoid confusion, let us consider as well the notation $\hat{x}^{(i)}_{k-n+1} = \omega^{(i)}_k$ and $Y_{[k-n+1,k]} = Y_k$, such that the Newton observer is

$$\omega_k^{(l+1)} = \omega_k^{(l)} + \left[\frac{\partial H\left(\omega_k^{(l)}\right)}{\partial \omega_k}\right]^{-1} \left(Y_k - H\left(\omega_k^{(l)}\right)\right), \quad l = 0, 1, 2, \dots, r-1 \tag{76}$$

The main disadvantage of the Newton observer is the computation of the Jacobian matrix and the corresponding inverse at each iteration and for each time instant. This continuous evaluation of the partial derivatives is time-consuming and increases the computational cost [18–20]. There are some strategies that allow avoiding the Jacobian matrix, such as the Finite Difference Approximations method, Variational Equation method and the Maximum Principle method (see [17,19,29] for further details). However, the first requires solving a system of n equations n times in the interval between time instants k and $k+n-1$. In addition, it requires identifying additional parameters, which depend on the time scale, the system parameters and the dynamics of the model. On the other hand, the second method requires solving n^2 equations between time instants k and $k+n-1$. Meanwhile, in the last method, n equations must be solved n times backward by considering different integration intervals.

Remark 9. If the overall change around the point of interest is zero, the inverse of the Jacobian matrix does not exist and therefore, the Newton observer fails. This scenario would also imply that the system is non-observable such that is not possible to estimate the unknown states from the available output.

4. Example and numerical results

In the following section, we present the implementation of the Fixed-Point observer to estimate the unknown states of a modified Chua chaotic attractor [30]. The Chua's system is the mathematical modelling of the well known Chua's circuit, a simple circuit that exhibits chaotic behaviour and was created to produce nonperiodic oscillations, which have, both theoretical and practical applications, such as in secure communications [31–34]. The equations that describe the attractor are the following:

$$\begin{aligned}
\dot{x}_1(t) &= \alpha\left[x_2(t) + b\sin\left(\frac{\pi x_1(t)}{2a} + d\right)\right] \\
\dot{x}_2(t) &= x_1(t) - x_2(t) + x_3(t) \\
\dot{x}_3(t) &= -\beta x_2(t) \\
y(t) &= x_1(t)
\end{aligned} \tag{77}$$

where the state vector is $x = (x_1, x_2, x_3)^T$. Meanwhile, the parameters α, β, a, b and d are constant values. An alternative discrete-time representation of system (77) can be obtained via a forward finite differences discretization scheme, such that the T-sampled system is:

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + \alpha T\left[x_{2,k} + b\sin\left(\frac{\pi x_{1,k}}{2a} + d\right)\right] = f_1(x_k) \\ x_{2,k+1} &= x_{2,k} + T\left(x_{1,k} - x_{2,k} + x_{3,k}\right) = f_2(x_k) \\ x_{3,k+1} &= x_{3,k} - \beta T x_{2,k} = f_3(x_k) \\ y_k &= x_{1,k} \end{aligned} \tag{78}$$

with $x_k = \left(x_{1,k}, x_{2,k}, x_{3,k}\right)^T$. Then, we can design a Fixed-Point estimator for the system (78).

4.1. Fixed-Point estimator design for modified Chua chaotic attractor

One can note that the system's dimension is $n = 3$. Therefore, we have

$$Y_k = \begin{bmatrix} y_{k-2} \\ y_{k-1} \\ y_k \end{bmatrix} \tag{79}$$

and on the other hand,

$$H(x_{k-2}) = \begin{bmatrix} x_{1,k-2} \\ x_{1,k-2} + \alpha T\left[x_{2,k-2} + b\sin\left(\frac{\pi x_{1,k-2}}{2a} + d\right)\right] \\ \left(\begin{array}{c} x_{1,k-2} + \alpha T\left[x_{2,k-2} + b\sin\left(\frac{\pi x_{1,k-2}}{2a} + d\right)\right] \\ +\alpha T\left\{x_{2,k-2} + T\left[x_{1,k-2} - x_{2,k-2} + x_{3,k-2}\right]\right. \\ \left. +b\sin\left(\frac{\pi}{2a}\left\{x_{1,k-2} + \alpha T\left[x_{2,k-2} + b\sin\left(\frac{\pi x_{1,k-2}}{2a} + d\right)\right]\right\} + d\right)\right\} \end{array} \right) \end{bmatrix} \tag{80}$$

Then, by considering the representation (24) of the Fixed-Point observer, we have that

$$\begin{aligned} \omega_k^{(l+1)} &= \omega_k^{(l)} + Y_k - H\left(\omega_k^{(l)}\right) = G(\omega_k^{(l)}) \\ &= \begin{bmatrix} w_{1,k}^{(l)} \\ w_{2,k}^{(l)} \\ w_{3,k}^{(l)} \end{bmatrix} + \begin{bmatrix} y_{k-2} \\ y_{k-1} \\ y_k \end{bmatrix} \\ &- \begin{bmatrix} w_{1,k}^{(l)} \\ w_{1,k}^{(l)} + \alpha T\left[w_{2,k}^{(l)} + b\sin\left(\frac{\pi w_{1,k}^{(l)}}{2a} + d\right)\right] \\ \left(\begin{array}{c} w_{1,k}^{(l)} + \alpha T\left[w_{2,k}^{(l)} + b\sin\left(\frac{\pi w_{1,k}^{(l)}}{2a} + d\right)\right] \\ +\alpha T\left\{w_{2,k}^{(l)} + T\left[w_{1,k}^{(l)} - w_{2,k}^{(l)} + w_{3,k}^{(l)}\right]\right. \\ \left. +b\sin\left(\frac{\pi}{2a}\left\{w_{1,k}^{(l)} + \alpha T\left[w_{2,k}^{(l)} + b\sin\left(\frac{\pi w_{1,k}^{(l)}}{2a} + d\right)\right]\right\} + d\right)\right\} \end{array} \right) \end{bmatrix} \end{aligned} \tag{81}$$

where

$$G\left(\omega_k^{(i)}\right) = \left[g_1\left(\omega_k^{(i)}\right), g_2\left(\omega_k^{(i)}\right), g_3\left(\omega_k^{(i)}\right)\right]^T \tag{82}$$

Then, to guarantee convergence of the iteration method we need to find acceptable initial conditions. To achieve this, we analyze the partial derivatives of G and select initial conditions such that the absolute value of each partial derivative is less than 1 (contraction map). It is easy to note that

$$\left|\frac{\partial g_1\left(\omega_k^{(i)}\right)}{\partial \mathsf{w}_{1,k}}\right| = 0, \quad \left|\frac{\partial g_1\left(\omega_k^{(i)}\right)}{\partial \mathsf{w}_{2,k}}\right| = 0, \quad \left|\frac{\partial g_1\left(\omega_k^{(i)}\right)}{\partial \mathsf{w}_{3,k}}\right| = 0, \quad \left|\frac{\partial g_2\left(\omega_k^{(i)}\right)}{\partial \mathsf{w}_{3,k}}\right| = 0 \tag{83}$$

On the other hand, we have that

$$\begin{aligned}
\left|\frac{\partial g_2\left(\omega_k^{(i)}\right)}{\partial \mathsf{w}_{1,k}}\right| &= \left|1 + \frac{\alpha T b\pi}{2a}\bar{z}_1\right| < 1, \\
\left|\frac{\partial g_2\left(\omega_k^{(i)}\right)}{\partial \mathsf{w}_{2,k}}\right| &= |1 - \alpha T| < 1, \\
\left|\frac{\partial g_3\left(\omega_k^{(i)}\right)}{\partial \mathsf{w}_{1,k}}\right| &= \left|1 + \frac{\alpha T b\pi}{2a}\left(\bar{z}_1 + \bar{z}_2 + \frac{\alpha T b\pi}{2a}\bar{z}_1\bar{z}_2\right) + \alpha T^2\right| < 1, \\
\left|\frac{\partial g_3\left(\omega_k^{(i)}\right)}{\partial \mathsf{w}_{2,k}}\right| &= \left|2\alpha T - \alpha T^2 + \frac{\alpha^2 T^2 \pi b}{2a}\bar{z}_2\right| < 1, \\
\left|\frac{\partial g_3\left(\omega_k^{(i)}\right)}{\partial \mathsf{w}_{3,k}}\right| &= \left|\alpha T^2 - 1\right| < 1
\end{aligned} \tag{84}$$

with

$$\begin{aligned}
\bar{z}_1 &= \cos\left(\frac{\pi \mathsf{w}_{1,k}^{(i)}}{2a} + d\right) \\
\bar{z}_2 &= \cos\left(\frac{\pi \mathsf{w}_{1,k}^{(i)}}{2a} + \frac{\alpha T \pi \mathsf{w}_{2,k}^{(i)}}{2a} + \frac{\alpha T \pi b}{2a}\sin\left(\frac{\pi \mathsf{w}_{1,k}^{(i)}}{2a} + d\right) + d\right)
\end{aligned}$$

The system of inequalities (84) can be solved by means of a mathematical software. Thus, with the help of Wolfram Mathematica we find that the inequalities are satisfied if

$$T \in \left(0, \frac{2}{\alpha}\right) \tag{85}$$

4.2. Newton observer design for modified Chua chaotic attractor

For comparison purposes, we implement the Newton observer to the Chua attractor. Thus, according to the Newton observer algorithm, we have that the Jacobian matrix is

$$\frac{\partial H(\omega_k^{(i)})}{\partial \omega_k} = \begin{bmatrix} 1 & 0 & 0 \\ \bar{H}_{2,1} & \alpha T & 0 \\ \bar{H}_{3,1} & \bar{H}_{3,2} & \alpha T^2 \end{bmatrix} \tag{86}$$

with

$$\bar{H}_{2,1} = 1 + \frac{\alpha T b\pi}{2a}\cos\left(\frac{\pi \mathrm{w}_{1,k}^{(i)}}{2a} + d\right)$$

$$\bar{H}_{3,1} = \bar{H}_{2,1} + \alpha T^2 + \left[\frac{\alpha T\pi b}{2a} + \frac{\alpha^2 T^2 b^2 \pi^2}{4a^2}\cos\left(\frac{\pi \mathrm{w}_{1,k}^{(i)}}{2a} + d\right)\right]C$$

$$\bar{H}_{3,2} = 2\alpha T - \alpha T^2 + \frac{\alpha^2 T^2 b\pi}{2a}C$$

and where

$$C = \cos\left(\frac{\pi}{2a}\left\{\mathrm{w}_{1,k}^{(i)} + \alpha T\left[\mathrm{w}_{2,k}^{(i)} + b\sin\left(\frac{\pi \mathrm{w}_{1,k}^{(i)}}{2a} + d\right)\right]\right\} + d\right)$$

Hence, the Newton observer is given by

$$\omega_k^{(l+1)} = \omega_k^{(l)} + \left[\frac{\partial H(\omega_k^{(l)})}{\partial \omega_k}\right]^{-1}\left[Y_k - H(\omega_k^{(l)})\right] \tag{87}$$

for $l = 0, 1, 2, \dots, r-1$ and with Y_k and $H(\omega_k^{(l)})$ as in the Fixed-Point observer. Note that we have already considered the representation (76) of the Newton observer. Finally, the state estimate at time instant k is

$$\hat{x}_k = \begin{pmatrix}\hat{\mathrm{x}}_{1,k}\\ \hat{\mathrm{x}}_{2,k}\\ \hat{\mathrm{x}}_{3,k}\end{pmatrix} = \begin{pmatrix} f_1^2\left(\mathbf{w}_{1,k}^{(r)}, \mathbf{w}_{2,k}^{(r)}, \mathbf{w}_{3,k}^{(r)}\right)\\ f_2^2\left(\mathbf{w}_{1,k}^{(r)}, \mathbf{w}_{2,k}^{(r)}, \mathbf{w}_{3,k}^{(r)}\right)\\ f_3^2\left(\mathbf{w}_{1,k}^{(r)}, \mathbf{w}_{2,k}^{(r)}, \mathbf{w}_{3,k}^{(r)}\right)\end{pmatrix} \tag{88}$$

4.3. Results and comparison discussion

Now, we show the numerical results obtained by the Fixed-Point estimator. These results are compared with those obtained by the Newton observer, a discrete-time Luenberger nonlinear observer and a discrete-time sliding mode observer.

Let us consider the following parameters for the modified Chua attractor: $\alpha = 10.82$, $\beta = 14.286$, $a = 1.3$, $b = 0.11$ and $d = 1$. Therefore, by considering (85), we set the sampling time as $T = 0.05$.

For the Fixed-Point estimator we consider $\omega_0^{(0)} = [5, 10, -5]^T$ as initial guess and $\bar{m} = 10^{-5}$ for its stop criterion. The same initial guess is considered for the Newton observer

On the other hand, we implement the following discrete-time Luenberger nonlinear observer:

$$\begin{aligned} {}_L\hat{x}_{k+1} &= f({}_L\hat{x}_k) + \bar{L}\left(y_k - {}_L\hat{y}_k\right) \\ {}_L\hat{y}_k &= {}_L\hat{x}_{1,k} \end{aligned} \tag{89}$$

where ${}_L\hat{x}_k$ and ${}_L\hat{y}_k$ are the Luenberger observer state vector and output, respectively. $\bar{L} \in \mathbb{R}^3$ is the Luenberger gain, which was selected as $\bar{L} = [0.5, 0.4, 0.2]^T$. The initial conditions for this observer are ${}_L\hat{x}_0 = [5, 10, -5]^T$.

In addition, we consider the following sliding mode observer:

$$\begin{aligned} {}_{SM}\hat{x}_{k+1} &= f({}_{SM}\hat{x}_k) + Q\big(y_k - {}_{SM}\hat{y}_k\big) + S(k) \\ {}_{SM}\hat{y}_k &= {}_{SM}\hat{x}_{1,k} \end{aligned} \tag{90}$$

where ${}_{SM}\hat{x}_k$ and ${}_{SM}\hat{y}_k$ are state vector and output of the observer, respectively. $Q \in \mathbb{R}^3$ is a gain vector and $S(k)$ is a saturation function given by

$$S(k) = R \quad sat\left(\frac{y_k - {}_{SM}\hat{y}_k}{\gamma}\right) \tag{91}$$

with $R = [r_1, r_2, r_3]^T$, $r_1 > 0, r_2 > 0, r_3 > 0$ and $\gamma > 0$. The following values are selected for these parameters: $Q = [0.1, 0.2, 0.1]^T$, $R = [4.1, 2.1, 3.9]^T$ and $\gamma = 10$. Meanwhile, the initial conditions are ${}_{SL}\hat{x}_0 = [5, 10, -5]^T$. It is worth mentioning that the gain parameters of this and the last observer, were chosen based on an LMI approach. All the following simulations were performed on a machine with an Intel Core i7-4770k, 8 GB memory RAM installed and Matlab 2020b.

Figs. 3 and 4 show the estimates made by each of the state observers. All of them are capable of an adequate state estimation. However, we can appreciate that the estimates of the Fixed-Point and the Newton estimators are closer to the real state values since the very first time instants. This is due to the nature of these. Remember that for any time instant a root search problem is solved, this is, an accurate estimation is obtained since the very first time instants, as is shown in the figure. On the other hand, the estimates made by the Luenberger and sliding mode observers present a transition behaviour, i.e., converge to the actual state after several time instants.

In Fig. 5 we can observe the estimation error of each observer. Note that the estimation error is defined for any time instant k as the difference between the real state and the estimated value ($e_k = x_k - \hat{x}_k$). In this figure we note again how the Luenberger and sliding mode observers require several time instants before converging to the real state. On the other hand, we can observe that the estimates of the Fixed-Point and Newton estimators present certain noise. Nonetheless, let us consider the mean square error (MSE):

$$MSE = \frac{1}{M}\sum_{i=1}^{M}\big(x_i - \hat{x}_i\big)^2 \tag{92}$$

where M is the total number of time instants of the simulation. Then, when we calculate the MSE (considering the whole interval) of each estimator, we find out that the Fixed-Point estimator has the best performance.

Fixed-Point observer:	$MSE = 0.0165$
Newton observer:	$MSE = 0.0180$
Luenberger observer:	$MSE = 0.3335$
Sliding-Mode observer:	$MSE = 0.3025$

It is clear that as the number of time instants considered in the simulation grows, the performance of all observers tends to be equal. Although, it is important to have in mind that the Fixed-Point estimator, does not require determining a great number of additional parameters and its estimates are close to the real state values since the very first-time instants.

Finally, in Fig. 6 we observe the sequences generated at time instant $k = 3$ by the Newton observer (a) and by the Fixed-Point estimator considering the different accelerated convergence

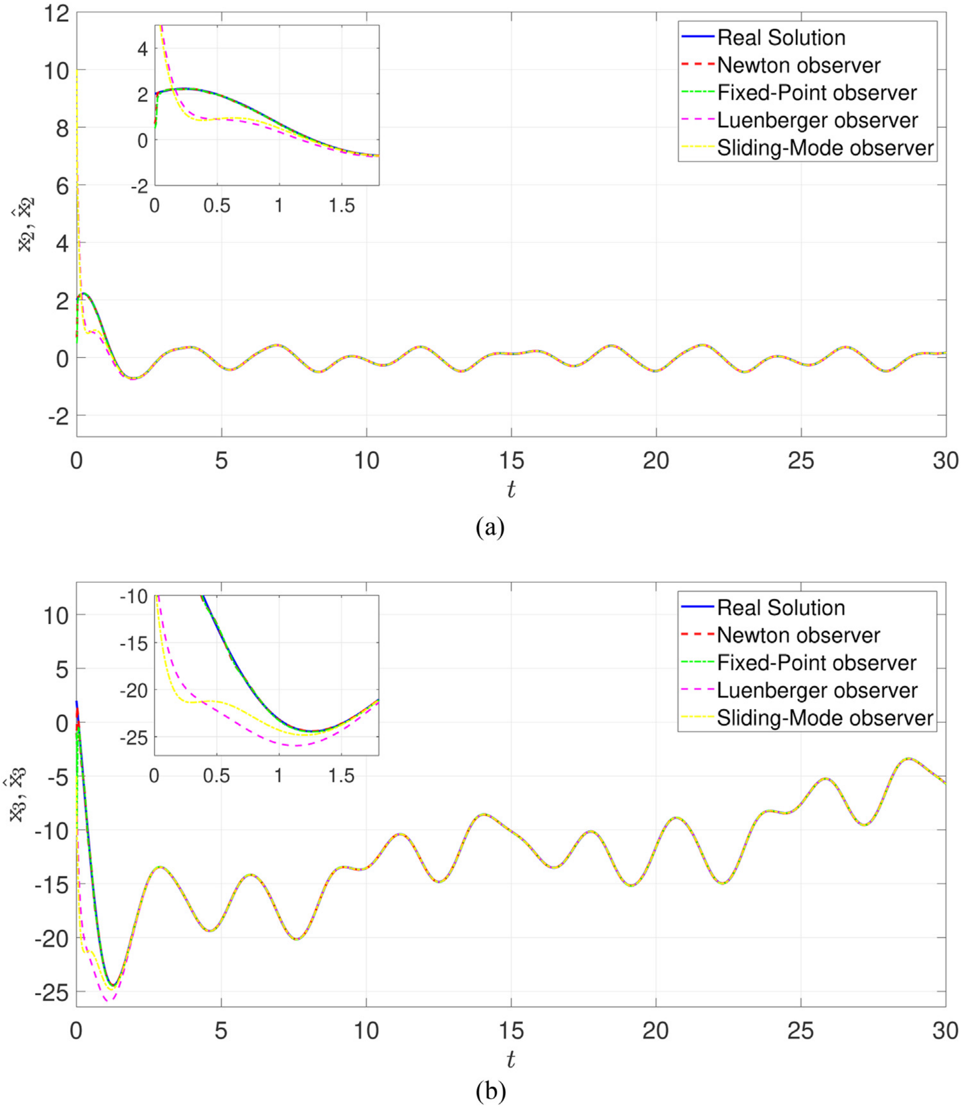


Fig. 3. Modified Chua chaotic attractor: estimates obtained by the different observers, a) $\hat{x}_2$ and b) $\hat{x}_3$.

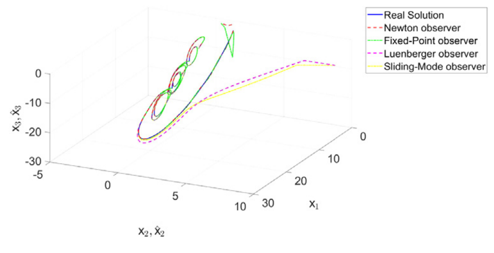
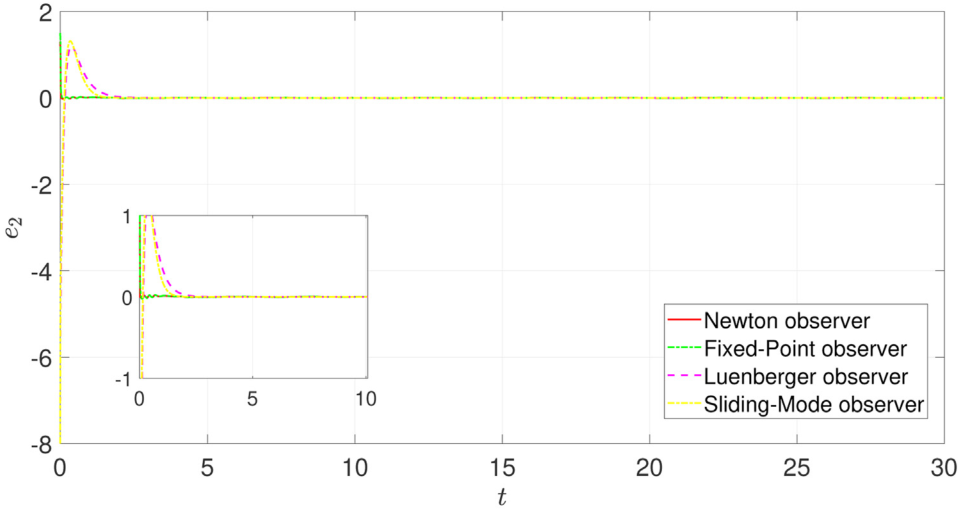
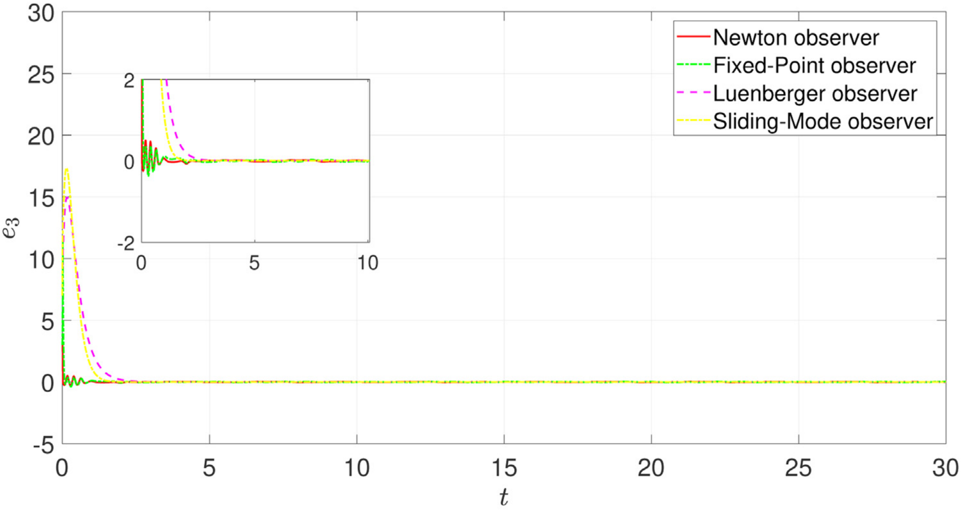


Fig. 4. Modified Chua chaotic attractor space state and the different state estimates.



(a)

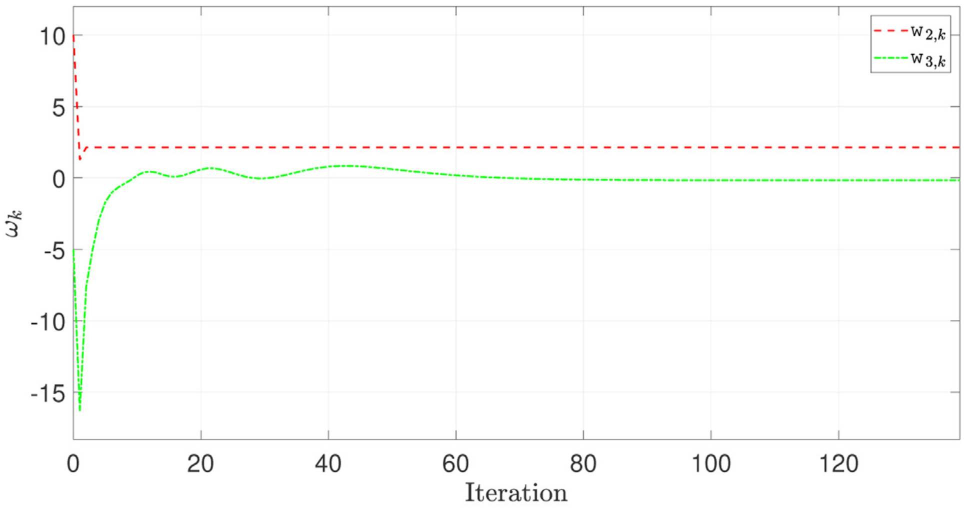


(b)

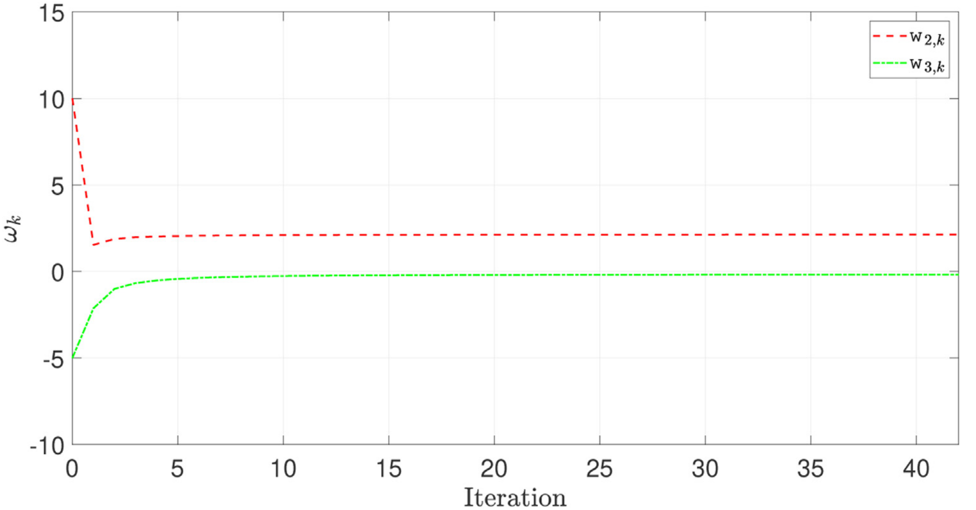
Fig. 5. Modified Chua chaotic attractor estimation error for the different state observers: a) $e_2 = \mathrm{x}_{2,k} - \hat{\mathrm{x}}_{2,k}$ and b) $e_3 = \mathrm{x}_{3,k} - \hat{\mathrm{x}}_{3,k}$.

schemes. The red line represents the sequence $\{\mathrm{w}_{2,k}^{(l)}\}_{l=0}^{\infty}$, while the green line corresponds to the sequence $\{\mathrm{w}_{3,k}^{(l)}\}_{l=0}^{\infty}$. We show the sequences at this time instant since it is when the initial guess is farthest from the fixed point.

Observe that all sequences converge to the fixed point $\left[y_k, 2.1428, -0.1648\right]^T$. However, the convergence speed of each is different. The Newton observer converges faster since it only requires three iterations. It is followed by the Fixed-Point estimator with Steffensen method (42 iterations), then with the Aitken method (139 iterations) and finally without any accelerated convergence scheme (284 iterations).



(a)



(b)

Fig. 6. Sequences convergence at time instant $k = 3$: a) Newton observer, b) Fixed-Point estimator with no accelerated convergence, c) Fixed-Point estimator with Δ^2-Aitken accelerated convergence and d) Fixed-Point estimator with Steffensen accelerated convergence.

Although the Newton observer requires very few iterations, compared to the Fixed-Point estimator, the latter has a lower computational cost. This is reflected in the elapsed time required to execute the algorithms: 0.040140 seconds for the Fixed-Point estimator (Steffensen method) and 0.250582 seconds for the Newton observer.

5. Conclusions

In this work, we have presented an alternative algorithm for the state estimation of discrete-time nonlinear systems. We named this algorithm as Fixed-Point observer with Steffensen-

Aitken accelerated convergence. We have addressed the state estimation problem as a set of consecutive fixed point iteration problems. We relied on the fixed point concept to design the observer and, in particular, to choose the sampling time and the initial guess of the iteration method. Additionally, we have used the Δ^2-Aitken method to increase the convergence speed. Two schemes were considered: 1) using only the Aitken method and 2) applying the Aitken method multiple times (Steffensen method).

Our proposal has a simple structure, does not require the identification of a great number of additional parameters (such as gains), its estimates are close to the real state since the very first-time instants and compared to a similar algorithm such as the Newton observer, does not require complex calculations. Due to all this, the Fixed-Point estimator is less time-consuming and easy to implement. In addition, to avoid unnecessary calculations, the algorithm incorporates a stop criterion for the iteration method. All the corresponding mathematical analyses have been presented.

The numerical example shows that our proposal is capable of reconstructing the unknown variables from a discrete-time sample. Moreover, the Fixed-Point estimator has the best performance. The last claim is validated by the MSE criterion. Besides, the results show that the second convergence acceleration scheme (Steffensen method) has indeed the best performance. As can be observed in Fig. 6c and d, with the Steffensen approach, the necessary number of iterations before satisfying the stop criterion and therefore the specified convergence error, decreases from 130 to 43. This of course traduces into a computational cost even lower than the Fixed-Point observer can have by itself.

For the Fixed-Point estimator, it is necessary to analyze the partial derivatives of its iteration method to find the conditions that ensure its convergence. Therefore, the initial guess is restricted to a specific set. However, this is a common issue in all state observers that demands more attention. On the other hand, numerical results show that the iteration method still requires a considerable amount of iterations before converging. Therefore, in the future we pretend to implement an iterative method able to further reduce the number of iterations and therefore, the computational cost. Also, we will try to extend the set of acceptable initial guesses of the algorithm. At this moment, our first approach is based on the iterative Gauss-Seidel method for systems of nonlinear equations.

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Competing Interests

The authors have no relevant financial or non-financial interests to disclose.

Disclosure of potential conflicts of interest

The authors declare that they have no conflict of interest.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

CRediT authorship contribution statement

Rafael Martínez-Guerra: Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing. **Juan Pablo Flores-Flores:** Conceptualization, Data curation, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

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