Nonlinear Feedback Design for Fixed-Time Stabilization of Linear Plants

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Outline

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$$\dot{x}(t) = -x^{1/3} - x^3, \quad x(0) = x_0$$
$$x \in \mathbb{R}, t > 0, \; T_{\max} = 2.5$$

Polynomial Feedbacks

I. Robustness (Pervozvanski 1971)

$$\dot{x} = \lambda x + u$$

where $x \in \mathbb{R}$ - state, the number $\lambda \in \mathbb{R}$ is **unknown**, $u \in \mathbb{R}$ - control.

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II. Real-life applications for automobile engine control in GMC

(Nikiforov et all, 2011, ACC)

$$u = -\alpha x - \beta \,\text{sign}[x] - \gamma x^3$$

Historical remarks

- **Finite-time stability**
  (Roxin 1966, Haimo 1986, Bhat & Bernstein 2000);
- **Polynomial feedbacks**
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- **Uniform 2SM-observer**
  (Cruz, Moreno & Fridman 2010).

System Description and Basic Assumptions

$$\dot{x} = Ax + Bu(x) + f(t,x) \tag{1}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $u \in \mathbb{R}^m$,the function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ describes uncertainties and disturbances.

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Assumption 1

The pair (A, B) is controllable, i.e.

$$\text{rank}[B, AB, A^2B, ..., A^{n-1}B] = n$$

Assumption 2

The uncertain function $f(t,x)$ satisfies the matching condition, i.e.

$$f(t,x) = B\gamma(t,x)$$

where $\gamma(t,x))$ is an unknown function. The function $\gamma(t,x)$ is assumed to be bounded by $\gamma_0(t,x) \geq 0$, i.e.

$$\|\gamma(t,x)\|_\infty \leq \gamma_0(t,x) \text{ for } \forall t > 0 \text{ and } \forall x \in \mathbb{R}^n$$

Definitions of stability

Definition 1

The equilibrium point $x = 0$ of the closed-loop system (1) is said to be globally **finite-time** stable if it is globally asymptotically stable and any solution $x(t, x_0)$ of (1) achieves the equilibria at some finite time moment and

$$x(t, x_0) = 0 \text{ for } \forall t \geq T(x_0)$$

where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is the so-called settling-time function.

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Definition 2

The equilibrium point $x = 0$ of the closed-loop system (1) is said to be globally **fixed-time** stable if it is globally finite-time stable and the settling-time function $T(x_0)$ is bounded by some positive number $T_{\max} > 0$, i.e. $T(x_0) \leq T_{\max}$ for $\forall x_0 \in \mathbb{R}^n$.

Definitions of attractivity

Definition 3

The set M is said to be globally **finite-time** attractive for the system (1) if any solution $x(t, x_0)$ of (1) reaches M in some finite time moment and

$$x(t, x_0) \in M \text{ for all } t \geq T(x_0)$$

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The set M is said to be globally **fixed-time** attractive for the system (1) if it is globally *finite-time* attractive and the settling-time function $T(x_0)$ is bounded by some positive number $T_{\max} > 0$, i.e.

$$T(x_0) \leq T_{\max} \text{ for } \forall x_0 \in \mathbb{R}^n$$

Problem Statement

Denote a closed ball of radius $r > 0$ with the center in the origin by B_r, i.e. $B_r := \{x \in \mathbb{R}^n : \|x\|_\infty \leq r\}$, $\|x\|_\infty := \max_{i=1,2,\ldots,n} |x_i|$.

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For given $T_{max} > 0$ and $r > 0$ we need to design a feedback control $u = u(x)$ for the system (1), which provides a fixed-time attractivity property of the ball B_r with the settling-time estimate $T_{\max}$.

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Problem 2

For given $T_{max} > 0$ we need to design a feedback control $u = u(x)$, which guarantees fixed-time stability of the origin of the closed-loop system (1) with the settling-time estimate $T_{\max}$.

Block-Controlability Form (Drakunov et all 1990)

$$\exists G \in \mathbb{R}^{n\times n} : \; y{=}Gx \text{ and } \begin{cases} \dot{y}_1 = A_{11}y_1 + A_{12}y_2 \\ \dot{y}_2 = A_{21}y_1 + A_{22}y_2 + A_{23}y_3 \\ \ldots \\ \dot{y}_k = A_{k1}y_1 + \ldots + A_{kk}y_k + A_{kk+1}(u+\gamma) \end{cases}$$

where $y{=}(y_1^T, \ldots, y_k^T)^T$, $y_i \in \mathbb{R}^{n_i}$, $[0 \quad A_{kk+1}^T]^T = GB$ and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ are blocks of the matrix GAG^T such that $\text{rank}(A_{ii+1}){=}n_i$

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Algorithm

Initialization. $A_0 = A$, $B_0 = B$, $T_0 = I_n$, $k = 0$

Recursion. While $\mathrm{rank}(B_k) < \mathrm{rown}(A_k)$ do

$$A_{k+1} = B_k^{\perp} A_k \left(B_k^{\perp}\right)^T, B_{k+1} = B_k^{\perp} A_k \tilde{B}_k, T_{k+1} = \begin{pmatrix} B_k^{\perp} \\ \tilde{B}_k \end{pmatrix}, k = k+1$$

where $B_k^{\perp} := \left(\mathrm{null}(B_k^T)\right)^T$, $\tilde{B}_k := \left(\mathrm{null}\left(B_k^{\perp}\right)\right)^T$, rown - number of rows

$$G{=}\begin{pmatrix} T_k & 0 \\ 0 & I_{w_k} \end{pmatrix}\begin{pmatrix} T_{r-1} & 0 \\ 0 & I_{w_{k-1}} \end{pmatrix}\dots\begin{pmatrix} T_2 & 0 \\ 0 & I_{w_2} \end{pmatrix} T_1, \quad w_i{=}n\text{-}\,\mathrm{rown}(T_i)$$

Fixed-time Attractivity: Coordinate Transformation

Denote $\rho^{[p]}=|\rho|^p \,\mathrm{sign}[\rho]$ for $\rho \in \mathbb{R}$ and $z^{[p]}=(z_1^{[p]},\ldots,z_k^{[p]})^T$ for $z \in \mathbb{R}^k$.

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$$\begin{cases} \dot{s}_1 = -\alpha_1 s_1 - \beta_1 s_1^{[3]} + A_{12} s_2 \\ \dot{s}_2 = -\alpha_2 s_2 - \beta_2 s_2^{[3]} + A_{23} s_3 \\ .... \\ \dot{s}_k = \xi(y_1, ..., y_k) + A_{k\ k+1}(u+\gamma) \end{cases} \tag{2}$$

$$s = \Phi(y), s = (s_1^T, ..., s_k^T)^T, s_i \in \mathbb{R}^{n_i}, y = (y_1^T, ..., y_k^T)^T, y_i \in \mathbb{R}^{n_i}$$

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$$s_i = y_i + \varphi_i(y_1, \ldots, y_i), \quad i = 1, 2, \ldots, k$$

$$\varphi_{i+1} = A_{i\ i+1}^{+}\left(\alpha_i(y_i+\varphi_i)+\beta_i(y_i+\varphi_i)^{[3]}+\sum_{j=1}^{i} A_{ij} y_j+\sum_{r=1}^{i} \frac{\partial \varphi_i}{\partial y_r} \sum_{j=1}^{r+1} A_{rj} y_j\right)$$

where $\varphi_1 = 0$, $\alpha_i > 0$, $\beta_i > 0$ and $A_{i\ i+1}^{+} = A_{i\ i+1}^{T}(A_{i\ i+1}A_{i\ i+1}^{T})^{-1}$.

$$\xi(y_1, \ldots, y_k) = \sum_{i=1}^{k} A_{ki} y_i + \sum_{i=1}^{k-1} \frac{\partial \varphi_{k-1}}{\partial y_i} \sum_{j=1}^{i+1} A_{ij} y_j$$

Fixed-time Attractivity: Feedback Design

$$u(y,s) = A_{k\,k+1}^{+}\left(-\alpha_k s_k - \beta_k s_k^{[3]} - \xi(y_1,\ldots,y_k)\right) \tag{3}$$

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$$u(y,s) = A^{+}_{k\ k+1}\left(-\alpha_k s_k - \beta_k s_k^{[3]} - \xi(y_1, ..., y_k)\right) \quad (3)$$

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Theorem (1)

If $u(Gx, \Phi(Gx))$ is the control of the form (3) with parameters

$$\alpha_i = \|A_{i\ i+1}\|_\infty, i = \overline{1, k-1}, \quad \alpha_k = \frac{\gamma_0 \|A_{k\ k+1}\|_\infty \|G\|_1 C_1}{r}$$

$$\beta_i = \frac{\|G\|_1^2 C_2}{2T_{\max} r^2}, i = \overline{1,k}$$

where $r > 0$ and $T_{\max} > 0$ are arbitrary positive numbers, then the ball B_r is the globally fixed-time attractive set of the closed-loop system (1) with the settling-time function bounded by $T_{\max}$.

Fixed-Time Stability: Sliding Mode Approach

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = u, \quad y_1, y_2 \in \mathbb{R}$$

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Nested Second Order Sliding Mode Controller(*Levant 1993*)

$$u = -\alpha \,\mathrm{sign}[y_2 + \beta y_1^{[1/2]}], \quad \alpha > 0, \quad \beta^2 \geq 2\alpha$$

If $s_1 := y_2 + \beta y_1^{[1/2]} = 0$ is the sliding surface then $y_2 = -\beta y_1^{[1/2]}$ and

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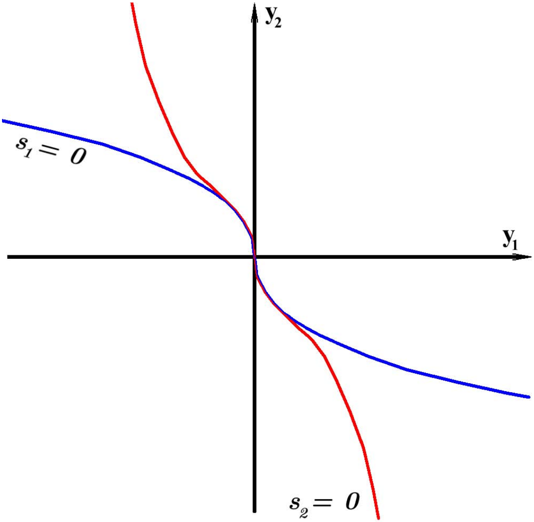
Fixed-time sliding surface

$$s_2 := y_2 + \left(y_2^{[2]} + \alpha_1 y_1 + \alpha_2 y_1^{[3]}\right)^{[1/2]} = 0$$

If $s_2 = 0$ is the sliding surface then $y_2 = -(0.5\alpha_1 y_1 + 0.5\beta_1 y_1^{[3]})^{[1/2]}$ and

$$\dot{y}_1 = -(0.5\alpha_1 y_1 + 0.5\beta_1 y_1^{[3]})^{[1/2]}$$

Fixed-Time Stability: Geometrical Interpretation



$$s_1 = y_2 + \beta y_1^{[1/2]} \text{ and } s_2 = y_2 + \left(y_2^{[2]} + \alpha_1 y_1 + \beta_1 y_1^{[3]}\right)^{[1/2]}$$

Fixed-Time Stability: Theorem

$$\begin{cases} \dot{y}_1 = A_{11}y_1 + A_{12}y_2 \\ \dot{y}_2 = A_{21}y_1 + A_{22}y_2 + A_{23}(u+\gamma) \end{cases} \tag{4}$$

$y_1 \in \mathbb{R}^{n_1}, y_2 \in \mathbb{R}^{n_2}$,$n_2$=rank$(B)$,$n_1$=$n$-$n_2$,rank$(A_{12})$=$n_1 \leq$ rank(A_{23})=n_2.

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Theorem (2)

The origin of the closed-loop system (4) *is globally fixed-time stable with the settling-time estimate* $T_{\max} > 0$ *if the control* $u(y_1, y_2)$ *has the form*

$$u(y_1, y_2) = -A_{23}^{+}\{u_{eq}(y_1, y_2) + u_d(y_1, y_2) + u_p(y_1, y_2)\} \quad (5)$$

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The origin of the closed-loop system (4) *is globally fixed-time stable with the settling-time estimate* $T_{\max} > 0$ *if the control* $u(y_1, y_2)$ *has the form*

$$u(y_1, y_2) = -A_{23}^{+}\left\{u_{eq}(y_1, y_2) + u_d(y_1, y_2) + u_p(y_1, y_2)\right\} \qquad (5)$$

$u_{eq}=A_{12}^{+}((A_{11}^2+A_{12}A_{21})y_1+(A_{11}A_{12}+A_{12}A_{22})y_2)+{A_{12}^{\perp}}^{+}A_{12}^{\perp}(A_{21}y_1+A_{22}y_2)$,
$u_d = \frac{\alpha_1+3\beta_1\|y_1\|_\infty^2+2\|A_{12}A_{23}\|_\infty\gamma_0}{2}A_{12}^{+}\,\text{sign}[s_1]+\gamma_0\|A_{12}^{\perp}\|_\infty {A_{12}^{\perp}}^{+}\,\text{sign}[s_2]$,
$u_p = A_{12}^{+}(\alpha_2 s_1 + \beta_2 s_1^{[3]})^{[\frac{1}{2}]} + {A_{12}^{\perp}}^{+}(\alpha_2 s_2 + \beta_2 s_2^{[3]})^{[\frac{1}{2}]}$, $A_{12}^{\perp} = \text{null}(A_{12})^T$
$s_1=A_{11}y_1+A_{12}y_2+\left((A_{11}y_1+A_{12}y_2)^{[2]}+\alpha_1 y_1+\beta_1 y_1^{[3]}\right)^{[\frac{1}{2}]}$, $s_2=A_{12}^{\perp}y_2$
with $0.5\alpha_1=\alpha_2=0.5\beta_1=\beta_2=64\,T_{\max}^{-2}$.

Benchmark Example

$$\dot{x} = \begin{pmatrix} 1 & -3 & 2 \\ -2 & 0 & 3 \\ 0 & -1 & 4 \end{pmatrix} x + \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & -3 \end{pmatrix} u + \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} \sin(t)$$

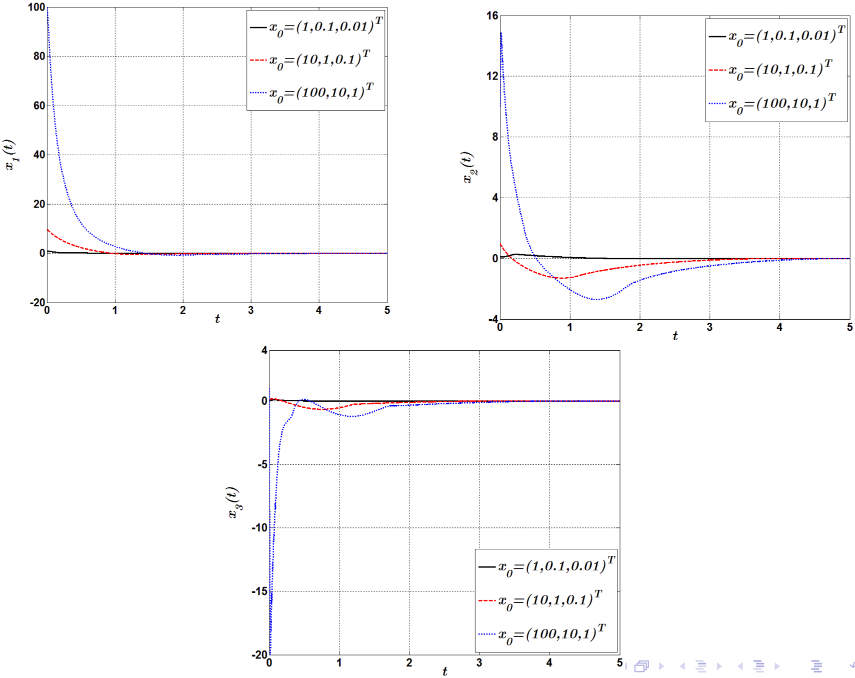
$$y = Gx,\ G = \begin{pmatrix} 0.4286 & 0.8571 & 0.2857 \\ -0.8571 & 0.4857 & -0.1714 \\ -0.2857 & -0.1714 & 0.9429 \end{pmatrix}$$

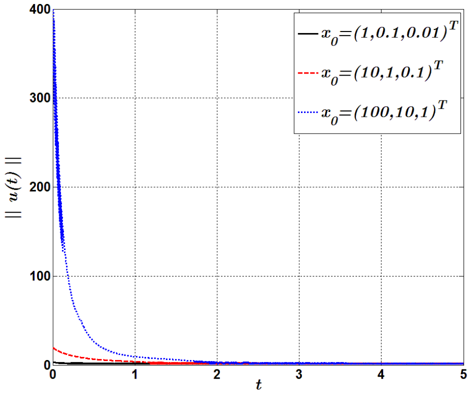
$$\begin{cases} \dot{y}_1 = A_{11}y_1 + A_{12}y_2 \\ \dot{y}_2 = A_{21}y_1 + A_{22}y_2 + A_{23}(u + (1\ 1)^T \sin(t)) \end{cases}$$

$$A_{11} = -0.5918, A_{12} = \begin{pmatrix} -0.4449 & 4.9469 \end{pmatrix}, A_{21} = \begin{pmatrix} 1.2980 & 0.7184 \end{pmatrix}^T,$$

$$A_{22} = \begin{pmatrix} 3.0612 & -0.8367 \\ -0.5510 & 2.5306 \end{pmatrix},\ A_{23} = \begin{pmatrix} -2.200 & 1.000 \\ -0.400 & -3.000 \end{pmatrix}$$

$$\alpha_1 = \beta_1 = 1, \alpha_2 = \beta_2 = 0.5 \quad \Rightarrow \quad T_{max} = 8$$


$x_0=(1,0.1,0.01)^T$
$x_0=(10,1,0.1)^T$
$x_0=(100,10,1)^T$
$x_1(t)$
$x_2(t)$
$x_3(t)$
t


400
300
200
100
0
0
1
2
3
4
5
t
|| u(t) ||
x0=(1,0.1,0.01)T
x0=(10,1,0.1)T
x0=(100,10,1)T

Thank you for your attention